J. Math. Soc. Japan
Vol. 36, No. 2, 1984

Invariant spherical distributions and the Fourier inversion formula on $GL(n, \boldsymbol{C})/GL(n, \boldsymbol{R})$

By Shigeru SANO

(Received Feb. 23, 1983)

Introduction.

The purpose of this paper is to study the invariant spherical distributions on the reductive symmetric space $M \cong GL(n, \boldsymbol{C})/GL(n, \boldsymbol{R})$, and prove the Fourier inversion formula (the expansion of the δ-function by invariant spherical distributions on M) by using the same method as in the case of semisimple Lie groups, in particular by using the gap relations.

In more detail, the content of this paper is as follows. In the first place, we give the Weyl integral formula for general semi-simple symmetric spaces in §1 and §2. From §3, we restrict ourselves to the space $GL(n, \boldsymbol{C})/GL(n, \boldsymbol{R})$. We define Harish-Chandra transforms on $\mathfrak{gl}(n, \boldsymbol{C})/\mathfrak{gl}(n, \boldsymbol{R})$ and on $M \cong GL(n, \boldsymbol{C})/GL(n, \boldsymbol{R})$ and prove the gap relations of these transforms in §3 and §4. Let A be a global Cartan subspace of M. Put $\mathfrak{g} = \mathfrak{gl}(n, \boldsymbol{C})$. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and $\mathfrak{Z}$ be the center of $U(\mathfrak{g})$. The radial component of any element $D \in \mathfrak{Z}$ on A under the transformation of $GL(n, \boldsymbol{R})$ is given in Proposition 5.3. After establishing this proposition, we define invariant spherical distributions on M.

The Fourier inversion formula for semi-simple Lie groups G can be regarded as the expansion of δ-function on G by characters of irreducible unitary representations of G. These characters are invariant eigendistributions on G. Hence the inversion formula is the expansion of δ-function by invariant eigendistributions on G. We will discuss harmonic analysis on M from this point of view.

Let M' be the set of all $\mathfrak{q}$-regular elements in M (cf. §1). The conditions that an analytic function on M' can be extended to an invariant spherical distribution on M are given in §6. And in §7 we construct the tempered invariant spherical distributions which contribute to the inversion formula for M. It is remarkable that these invariant spherical distributions are fairly different from the characters of representations of semi-simple Lie groups. In §8, we give the

This research was partially supported by Grant-in-Aid for Scientific Research (No. 56340004), Ministry of Education.

expansion of δ-function on M by the invariant spherical distributions (Theorem 8.7).

The author expresses his hearty thanks to Prof. M. Sugiura, Prof. T. Hirai and Prof. T. Oshima for their kind discussion, and to Prof. S. Maruyama and Prof. M. Okuzumi for their constant encouragement.

§1. Cartan subspaces for semi-simple symmetric spaces.

Let G be a connected semi-simple Lie group, and σ be an involutive automorphism of G. Put $G_\sigma=\{x\in G;\ \sigma(x)=x\}$. Let φ be the mapping of G into G defined by $\varphi(x)=x\sigma(x)^{-1}$ for $x\in G$. Then G/G_σ and $\varphi(G)$ are diffeomorphic by this mapping. Put $M=\varphi(G)$, $H=G_\sigma$. The pair (G, H) is called a semi-simple symmetric pair.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H respectively. The automorphism of $\mathfrak{g}$ induced by the automorphism σ of G is denoted by the same letter σ. Put $\mathfrak{q}=\{X\in\mathfrak{g};\ \sigma(X)=-X\}$. Then $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$ (direct sum). For a subspace $\mathfrak{a}$ of $\mathfrak{g}$, its complexification is denoted by $\mathfrak{a}_c$. For a subset $\mathfrak{a}$ of $\mathfrak{g}$ and a subset A of G, let $N_A(\mathfrak{a})$ be the normalizer of $\mathfrak{a}$ in A, and $Z_A(\mathfrak{a})$ the centralizer of $\mathfrak{a}$ in A.

Definition 1.1. A subspace $\mathfrak{a}\subset\mathfrak{q}$ is called a *Cartan subspace* if the following two conditions are satisfied:

(i) $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{q}$;

(ii) For each $Y\in\mathfrak{a}$, $\mathrm{ad}(Y)$ is a semi-simple endomorphism of $\mathfrak{g}$.

For a Cartan subspace $\mathfrak{a}$ of $\mathfrak{q}$, define the Weyl group $W(\mathfrak{a};H)=N_H(\mathfrak{a})/Z_H(\mathfrak{a})$. And $Z_M(\mathfrak{a})$ is called a global Cartan subspace of M.

For every $X\in\mathfrak{q}$ consider the characteristic polynomial

$$\det(t-\mathrm{ad}(X))=\sum_{i=0}^{n} d_i(X)t^i \tag{1.1}$$

of the endomorphism $\mathrm{ad}(X)$ of $\mathfrak{g}$ where t is an indeterminate, $n=\dim\mathfrak{g}$ and the d_i's are polynomial functions on $\mathfrak{q}$. Let l be the least integer such that $d_l\not\equiv 0$.

Definition 1.2. An element $X\in\mathfrak{q}$ is said to be $\mathfrak{q}$-*regular* if $d_l(X)\neq 0$. Let $\mathfrak{b}$ be any subspace of $\mathfrak{q}$. The set of all $\mathfrak{q}$-regular elements in $\mathfrak{b}$ is denoted by $\mathfrak{b}'$.

For every $x\in M$, put

$$\det(t+1-\mathrm{Ad}(x))=\sum_{i=0}^{n} D_i(x)t^i. \tag{1.2}$$

Then D_i are analytic functions on M and $D_l\not\equiv 0$.

Definition 1.3. An element $x\in M$ is said to be $\mathfrak{q}$-*regular* if $D_l(x)\neq 0$. Let

B be any subspace of M. The set of all $\mathfrak{q}$-regular elements in B is denoted by B'.

The following proposition is due to Oshima and Matsuki [**19**].

PROPOSITION 1.1. *Let (G, H) be a semi-simple symmetric pair and $(\mathfrak{g}, \mathfrak{h})$ be the corresponding semi-simple Lie algebras. Let $\{\mathfrak{a}_i\,;\, i\in I\}$ be a maximal set of Cartan subspaces which are not conjugate to each other under H-conjugations.*

(i) *Then we get*

$$\mathfrak{q}'=\bigcup_{i\in I}\bigcup_{y\in H}\mathrm{Ad}(y)\mathfrak{a}_i' \qquad (\textit{disjoint union}).$$

And the mapping $\zeta_i : H/Z_H(\mathfrak{a}_i)\times\mathfrak{a}_i'\to\mathfrak{q}'$ defined by $\zeta_i(y^*, X)=\mathrm{Ad}(y)X$ $(y^*=yZ_H(\mathfrak{a}_i)$, $y\in H$, $X\in\mathfrak{a}_i')$ is an everywhere regular $|W(\mathfrak{a}_i\,;H)|$-fold covering mapping.

(ii) *Put $A_i=Z_M(\mathfrak{a}_i)$, $W(A_i\,;H)=N_H(A_i)/Z_H(A_i)$. Set $M_i'=\bigcup_{y\in H} yA_i'y^{-1}$. Then $M'=\bigcup_{i\in I} M_i'$ (disjoint union). And the mapping $\eta_i : H/Z_H(A_i)\times A_i'\to M'$ defined by $\eta_i(y^*, a)=yay^{-1}$ $(y\in H,\ a\in A_i')$ is an everywhere regular $|W(A_i\,;H)|$-fold covering mapping.*

§ 2. The Weyl integral formula for semi-simple symmetric spaces.

We regard an element X in $\mathfrak{g}$ as a tangent vector of $T_e(G)$ (e identity element of G) by the following formula:

$$Xf=\frac{d}{dt}\Big|_{t=0}f(\exp tX) \qquad f\in C^\infty(G)\,. \tag{2.1}$$

Similarly, we regard $X\in\mathfrak{q}$ as a tangent vector $\tilde{X}$ of $T_e(M)$ as follows:

$$\tilde{X}f=\frac{d}{dt}\Big|_{t=0}f\Big(\exp\frac{tX}{2}\,\sigma\Big(\exp\frac{tX}{2}\Big)^{-1}\Big) \qquad f\in C^\infty(M)\,. \tag{2.2}$$

Define differentiable mappings l_x $(x\in G)$ from G onto G by $l_x(y)=xy$ $(y\in G)$. Define differentiable mappings B_y $(y\in G)$ from M onto M by $B_y(x)=yx\sigma(y)^{-1}$ $(x\in M)$.

Let $\mathfrak{a}$ be a Cartan subspace of $\mathfrak{q}$. Put $\mathfrak{m}=Z_{\mathfrak{h}}(\mathfrak{a})$. For $\lambda\in(\mathfrak{a})_c^*$, define $\mathfrak{g}_c(\mathfrak{a}\,;\lambda)=\{X\in\mathfrak{g}_c\,;\,\mathrm{ad}(Y)X=\lambda(Y)X$ for any $Y\in\mathfrak{a}\}$ and $\Sigma(\mathfrak{a})=\{\lambda\in(\mathfrak{a})_c^*-(0)\,;\,\mathfrak{g}_c(\mathfrak{a}\,;\lambda)\neq(0)\}$. Put $\mathfrak{n}_c=\sum_{\lambda\in\Sigma(\mathfrak{a})}\mathfrak{g}_c(\mathfrak{a}\,;\lambda)$. Then we have $\mathfrak{g}_c=\mathfrak{m}_c+\mathfrak{a}_c+\mathfrak{n}_c$.

Let $X_1, X_2, \cdots, X_n$ be elements of $\mathfrak{n}_c$ satisfying the following three conditions.

(i) For any j $(1\leqq j\leqq n)$, there exists $\lambda(j)\in\Sigma(\mathfrak{a})$ such that $X_j\in\mathfrak{g}_c(\mathfrak{a}\,;\lambda(j))$.

(ii) $\{X_1+\sigma(X_1), X_2+\sigma(X_2), \cdots, X_n+\sigma(X_n)\}$ is a base of $\mathfrak{h}_c\cap\mathfrak{n}_c$.

(iii) $\{X_1-\sigma(X_1), X_2-\sigma(X_2), \cdots, X_n-\sigma(X_n)\}$ is a base of $\mathfrak{q}_c\cap\mathfrak{n}_c$.

Let $\gamma : \mathfrak{h}_c\cap\mathfrak{n}_c\to\mathfrak{q}_c\cap\mathfrak{n}_c$ be the bijective linear mapping determined by

$$\gamma(X_j+\sigma(X_j))=X_j-\sigma(X_j) \qquad (1\leqq j\leqq n)\,. \tag{2.3}$$

We fix a $B_y(y\in G)$-invariant measure dx on M. The corresponding B_y-invariant differential form is denoted by ω. Let $A_{\mathfrak{a}}=Z_M(\mathfrak{a})$ and $M_{\mathfrak{a}}=\bigcup_{y\in H} yA_{\mathfrak{a}}y^{-1}$. Let dy^* be an H-invariant measure on $H/Z_H(A_{\mathfrak{a}})$. The corresponding differential form is denoted by ν. Let da be a $B_{\exp X}(X\in\mathfrak{a})$-invariant measure on $A_{\mathfrak{a}}$ and μ the corresponding differential form.

Let $\{X_1, \cdots, X_r\}$ be a base of $\mathfrak{h}_c \pmod{\mathfrak{m}_c}$. And take $Y_1, \cdots, Y_m\ (\in\mathfrak{a})$ such that $\{\tilde{Y}_1, \cdots, \tilde{Y}_m\}$ is a base of $T_e(A_{\mathfrak{a}})$. We assume that ω, μ and ν satisfy the following conditions:

$$\omega_e(\widetilde{\gamma(X_1)}, \cdots, \widetilde{\gamma(X_r)}, \tilde{Y}_1, \cdots, \tilde{Y}_m)=\nu_{e^*}(X_1^*, \cdots, X_r^*)\,\mu_e(\tilde{Y}_1, \cdots, \tilde{Y}_m).$$

PROPOSITION 2.1. *There exists p $(=0, 1)$ satisfying the equality, for any $f\in C_c(M_{\mathfrak{a}})$*

$$(2.4)\quad |W(A_{\mathfrak{a}}; H)|\int_{M_{\mathfrak{a}}} f(x)dx=\int_{H/Z_H(A_{\mathfrak{a}})}\int_{A_{\mathfrak{a}}} f(yay^{-1})(\sqrt{-1})^p|D_l(a)|^{1/2}da\,dy^*.$$

PROOF. $M'_{\mathfrak{a}}$ is an open dense subset of $M_{\mathfrak{a}}$ and differs from $M_{\mathfrak{a}}$ up to a set of measure 0.

Let η be the mapping $\eta: H/Z_H(A_{\mathfrak{a}})\times A'_{\mathfrak{a}}\to M'_{\mathfrak{a}}$ defined by $\eta(y^*, a)=yay^{-1}$ $(y\in H,\ a\in A'_{\mathfrak{a}})$. Then for $f\in C^\infty(M)$

$$\begin{aligned}
d\eta_{(y^*,a)}(dl_yX, 0)f&=(dl_yX, 0)_{(y^*,a)}f\circ\eta\\
&=\frac{d}{dt}\Big|_{t=0} f\circ\eta(y\exp tX, a)\\
&=\frac{d}{dt}\Big|_{t=0} f(yx\exp t\,\mathrm{Ad}(x^{-1})X\sigma(yx\exp t\,\mathrm{Ad}(x^{-1})X)^{-1})\\
&\qquad (a=x\sigma(x)^{-1}\in A_{\mathfrak{a}},\ x\in G,\ y\in H),
\end{aligned}$$

$$\begin{aligned}
\mathrm{Ad}(x^{-1})X&=\frac{1}{2}\{\mathrm{Ad}(x^{-1})+\mathrm{Ad}(\sigma(x)^{-1})\}X\\
&\quad+\frac{1}{2}\{\mathrm{Ad}(x^{-1})-\mathrm{Ad}(\sigma(x)^{-1})\}X.
\end{aligned}$$

Since $(1/2)\{\mathrm{Ad}(x^{-1})+\mathrm{Ad}(\sigma(x)^{-1})\}X$ is in $\mathfrak{h}_c$, we can omit this term. Let $\mathfrak{j}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$ and put $J=Z_G(\mathfrak{j})$. Let α be a root of $\mathfrak{g}$ with respect to $\mathfrak{j}$ satisfying $\alpha|_{\mathfrak{a}}=\lambda$ and X_α be a root vector in $\mathfrak{g}_c$. Define $\xi_\alpha(x)$ by

$$\mathrm{Ad}(x)X_\alpha=\xi_\alpha(x)X_\alpha \qquad (x\in J).$$

For any $x=\sigma(x)^{-1}\in J$, the value $\xi_\alpha(x)$ does not depend on the choice of α and X_α, and then it is denoted by $\xi_\lambda(x)$. We take $x\in J$ satisfying $a=x\sigma(x)^{-1}$, $x=\sigma(x)^{-1}$. For $X=X_\lambda+\sigma(X_\lambda)\in\mathfrak{h}_c$ $(X_\lambda\in\mathfrak{g}_c(\mathfrak{a};\lambda))$, we have

$$\frac{1}{2}\{\mathrm{Ad}(x^{-1})-\mathrm{Ad}(\sigma(x)^{-1})\}(X_\lambda+\sigma(X_\lambda))$$

$$=\frac{1}{2}\{\mathrm{Ad}(x^{-1})-\mathrm{Ad}(\sigma(x)^{-1})\}X_\lambda+\frac{1}{2}\{\mathrm{Ad}(x^{-1})-\mathrm{Ad}(\sigma(x)^{-1})\}\sigma(X_\lambda)$$

$$=\{\xi_{-\lambda}(x)-\xi_\lambda(x)\}\frac{1}{2}(X_\lambda-\sigma(X_\lambda)).$$

We put $p\equiv{}^{\#}\{j\,;\,\lambda(j)(\mathfrak{a})\subset\sqrt{-1}\mathfrak{a},\ 1\leqq j\leqq n\}\pmod 2$.

On the other hand, we get

$$|D_l(a)|^{1/2}=\prod_{j=1}^{n}|\xi_{\lambda(j)}(a)-1|^{1/2}|\xi_{-\lambda(j)}(a)-1|^{1/2}$$

$$=\prod_{j=1}^{n}|\xi_{\lambda(j)}(x)-\xi_{-\lambda(j)}(x)|.$$

Q. E. D.

COROLLARY 2.2. *Let* $\{\mathfrak{a}_i\}_{i\in I}$ *be a maximal set of Cartan subspaces which are not conjugate to each other under H-conjugations. Put* $A_i=Z_M(\mathfrak{a}_i)$ $(i\in I)$. *Let* d_ia *be a* $B_{\exp X}(X\in\mathfrak{a}_i)$*-invariant measure on* A_i. *Let* d_iy^* *be an H-invariant measure on* $H/Z_H(A_i)$. *Then there exist positive constants* γ_i *and* p_i $(=0, 1)$ $(i\in I)$ *satisfying the equality*

$$\int_M f(x)dx=\sum_{i\in I}\gamma_i\int_{H/Z_H(A_i)}\int_{A_i}f(yay^{-1})(\sqrt{-1})^{p_i}|D_l(a)|^{1/2}d_ia\ d_iy^* \tag{2.5}$$

for any $f\in C_c(M)$.

PROOF. $M'=\bigcup_{i\in I}\bigcup_{y\in H}yA_i'y^{-1}$ is an open dense subset of M and differs from M up to a set of measure 0. We can apply Proposition 2.1 to each $M_i=\bigcup_{y\in H}yA_iy^{-1}$ and get the corollary. Q. E. D.

REMARK. (1) Let G be a connected real semi-simple Lie group with finite center. Let K be a maximal compact subgroup of G. If we take a Weyl base of $\mathfrak{g}$ for the base (2.3), then the Weyl integral formula (2.5) for the symmetric pair (G, K) gives the integral formula for the Cartan decomposition KAK of G.

(2) Let G be a reductive Lie group and let σ be an involutive automorphism of G. Proposition 2.1 and Corollary 2.2 can be applied to the reductive symmetric pair (G, G_σ) without essential change.

§ 3. Harish-Chandra transforms on $\mathfrak{gl}(n, \boldsymbol{C})/\mathfrak{gl}(n, \boldsymbol{R})$.

Put $G=GL(n, \boldsymbol{C})$. And let σ be the involutive automorphism of G defined by $\sigma(x)=\mathrm{conj}\,x$ $(x\in G)$. Put $H=\{x\in G\,;\,\sigma(x)=x\}$, then $H\doteqdot GL(n, \boldsymbol{R})$. Let $\mathfrak{g}$ and

$\mathfrak{h}$ be the Lie algebras of G and H respectively. We also denote the real automorphism of $\mathfrak{g}$ induced by the automorphism σ of G by the same letter σ. Put $\mathfrak{q}=\{X\in\mathfrak{g}\,;\,\sigma(X)=-X\}$, then $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$ (direct sum). Let θ be the Cartan involution of $\mathfrak{g}$ defined by $\theta(X)=-{}^t\bar{X}$ $(X\in\mathfrak{g})$ and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the corresponding Cartan decomposition of $\mathfrak{g}$, where $\mathfrak{k}=\mathfrak{o}(n)$. The following Cartan subspaces $\mathfrak{a}_k$ $\left(0\leqq k\leqq\left[\frac{n}{2}\right]\right)$ of $\mathfrak{q}$ form a maximal set of Cartan subspaces which are not conjugate to each other under adjoint actions of H. For each integer $k\in\left[0,\left[\frac{n}{2}\right]\right]$, let $\mathfrak{a}_k=\mathfrak{a}_k^+ +\mathfrak{a}_k^-$ $(\mathfrak{a}_k^+=\mathfrak{a}_k\cap\mathfrak{k},\ \mathfrak{a}_k^-=\mathfrak{a}_k\cap\mathfrak{p})$ and $\mathfrak{a}_k^+$ and $\mathfrak{a}_k^-$ be the subspaces of $\mathfrak{a}_k$ consisting of all matrices of the following form respectively:

$$\mathfrak{a}_k^+:\ D(i\varphi_1,\cdots,i\varphi_{n-2k},i\theta_k,i\theta_k,\cdots,i\theta_1,i\theta_1)\qquad(i=\sqrt{-1}),$$

$$\mathfrak{a}_k^-:\ \begin{pmatrix}0_{n-2k} & & & \\ & \alpha_2(t_k) & & \\ & & \ddots & \\ & & & \alpha_2(t_1)\end{pmatrix}\qquad\left(\alpha_2(t)=\begin{pmatrix}0 & it\\ -it & 0\end{pmatrix}\right)$$

where φ_p, θ_q, t_r are real numbers and the blank of the above matrix must be filled up by 0, and $D(a_1,a_2,\cdots,a_n)$ denotes the diagonal matrix with diagonal elements $a_1,a_2,\cdots,a_n$. Let $X\in\mathfrak{a}_k$ be the sum of the above two matrices, then its eigenvalues are $i\varphi_1,\cdots,i\varphi_{n-2k},i\theta_k+t_k,i\theta_k-t_k,\cdots,i\theta_1+t_1,i\theta_1-t_1$. Let us denote these eigenvalues by $\mu_1,\mu_2,\cdots,\mu_n$ according to the above order. Let $\alpha_{p,q}$ $(1\leqq p,\,q\leqq n,\,p\neq q)$ be the linear mapping from $\mathfrak{a}_k$ into $\boldsymbol{C}$ defined by $\alpha_{p,q}(X)=\mu_p-\mu_q$. The set $\Sigma(\mathfrak{a}_k)=\Sigma(\mathfrak{a}_k,\mathfrak{q})=\{\alpha_{p,q}\,;\,1\leqq p,\,q\leqq n,\,p\neq q\}$ is the root system of $(\mathfrak{a}_k,\mathfrak{q})$. Take $\Sigma^+(\mathfrak{a}_k)=\{\alpha_{p,q}\,;\,1\leqq p<q\leqq n\}$ as a positive root system. Let $B(\ ,\ )$ be the Killing form of $\mathfrak{g}$. For $\alpha\in\Sigma(\mathfrak{a}_k)$, there exists a unique element $H_\alpha\in\mathfrak{a}_k+i\mathfrak{a}_k$ satisfying $B(X,H_\alpha)=\alpha(X)$ for all $X\in\mathfrak{a}_k$. If V is a real Euclidean space of finite dimension, we denote by $\mathcal{S}(V)$ the Schwartz space of rapidly decreasing functions on V. For an element $X\in\mathfrak{a}_k'$, put

$$\pi^k(X)=\prod_{\alpha>0}\alpha(X),\qquad \varepsilon_F^k(X)=\prod_{1\leqq p<q\leqq n-2k}\mathrm{sgn}(\varphi_p-\varphi_q).$$

Put $B_k=Z_H(\mathfrak{a}_k)$, then B_k is a Cartan subgroup of H. Let d_ky^* $(y^*=yB_k)$ be an H-invariant measure on H/B_k. We define the Harish-Chandra transform for any $f\in\mathcal{S}(\mathfrak{q})$ by

$$\psi_f^k(X)=\varepsilon_F^k(X)\pi^k(X)\int_{H/B_k}f(y^*X)d_ky^*.\tag{3.1}$$

We fix an element X_0 of $\mathfrak{a}_{k+1}$ which satisfies $\alpha(X_0)\neq 0$ for all $\alpha\in\Sigma^+(\mathfrak{a}_{k+1})$ except $\alpha_0=\alpha_{n-2k-1,n-2k}$ and $\alpha_0(X_0)=0$. Put $\mathfrak{z}^-=Z_{\mathfrak{q}}(X_0)$ (the centralizer of X_0 in $\mathfrak{q}$) and $\mathfrak{z}^+=Z_{\mathfrak{h}}(X_0)$. Set $\mathfrak{l}_-=[\mathfrak{z}^-,\mathfrak{z}^+]$ and $\mathfrak{l}_+=[\mathfrak{z}^+,\mathfrak{z}^+]$. Let $i\sigma_-$ be the center of $\mathfrak{z}^+$, then $\mathfrak{z}^-=\mathfrak{l}_-+\sigma_-$ (direct sum). $\{\mathfrak{a}_k,\mathfrak{a}_{k+1}\}$ is a maximal set of Cartan subspaces of $\mathfrak{z}^-$ which are not conjugate to each other under the adjoint actions of $\mathfrak{z}^+$. Put $\mathfrak{a}=\mathfrak{a}_k$ and $\mathfrak{b}=\mathfrak{a}_{k+1}$. Let X', H', Y' be a base of $\mathfrak{l}_+$ defined by

$$X'=\begin{pmatrix} O_{n-2(k-1)} & & \\ & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \\ & & O_{2k} \end{pmatrix},\quad H'=\begin{pmatrix} O_{n-2(k-1)} & & \\ & \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \\ & & O_{2k} \end{pmatrix},\quad Y'=\begin{pmatrix} O_{n-2(k-1)} & & \\ & \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} & \\ & & O_{2k} \end{pmatrix}.$$

Let $\boldsymbol{C}\otimes_{\boldsymbol{R}}\mathfrak{g}$ be the tensor product of $\boldsymbol{C}$ and $\mathfrak{g}$. For a subspace $V\subset\mathfrak{g}$, put its complexification $V_c=\boldsymbol{C}\otimes_{\boldsymbol{R}}V$. Let ν be the linear map of $\mathfrak{b}_c$ onto $\mathfrak{a}_c$ given by $\nu(1\otimes i(X'-Y'))=i\otimes iH'$ and $\nu|_{\mathfrak{a}_c\cap\mathfrak{b}_c}=\text{identity}$. Put $'\sigma_-=\{X\in\sigma_-\,;\ \alpha(X)\neq 0$ for all $\alpha\in\Sigma(\mathfrak{a}_{k+1})$ except $\pm\alpha_0$ and $\alpha_0(X)=0\}$.

THEOREM 3.1. *If D belongs to* Diff$(\mathfrak{b}_c)$ (*the algebra of differential operators on $\mathfrak{b}_c$ with polynomial coefficients*), *then the limits*

$$(D\phi_f^{k+1})^{\pm}(C)=\lim_{t\to\pm 0}(D\phi_f^{k+1})(C-ti(Y'-X')) \tag{3.2}$$

*exist for all $f\in\mathcal{S}(\mathfrak{q})$, $C\in{}'\sigma_-$ and $(D\phi_f^{k+1})^{\pm}$ are continuous on each connected component of $'\sigma_-$. And there exists a positive constant s such that for any $D\in$*Diff$(\mathfrak{a}_c)$, *$f\in\mathcal{S}(\mathfrak{q})$ and $C\in{}'\sigma_-$*

$$(D\phi_f^{k+1})^{+}(C)-(D\phi_f^{k+1})^{-}(C)=s(D^{\nu}\phi_f^{k})(C) \tag{3.3}$$

where D^{ν} is the element of Diff$(\mathfrak{a}_c)$ *that corresponds to D under the isomorphism ν of $\mathfrak{b}_c$ with $\mathfrak{a}_c$.*

To prove this theorem, we reduce the problem to the one for the space $\mathfrak{z}^-$ of rank one. We shall prove theorem 3.1 after preparing some lemmas which are proved by the same method as for reductive Lie groups.

Set $Z=Z_H(X_0)$. We replace $(H,\mathfrak{q})$ by $(Z,\mathfrak{z}^-)$, and define Harish-Chandra transforms $\phi_{g,Z}^{k}$ and $\phi_{g,Z}^{k+1}$ for any function $g\in\mathcal{S}(\mathfrak{z}^-)$. We consider the mapping ϕ_- from $H/Z\times\mathfrak{z}^-$ into $\mathfrak{q}$ defined by $\phi_-(\bar{y},X)=yX$ $(y\in H,\ X\in\mathfrak{z}^-)$. For $\tau>0$, put $\mathfrak{l}_-(\tau)=\{X\in\mathfrak{l}_-\,;$ any eigenvalue λ of ad$X$ is $|\lambda|<\tau\}$. Let γ be an open neighborhood of X_0 in σ_-. Define for τ and γ, $V=V_{\gamma,\tau}=\gamma+\mathfrak{l}_-(\tau)$. If we take sufficiently small τ and γ, there exists a regularly imbedded analytic submanifold N of H including the unit element e which has the following properties:

(3.4)
- (i) Put $B=H\cdot X_0$. Then $N\cdot X_0$ is an open neighborhood of X_0 in B.
- (ii) The mapping $y\mapsto yX_0$ is an analytic diffeomorphism of N onto $N\cdot X_0$.
- (iii) The mapping $N\cdot X_0\times V\to\phi_-(N\times V)$ given by $(yX_0,Y)\mapsto yY$ is an analytic diffeomorphism.

For two functions $g\in C_c^{\infty}(V)$ and $h\in C_c^{\infty}(N\cdot X_0)$, put

$$f(y\cdot Y)=h(yX_0)g(Y)\qquad(y\in N,\ Y\in V). \tag{3.5}$$

Then f belongs to $C_c^\infty(\psi_-(N\times V))$. On the other hand $C_c^\infty(V)\otimes C_c^\infty(N\cdot X_0)$ is open dense in $C_c^\infty(\psi_-(N\times V))$. Then it is sufficient to prove Theorem 3.1 for such functions f. Put $\bar{N}=N/Z$ and $h(\bar{y})=h(yX_0)$. For $\Sigma^+(\mathfrak{a}, \mathfrak{z}^-)=\{\alpha_0\}$, put $\pi_Z^k=\alpha_0$ and $\varepsilon_{F,Z}^k(X)=\mathrm{sgn}(\varphi_{n-2k-1}-\varphi_{n-2k})$. For $\Sigma^+(\mathfrak{b}, \mathfrak{z}^-)=\{\beta_0=\alpha_0\nu\}$, put $\pi_Z^{k+1}=\beta_0$ and $\varepsilon_{F,Z}^{k+1}=1$. Put $\pi_{/Z}^p=\pi^p/\pi_Z^p$ ($p=k$ or $k+1$).

Lemma 3.2. *Let $d_\mathfrak{a}z^*$ (resp. $d_\mathfrak{b}z^*$) be an invariant measure on Z/B_k (resp. Z/B_{k+1}) and $d\bar{y}$ be an invariant measure on H/Z. The invariant measure $d_\mathfrak{a}y^*$ (resp. $d_\mathfrak{b}y^*$) on H/B_k (resp. H/B_{k+1}) is defined by $d_\mathfrak{a}y^*=d\bar{y}d_\mathfrak{a}z^*$ (resp. $d_\mathfrak{b}y^*=d\bar{y}d_\mathfrak{b}z^*$). For $g\in C_c^\infty(V)$ and $h\in C_c^\infty(N\cdot X_0)$, define $f\in C_c^\infty(\psi_-(\bar{N}\times V))$ by (3.5). Then we get for $p=k$ or $k+1$,*

$$\psi_f^p(X)=a(p)\,c[h]\,\pi_{/Z}^p\psi_{g,Z}^p(X) \qquad (X\in\mathfrak{a}_p'\cap V) \tag{3.6}$$

where $c[h]=\int_{H/Z}h(\bar{y})d\bar{y}$ and $a(p)$ is the constant 1 or -1.

We denote the adjoint group of $\mathfrak{z}^+$ by Z_0. Put $L_\mathfrak{a}=Z_{Z_0}(\mathfrak{a})$. We define an invariant integral for any $g\in\mathcal{S}(\mathfrak{z}^-)$ by

$$\psi_{g,Z_0}^k(X)=\varepsilon_{F,Z}^k(X)\pi_Z^k(X)\int_{Z_0/L_\mathfrak{a}}g(u^*X)du^*. \tag{3.7}$$

Put $L_\mathfrak{b}=Z_{Z_0}(\mathfrak{b})$. We define an invariant integral for any function $g\in\mathcal{S}(\mathfrak{z}^-)$,

$$\psi_{g,Z_0}^{k+1}(X)=\pi_Z^{k+1}(X)\int_{Z_0/L_\mathfrak{b}}g(u^*X)du^*. \tag{3.8}$$

Let $Z_0=K_0N_0A_0$ be the Iwasawa decomposition of Z_0, where $K_0=SO(2)$, $N_0=\{\exp sX'\,;\, s\in\boldsymbol{R}\}$ and $A_0=\{D(\varepsilon, \varepsilon)\exp tH'\,;\, \varepsilon=\pm1,\, t\in\boldsymbol{R}\}$. Let dk be the Haar measure of K_0 normalized by $\int_{K_0}dk=1$. For $n=\exp sX'\in N_0$, $dn=ds$ is a Euclidean measure on N. We define a Z_0-invariant measure $d_\mathfrak{a}z_0^*$ on $Z_0/L_\mathfrak{a}$ by the equality

$$\int_{Z_0/L_\mathfrak{a}}f(z_0^*)d_\mathfrak{a}z_0^*=\int_{K_0}\int_{-\infty}^{\infty}f((k\exp sX')^*)dk\,ds \tag{3.9}$$

for any $f\in C_c^\infty(Z_0/L_\mathfrak{a})$.

Let $Z_0=K_0A_0^+K_0$ be the Cartan decomposition Z_0 where $A_0^+=\{\exp tH'\,;\, t\geqq0\}$. We define a Z_0-invariant measure $d_\mathfrak{b}z_0^*$ on $Z_0/L_\mathfrak{b}$ such that for any $f\in C_c^\infty(Z_0/L_\mathfrak{b})$,

$$\int_{Z_0/L_\mathfrak{b}}f(z_0^*)d_\mathfrak{b}z_0^*=\int_{K_0}\int_0^\infty f((k\exp tH')^*)(e^{2t}-e^{-2t})dk\,dt. \tag{3.10}$$

Let η be the automorphism of $\mathfrak{z}^-$ such that $\eta|_{\sigma_-}=$identity and $\eta(iH)=iH$, $\eta(iX)=-iX$, $\eta(iY)=-iY$. Because $Z/B_k\cong Z_0/L_\mathfrak{a}$ and $Z/B_{k+1}\cong Z_0/L_\mathfrak{b}\cup\eta(Z_0)/L_\mathfrak{b}$, we can normalize invariant measures d_pz^* on Z/B_p ($p=k$, $k+1$) in such a manner that for all $g\in\mathcal{S}(\mathfrak{z}^{-})$

$$\psi_{g,Z}^{k}=\psi_{g,Z_0}^{k} \tag{3.11}$$

and

$$\psi_{g,Z}^{k+1}=\psi_{g,Z_0}^{k+1}-(\psi_{g,Z_0}^{k+1})^{\eta}\ . \tag{3.12}$$

LEMMA 3.3. *Let $d_{\mathfrak{a}}z_0^*$ (resp. $d_{\mathfrak{b}}z_0^*$) be the invariant measure on $Z_0/L_{\mathfrak{a}}$ (resp. $Z_0/L_{\mathfrak{b}}$) defined by* (3.9) (*resp.* (3.10)). *Put ψ_{g,Z_0}^{k} (resp. ψ_{g,Z_0}^{k+1}) as in* (3.7) (*resp.* (3.8)). *And we define $\psi_{g,Z}^{k}$ (resp. $\psi_{g,Z}^{k+1}$) by* (3.11) (*resp.* (3.12)). *Then we obtain the following properties.*

[$\mathfrak{a}$] (i) *For any $g\in C_c^{\infty}(\mathfrak{z}^{-})$, $\psi_{g,Z}^{k}$ extends to an element of $C_c^{\infty}(\mathfrak{a})$.*

(ii) *$\psi_{g,Z}^{k}$ is invariant under the reflection S_{α_0} in $\mathfrak{a}$ corresponding to α_0.*

(iii) *For any $E\in$ Diff$(\mathfrak{a}_c)$ satisfying $S_{\alpha_0}(E)=-E$, we get*

$$E\psi_{g,Z}^{k}=0 \qquad on\ \sigma_{-}\,.$$

[$\mathfrak{b}$] (i) *Put*

$$\mathfrak{b}^{\pm}=\{C\pm ti(Y'-X')\,;\,t>0,\ C\in\sigma_{-}\}\,.$$

Then $\mathfrak{b}'=\mathfrak{b}^{+}\cup\mathfrak{b}^{-}$. For $g\in C_c^{\infty}(\mathfrak{z}^{-})$, $\psi_{g,Z}^{k+1}$ is in $C^{\infty}(\mathfrak{b}')$.

(ii) *If D belongs to* Diff$(\mathfrak{b}_c)$, *then the restrictions of $(D\psi_{g,Z}^{k+1})$ on $\mathfrak{b}^{\pm}$ can be extended to continuous functions $(D\psi_{g,Z}^{k+1})^{\pm}$ on* Cl$(\mathfrak{b}^{\pm})$ *respectively. And we obtain*

$$(D\psi_{g,Z}^{k+1})^{+}(C)-(D\psi_{g,Z}^{k+1})^{-}(C)=-2i(D^{\nu}\psi_{g,Z}^{k})(C) \qquad (C\in{}'\sigma_{-})\,. \tag{3.13}$$

Especially, $D\psi_{g,Z}^{k+1}$ can be extended to a continuous function on $\mathfrak{b}$ if $S_{\beta_0}D=-D$.

Now, we prove Theorem 3.1. Let γ, τ be as in (3.4). We assume that γ is star-like at X_0. Put $V=V_{\gamma,\tau}$ as before. From the definition of $V_{\gamma,\tau}$, we see that

$$\mathfrak{a}\cap V=\gamma+\left\{\theta iH'\,;\,|\theta|<\frac{\tau}{2}\right\},\quad \mathfrak{b}\cap V=\gamma+\left\{ti(X'-Y')\,;\,|t|<\frac{\tau}{2}\right\}.$$

Recall that $\Sigma^{+}(\mathfrak{a},\mathfrak{z}^{-})=\{\alpha_0\}$, $\Sigma^{+}(\mathfrak{b},\mathfrak{z}^{-})=\{\beta_0\}$. If we take γ, τ sufficiently small, then there exist $a(k+1)$, $a(k)=\pm 1$ such that $\varepsilon_{F,Z}^{k}=a(k)\varepsilon_F^{k}$ on $\mathfrak{a}_k'\cap V$, $\varepsilon_{F,Z}^{k+1}=1=a(k+1)\varepsilon_F^{k+1}$ on $\mathfrak{a}_{k+1}'\cap V$ and $a(k+1)a(k)=1$. Let N be as in (3.4) and $f_{h,g}$ as in (3.5). Then we have

$$\psi_{f_{h,g}}^{k}(X)=a(k)c[h]\pi_{/Z}^{k}\psi_{g,Z}^{k}(X) \qquad X=C+\theta iH'\in\mathfrak{a}_k'\cap V\,,$$

$$\psi_{f_{h,g}}^{k+1}(X_1)=a(k+1)c[h]\pi_{/Z}^{k+1}\psi_{g,Z}^{k+1}(X_1) \qquad X_1=C+ti(X'-Y')\in\mathfrak{a}_{k+1}'\cap V$$

where $c[h]=\int_{H/Z}h(\dot{y})d\dot{y}$.

For $D\in$ Diff$(\mathfrak{b}_c)$, put $D_1=D\pi_{/Z}^{k+1}$. Then $D_1^{\nu}=D^{\nu}\pi_{/Z}^{k+1}$ and

$$(D^{\nu}\psi_f^k)(X)=a(k)c[h](D_1^{\nu}\psi_{g,Z}^k)(X) \qquad X=C+\theta iH',$$

$$(D\psi_f^{k+1})(X_1)=a(k+1)c[h](D_1\psi_{g,Z}^{k+1})(X_1) \qquad X_1=C+ti(X'-Y'),\quad C\in\gamma.$$

If we normalize measures as in Lemma 3.3, then we get for any $E\in\mathrm{Diff}(\mathfrak{b}_c)$

$$(E\psi_{g,Z}^{k+1})^+(C)-(E\psi_{g,Z}^{k+1})^-(C)=-2i(E^{\nu}\psi_{g,Z}^k)(C).$$

Hence we obtain the following equality

$$(D\psi_f^{k+1})^+(C)-(D\psi_f^{k+1})^-(C)=-2i(D^{\nu}\psi_f^k)(C) \qquad C\in{}'\sigma_-. \qquad \text{Q.E.D.}$$

§ 4. Harish-Chandra transforms on $GL(n, \boldsymbol{C})/GL(n, \boldsymbol{R})$.

Let φ be the mapping from G/H into G defined by $\varphi(xH)=x\sigma(x)^{-1}$. Put $M=\{x\sigma(x)^{-1}; x\in G\}$. Then the mapping φ is an analytic diffeomorphism of G/H onto M. Put $A_k=Z_M(\mathfrak{a}_k)$ $\left(0\leqq k\leqq\left[\frac{n}{2}\right]\right)$. Then $A_k=A_k^+A_k^-$ and A_k^+ and A_k^- are the subsets of M consisting of all matrices of the following form, respectively:

$$A_k^+:\ D(e^{i\varphi_1},\cdots,e^{i\varphi_{n-2k}},e^{i\theta_k},e^{i\theta_k},\cdots,e^{i\theta_1},e^{i\theta_1}),$$

$$(4.1)\qquad A_k^-:\ \begin{pmatrix}1_{n-2k} & & & \\ & v_2(t_k) & & \\ & & \ddots & \\ & & & v_2(t_1)\end{pmatrix} \qquad \left(v_2(t)=\begin{pmatrix}\cosh t & i\sinh t\\ -i\sinh t & \cosh t\end{pmatrix}\right).$$

$\left\{A_k; 0\leqq k\leqq\left[\frac{n}{2}\right]\right\}$ is a maximal set of global Cartan subspaces of M which are not conjugate to each other under adjoint actions of H. Let $a\in A_k$ be the product of the above two matrices, then its eigenvalues are $e^{i\varphi_1},\cdots,e^{i\varphi_{n-2k}},e^{z_k},e^{-\bar{z}_k},\cdots,e^{z_1},e^{-\bar{z}_1}$ $(z_p=i\theta_p+t_p)$. We take $(e^{\mu_1},e^{\mu_2},\cdots,e^{\mu_n})$ as its coordinates with its order. For each A_k $\left(0\leqq k\leqq\left[\frac{n}{2}\right]\right)$, we define the Weyl group $W(A_k;H)$ by $N_H(A_k)/Z_H(A_k)$. Then $W(A_k;H)$ contains the following transformations:

(i) all permutations of $e^{i\varphi_1},\cdots,e^{i\varphi_{n-2k}}$;

(ii) all permutations of k-pairs $(e^{z_1},e^{-\bar{z}_1}),\cdots,(e^{z_k},e^{-\bar{z}_k})$;

(iii) the permutations of z_p and $-\bar{z}_p$, i.e. t_p and $-t_p$ for any fixed p $(1\leqq p\leqq k)$.

LEMMA 4.1. *The Weyl group $W(A_k;H)$ is generated by the transformations of A_k listed above. And its order is given by*

$$|W(A_k;H)|=(n-2k)!\,k!\,2^k.$$

PROOF. Any transformation in the Weyl group keeps the set of eigenvalues stable and is one of the above three kinds of permutations. Q.E.D.

For a subset V of G, put $V'=\{x\in V\,;\text{ regular element}\}$. Then $(V\cap M)'=V'\cap M$. For an element a of A_k', we put

$$
\begin{aligned}
\Delta_k(a)&=\prod_{1\le p<q\le n}\{\exp(\mu_p-\mu_q)/2-\exp(\mu_q-\mu_p)/2\}, \\
\varepsilon_F^k(a)&=\prod_{1\le p<q\le n-2k}\operatorname{sgn}\{\sin(\varphi_p-\varphi_q)/2\}.
\end{aligned}
\tag{4.2}
$$

$\Delta_k(a)$ and $\varepsilon_F^k(a)$ are one-valued or two-valued functions on A_k according as n is odd or even. But $\varepsilon_F^k(a)\Delta_k(a)$ is a one-valued function on A_k. We define the Harish-Chandra transform for a function $f\in C_c^\infty(M)$ by

$$
F_f^k(a)=\varepsilon_F^k(a)\Delta_k(a)\int_{H/B_k}f(yay^{-1})d_ky^* \qquad (a\in A_k') \tag{4.3}
$$

where d_ky^* is an invariant measure on H/B_k.

We fix an element a_0 in A_{k+1} which satisfies $\xi_\alpha(a_0)\neq 1$ for all $\alpha\in\Sigma^+(\mathfrak{a}_{k+1})$ except α_0 and $\xi_{\alpha_0}(a_0)=1$, where each ξ_α is the homomorphism from $A_{k+1}^c=Z_G(\mathfrak{a}_{k+1})$ into $\boldsymbol{C}^*$ defined by $\xi_\alpha(a)=e^{\alpha(\log a)}$. Then we get $\mathfrak{z}^-=Z_{\mathfrak{q}}(a_0)$. For $\gamma\subset\sigma_-$ and $\tau>0$, we put $\mathfrak{z}^-(\tau)$ and $\mathfrak{z}_{\gamma,\tau}^-$ as in § 3. Set $Z_{\gamma,\tau}=\{\exp(X/2)a_0\sigma(\exp(X/2))^{-1}\,;\, X\in\mathfrak{z}_{\gamma,\tau}^-\}$ and $Z_M=Z_M(a_0)$. $\{A_k,\, A_{k+1}\}$ is a maximal set of global Cartan subspaces of Z_M which are not conjugate to each other under the adjoint actions of Z. Let $\mathfrak{u}$ be the subspace of $\mathfrak{g}$ defined by

$$
\mathfrak{u}=\left\{\begin{pmatrix}
0 & & & & & & & \\
 & 0 & & & & & * & \\
 & & \ddots & & & & & \\
 & & & 0 & & & & \\
 & & & & \begin{matrix}0&0\\0&0\end{matrix} & & & \\
 & & & & & \begin{matrix}a_k & ia_k\\ ia_k & -a_k\end{matrix} & & \\
 & 0 & & & & & \ddots & \\
 & & & & & & & \begin{matrix}a_1 & ia_1\\ ia_1 & -a_1\end{matrix}
\end{pmatrix};\ a_p\in\boldsymbol{R}\right\}. \tag{4.4}
$$

Then $\mathfrak{y}=\mathfrak{u}+\mathfrak{z}^-$ is a direct sum and $\mathfrak{u}$ is orthogonal to $\mathfrak{z}^-$ with respect to the positive Hermitian form $\langle X, Y\rangle=-B(X,\theta(Y))$ for $X, Y\in\mathfrak{g}$. For $x\in Z_M$, we define

$$
\nu_{a_0}(x)=\det(1-\operatorname{Ad}(x)^{-1})|_{\mathfrak{u}_c} \tag{4.5}
$$

and put $'Z_M=\{x\in Z_M\,;\,\nu_{a_0}(x)\neq 0\}$. If we take sufficiently small γ and τ, then the set $Z_{\gamma,\tau}$ has the following properties.

(4.6)
(i) $Z_{\gamma,\tau}\subset' Z_M$ is an open subset and the map $\mathfrak{z}^-_{\gamma,\tau}\to Z_{\gamma,\tau}$ given by $Y\mapsto \exp(Y/2)a_0\sigma(\exp(Y/2))^{-1}$ is an analytic diffeomorphism;
(ii) $Z_{\gamma,\tau}$ is Z-invariant. If $yZ_{\gamma,\tau}y^{-1}\cap Z_{\gamma,\tau}\neq\emptyset$ $(y\in H)$, then $y\in Z$ and $yZ_{\gamma,\tau}y^{-1}=Z_{\gamma,\tau}$;
(iii) If a subset Z_1 of $Z_{\gamma,\tau}$ is Z-invariant and closed in Z_M, then $\{yxy^{-1};\ y\in H,\ x\in Z_1\}$ is closed in M.

For $X\in\mathfrak{z}^-$, put

$$j(X)=\left|\det\left(\frac{\sinh(\operatorname{ad}(X/2))}{\operatorname{ad}(X/2)}\right)\Big|_{\mathfrak{z}_c^-}\right|^{1/2}. \tag{4.7}$$

LEMMA 4.2. *If we take sufficiently small γ, τ as in* (4.6), *then we obtain the following equalities*

$$\varepsilon_F^k(a)\Delta_k(a)=\varepsilon_F^k(a)\xi_{\rho-\lambda}(a)\nu_{a_0}(a)j(X)a_0(X)\quad (a=a_0\exp X,\ X\in\mathfrak{a}\cap\mathfrak{z}^-_{\gamma,\tau})$$

and

$$\varepsilon_F^{k+1}(b)\Delta_{k+1}(b)=\varepsilon_F^{k+1}(b)\xi_{\rho-\lambda}(b)\nu_{a_0}(b)j(Y)\beta_0(Y)\quad (b=a_0\exp Y,\ Y\in\mathfrak{b}\cap\mathfrak{z}^-_{\gamma,\tau})$$

where λ is the linear mapping from $\mathfrak{a}_p$ into $\boldsymbol{C}$ defined by $\lambda(X)=\mu_{n-2k-1}$ $(p=k,\ k+1)$.

PROOF. We get for $a\in A'_k$

$$\varepsilon_F^k(a)\xi_{\rho-\lambda}(a)\nu_{a_0}(a)=\frac{\varepsilon_F^k(a)\Delta_k(a)}{e^{i\varphi_{n-2k-1}}-e^{i\varphi_{n-2k}}},\quad j(X)=\frac{e^{i\varphi_{n-2k-1}}-e^{i\varphi_{n-2k}}}{i(\varphi_{n-2k-1}-\varphi_{n-2k})}$$

and for $b\in A'_{k+1}$

$$\varepsilon_F^{k+1}(b)\xi_{\rho-\lambda}(b)\nu_{a_0}(b)=\frac{\varepsilon_F^{k+1}(b)\Delta_{k+1}(b)}{e^{t_{k+1}}-e^{-t_{k+1}}},\quad j(Y)=\frac{e^{t_{k+1}}-e^{-t_{k+1}}}{2t_{k+1}}.$$

Then we obtain the assertion of the lemma. Q.E.D.

For the $Z_{\gamma,\tau}$ in (4.6), we can take a regularly imbedded analytic submanifold R of H which contains the unit element e as follows:

(4.8)
(i) $\{xa_0x^{-1};\ x\in R\}\subset\{xa_0x^{-1};\ x\in H\}$ open subset;
(ii) The map $R\to\{xa_0x^{-1};\ x\in R\}$ defined by $x\mapsto xa_0x^{-1}$ $(x\in R)$ is an analytic diffeomorphism;
(iii) The mapping ψ_M of $R\times Z_{\gamma,\tau}$ into M defined by $\psi_M(x,a)=xax^{-1}$ is an analytic diffeomorphism.

Put $\bar{R}=\{\bar{y}=yZ;\ y\in R\}$. Using these notations, we define for $h\in C_c^\infty(\bar{R})$, $g\in C_c^\infty(\mathfrak{z}^-_{\gamma,\tau})$,

$$f_{h,g}(y\exp(X/2)a_0\sigma(\exp(X/2))^{-1}y^{-1})=h(\bar{y})g(X)\quad (y\in R,\ X\in\mathfrak{z}^-_{\gamma,\tau}). \tag{4.9}$$

LEMMA 4.3. *Let $d_{\mathfrak{a}}z^*$ (resp. $d_{\mathfrak{b}}z^*$) be an invariant measure on Z/B_k (resp. Z/B_{k+1}) and $d\bar{y}$ be an invariant measure on H/Z. Define the invariant measure*

$d_{\mathfrak{a}}y^*$ (*resp.* $d_{\mathfrak{b}}y^*$) *on* H/B_k (*resp.* H/B_{k+1}) *by* $d_{\mathfrak{a}}y^*=d\bar{y}\, d_{\mathfrak{a}}z^*$ (*resp.* $d_{\mathfrak{b}}y^*=d\bar{y}\, d_{\mathfrak{b}}z^*$). *For* $h\in C_c^\infty(\bar{R})$ *and* $g\in C_c^\infty(\mathfrak{z}_{\bar{r},\tau})$, *define* $f_{h,g}$ *as in* (4.9). *Let* Y_1 *be an element of* $\mathfrak{z}_{\bar{r},\tau}\cap\mathfrak{a}$ *with coordinates* $(\xi_1, \cdots, \xi_{n-2(k+1)}, i\theta, -i\theta, \xi_{n-2k+1}, \cdots, \xi_n)$. *Put* $X_0=\lim_{\theta\to 0} Y_1 \in\mathfrak{z}_{\bar{r},\tau}\cap\mathfrak{a}\cap\mathfrak{b}$. *Then we get*

$$\lim_{\theta\to 0} F^k_{f_{h,g}}(a_0\exp Y_1)=\varepsilon_F^{k+1}(a_0\exp X_0)\xi_{\rho-\lambda}(a_0\exp X_0)c[h]\lim_{\theta\to 0}\psi^k_{ug,Z}(Y_1)\,.$$

Let Y_2 *be an element of* $\mathfrak{z}_{\bar{r},\tau}\cap\mathfrak{b}$ *with coordinates* $(\xi_1, \cdots, \xi_{n-2(k+1)}, t, -t, \xi_{n-2k+1}, \cdots, \xi_n)$. *Then we get*

$$\begin{aligned}(F^{k+1}_{f_{h,g}})^{\pm}(a_0)&=\lim_{t\to\pm 0} F^{k+1}_{f_{h,g}}(a_0\exp Y_2)\\ &=\varepsilon_F^{k+1}(a_0\exp X_0)\xi_{\rho-\lambda}(a_0\exp X_0)c[h]\lim_{t\to\pm 0}\psi^{k+1}_{ug,Z}(Y_2)\end{aligned}$$

where

$$c[h]=\int_{H/Z} h(\bar{y})d\bar{y}$$

and

$$u(X)=\nu_{a_0}(\exp(X/2)a_0\sigma(\exp(X/2))^{-1})j(X) \qquad (X\in\mathfrak{z}_{\bar{r},\tau})\,.$$

PROOF. The function $u(x)$ is a Z-invariant function. From Lemma 4.2 and the normalization of measures, we obtain the assertion. Q.E.D.

THEOREM 4.4. *Let* $d_{\mathfrak{a}}z_0^*$ (*resp.* $d_{\mathfrak{b}}z_0^*$) *be the invariant measure on* $Z_0/L_{\mathfrak{a}}$ (*resp.* $Z_0/L_{\mathfrak{b}}$) *as in* (3.9) (*resp.* (3.10)). *Let* $d_{\mathfrak{a}}z^*$ (*resp.* $d_{\mathfrak{b}}z^*$) *be the invariant measure on* Z/B_k (*resp.* Z/B_{k+1}) *given by* (3.11) (*resp.* (3.12)). *Take an invariant measure dy on* H/Z. *Let* $d_{\mathfrak{a}}y^*$ (*resp.* $d_{\mathfrak{b}}y^*$) *be the invariant measure on* H/B_k (*resp.* H/B_{k+1}) *defined by* $d_{\mathfrak{a}}y^*=d\bar{y}d_{\mathfrak{a}}z^*$ (*resp.* $d_{\mathfrak{b}}y^*=d\bar{y}d_{\mathfrak{b}}z^*$). *Using these measures, define Harish-Chandra transforms* F_f^k, F_f^{k+1} $\left(0\leq k\leq\left[\frac{n}{2}\right]-1\right)$ *for a function* $f\in C_c^\infty(M)$ *as in* (4.3). *Then we get*

$$(4.10)\qquad \begin{aligned}&(DF_f^{k+1})^+(a_0\exp Y)-(DF_f^{k+1})^-(a_0\exp Y)\\ &\qquad=-2i(D^\nu F_f^k)(a_0\exp Y) \qquad (Y\in\gamma)\end{aligned}$$

for any $f\in C_c^\infty(M)$ *and* $D\in U(\mathfrak{b}_c)$ (*the universal enveloping algebra of* $\mathfrak{b}_c$).

PROOF. From Lemma 4.3 and (3.13), we obtain the assertion of the theorem. Q.E.D.

§5. Invariant spherical distributions.

For a subalgebra V of $\mathfrak{g}$, denote its complexification $C\otimes V$ by V_c. Let $T(V_c)$ (resp. $U(V_c)$) be the tensor algebra (resp. the universal enveloping algebra) of V_c. Denote the set of all left invariant differential operators on G (resp.

G/H) by $\mathcal{D}_l(G)$ (resp. $\mathcal{D}_l(G/H)$) and the set of all B_G-invariant differential operators on M by $\mathcal{D}_B(M)$. $U(\mathfrak{g}_c)$ is naturally isomorphic with $\mathcal{D}_l(G)$ by the mapping $D \mapsto \partial_l(D)$, where

$$[\partial_l(c \otimes X)]_x f = c \frac{d}{dt}\Big|_{t=0} f(x \exp tX) \qquad \text{for} \quad x \in G,\ c \otimes X \in \mathfrak{g}_c . \tag{5.1}$$

Put

$$U(\mathfrak{g}_c)^H = \{D \in U(\mathfrak{g}_c)\ ;\ \mathrm{Ad}(h)D = D \text{ for all } h \in H\}.$$

Then $U(\mathfrak{g}_c)^H / \langle U(\mathfrak{g}_c)^H \cap U(\mathfrak{g}_c)\mathfrak{h}_c \rangle$ is isomorphic with $\mathcal{D}_l(G/H)$ by the map (5.1) (cf. Oshima and Sekiguchi [**20**]). For an element $c \otimes X$ in $\mathfrak{g}_c$ define the tangent vector $\widetilde{c \otimes X}$ in $T_e(M)$ by

$$(\widetilde{c \otimes X}) f = c \frac{d}{dt}\Big|_{t=0} f(\exp tX \sigma(\exp tX)^{-1}) . \tag{5.2}$$

And for an X in $\mathfrak{g}_c$, we define B_G-invariant vector field $\partial_B(X)$ on M by $[\partial_B(X)]_{x\sigma(x)^{-1}} f = dB_x \circ \tilde{X} f$. By this correspondence, any differential operator $D \in U(\mathfrak{g}_c)$ defines the B_G-invariant differential operator $\partial_B(D)$ on M.

For any $X \in \mathfrak{g}_c$, let L_X (resp. R_X) be the endomorphism in $U(\mathfrak{g}_c)$ given by left (resp. right) multiplication by X. Let Γ denote the linear mapping of $U(\mathfrak{g}_c)$ into itself given by

$$\Gamma(X_1 X_2 \cdots X_n) = (L_{X_1} - R_{\sigma(X_1)})(L_{X_2} - R_{\sigma(X_2)}) \cdots (L_{X_{n-1}} - R_{\sigma(X_{n-1})})(X_n - \sigma(X_n))$$

for $X_1, X_2, \cdots, X_n \in \mathfrak{g}_c$ and $\Gamma(1) = 1$.

LEMMA 5.1. *For any $D \in U(\mathfrak{g}_c)$, we obtain*

$$[\partial_B(D)]_{x\sigma(x)^{-1}} = [\partial_l(\mathrm{Ad}(\sigma(x))\Gamma(D))]_{x\sigma(x)^{-1}} \qquad (x \in G) .$$

If $D \in U(\mathfrak{g}_c)^H$, $\partial_B(D)$ is in $\mathcal{D}_B(M)$.

PROOF. Let $\otimes^k \mathfrak{g}_c$ denote the k-fold tensor product of $\mathfrak{g}_c$. Put $T^p = \sum_{k \leq p} \otimes^k \mathfrak{g}_c$. Let η be the natural homomorphism of $T(\mathfrak{g}_c)$ onto $U(\mathfrak{g}_c)$ and put $\eta(T^p) = U^p$. We shall prove the lemma by the induction on p. Clearly the assertion is true for U^0. Suppose that we have proved the result for p. For any $D \in U^p$ and any $X \in \mathfrak{g}_c$, we get

$$\begin{aligned}[\partial_B(XD)]_{x\sigma(x)^{-1}} f &= [\partial_B(X)]_{x\sigma(x)^{-1}} (\partial_l(\mathrm{Ad}\,\sigma(x))\Gamma(D) f) \\ &= \frac{d}{dt}\Big|_{t=0} (\partial_l(\mathrm{Ad}\,\sigma(x))\Gamma(D) f) \cdot B_x(\exp tX \sigma(\exp tX)^{-1})\end{aligned}$$

$$=\frac{d}{dt}\Big|_{t=0}(\partial_l(\mathrm{Ad}\,\sigma(x))\Gamma(D)f)$$
$$\cdot(x\sigma(x)^{-1}\exp(t\,\mathrm{Ad}(\sigma(x))X)\exp(-t\,\mathrm{Ad}(\sigma(x))\sigma X))$$
$$=[\partial_l(\mathrm{Ad}\,\sigma(x))\Gamma(XD)]_{x\sigma(x)^{-1}}f\,.$$

Hence the assertion is true for $p+1$. Q.E.D.

Let $\mathcal{J}$ be the subalgebra of $U(\mathfrak{g}_c)$ generated by $\{1\otimes X-i\otimes iX\,;\,X\in\mathfrak{g}\}$. Then $U(\mathfrak{g})$ is isomorphic with $\mathcal{J}$ under the linear mapping

$$\psi:\ X_1X_2\cdots X_n\ \longmapsto\ \frac{1}{2}(1\otimes X_1-i\otimes iX_1)\frac{1}{2}(1\otimes X_2-i\otimes iX_2)$$
$$\cdots\frac{1}{2}(1\otimes X_n-i\otimes iX_n)\quad(X_1,X_2,\cdots,X_n\in\mathfrak{g})\,.$$

Let $\mathfrak{Z}$ be the center of $U(\mathfrak{g})$. $\mathfrak{Z}$ is isomorphic with $U(\mathfrak{g}_c)^H/\langle U(\mathfrak{g}_c)^H\cap U(\mathfrak{g}_c)\mathfrak{h}_c\rangle$ by the map $D\mapsto[\psi(D)]$ $(D\in\mathfrak{Z})$. If we regard $D\in U(\mathfrak{g}_c)$ as the differential operator on G, we can identify $U(\mathfrak{g}_c)$ with $U(\mathfrak{g})$ by the linear mapping $\zeta: c\otimes X\mapsto cX$.

LEMMA 5.2. *The linear mapping $\zeta\circ\Gamma\circ\psi$ is equal to the identity mapping of* $U(\mathfrak{g})$.

PROOF. For any $X_1,\cdots,X_{n-1},X_n\in\mathfrak{g}$, we get

$$\zeta\circ\Gamma\circ\psi(X_1\cdots X_{n-1}X_n)$$
$$=\zeta\circ\Gamma\Big\{\frac{1}{2}(1\otimes X_1-i\otimes iX_1)\cdots\frac{1}{2}(1\otimes X_{n-1}-i\otimes iX_{n-1})\frac{1}{2}(1\otimes X_n-i\otimes iX_n)\Big\}$$
$$=\zeta\Big\{(L_{(1/2)(1\otimes X_1-i\otimes iX_1)}-R_{(1/2)(1\otimes X_1+i\otimes iX_1)})$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$(L_{(1/2)(1\otimes X_{n-1}-i\otimes iX_{n-1})}-R_{(1/2)(1\otimes X_{n-1}+i\otimes iX_{n-1})})$$
$$\Big(\frac{1}{2}(1\otimes X_n-i\otimes iX_n)-\frac{1}{2}(1\otimes X_n+i\otimes iX_n)\Big)\Big\}$$
$$=X_1\cdots X_{n-1}X_n$$

and

$$\zeta\circ\Gamma\circ\psi(1)=1\,.$$ Q.E.D.

Let $a\in A_k$ be the product of the two elements as in (4.1). We take $(i\varphi_1,\cdots,i\varphi_{n-2k},z_k,-\bar{z}_k,\cdots,z_1,-\bar{z}_1)$ as its coordinates where $z_p=i\theta_p+t_p$. Let $\mathfrak{a}_k^c$ be the complexification of $\mathfrak{a}_k$ in $\mathfrak{g}$. Let $I(\mathfrak{a}_k^c)$ be the set of invariants of $U(\mathfrak{a}_k^c)$ with respect to the Weyl group W of $(\mathfrak{g},\mathfrak{a}_k^c)$. Every element of $U(\mathfrak{a}_k^c)$ may be considered as a differential operator on A_k as in (5.1). $I(\mathfrak{a}_k^c)$ is isomorphic to the algebra $\mathfrak{S}_n$ of all symmetric polynomials of n indeterminates $X_1,X_2,\cdots,X_n$ in the following way. Take any element $S(X_1,X_2,\cdots,X_n)\in\mathfrak{S}_n$ and replace X_1, X_2,

$\cdots, X_n$ by the following operators respectively

$$(5.3)\qquad \frac{\partial}{i\partial\varphi_1},\ \cdots,\ \frac{\partial}{i\partial\varphi_{n-2k}},\ \frac{1}{2}\Big(\frac{\partial}{\partial t_k}+\frac{\partial}{i\partial\theta_k}\Big),\ \frac{1}{2}\Big(\frac{\partial}{\partial t_k}-\frac{\partial}{i\partial\theta_k}\Big),$$
$$\cdots,\ \frac{1}{2}\Big(\frac{\partial}{\partial t_1}+\frac{\partial}{i\partial\theta_1}\Big),\ \frac{1}{2}\Big(\frac{\partial}{\partial t_1}-\frac{\partial}{i\partial\theta_1}\Big).$$

Then we obtain an element of $I(\mathfrak{a}_k^c)$ and by this correspondence, every $I(\mathfrak{a}_k^c)$ $\left(0\leqq k\leqq\left[\frac{n}{2}\right]\right)$ is isomorphic to $\mathfrak{S}_n$. Put $M_k=\bigcup_{y\in H} yA_ky^{-1}$.

PROPOSITION 5.3. *There exists the linear mapping* $D\mapsto S_D(X_1, \cdots, X_n)$ *of* $\mathfrak{Z}$ *onto* $\mathfrak{S}_n$ *satisfying*

$$\overline{(\partial_B(\psi(D))f)}(a)=\Delta_k^{-1}(a)S_D(X_1, X_2, \cdots, X_n)(\Delta_k(a)\bar{f}(a))\qquad (a\in A_k')$$

for any function $f\in C^\infty(M_k)$ *which is invariant under the mapping* $x\mapsto yxy^{-1}$ *(any* $y\in H$*) where* $X=(X_1, X_2, \cdots, X_n)$ *are the differential operators on* A_k *defined as in* (5.3) *and* $\bar{f}$ *is the restriction of* f *to* A_k.

PROOF. We first complexify M and discuss the proposition on G. $\mathfrak{a}_k^c$ is a Cartan subalgebra of $\mathfrak{g}$ and A_k^c is the corresponding Cartan subgroup of G. If we regard $\partial_B(\psi(D))$ as a differential operator on G, then from Lemmas 5.1 and 5.2 it is sufficient to consider $\partial_l(D)$ $(D\in\mathfrak{Z})$. Let $\mathfrak{F}(G)$ be the space of functions spanned by the characters of finite dimensional representations of G. Let C be a Weyl chamber. The map $(w, a)\mapsto wa$ of $W\times C$ onto $(A_k^c)'$ is a regular mapping. The space of restrictions $\mathfrak{F}(G)|_C$ is dense in $C^\infty(C)$. Any W-invariant analytic differential operator on A_k^c defines the unique analytic differential operator on C and it is determined by the operation for $\mathfrak{F}(A_k^c)$. The coordinates of the following element,

$$X=\begin{pmatrix} a_1 & & & & & & \\ & \ddots & & & & & \\ & & a_{n-2k} & & & & \\ & & & b_k & ic_k & & \\ & & & -ic_k & b_k & & \\ & & & & & \ddots & \\ & & & & & b_1 & ic_1 \\ & & & & & -ic_1 & b_1 \end{pmatrix}\qquad (a_p, b_q, c_q\in \boldsymbol{C})$$

in $\mathfrak{a}_k^c$, is given by $(a_1, \cdots, a_{n-2k}, b_k+c_k, b_k-c_k, \cdots b_1+c_1, b_1-c_1)$ and denoted simply by $(\mu_1, \mu_2, \cdots, \mu_n)$. Let λ_p $(1\leqq p\leqq n)$ be the linear form on $\mathfrak{a}_k^c$ defined by $\lambda_p(X)=\mu_p$. For an element $a=e^x\in A_k^c$, we put $\Delta(a)=\prod_{1\leq p<q\leq n}(e^{(\mu_p-\mu_q)/2}-e^{(\mu_q-\mu_p)/2})$. Let Σ be the root system of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{a}_k^c$ and $\Sigma^+=\{\lambda_p-\lambda_q\,;1\leqq p<q\leqq n\}$ be a set of positive roots. Put $\rho=\frac{1}{2}\sum_{\alpha\in\Sigma^+}\alpha$. $D\in\mathfrak{Z}$ can be expressed by

$$D=\sum(X_-)(H)(X_+)$$

where $(X_-)=X_-^1X_-^2\cdots X_-^p$ $(X_-^1, X_-^2, \cdots, X_-^p\in\sum_{-\alpha>0}\mathfrak{g}(\mathfrak{a}_k^c;\alpha))$, $(X_+)=X_+^1X_+^2\cdots X_+^q$ $(X_+^1, X_+^2, \cdots, X_+^q\in\sum_{\alpha>0}\mathfrak{g}(\mathfrak{a}_k^c;\alpha))$ and $(H)=H^1H^2\cdots H^r$ $(H^1, H^2, \cdots, H^r\in\mathfrak{a}_k^c)$. Let (π_Λ, V_Λ) be the irreducible finite dimensional representation of G with a highest weight $\Lambda=m_1\lambda_1+m_2\lambda_2+\cdots+m_n\lambda_n$ $(m_1>m_2>\cdots>m_n)$ (it is corresponding to the Young diagram with the partition $(m_1, m_2, \cdots, m_n)$). Let v_Λ be a highest weight vector of V_Λ. D operates on V_Λ as a scalar multiplication. Then from $X_+v_\Lambda=0$, the term actually including (X_+) sends v_Λ to 0. $X_{-\alpha}v_\Lambda$ $(X_{-\alpha}\in\mathfrak{g}(\mathfrak{a}_k^c;\alpha), \alpha>0)$ has the weight $\Lambda-\alpha$ and then the contribution of terms including (X_-) is equal to 0. It is sufficient to consider the contribution of (H) parts including no (X_+), (X_-). Let H_p be the element of $\mathfrak{a}_k^c$ with the coordinate $(0, \cdots, 0, \overset{p}{1}, 0, \cdots, 0)$. There exists a polynomial $P_D(x_1, x_2, \cdots, x_n)$, such that $Dv_\Lambda=P_D(m_1, m_2, \cdots, m_n)v_\Lambda=P_D(H_1, H_2, \cdots, H_n)v_\Lambda$. And for the character χ_Λ of the representation (π_Λ, V_Λ), we have $D\chi_\Lambda=P_D(m_1, m_2, \cdots, m_n)\chi_\Lambda$. The character χ_Λ is given explicitly by

$$\chi_\Lambda(a)=\frac{\sum_{w\in W}\operatorname{sgn} w\, e^{w(\Lambda+\rho)X}}{\Delta(a)} \qquad (a=e^X\in(A_k^c)').$$

For any $P(\ )\in\mathfrak{S}_n$, we have $\Delta^{-1}P(H_1, H_2, \cdots, H_n)\Delta\chi_\Lambda=P\Big(m_1+\frac{n-1}{2}, m_2+\frac{n-3}{2}, \cdots, m_n-\frac{n-1}{2}\Big)\chi_\Lambda$. Put $S_D(x_1, x_2, \cdots, x_n)=P_D\Big(x_1-\frac{n-1}{2}, x_2-\frac{n-3}{2}, \cdots, x_n+\frac{n-1}{2}\Big)$, then $S_D(\ )$ is determined independently on the choice of $\mathfrak{a}_k$ $\Big(k=0, 1, \cdots, \Big[\frac{n}{2}\Big]\Big)$. Since $\Delta^{-1}S_D(H_1, H_2, \cdots, H_n)\Delta\chi_\Lambda=S_D\Big(m_1+\frac{n-1}{2}, m_2+\frac{n-3}{2}, \cdots, m_n-\frac{n-1}{2}\Big)\chi_\Lambda$ for any Λ, $S_D(\ )$ is in $\mathfrak{S}_n$. For any function $f\in C^\infty(G)$ which is invariant under inner automorphisms of G, we get $\Delta^{-1}S_D(H_1, H_2, \cdots, H_n)(\Delta\bar{f})=\overline{Df}$, where $\bar{f}$ is the restriction of f to A_k^c. Hence from the restrictions of both sides of the above equality to M_k, we obtain the assertion of the proposition. Q.E.D.

Now we define invariant spherical distributions on M.

DEFINITION 5.1. A distribution Φ on M is called an *invariant spherical distribution* if it has the following properties:

(i) $\Phi(yxy^{-1})=\Phi(x)$ for all $x\in M$, $y\in H$;

(ii) There exists a homomorphism $\lambda:\mathfrak{Z}\to\boldsymbol{C}$ such that $\partial_l(D)\Phi=\lambda(D)\Phi$ for all $D\in\mathfrak{Z}$.

§ 6. The construction of invariant spherical distributions.

We define for any $f\in C_c^\infty(M)$ a function K_f^k on A_k' $\Big(0\leqq k\leqq\Big[\frac{n}{2}\Big]\Big)$ by

$$K_f^k(a)=\varepsilon_F^k(a)\operatorname{conj}(\Delta_k(a))\int_{H/B_k} f(yay^{-1})d_k y^* \qquad (a\in A_k') \tag{6.1}$$

where $\operatorname{conj}(\Delta_k(a))$ is the complex conjugate of $\Delta_k(a)$ and $y^*=yB_k$. From a direct calculation, $\operatorname{conj}(\Delta_k(a))=(-1)^{n(n-1)/2-k}\Delta_k(a)$. Then we get $K_f^k(a)=(-1)^{n(n-1)/2-k}F_f^k(a)$. Put $A_k'(T)=\{a\in A_k\,;\,\prod_{p=1}^{k}(1-e^{-t_p})\neq 0\}$. Define $\varepsilon(w)$ for any $w\in W(A_k\,;H)$, $(\varepsilon_F^k\Delta_k)(wa)=\varepsilon(w)(\varepsilon_F^k\Delta_k)(a)$. Set $A_k'(F)=\{a\in A_k\,;\prod_{p\neq q}(e^{i\varphi_p}-e^{i\varphi_q})\neq 0\}$.

LEMMA 6.1. *Every K_f^k $\left(f\in C_c^\infty(M),\ 0\leqq k\leqq\left[\frac{n}{2}\right]\right)$ has the following properties.*

(i) *The function K_f^k can be extended to a C^∞-function on the closure of every connected component of $A_k'(T)$.*

(ii) *Let $a\in A_k$ and $X\in U(\mathfrak{a}_k^c)$. If $S_\alpha(X)=-X$ for any $\alpha=\alpha_{n+2(p-k)-1,\,n+2(p-k)}$ $(1\leqq p\leqq k)$ for which $\xi_\alpha(a)=1$, then XK_f^k can be extended to a continuous function on some neighborhood of a.*

(iii) $K_f^k(wa)=\varepsilon(w)K_f^k(a)$ $\quad w\in W(A_k\,;H)$.

(iv) *For any $D\in\mathfrak{Z}$,*

$$K^k_{\partial_l(D)f}(a)=S_D(X_1, X_2, \cdots, X_n)K_f^k(a) \qquad (a\in A_k') \tag{6.2}$$

where $X_1, X_2, \cdots, X_n$ are differential operators defined in (5.3).

PROOF. From Theorem 4.4, we obtain the assertions (i) and (ii). The definition of $\varepsilon(w)$ implies (iii). Applying Proposition 5.3 to K_f^k, we obtain the assertion (iv). Q.E.D.

Let dx be a fixed H-invariant measure on M. Let $d_k a$ be a Haar measure on A_k defined by

$$d_k a=\prod_{p=1}^{n-2k} d\varphi_p \prod_{q=1}^{k} dt_q d\theta_q \tag{6.3}$$

where φ_p, t_q, θ_q are the coordinates of $a\in A_k$ as defined in §4. These dx and $d_k a$ define the normalized H-invariant measure $d_k y^*$ $(y^*=yB_k)$ on H/B_k as in §2. Then by Proposition 2.1, we get for any $f\in C_c^\infty(M_k)$

$$\int_{M_k} f(x)dx=\gamma_k\int_{H/B_k}\int_{A_k} f(yay^{-1})|\Delta_k(a)|^2 d_k a\, d_k y^* \tag{6.4}$$

where $\gamma_k=\dfrac{1}{|W(A_k\,;H)|}$. And by Corollary 2.2, we obtain

$$\int_M f(x)dx=\sum_{k=0}^{[n/2]}\gamma_k\int_{H/B_k}\int_{A_k} f(yay^{-1})|\Delta_k(a)|^2 d_k a\, d_k y^* \tag{6.5}$$

for any $f\in C_c^\infty(M)$.

Let Λ_0 be the linear mapping from $\mathfrak{a}_k$ into $\boldsymbol{C}$ defined by $\Lambda_0(X)=$

$\frac{n-1}{2}(\mu_1+\mu_2+\cdots+\mu_n)$ $(X\in\mathfrak{a}_k)$. And define a function $\xi_{\Lambda_0}(a)$ on A_k by

$$\xi_{\Lambda_0}(a)=\exp\Big(\frac{n-1}{2}(\mu_1+\mu_2+\cdots+\mu_n)\Big).$$

ξ_{Λ_0} is a one-valued or two-valued function according to n is odd or even. But $\varepsilon_F^k(a)\xi_{\Lambda_0}(a)$ is a one-valued function on A_k.

Let $\tilde{\kappa}_k(a)$ $\left(0\leqq k\leqq\left[\frac{n}{2}\right]\right)$ be analytic functions on A_k' satisfying the following conditions:

(a-1) $\tilde{\kappa}_k(a)$ can be extended to an analytic function on $A_k'(F)$ and satisfies

$$(\varepsilon_F^k\xi_{\Lambda_0}^{-1}\tilde{\kappa}_k)(wa)=\varepsilon(w)(\varepsilon_F^k\xi_{\Lambda_0}^{-1}\tilde{\kappa}_k)(a)\qquad\text{for all } w\in W(A_k\,;H);$$

(a-2) There exist complex numbers $a_1, a_2, \cdots, a_n$ such that for any symmetric polynomial $S(x_1, x_2, \cdots, x_n)$ in n-indeterminates

$$S(X_1, X_2, \cdots, X_n)\tilde{\kappa}_k(a)=S(a_1, a_2, \cdots, a_n)\tilde{\kappa}_k(a)$$

where $X_1, X_2, \cdots, X_n$ are differential operators on A_k defined in (5.3);

(a-3)
$$\left[\left(\frac{\partial}{i\partial\varphi_{n-(2k+1)}}-\frac{\partial}{i\partial\varphi_{n-2k}}\right)\tilde{\kappa}_k(a)\right]_{\varphi_{n-(2k+1)}=\varphi_{n-2k}+0}=\frac{\partial}{\partial t_{k+1}}\tilde{\kappa}_{k+1}(a)$$

$$(a\in\{a\in A_k\cap A_{k+1}\,;\ \xi_\alpha(a)\neq 1\text{ for all }\alpha\in\Sigma^+(\mathfrak{a}_k)\text{ except }\alpha_0\})$$

where the both sides denote the limit values at a which exist under the conditions (a-1) and (a-2).

Put $\kappa_k(a)=\varepsilon_F^k(a)\xi_{\Lambda_0}(a)^{-1}\tilde{\kappa}_k(a)$ $\left(a\in A_k',\ 0\leqq k\leqq\left[\frac{n}{2}\right]\right)$.

Let S_n be the symmetric group of n letters. Let $S(x_1, x_2, \cdots, x_n)$ and $T(x_1, x_2, \cdots, x_n)$ be two polynomials in n-indeterminates such that there exists a homomorphism $\zeta: S_n\to\{\pm 1\}$

$$\begin{aligned}&S(x_{w(1)}, x_{w(2)}, \cdots, x_{w(n)})=\zeta(w)S(x_1, x_2, \cdots, x_n)\\ &T(x_{w(1)}, x_{w(2)}, \cdots, x_{w(n)})=\zeta(w)T(x_1, x_2, \cdots, x_n)\quad\text{for all } w\in S_n.\end{aligned}\tag{6.6}$$

LEMMA 6.2. *Let* $\tilde{\kappa}_k$ $\left(0\leqq k\leqq\left[\frac{n}{2}\right]\right)$ *be the analytic functions on* A_k' *satisfying the conditions* (a-1), (a-2) *and* (a-3). *And let* $S(x_1, \cdots, x_n)$ *and* $T(x_1, \cdots, x_n)$ *be two polynomials with the property* (6.6). *Then for any* $f\in C_c^\infty(M)$, *we have the equality*

$$0=\sum_{k=0}^{[n/2]}\gamma_k\int_{A_k}\{S(X)K_f^k\cdot T(X)\kappa_k-K_f^k\cdot S(-X)T(X)\kappa_k\}\,d_ka$$

where $\kappa_k(a)=\varepsilon_F^k(a)\xi_{\Lambda_0}^{-1}(a)\tilde{\kappa}_k(a)$ $(a\in A_k')$ *and* $X=(X_1, X_2, \cdots, X_n)$ *are differential* *operators on* A_k *defined by* (5.3).

PROOF. $S(X)$ is the sum of the following forms

$$\sum_{w\in W(\mathfrak{a}_k)}\zeta(w)X_{j_1}^w X_{j_2}^w\cdots X_{j_r}^w \qquad (1\leqq j_1,\ j_2,\ \cdots,\ j_r\leqq n)$$

where $w(X_{j_p})$ is denoted by $X_{j_p}^w$ $(1\leqq p\leqq r)$. So we assume that $S(X)$ has the above form. And put

$$I_k(S(X),\ T(X))=\int_{A_k}\{S(X)K_f^k\cdot T(X)\kappa_k-K_f^k\cdot S(-X)T(X)\kappa_k\}\,d_k a\,.$$

Set $\alpha_F=\alpha_{n-2k-1,n-2k}$, $\alpha_T=\alpha_{n-2k,n-2k+1}\in\Sigma(\mathfrak{a}_k)$. For any element α of $\Sigma(\mathfrak{a}_k)$, put $W(A_k\,;H:\alpha)=\{w\in W(A_k\,;H)\,;\,w\{\pm\alpha\}=\{\pm\alpha\}\}$, $N_\alpha^k=|W(A_k\,;H)|/|W(A_k\,;H:\alpha)|$ and $\Pi_\alpha^k=\{a\in A_k\,;\,\xi_\alpha(a)=1\}$. Let $d_F a$ (resp. $d_T a$) be the Haar measure on $\Pi_{\alpha_F}^k$ (resp. $\Pi_{\alpha_T}^k$) satisfying $d_F a\,d\varphi_{n-2k}=d_k a$ (resp. $d_T a\,dt_k=d_k a$). Using these notations, define

$$I_{F,k}^p=\frac{N_{\alpha_F}^k}{i|\alpha_F|}\sum_{w\in W(\mathfrak{a}_k)}\zeta(w)\alpha_F(X_{j_p}^w)$$
$$\times\int_{\Pi_{\alpha_F}^k}(X_{j_{p+1}}^w X_{j_{p+2}}^w\cdots X_{j_r}^w K_f^k)[X_{j_{p-1}}^w X_{j_{p-2}}^w\cdots X_{j_1}^w T(X)\kappa_k]_{-0}^{\alpha_F=+0}\,d_F a$$

and

$$I_{T,k}^p=\frac{N_{\alpha_T}^k}{|\alpha_T|}\sum_{w\in W(\mathfrak{a}_k)}\zeta(w)\alpha_T(X_{j_p}^w)$$
$$\times\int_{\Pi_{\alpha_T}^k}[X_{j_{p+1}}^w X_{j_{p+2}}^w\cdots X_{j_r}^w K_f^k]_{-0}^{\alpha_T=+0}(X_{j_{p-1}}^w X_{j_{p-2}}^w\cdots X_{j_1}^w T(X)\kappa_k)\,d_T a$$

where

$$[P(X)\kappa_k]_{-0}^{\alpha_F=+0}=[P(X)\kappa_k]_{\varphi_{n-2k-1}=\varphi_{n-2k}+0}-[P(X)\kappa_k]_{\varphi_{n-2k-1}=\varphi_{n-2k}-0}\,,$$
$$[Q(X)K_f^k]_{-0}^{\alpha_T=+0}=[Q(X)K_f^k]_{t_k=+0}-[Q(X)K_f^k]_{t_k=-0}\,.$$

Then, we get

$$\sum_{k=0}^{[n/2]}\gamma_k I_k(S(X),\ T(X))=\sum_{p=1}^{r}(-1)^p\sum_{k=0}^{[n/2]}\gamma_k(I_{F,k}^p+I_{T,k}^p)$$
$$=\sum_{p=1}^{r}(-1)^p\sum_{k=0}^{[n/2]-1}(\gamma_k I_{F,k}^p+\gamma_{k+1}I_{T,k+1}^p)\,.$$

Let ν be the linear map from $(\mathfrak{a}_{k+1})_c$ onto $(\mathfrak{a}_k)_c$ defined in §3. Put $\alpha=\alpha_F$ and $\beta=\alpha\cdot\nu$. Then $\Pi_\alpha^k=\Pi_\beta^{k+1}=A_k\cap A_{k+1}$, $|\beta|=|\alpha|$ and $\gamma_k N_\alpha^k=\gamma_{k+1}N_\beta^{k+1}$. From Theorem 4.4, we get

$$[X_{j_{p+1}}^{w'}X_{j_{p+2}}^{w'}\cdots X_{j_r}^{w'}K_f^{k+1}]_{-0}^{\beta=+0}(a)=2i(X_{j_{p+1}}^w X_{j_{p+2}}^w\cdots X_{j_r}^w K_f^k)(a) \qquad (a\in(\Pi_\alpha^k)')$$

where $w'=\nu^{-1}\cdot w\cdot\nu$ and $(\Pi_\alpha^k)'=\{a\in\Pi_\alpha^k\ ;\ \xi_\lambda(a)\neq 0$ for any $\lambda\in\Sigma(\mathfrak{a}_k),\ \lambda\neq\pm\alpha\}$. Hence we have

$$\begin{aligned}&\gamma_k I_{F,k}^p+\gamma_{k+1}I_{T,k+1}^p\\&=\frac{\gamma_k N_\alpha^k}{i|\alpha|}\sum_{w\in W(\mathfrak{a}_k)}\zeta(w)\alpha(X_{j_p}^w)\\&\quad\times\int_{\Pi_\alpha^k}(X_{j_{p+1}}^w X_{j_{p+2}}^w\cdots X_{j_r}^w K_f^k)\{[X_{j_{p-1}}^w X_{j_{p-2}}^w\cdots X_{j_1}^w\cdot T(X)\kappa_k]_{-0}^{\alpha=+0}\\&\qquad\qquad -2(X_{j_{p-1}}^{w'}X_{j_{p-2}}^{w'}\cdots X_{j_1}^{w'}T(X)\kappa_{k+1})\}\,d_F a\,.\end{aligned}$$

Among the terms of the right-hand side, we consider the sum of the two terms corresponding to w and $S_\alpha w$. From the condition (a-3) of $\tilde{\kappa}_k$

$$[(P(X)-P(S_\alpha X))\kappa_k]_{-0}^{\alpha=+0}(a)=2(P(X^{\nu^{-1}})-P((S_\alpha X)^{\nu^{-1}}))\kappa_{k+1}(a)\qquad(a\in(\Pi_\alpha^k)')$$

where $P(X)=X_{j_{p-1}}^w X_{j_{p-2}}^w\cdots X_{j_1}^w T(X)$. And from Lemma 6.1 (ii), we get

$$Q(X)K_f^k(a)=Q(S_\alpha(X))K_f^k(a)\qquad(a\in(\Pi_\alpha^k)')$$

where $Q(X)=X_{j_{p+1}}^w X_{j_{p+2}}^w\cdots X_{j_r}^w$. Then the sum is equal to 0 and $\gamma_k I_{F,k}^p+\gamma_{k+1}I_{T,k+1}^p=0$. Hence we obtain $\sum_{k=0}^{[n/2]}\gamma_k I_k(S(X),T(X))=0$. Q.E.D.

Let $\tilde{\kappa}_k(a)$ $\left(0\leqq k\leqq\left[\frac{n}{2}\right]\right)$ be analytic functions on A_k'. Define an analytic function $\Phi(x)$ on M' by

$$\Phi(x)=\xi_{\Lambda_0}(a)^{-1}\Delta_k(a)^{-1}\tilde{\kappa}_k(a)\qquad\text{for } x=yay^{-1}\in M'\quad(y\in H,\ a\in A_k')\,.\tag{6.7}$$

THEOREM 6.3. *Let* $\tilde{\kappa}_k$ $\left(0\leqq k\leqq\left[\frac{n}{2}\right]\right)$ *be analytic functions on* A_k' *satisfying conditions* (a-1), (a-2) *and* (a-3). *Let* $\Phi(x)$ *be an analytic function on* M *given by* (6.7). *Then* $\Phi(x)$ *defines an invariant spherical distribution on* M *as follows: for* $f\in C_c^\infty(M)$

$$f\ \longmapsto\ \int_{M'}f(x)\Phi(x)dx=\sum_{k=0}^{[n/2]}\gamma_k\int_{A_k'}K_f^k(a)\kappa_k(a)d_k a\,.$$

PROOF. The H-invariance of Φ is derived from the definition (6.7). Then we consider the property (ii) of Definition 5.1. From Lemma 6.1, for any $D\in\mathfrak{Z}$, there exists a symmetric polynomial $S_D(x_1,\cdots,x_n)$ in n-indeterminates satisfying

$$K_{\delta_l(D)f}^k(a)=S_D(X)K_f^k(a)\qquad(a\in A_k')\,.$$

Let $E\to\hat{E}$ $(E\in U(\mathfrak{a}_k^c))$ be the anti-automorphism of $U(\mathfrak{a}_k^c)$ induced by $X\mapsto -X$ $(X\in\mathfrak{a}_k^c)$. Then $S_{\hat{D}}(X)=S_D(-X)$. From the condition (a-3) of $\tilde{\kappa}_k$, there exists a row of complex constants $l=(l_1,l_2,\cdots,l_n)$ such that for any $D\in\mathfrak{Z}$

$$S_D(X)\kappa_k(a)=S_D(l)\kappa_k(a)\,.$$

Define the homomorphism $\lambda: \mathfrak{Z}\to \boldsymbol{C}$ by $\lambda(D)=S_D(l)$ $(D\in\mathfrak{Z})$. Using these properties and Lemma 6.2, we get for any $f\in C_c^\infty(M)$

$$
\begin{aligned}
(\Phi, \partial_l(\hat{D})f) &= \sum_{k=0}^{[n/2]}\gamma_k\int_{A_k} S_{\hat{D}}(X)K_f^k(a)\cdot\kappa_k(a)d_ka \\
&= \sum_{k=0}^{[n/2]}\gamma_k\int_{A_k} S_D(-X)K_f^k(a)\cdot\kappa_k(a)d_ka \\
&= \sum_{k=0}^{[n/2]}\gamma_k\int_{A_k} K_f^k(a)\cdot S_D(X)\kappa_k(a)d_ka \\
&= \sum_{k=0}^{[n/2]}\gamma_k\int_{A_k} K_f^k(a)\cdot S_D(l)\kappa_k(a)d_ka \\
&= \lambda(D)\sum_{k=0}^{[n/2]}\gamma_k\int_{A_k} K_f^k(a)\cdot\kappa_k(a)d_ka \\
&= \lambda(D)(\Phi, f).
\end{aligned}
$$

Hence we obtain $\partial_l(D)\Phi=\lambda(D)\Phi$. Q.E.D.

§ 7. Tempered invariant spherical distributions of height r.

Let Φ be any invariant spherical distribution on M, then Φ coincides with an analytic function on M'. Define analytic functions $\tilde{\kappa}_k(a)$ $\left(0\leqq k\leqq\left[\frac{n}{2}\right]\right)$ on A_k' by

$$\tilde{\kappa}_k(a)=\Delta_k(a)\xi_{A_0}(a)\Phi(a) \qquad (a\in A_k'). \tag{7.1}$$

Define an order on $\left\{A_k\,;\,0\leqq k\leqq\left[\frac{n}{2}\right]\right\}$ as $A_p<A_q$ if $p<q$. We call an invariant spherical distribution Φ is of height r if $\tilde{\kappa}_k=0$ for any $k>r$ and $\tilde{\kappa}_r\neq 0$. If Φ is of height r, then $\tilde{\kappa}_r(a)$ is expressed as

$$\tilde{\kappa}_r(a)=\sum_{\sigma\in S_n} p(x:A)e^{(x,\sigma c)} \qquad (x=\log a,\ a\in A_r') \tag{7.2}$$

for some $c=(c_1,\cdots,c_n)\in\boldsymbol{C}$, where $p(x:A)$ is a polynomial function on a connected component A of $A_r'(F)$ and S_n is the symmetric group of n letters. We call Φ exponential if all $p(x:A)$ are constants. We define exponential tempered invariant spherical distributions of height r which will contribute to the Fourier inversion formula on M as follows.

For $l=(m-i\lambda)/2$ $(m\in\boldsymbol{Z},\ \lambda>0)$, define a function on $\boldsymbol{C}^*=\boldsymbol{C}-(0)$ by

$$\xi(l\,;e^z)=e^{im\theta}(e^{i\lambda t}-e^{-i\lambda t}) \qquad z=t+i\theta \quad (t,\ \theta\in\boldsymbol{R}). \tag{7.3}$$

Set $\boldsymbol{T}=\{e^{i\varphi}\,;\varphi\in\boldsymbol{R}\}$. We define a function $D(l:\ )$ on $\boldsymbol{T}\times\boldsymbol{T}$ according to the following two cases. For $l=(m-i\lambda)/2$ ($m\in 2\boldsymbol{Z}$, $\lambda>0$), put

$$D(l: e^{i\varphi_1}, e^{i\varphi_2})=\frac{2e^{im\theta}\cosh\lambda(|\phi|-\pi/2)}{\sinh\pi\lambda/2} \tag{7.4}$$

where $\varphi_1, \varphi_2\in\boldsymbol{R}$, $\theta=(\varphi_1+\varphi_2)/2$ and $\phi\equiv(\varphi_1-\varphi_2)/2$ ($\bmod\,\pi$, $-\pi/2<\phi<\pi/2$). For $l=(m-i\lambda)/2$ ($m\in 2\boldsymbol{Z}+1$, $\lambda>0$), put

$$D(l: e^{i\varphi_1}, e^{i\varphi_2})=-\frac{2e^{im\theta}\sinh\lambda(|\phi|-\pi/2)}{\cosh\pi\lambda/2} \tag{7.5}$$

where $\varphi_1, \varphi_2\in\boldsymbol{R}$, $\theta=(\varphi_1+\varphi_2)/2$ and $\phi\equiv(\varphi_1-\varphi_2)/2$ ($\bmod 2\pi$, $0<|\phi|<\pi$).

Fix an integer r $\left(0\leqq r\leqq\left[\frac{n}{2}\right]\right)$. Let $c=(c_1, c_2, \cdots, c_{n-2r})$ be a row of integers satisfying $c_1>c_2>\cdots>c_{n-2r}$. Let $(m_1, m_2, \cdots, m_r)$ be a row of any integers. And let $(\lambda_1, \lambda_2, \cdots, \lambda_r)$ be a row of real numbers satisfying $\lambda_1>\lambda_2>\cdots>\lambda_r$. Put $l_p=(m_p-i\lambda_p)/2$ ($1\leqq p\leqq r$) and set $l=(l_1, l_2, \cdots, l_r)$. We call such a (c, l) is of type r. For (c, l) of type r, define analytic functions $\tilde{\kappa}_k(c, l: a)$ on A_k' $\left(0\leqq k\leqq\left[\frac{n}{2}\right]\right)$ by

$$\tilde{\kappa}_k(c, l: a)=\begin{cases} 0 & a\in A_k' \left(r<k\leqq\left[\frac{n}{2}\right]\right), \\ \displaystyle\sum_{\substack{\sigma\in S_{n-2k} \\ \sigma(n-2r+1)<\sigma(n-2r+2) \\ \cdots \\ \sigma(n-2k-1)<\sigma(n-2k)}} \sum_{\tau\in S_k} \sum_{\substack{\nu\in S_r \\ \nu(1)<\cdots<\nu(r-k) \\ \nu(r-k+1)<\cdots<\nu(r)}} \varepsilon_F^k(a)\xi_{\Lambda_0}(a) & \\ \quad\times\exp(i(c_1\varphi_{\sigma(1)}+\cdots+c_{n-2r}\varphi_{\sigma(n-2r)})) & \\ \quad\times\displaystyle\prod_{p=1}^{r-k} D(l_{\nu(p)}: e^{i\varphi_{\sigma(n-2r+2p-1)}}, e^{i\varphi_{\sigma(n-2r+2p)}}) & \\ \quad\times\displaystyle\prod_{p=1}^{k}\xi(l_{\nu(r-k+p)}: e^{z_{\tau(p)}}) & a\in A_k' \ (0\leqq k\leqq r). \end{cases} \tag{7.6}$$

These functions $\tilde{\kappa}_k(c, l: a)$ satisfy conditions (a-1), (a-2) and (a-3). Then from Theorem 6.3, we get an invariant spherical distribution $\Phi^r(c, l)$ on M. Put $\kappa_k(c, l: a)=(\varepsilon_F^k\xi_{\Lambda_0}^{-1}\tilde{\kappa}(c, l: \ ))(a)$.

§ 8. The expansion formula on M.

Let $L(\ )$ be the elementary alternating function in n-indeterminates defined by $L(x_1, x_2, \cdots, x_n)=\prod_{1\leqq p<q\leqq n}(x_p-x_q)$. Denote a differential operator $L(X_1, X_2, \cdots, X_n)$ on A_k simply by L where $X_1, X_2, \cdots, X_n$ are given by (5.3). For any $f\in C_c^\infty(M)$, define

$$G_f^k(a)=\gamma_k L K_f^k(a) \qquad a\in A_k'. \tag{8.1}$$

From Lemma 6.1, each G_f^k $\left(0 \leqq k \leqq \left[\frac{n}{2}\right]\right)$ has the following properties.

(8.2)
(i) G_f^k is zero outside some relatively compact subset of $A_k'(T)$ and can be extended to a continuous function on the whole A_k.

(ii) G_f^k can be extended to a C^∞-function on the closure of every connected component of $A_k'(T)$.

(iii) For $w \in W(A_k; H)$

$$G_f^k(wa) = \varepsilon'(w) G_f^k(a) \qquad a \in A_k'$$

where $\varepsilon'(w)$ is defined by $(\varepsilon_F^k \xi_{A_0})(wa) = \varepsilon'(w)(\varepsilon_F^k \xi_{A_0})(a)$.

LEMMA 8.1. *There exists a non zero constant γ^* such that for any $f \in C_c^\infty(M)$*

$$G_f^{[n/2]}(e) = \gamma^* f(e) \tag{8.3}$$

where e is the unit element of G.

PROOF. This property depends on the local structure around e. Let $h \in C_c^\infty(H)$ and $\Phi_h^{[n/2]}$ be the Harish-Chandra transform relative to the fundamental Cartan subgroup $B_{[n/2]}$ of H (Warner [30], § 8.5). $K_f^{[n/2]}$ and $\Phi_h^{[n/2]}$ have the same local properties around e. From Theorem 8.5.1.6 in Warner [30], we obtain the assertion. Q.E.D.

For $l = (m - i\lambda)/2$ $(m \in \boldsymbol{Z},\ \lambda > 0)$, define a function $\xi'(l: \ )$ on $\boldsymbol{C}^*$ by

$$\xi'(l: e^z) = -e^{im\theta}(e^{i\lambda t} + e^{-i\lambda t}) \qquad z = t + i\theta \ (t,\ \theta \in \boldsymbol{R}). \tag{8.4}$$

And we define a function $D'(l: \ )$ on $\boldsymbol{T} \times \boldsymbol{T}$ according to the following two cases. For $l = (m - i\lambda)/2$ $(m \in 2\boldsymbol{Z},\ \lambda > 0)$

$$D'(l: e^{i\varphi_1}, e^{i\varphi_2}) = \frac{2e^{im\theta} \sinh \lambda(|\phi| - \pi/2)}{\sinh \pi\lambda/2} \operatorname{sgn} \phi \tag{8.5}$$

where $\varphi_1, \varphi_2 \in \boldsymbol{R}$, $\theta = (\varphi_1 + \varphi_2)/2$ and $\phi \equiv (\varphi_1 - \varphi_2)/2$ $(\text{mod}\ \pi,\ 0 < |\phi| < \pi/2)$. For $l = (m - i\lambda)/2$ $(m \in 2\boldsymbol{Z} + 1,\ \lambda > 0)$, set

$$D'(l: e^{i\varphi_1}, e^{i\varphi_2}) = \frac{-2e^{im\theta} \cosh \lambda(|\phi| - \pi/2)}{\cosh \pi\lambda/2} \operatorname{sgn} \phi \tag{8.6}$$

where $\varphi_1, \varphi_2 \in \boldsymbol{R}$, $\theta = (\varphi_1 + \varphi_2)/2$ and $\phi \equiv (\varphi_1 - \varphi_2)/2$ $(\text{mod}\ 2\pi,\ 0 < |\phi| < \pi)$.

Fix an integer r $\left(0 \leqq r \leqq \left[\frac{n}{2}\right]\right)$ and take (c, l) as in § 7. Using the above notations, define functions $\kappa_k'(c, l: a)$ on A by

$$\kappa'_k(c, l : a) = \begin{cases} 0 & a \in A'_k \left(r < k \leqq \left[\frac{n}{2}\right]\right), \\ \displaystyle\sum_{\substack{\sigma \in S_{n-2k} \\ \sigma(n-2r+1)<\sigma(n-2r+2) \\ \cdots \\ \sigma(n-2k-1)<\sigma(n-2k)}} \sum_{\tau \in S_k} \sum_{\substack{\nu \in S_r \\ \nu(1)<\cdots<\nu(r-k) \\ \nu(r-k+1)<\cdots<\nu(r)}} \operatorname{sgn}\sigma & \\ \quad \times \exp(i(c_1\varphi_{\sigma(1)} + \cdots + c_{n-2r}\varphi_{\sigma(n-2r)})) & \\ \quad \times \prod_{p=1}^{r-k}{}' D'(l_{\nu(p)} : e^{i\varphi_{\sigma(n-2r+2p-1)}}, e^{i\varphi_{\sigma(n-2r+2p)}}) & \\ \quad \times \prod_{p=1}^{k} \xi'(l_{\nu(r-k+p)} ; e^{z_{\tau(p)}}) & a \in A'_k \ (0 \leqq k \leqq r). \end{cases} \tag{8.7}$$

Then we obtain $L\kappa_k(c, l : a) = L(c, l)\kappa'_k(c, l : a)$ $\left(0 \leqq k \leqq \left[\frac{n}{2}\right]\right)$ where $L(c, l) = L(c_1, \cdots, c_{n-2r}, l_1, \bar{l}_1, \cdots, l_r, \bar{l}_r)$.

LEMMA 8.2. *For* $0 < |\phi| < \pi/2$, *we get*

(i) $\displaystyle\int_0^\infty \frac{\sinh \lambda(\pi/2 - |\phi|)}{\sinh \pi\lambda/2} \operatorname{sgn}\phi \, d\lambda = \cot\phi$,

(ii) $\displaystyle\cot\phi = i\sum_{n=1}^{\infty} (e^{-2in\phi} - e^{2in\phi})$ (*as a periodic distribution*).

PROOF. The assertion (i) is given by [18]. To prove (ii), we take a C^∞-function $U(t)$ on $(-\infty, \infty)$ satisfying

$$\operatorname{supp}(U(t)) \subset [-1, 1],$$

$$\sum_{n=-\infty}^{\infty} U(t+n) = 1.$$

We have

$$\frac{2}{\pi}\int_{-\infty}^{\infty} \cot t \, U\left(\frac{t}{\pi}\right) \cos mt \, dt = 0 \qquad (m = 0, 1, 2, \cdots),$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty} \cot t \, U\left(\frac{t}{\pi}\right) \sin nt \, dt = \begin{cases} 0 & (n = 1, 3, 5, \cdots) \\ 2 & (n = 2, 4, 6, \cdots). \end{cases}$$

Then the Fourier series

$$\cot t = 2\sum_{p=1}^{\infty} \sin 2pt = i\sum_{p=1}^{\infty} (e^{-2ipt} - e^{2ipt})$$

is convergent as a distribution. Q.E.D.

LEMMA 8.3. *For* $0 < |\phi| < \pi$ $(|\phi| \neq \pi/2)$, *we obtain*

(i) $\displaystyle\int_0^\infty \frac{\cosh(|\phi| - \pi/2)\lambda}{\cosh \pi\lambda/2} \operatorname{sgn}\phi \, d\lambda = \operatorname{sgn}\phi \sec(|\phi| - \pi/2)$.

(ii) $\operatorname{sgn}\phi \sec(|\phi|-\pi/2)=i\sum_{n=0}^{\infty}(e^{-i(2n+1)\phi}-e^{i(2n+1)\phi})$ *(as a periodic distribution).*

PROOF. The formula (i) is also given by [18]. Put $f(t)=\operatorname{sgn}s \sec(|s|-\pi/2)$ $(t\equiv s \bmod 2\pi,\ 0<|s|<\pi)$. And from

$$\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\,U\Big(\frac{t}{2\pi}\Big)\cos mt\,dt=0 \qquad (m=0,1,2,\cdots),$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\,U\Big(\frac{t}{2\pi}\Big)\sin nt\,dt=\begin{cases}0 & (n=2,4,6,\cdots)\\ 2 & (n=1,3,5,\cdots),\end{cases}$$

we get the assertion (ii). Q.E.D.

LEMMA 8.4. *For any integer m and any positive number λ, put $l=(m-i\lambda)/2$. Let $G(e^{i\varphi_1}, e^{i\varphi_2})$ be a C^∞-function on $\boldsymbol{T}\times\boldsymbol{T}$ satisfying $G(e^{i\varphi_1}, e^{i\varphi_2})=-G(e^{i\varphi_2}, e^{i\varphi_1})$. Then we obtain*

$$\sum_m\int_0^\infty\int_{\varphi_1,\varphi_2\in[-\pi,\pi]}G(e^{i\varphi_1}, e^{i\varphi_2})D'(l:e^{i\varphi_1}, e^{i\varphi_2})d\varphi_1 d\varphi_2 d\lambda$$

$$=4i\sum_{m_1>m_2}\int_{\varphi_1,\varphi_2\in[-\pi,\pi]}G(e^{i\varphi_1}, e^{i\varphi_2})\exp(i(m_1\varphi_1+m_2\varphi_2))d\varphi_1 d\varphi_2\,.$$

PROOF. Applying Lemma 8.2 (resp. Lemma 8.3) to even integers m (resp. odd integers m), we get the assertion of the lemma. Q.E.D.

For (c, l) of type r, put

$$|(c, l)|=\sum_{p=1}^{n-2r}c_p^2+\sum_{q=1}^{r}(m_q^2+\lambda_q^2)/4\,.$$

LEMMA 8.5. *Let $f\in C_c^\infty(M)$. For any positive integer N, there exists a positive constant $M_{N,f}$ such that for any (c, l) of type r the following equality holds:*

$$|1+|(c, l)||^N|(\Phi^r(c, l), f)|\leqq M_{N,f}\,. \tag{8.8}$$

PROOF. For any $D\in\mathfrak{Z}$, we get

$$S_D(c, l)(\Phi^r(c, l), f)=(\partial_l(D)\Phi^r(c, l), f)$$
$$=(\Phi^r(c, l), \partial_l(\hat{D})f)\,.$$

For any $h\in C_c^\infty(M)$, put $M_h=\sum_{k=0}^{[n/2]}\sup_{a\in A_k'}|K_h^k(a)|$. Then there exists a positive constant s such that for any (c, l)

$$|(\Phi^r(c, l), h)|\leqq sM_h\,.$$

Then, by the relation

$$|S_D(c, l)||(\Phi^r(c, l), f)|=|(\Phi^r(c, l), \partial_l(\hat{D})f)|$$
$$=sM_{\partial_l(\hat{D})f}$$

and Lemma 3.2 in [**10**], we obtain the assertion. Q.E.D.

COROLLARY 8.6. *The series*

$$\sum_{c,m}(\Phi^r(c,l),f)=\sum_{c_1=-\infty}^{\infty}\cdots\sum_{c_{n-2r}=-\infty}^{\infty}\sum_{m_1=-\infty}^{\infty}\cdots\sum_{m_r=-\infty}^{\infty}(\Phi^r(c,l),f)$$

is absolutely covergent and the convergence is uniform with respect to $(\lambda_1,\lambda_2,\cdots,\lambda_r)\in \boldsymbol{R}^r$.

Let (c,l) be of type r $\left(0\leqq r\leqq\left[\frac{n}{2}\right]\right)$. Put for $f\in C_c^\infty(M)$

$$I_r=\sum_{\substack{c_1>\cdots>c_{n-2r}\\ m_1,\cdots,m_r}}\int_{\lambda_1>\cdots>\lambda_r>0}(\Phi^r(c,l),f)L(c,l)d\lambda_1\cdots d\lambda_r. \tag{8.9}$$

The Fourier transform of $G_f^k(a)$ is given by

$$J_k=\sum_{\substack{c_1,\cdots,c_{n-2k}\\ m_1,\cdots,m_r}}\int_{\lambda_1,\cdots,\lambda_k}\int_{A_k}G_f^k(a)\exp(i(c_1\varphi_1+\cdots+c_{n-2k}\varphi_{n-2k}+m_1\theta_1+\cdots+m_k\theta_k+\lambda_1t_1+\cdots+\lambda_kt_k))d_ka\,d\lambda_1\cdots d\lambda_k. \tag{8.10}$$

Using above lemmas and Lemma 6.2, we get

$$I_r=\sum_{k=0}^{r}\gamma_{r,k}J_k\qquad\left(0\leqq r\leqq\left[\frac{n}{2}\right]\right) \tag{8.11}$$

where $\gamma_{r,k}=\dfrac{(n-2k)!\,i^{r-k}(-1)^k}{(n-2r)!\,(r-k)!}$ $(0\leqq k\leqq r)$.

Define an upper triangular matrix A by

$$A=\begin{pmatrix}\gamma_{[n/2],[n/2]} & \cdots & \gamma_{[n/2],1} & \gamma_{[n/2],0}\\ & \ddots & \vdots & \vdots\\ 0 & & \gamma_{1,1} & \gamma_{1,0}\\ & & & \gamma_{0,0}\end{pmatrix}. \tag{8.12}$$

Then by the relation $A=A^{-1}$, we have

$$J_{[n/2]}=\sum_{r=0}^{[n/2]}\gamma_{[n/2],r}I_r. \tag{8.13}$$

THEOREM 8.7. *For any* $f\in C_c^\infty(M)$, *we obtain*

$$c_0f(e)=\sum_{r=0}^{[n/2]}\gamma_{[n/2],r}\sum_{\substack{c_1>\cdots>c_{n-2r}\\ m_1,\cdots,m_r}}\int_{\lambda_1>\cdots>\lambda_r>0}(\Phi^r(c,l),f)L(c,l)d\lambda_1\cdots d\lambda_r \tag{8.14}$$

where $c_0=(2\pi)^n\gamma^*$ *and* γ^* *is a constant in Lemma* 8.1 *which is independent on* f.

PROOF. From Lemma 8.1 and (8.13), we get the assertion of the theorem. Q.E.D.

References

[1] J. Faraut, Distributions sphériques sur les espaces hyperboliques, J. Math. Pures Appl., **58** (1978), 369-444.

[2] I. M. Gel'fand, M. I. Graev and N. Ya. Vilenkin, Generalized functions, Vol. 5, Academic Press, 1966.

[3] Harish-Chandra, Discrete series for semisimple Lie groups I, II, Acta Math., **113** (1965), 241-318; Ibid., **116** (1966), 1-111.

[4] Harish-Chandra, Harmonic analysis on real reductive groups I, J. Functional Analysis. **19** (1975), 104-204.

[5] Harish-Chandra, Harmonic analysis on real reductive groups II, Invent. Math., **36** (1976), 1-55.

[6] Harish-Chandra, Harmonic analysis on real reductive groups III, Ann. of Math., **104** (1976), 117-201.

[7] S. Helgason, Analysis on Lie groups and homogeneous spaces, Regional Conference Series in Mathematics, Amer. Math. Soc., **14**, 1972.

[8] T. Hirai, The characters of some induced representations of semisimple Lie groups, J. Math. Kyoto Univ., **8** (1968), 313-363.

[9] T. Hirai, Invariant eigendistributions of Laplace operators on real simple Lie groups I, Japan. J. Math., **39** (1970), 1-68.

[10] T. Hirai, The Plancherel formula for $SU(p,q)$, J. Math. Soc. Japan, **22** (1970), 134-179.

[11] T. Hirai, Invariant eigendistributions of Laplace operators on real simple Lie groups II, Japan. J. Math. New Series, **2** (1976), 27-89.

[12] T. Hirai, Invariant eigendistributions of Laplace operators on real simple Lie groups III, Japan. J. Math. New Series, **2** (1976), 269-341.

[13] T. Hirai, Invariant eigendistributions of Laplace operators on real simple Lie groups IV, Japan. J. Math. New Series, **3** (1977), 1-48.

[14] T. Hirai, The characters of the discrete series for semisimple Lie groups, J. Math. Kyoto Univ., **21** (1981), 418-500.

[15] R. Lipsman, On the characters and equivalence of continuous series representations, J. Math. Soc. Japan., **23** (1971), 452-480.

[16] H. Midorikawa, On the explicit formula of characters in the discrete series, Japan. J. Math. New Series, **3** (1977), 313-368.

[17] U. F. Molčanov, Analogue of the Plancherel formula for hyperboloids, Soviet Math. Dokl., **9** (1968), 1382-1385.

[18] S. Moriguchi, K. Udagawa and S. Hitotumatu, Mathematical formulas, II, Iwanami, 1956.

[19] T. Oshima and T. Matsuki, Orbits on affine symmetric spaces under the actions of the isotropy subgroups, J. Math. Soc. Japan, **32** (1980), 399-414.

[20] T. Oshima and J. Sekiguchi, Eigenspaces of invariant differential operators on an affine symmetric space, Invent. Math., **57** (1980), 1-81.

[21] B. D. Romm, Analogue to the Plancherel formula for the real unimodular group of the n-th order, Amer. Math. Soc. Transl. (2), **58** (1966), 155-215.

[22] S. Sano and J. Sekiguchi, The Plancherel formula for $SL(2,C)/SL(2,R)$, Sci. Papers College Gen. Ed. Univ. Tokyo, **30** (1980), 93-105.

[23] S. Sano, The inversion formula for $Sp(p,q)$, Bull. Inst. Vocational Training, **11-A** (1982), 35-52.

[24] S. Sano, The Plancherel formula for $Sp(n,R)$, Lecture in Math. Kyoto Univ., **14**

(1982), 223-259.
[25] T. Shintani, On the decomposition of regular representation of the Lorentz group on a hyperboloid of one sheet, Proc. Japan Acad., **43** (1967), 1-5.
[26] R. S. Strichartz, Harmonic analysis on hyperboloids, J. Functional Analysis, **12** (1973), 341-383.
[27] M. Sugiura, Conjugate classes of Cartan subalgebras in real semi-simple Lie algebras, J. Math. Soc. Japan, **11** (1959), 374-434.
[28] M. Sugiura, Unitary representations and harmonic analysis, Kodansha, Tokyo, 1975.
[29] R. Takahashi, Sur les functions spheriques et la formule de Plancherel dans le groupe hyperbolique, Japan. J. Math., **31** (1961), 55-90.
[30] G. Warner, Harmonic analysis on semi-simple Lie groups I, II, Springer-Verlag, New York, 1972.

Shigeru SANO
Department of Mathematics
The Institute of Vocational Training
Sagamihara, Kanagawa 229
Japan