J. Math. Soc. Japan
Vol. 36, No. 1, 1984

A remark of decompositions of the regular representations of semi-direct product groups

Dedicated to Professor Hisaaki Yoshizawa on his 60th birthday

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(Received Jan. 26, 1983)

Introduction.

The aim of the present paper is to show that the regular representations of some non-type I semi-direct product groups can be decomposed into direct integrals of irreducible representations in an uncountably infinite number of completely different ways. This is related with some cohomology groups.

The non-uniqueness of irreducible decompositions of a non-type I representation has been pointed out by several authors, for example, [3], [4], [7], [8], [9], [10], [11], [12], [13], [18], [19] and [20]. Concerning the regular representations λ of non-type I semi-direct product groups G, [4], [12] and [13] gave two kinds of entirely different irreducible decompositions of λ under some restrictions. In the present paper, we shall establish similar facts in a more general situation. We have studied in [7] and [10] that it is possible to give various kinds of irreducible decompositions of certain non-type I factor representations, related with some cohomology groups. In the present paper, we shall show that similar results may be obtained even for the regular representation λ of G and that there are an uncountably infinite number of completely different irreducible decompositions of λ in some cases.

Our main result is as follows. Let G be a semi-direct product $N\times_s K$ of N with K where N and K are assumed to be separable locally compact abelian groups. Then, the left regular representation λ of G is decomposed into irreducible components as

$$\lambda\cong\int_{\hat{N}}^{\oplus}\int_{\hat{H}_\chi}^{\oplus} U^{(\chi,\theta)}d\tau_\chi(\theta)d\mu(\chi) \qquad \text{(I)}$$

$$\cong\int_{Z}^{\oplus}\int_{\hat{K}}^{\oplus} V^{(a,\eta,\zeta)}d\nu(\eta)d\sigma(\zeta) \qquad \text{(II)}$$

where a is a cocycle of the double transformation group $(K;\hat{N}\times K;K)$. Further, we describe a maximal abelian von Neumann subalgebra A^a in $\lambda(G)'$ explicitly, which will give rise to the decomposition in (II). We state also the unitary inequivalence among the component representations and the discrepancy of different decompositions. See Proposition 1 and Theorem 4.

I would like to express my hearty thanks to Professor O. Takenouchi for various advices and careful reading of the manuscript, and to all the members of his seminar for stimulating discussions and valuable suggestions.

Preliminaries.

Let G be a semi-direct product group $N\times_s K$, where K acts on N as an automorphism group. We consider the case where N and K are separable locally compact abelian groups. The action is denoted by $z\in N\rightarrow k\cdot z\in N$ for $k\in K$. The element of G is written as (z, k) $(z\in N, k\in K)$ and the multiplication is given by $(z, k)(z', k')=(z+k\cdot z', k+k')$. The subgroups $\{(z, 0); z\in N\}$ and $\{(0, k); k\in K\}$ of G are identified with N and K respectively.

Via the action of K on N, an action of K on the topological space $\hat{N}$ (the dual of N) is defined: for $k\in K$ and $\chi\in\hat{N}$, $\langle z, k\cdot\chi\rangle=\langle k\cdot z, \chi\rangle$ for all $z\in N$. We get in this way a topological transformation group $(K; \hat{N})$ which satisfies $k_2\cdot(k_1\cdot\chi)=(k_1+k_2)\cdot\chi$ for $k_1, k_2\in K$ and $\chi\in\hat{N}$. When $(K; \hat{N})$ is smooth ([**2**]), $G=N\times_s K$ is called a "regular" semi-direct product group ([**14**]). We note that G is of type I if and only if G is a regular semi-direct product group of abelian groups ([**15**]). Our concern will be centered around to the case where G is a non-regular semi-direct product group (therefore, of non-type I).

When the topological transformation group $(K; \hat{N})$ is non-smooth, following two facts are known.

(i) There are various kinds of quasi-orbits on $\hat{N}$ under the action of K ([**5**], [**3**]).

(ii) For each non-transitive quasi-orbit μ, the one-cohomology group $H^1_\mu(K; \hat{N})$ is large (i.e. uncountably infinite) because the action of K is amenable.

In the present paper, we shall give different irreducible decompositions of the left regular representation λ of G, in relation with these facts.

Throughout the paper, we assume that a Haar measure of N is invariant under the action of K for simplicity. By the assumption, we see that $G=N\times_s K$ is a unimodular group and that a Haar measure of $\hat{N}$ is also invariant under the action of K.

Canonical decomposition of λ.

Let λ be the left regular representation of G and ι be the trivial representation of the subgroup $\{e\}$ of G where e denotes the unit element of G. Then, we see, by general considerations of induced representations [**15**], λ is decomposed as follows.

$$\begin{aligned}\lambda &\cong \mathrm{Ind}_{\{e\}}^{G}\iota \\ &\cong \mathrm{Ind}_{N}^{G}\,\mathrm{Ind}_{\{e\}}^{N}\iota \\ &\cong \mathrm{Ind}_{N}^{G}\int_{\hat{N}}^{\oplus}\chi d\mu(\chi) \\ &\cong \int_{\hat{N}}^{\oplus}\mathrm{Ind}_{N}^{G}\chi d\mu(\chi)\end{aligned}$$

where μ is a suitable Haar measure of $\hat{N}$ (the dual of N).

For $\chi\in\hat{N}$, let H_χ denote the stabilizer of K at χ. Put $G_\chi=N\times_s H_\chi$. For $\theta\in\hat{H}_\chi$, a unitary representation $L^{(\chi,\theta)}$ of G_χ is defined by $L^{(\chi,\theta)}_{(z,h)}=\langle z,\chi\rangle\langle h,\theta\rangle$ for $(z,h)\in N\times_s H_\chi=G_\chi$. Thus, we get a unitary representation $U^{(\chi,\theta)}$ of G by $U^{(\chi,\theta)}=\mathrm{Ind}_{G_\chi}^{G}L^{(\chi,\theta)}$. Then, each component $\mathrm{Ind}_N^G\chi$ of the above decomposition is further decomposed as

$$\begin{aligned}\mathrm{Ind}_N^G\chi &\cong \mathrm{Ind}_{G_\chi}^{G}\,\mathrm{Ind}_{N}^{G_\chi}\chi \\ &\cong \mathrm{Ind}_{G_\chi}^{G}\int_{\hat{H}_\chi}^{\oplus}L^{(\chi,\theta)}d\tau_\chi(\theta) \\ &\cong \int_{\hat{H}_\chi}^{\oplus}\mathrm{Ind}_{G_\chi}^{G}L^{(\chi,\theta)}d\tau_\chi(\theta) \\ &= \int_{\hat{H}_\chi}^{\oplus}U^{(\chi,\theta)}d\tau_\chi(\theta)\end{aligned}$$

where τ_χ is a Haar measure of $\hat{H}_\chi$. Therefore, we get the following.

PROPOSITION 1. *The left regular representation λ of $G=N\times_s K$ is decomposed as*

$$\lambda\cong\int_{\hat{N}}^{\oplus}\int_{\hat{H}_\chi}^{\oplus}U^{(\chi,\theta)}d\tau_\chi(\theta)d\mu(\chi)\,.$$

The components $U^{(\chi,\theta)}$ ($\chi\in\hat{N}$, $\theta\in\hat{H}_\chi$) have the following properties.

(i) *All $U^{(\chi,\theta)}$ are irreducible representations of G.*

(ii) *$U^{(\chi,\theta)}$ is unitarily equivalent to $U^{(\chi',\theta')}$ if and only if $\chi'\in\mathrm{Orb}_K(\chi)$ and $\theta'=\theta$.*

PROOF. These are easily verified by using Mackey's theory of induced representations [14]. So we omit the detail. [Q.E.D.]

Other decompositions of λ.

Now, we shall describe the possibility of other irreducible decompositions of λ. To do this, at first, we realize λ on $L^2(\hat{N}\times K)$ as follows.

LEMMA 2. *The left regular representation λ of G is realized on $L^2(\hat{N}\times K)$ as*

$$(\lambda_{(z,k)}\xi)(\chi,s)=\langle z,\chi\rangle\xi(k\cdot\chi,s-k)$$

for $(z, k)\in N\times_s K=G$ *and* $\xi(\chi, s)\in L^2(\hat{N}\times K)$.

PROOF. Transform the representation space of λ from $L^2(N\times K)$ to $L^2(\hat{N}\times K)$ by the unitary operator $F\otimes I$ where F is the Fourier transformation from $L^2(N)$ to $L^2(\hat{N})$ and I is the identity operator on $L^2(K)$. Then, we get the desired conclusion. [Q.E.D.]

Here, we may consider two actions of K on the space $\hat{N}\times K$, defined by

$$k\cdot(\chi, s)=(k\cdot\chi, s-k)$$

$$(\chi, s)\cdot t=(\chi, s+t)$$

for $(\chi, s)\in\hat{N}\times K$ and $k, t\in K$. Then, we get a double transformation group $(K;\hat{N}\times K;K)$. Let $\boldsymbol{T}$ be the one-dimensional torus group. As in [**6**], a $\boldsymbol{T}$-valued Borel function $a(\chi, s)$ on $\hat{N}\times K$ is called a cocycle of $(K;\hat{N}\times K;K)$ if $a(\chi, s)$ satisfies the following condition. For each $k, t\in K$,

$$a(k\cdot(\chi, s)t)=a(k\cdot(\chi, s))\overline{a(\chi, s)}a((\chi, s)\cdot t),$$

namely,

$$a(k\cdot\chi, s-k+t)=a(k\cdot\chi, s-k)\overline{a(\chi, s)}a(\chi, s+t).$$

$Z^1(K;\hat{N}\times K;K)$ denotes the abelian group of all such cocycles.

For $a\in Z^1(K;\hat{N}\times K;K)$, put

$$(\rho_t^a\xi)(\chi, s)=a(\chi, s)\overline{a(\chi, s+t)}\xi(\chi, s+t)$$

for $t\in K$ and $\xi(\chi, s)\in L^2(\hat{N}\times K)$. Then, ρ^a is a unitary representation of K on $L^2(\hat{N}\times K)$.

Let $L^\infty(\hat{N})$ denote the algebra of all μ-essentially bounded measurable functions, where μ is the Haar measure of $\hat{N}$ and $L^\infty(\hat{N})^K$ denotes the fixed point subalgebra of $L^\infty(\hat{N})$ under the action of K, namely, the set of elements $f\in L^\infty(\hat{N})$ satisfying that, for each $k\in K$, $f(k\cdot\chi)=f(\chi)$ μ-a.a. $\chi\in\hat{N}$. When we regard $L^\infty(\hat{N})$ as a von Neumann algebra on $L^2(\hat{N})$, we denote the operator of $L^\infty(\hat{N})$ by T_f for $f\in L^\infty(\hat{N})$.

Now, we take a von Neumann algebra A^a on $L^2(\hat{N})\otimes L^2(K)$ for $a\in Z^1(K;\hat{N}\times K;K)$, defined by

$$A^a=\{T_f\otimes\rho_t^a\,;\, f\in L^\infty(\hat{N})^K \text{ and } t\in K\}'',$$

where $T_f\otimes\rho_t^a$ means $(T_f\otimes I)\rho_t^a$. When the regular representation λ of G is realized on $L^2(\hat{N})\otimes L^2(K)$ as Lemma 2, we get the following lemma.

LEMMA 3. *A^a is a maximal abelian von Neumann algebra in $\lambda(G)'$ for each $a\in Z^1(K;\hat{N}\times K;K)$.*

PROOF. To show that $A^a\subset\lambda(G)'$, it is sufficient to verify that

$$\lambda_{(z,k)}(T_f\otimes\rho_t^a)=(T_f\otimes\rho_t^a)\lambda_{(z,k)}$$

for each $(z, k)\in G$, $t\in K$ and $f\in L^{\infty}(\hat{N})^{K}$. This can be seen as follows. For $\xi(\chi, s)\in L^{2}(\hat{N})\otimes L^{2}(K)$,

$$\begin{aligned}&(\lambda_{(z,k)}(T_{f}\otimes\rho_{t}^{a})\xi)(\chi, s)\\&=\langle z, \chi\rangle f(k\cdot\chi)a(k\cdot\chi, s-k)\overline{a(k\cdot\chi, s-k+t)}\xi(k\cdot\chi, s-k+t)\\&=\langle z, \chi\rangle f(\chi)a(\chi, s)\overline{a(\chi, s-t)}\xi(k\cdot\chi, s-k+t)\\&=((T_{f}\otimes\rho_{t}^{a})\lambda_{(z,k)}\xi)(\chi, s).\end{aligned}$$

The maximality of A^{a} in $\lambda(G)'$ will be shown later. [Q. E. D.]

Now, we shall consider the irreducible decomposition of λ corresponding to the maximal abelian von Neumann subalgebra A^{a} in $\lambda(G)'$ [**17**].

For the abelian von Neumann algebra $L^{\infty}(\hat{N})^{K}$ on $L^{2}(\hat{N})$, there exists a compact Hausdorff space Z and a positive finite measure σ on Z such that supp $\sigma=Z$ and $L^{\infty}(\hat{N})^{K}$ is algebraically isomorphic with $L^{\infty}(Z, \sigma)$ [**1**]. At the same time, the Haar measure μ on $\hat{N}$ is decomposed to ergodic measures μ_{ζ} $(\zeta\in Z)$ as

$$\mu=\int_{Z}^{\oplus}\mu_{\zeta}d\sigma(\zeta)$$

where the component measures μ_{ζ} on $\hat{N}$ are chosen to be invariant under the action of K for σ-a. a. $\zeta\in Z$ by the invariance of μ and the uniqueness of decompositions. Associated with this decomposition, we have that

$$L^{2}(\hat{N}, \mu)\cong\int_{Z}^{\oplus}L^{2}(\hat{N}, \mu_{\zeta})d\sigma(\zeta)$$

and

$$L^{\infty}(\hat{N}, \mu)=\int_{Z}^{\oplus}L^{\infty}(\hat{N}, \mu_{\zeta})d\sigma(\zeta)$$

and that $L^{\infty}(\hat{N})^{K}$ is transformed to the diagonal algebra.

For $a\in Z^{1}(K;\hat{N}\times K;K)$, a cocycle $c^{a}(k, \chi)$ of $(K;\hat{N})$ is obtained by

$$c^{a}(k, \chi)=\overline{a(\chi, s)}a(k\cdot\chi, s-k)$$

which is well-defined by the cocycle condition of a. Then, we may define a unitary representation $V^{(a,\eta,\zeta)}$ $(\zeta\in Z,\ \eta\in\hat{K})$ by

$$(V_{(z,k)}^{(a,\eta,\zeta)}\xi)(\chi)=c^{a}(k, \chi)\langle z, \chi\rangle\langle k, \eta\rangle\xi(k\cdot\chi)$$

for $\xi(\chi)\in L^{2}(N, \mu_{\zeta})$ and $(z, k)\in N\times_{s}K=G$.

Thus, we get the following theorem.

THEOREM 4. *The left regular representation λ of $G=N\times_{s}K$ is decomposed as follows, corresponding to the abelian von Neumann algebra A^{a} $(a\in Z^{1}(K;\hat{N}\times K;K))$ in $\lambda(G)'$.*

$$\lambda\cong\int_{Z}^{\oplus}\int_{\hat{K}}^{\oplus}V^{(a,\eta,\zeta)}d\nu(\eta)d\sigma(\zeta)$$

where ν *is a Haar measure of* $\hat{K}$ *and* $V^{(a,\eta,\zeta)}$ *is given as above. Moreover,* $V^{(a,\eta,\zeta)}$ $(\zeta\in Z,\ \eta\in\hat{K})$ *have the following properties.*

(i) $V^{(a,\eta,\zeta)}$ *is irreducible for each* $a\in Z^1(K;\hat{N}\times K;K)$, $\zeta\in Z$, *and* $\eta\in\hat{K}$.

(ii) $V^{(a,\eta,\zeta)}$ *is unitarily equivalent to* $V^{(a',\eta',\zeta')}$ *if and only if* $\zeta=\zeta'$ *and* $c^a+\eta$ *is* μ_ζ*-cohomologous to* $c^{a'}+\eta'$.

(iii) $V^{(a,\eta,\zeta)}$ *is unitarily equivalent to* $U^{(\chi,\theta)}$ *if and only if the measure* μ_ζ *concentrates on* $\mathrm{Orb}_K(\chi)$ *and* $c^a+\eta$ *is* μ_ζ*-cohomologous to an extension of* θ *to* K.

PROOF. For $a\in Z^1(K;\hat{N}\times K;K)$, define a unitary operator T_a on $L^2(\hat{N}\times K)$ by

$$(T_a\xi)(\chi,s)=\overline{a(\chi,s)}\xi(\chi,s)$$

for $\xi(\chi,s)\in L^2(\hat{N}\times K)$. Then, by simple calculations, we know that, for $\xi(\chi,s)\in L^2(\hat{N}\times K)$,

$$\begin{aligned}T_a\lambda_{(z,k)}T_a^*:\xi(\chi,s)&\longrightarrow\overline{a(\chi,s)}\,a(k\cdot\chi,s-k)\langle z,\chi\rangle\xi(k\cdot\chi,s-k)\\&=c^a(k,\chi)\langle z,\chi\rangle\xi(k\cdot\chi,s-k)\end{aligned}$$

and

$$T_a(T_f\otimes\rho_t^a)T_a^*:\xi(\chi,s)\longrightarrow f(\chi)\xi(\chi,s+t)$$

for $(z,k)\in G$, $f\in L^\infty(\hat{N})^K$, and $t\in K$.

Next, take a unitary operator $I\otimes F$ from $L^2(\hat{N})\otimes L^2(K)$ to $L^2(\hat{N})\otimes L^2(\hat{K})$ where I is the identity operator on $L^2(\hat{N})$ and F is the Fourier transformation from $L^2(K)$ to $L^2(\hat{K})$, and put $W_a=(I\otimes F)T_a$. Then, we get, for $\xi(\chi,\eta)\in L^2(\hat{N}\times\hat{K})$,

$$W_a\lambda_{(z,k)}W_a^*:\xi(\chi,\eta)\longrightarrow c^a(k,\chi)\langle z,\chi\rangle\langle k,\eta\rangle\xi(k\cdot\chi,\eta)$$

and

$$W_a(T_f\otimes\rho_t^a)W_a^*:\xi(\chi,\eta)\longrightarrow f(\chi)\overline{\langle t,\eta\rangle}\xi(\chi,\eta)\,.$$

Hence, we see that, corresponding to the abelian von Neumann algebra $W_aA^aW_a^*$, the Hilbert space $L^2(\hat{N}\times\hat{K})$ is decomposed as

$$L^2(\hat{N}\times\hat{K})\cong\int_Z^\oplus\int_{\hat{K}}^\oplus H^{(\eta,\zeta)}d\nu(\eta)d\sigma(\zeta)$$

where $H^{(\eta,\zeta)}\cong L^2(\hat{N},\mu_\zeta)$ for σ-a.a. $\zeta\in Z$,

$$\lambda_{(z,k)}\cong\int_Z^\oplus\int_{\hat{K}}^\oplus V_{(z,k)}^{(a,\eta,\zeta)}d\nu(\eta)d\sigma(\zeta)$$

and

$$T_f\otimes\rho_t^a\cong\int_Z^\oplus\int_{\hat{K}}^\oplus\tilde{f}(\zeta)\overline{\langle t,\eta\rangle}d\nu(\eta)d\sigma(\zeta)$$

where $\tilde{f}\in L^\infty(Z,\sigma)$ for $f\in L^\infty(\hat{N})^K$. Thus, we get the desired decomposition of λ.

(i) Suppose that there exists an operator S on $L^2(\hat{N},\mu_\zeta)$ such that $V_{(z,k)}^{(a,\eta,\zeta)}S=SV_{(z,k)}^{(a,\eta,\zeta)}$ for all $(z,k)\in G$. Then, the equality $V_{(z,0)}^{(a,\eta,\zeta)}S=SV_{(z,0)}^{(a,\eta,\zeta)}$ for all $z\in N$

implies that $S=T_g$ for some $g\in L^\infty(\hat{N}, \mu_\zeta)$ because the set $\{V_{(z,0)}^{(a,\eta,\zeta)}; z\in N\}$ generates a maximal abelian von Neumann algebra $L^\infty(\hat{N}, \mu_\zeta)$ on $L^2(\hat{N}, \mu_\zeta)$. Next, $V_{(0,k)}^{(a,\eta,\zeta)}S=SV_{(0,k)}^{(a,\eta,\zeta)}$ implies that, for each $k\in K$, $g(k\cdot\chi)=g(\chi)$ μ_ζ-a.a. $\chi\in\hat{N}$. By the ergodicity of μ_ζ, $g(\chi)=$constant (μ_ζ-a.a. $\chi\in\hat{N}$), and so S must be a constant multiplication operator. This means the irreducibility of $V^{(a,\eta,\zeta)}$.

(ii) Suppose that $V^{(a,\eta,\zeta)}$ is unitarily equivalent to $V^{(a',\eta',\zeta')}$. The restriction of each representation to the abelian subgroup N of G is decomposed as

$$_N|V^{(a,\eta,\zeta)}\cong\int_{\hat{N}}^{\oplus}\chi d\mu_\zeta(\chi)$$

and

$$_N|V^{(a',\eta',\zeta')}=\int_{\hat{N}}^{\oplus}\chi d\mu_{\zeta'}(\chi)\,.$$

Then, the unitary equivalency of these representations implies that the measures μ_ζ and $\mu_{\zeta'}$ on $\hat{N}$ are mutually equivalent [**16**] and so $\zeta=\zeta'$.

Thus, we may assume that there exists a unitary operator S on $L^2(N, \mu_\zeta)$ such that

$$V_{(z,k)}^{(a,\eta,\zeta)}S=SV_{(z,k)}^{(a',\eta',\zeta)}$$

for all $(z, k)\in G$. Similarly as in the proof of (i), we see that $S=T_g$ for some $\boldsymbol{T}$-valued Borel function g on $\hat{N}$. Next, by the equality

$$V_{(0,k)}^{(a,\eta,\zeta)}=T_gV_{(0,k)}^{(a',\eta',\zeta)}T_g^*\,,$$

we get, for each $k\in K$,

$$c^a(k, \chi)\langle k, \eta\rangle=g(\chi)c^{a'}(k, \chi)\langle k, \eta'\rangle\overline{g(k\cdot\chi)}$$

for μ_ζ-a.a. $\chi\in\hat{N}$ so that $c^a+\eta$ is μ_ζ-cohomologous to $c^{a'}+\eta'$, where we regard η and η' $(\in\hat{K})$ as elements of $Z^1(K;\hat{N})$. The converse is easily verified.

(iii) For $\chi\in\hat{N}$, let ω_χ be the canonical transitive invariant measure on $\hat{N}$ concentrated on $\mathrm{Orb}_K(\chi)$. Then, the unitary representation $U^{(\chi,\theta)}$ of G is realized on $L^2(\hat{N};\omega_\chi)$ as

$$(U_{(z,k)}^{(\chi,\theta)}\xi)(\chi)=\langle z, \chi\rangle\langle k, \tilde{\theta}\rangle\xi(k\cdot\chi)$$

for $\xi(\chi)\in L^2(\hat{N}, \omega_\chi)$, where $\tilde{\theta}$ is an extension character of θ to K. Hence, it is easy to deduce the desired conclusion by similar arguments as in the proof of (ii). So we omit the detail. [Q.E.D.]

REMARK 5. (a) By the irreducibility of $V^{(a,\eta,\zeta)}$, we see that A^a was a "maximal" abelian von Neumann subalgebra in $\lambda(G)'$ (see [**17**]).

(b) When the measure μ_ζ on $\hat{N}$ is not transitive for σ-a.a. $\zeta\in Z$, by (iii) in Theorem 4, we see that the regular representation λ of G is decomposed to irreducible components at least in two completely different ways. We note that this fact is connected with the existence of non-transitive quasi-orbits on $\hat{N}$ for the non-smooth topological transformation group $(K; \hat{N})$. This result is interpreted as a generalization of examples obtained by several authors, for example, by G. W. Mackey [**13**] (1951; some discrete semi-direct product groups), A. A. Kirillov [**12**] (1972; the Mautner group), and S. Funakosi [**4**] (1981; some general cases).

(c) When a cocycle c^a is not weakly μ_ζ-cohomologous to a cocycle $c^{a'}$ (this means that $c^a+\eta$ is not μ_ζ-cohomologous to $c^{a'}+\eta'$ for any $\eta, \eta' \in \hat{K}$) for σ-a.a. $\zeta \in Z$, by (ii) in Theorem 4, we see that the regular representation λ of G has completely different irreducible decompositions. This fact is connected with the weak cohomology group of $(K; \hat{N})$ for each quasi-orbit $[\mu_\zeta]$ and it is a new result for the regular representation. For a particular factor representation π, we have studied the relation between decompositions of π and the weak cohomology group associated with π in [**7**], [**8**], [**10**], [**11**]. Applying the arguments described there to this fact, we see that there are an uncountable infinite number of completely different irreducible decompositions of the regular representation for some concrete non-type I groups, for example, the discrete Mautner group, the Mautner group, the discrete Heisenberg group, and the Dixmier group.

EXAMPLE 6. $G=\boldsymbol{Z}^2\times_s\boldsymbol{Z}$. Let $\boldsymbol{Z}$ be the additive group of integers and $\boldsymbol{Z}^2$ be the product of two copies of $\boldsymbol{Z}$. Let G be a semi-direct product of $\boldsymbol{Z}^2$ $(=N)$ by $\boldsymbol{Z}$ $(=K)$, where the action of $\boldsymbol{Z}$ on $\boldsymbol{Z}^2$ as automorphism groups of $\boldsymbol{Z}^2$ is defined by

$$n\cdot z=\begin{pmatrix} m+1 & m \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} a \\ b \end{pmatrix} \qquad (m \in \boldsymbol{N})$$

for $n \in \boldsymbol{Z}$ and $z=(a, b) \in \boldsymbol{Z}^2$. It is well-known that the transformation group $(K; \hat{N})$ is non-smooth so that G is of non-type I and that the normalized Haar measure μ on $\hat{N}$ is ergodic under the action of K. We note that G does not satisfy the imbedding assumption $(*)$ which was crucial in the decomposition theory in [**7**].

Let $c^{(p,q)}$ $(p, q \in \boldsymbol{Z})$ be a concrete cocycle of $(K; \hat{N})$ canonically obtained by

$$c^{(p,q)}(1, (s, t))=e^{ips}s^{iqt}$$

for $(s, t) \in [0, 2\pi) \times [0, 2\pi) \cong \hat{N}$. Then, by similar arguments as in Lemma 4.2 of [**6**], we see that $c^{(p,q)}$ is weakly cohomologous to $c^{(p',q')}$ if and only if $p=p'$ and $q=q'$. Next, using the technique in [**11**], we know that the cardinal number of the weak cohomology group of $(K; \hat{N})$ is uncountably infinite. Thus, by (c) of Remark 5, the left regular representation λ of G has an uncountably infinite number of completely different irreducible decompositions in the following form.

$$\lambda \cong \int_0^{2\pi} \oplus V^{(c,r)} dr \qquad c \in Z^1(K, \hat{N}).$$

Here we note that there is another technique which gives rise to different decompositions of λ. Let K_z be a subgroup of G generated by $(z, 1)\in G$ and η^r be a unitary character of K_z obtained by $\langle(z, 1), \eta^r\rangle=e^{ir}$ for $r\in[0, 2\pi)$. Put $W^{(z,r)}=\mathrm{Ind}_{K_z}^G\eta^r$. Then, we get

$$\lambda\cong\int_0^{2\pi}\oplus W^{(z,r)}dr\,.$$

Moreover, the following facts hold.

(i) $W^{(z,r)}$ is irreducible for any $z\in N$ and $r\in[0, 2\pi)$.

(ii) If K_z is not cojugate to $K_{z'}$, $W^{(z,r)}$ is never unitarily equivalent to $W^{(z',r)}$ for arbitrary choices of $r, r'\in[0, 2\pi)$.

(iii) The number of the conjugacy classes of $\{K_z\,;z\in N\}$ is finite, exactly m.

Therefore, we see that λ has m kinds of completely different irreducible decompositions. This is a technique similar to the one used in [**9**], [**19**] and [**20**]. However, we can show that these decompositions are all contained in ours given in Theorem 4. In a subsequent paper, we will observe the diverse possibility of decompositions of representations, concerned with automorphisms of G. There, we will clarify the relationship between the conjugacy classes of some family of closed subgroups of G and the weak cohomology group of $(K\,;\hat{N})$.

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