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A remark on the values of the zeta functions associated with cusp forms

By Yoshitaka MAEDA

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Introduction.

For two primitive cusp forms $f(z)=\sum_{n=1}^{\infty}a(n)e(nz)$ and $g(z)=\sum_{n=1}^{\infty}b(n)e(nz)$ ($e(z)=\exp(2\pi iz)$, $z\in\mathfrak{H}$: the upper half complex plane), we define a zeta function by

$$D(s,\ f,\ g)=\sum_{n=1}^{\infty}a(n)b(n)n^{-s}\qquad(s\in\boldsymbol{C}),$$

and denote by K the field generated over $\boldsymbol{Q}$ by $a(n)$ and $b(n)$ for all n. If the weight k of f is greater than the weight l of g, Shimura [4] proved that $\pi^{-k}\langle f,\ f\rangle^{-1}D(m,\ f,\ g)$ belongs to K for an integer m with $(1/2)(k+l-2)<m<k$, where $\langle\ ,\ \rangle$ denotes the normalized Petersson inner product as in [4]. When K is a *CM-field*, namely, a totally imaginary quadratic extension over a totally real field F, we are going to show the divisibility of these special values by a certain polynomial of the Fourier coefficients $a(p)$ and $b(p)$ at prime divisors p of the level of these forms. Roughly speaking, $a(p)-\overline{b(p)}p^{e}$ with a certain integer e depending on k, m and p divides the numerator of $\pi^{-k}\langle f,\ f\rangle^{-1}D(m,\ f,\ g)$. More precisely, we have

THEOREM 1. *Let χ be the character of f and N the conductor of f. Assume that the character of g is the complex conjugate $\bar{\chi}$ of χ and g has the same conductor N as f. Write M for the conductor of χ. Let A be the set of prime divisors of N satisfying one of the following conditions:*

($\mathrm{C_a}$) *The p-primary part of N is equal to that of M; or,*

($\mathrm{C_b}$) *$p\mid N$, $p^{2}\nmid N$ and $p\nmid M$.*

Put

$$C=N\times\prod_{p\in A}[a(p)^{\rho}\{a(p)-b(p)^{\rho}p^{k-\delta(p)-m}\}],$$

where

$$\delta(p)=\begin{cases}1 & \textit{if } p \textit{ satisfies Condition } (\mathrm{C_a}),\\ 2 & \textit{if } p \textit{ satisfies Condition } (\mathrm{C_b}),\end{cases}$$

and ρ *denotes the complex conjugation. Then*

(1) $\pi^{-k}\langle f, f\rangle^{-1}D(m, f, g)/C$ *belongs to the maximal real subfield* F *of* K;

(2) *Let us write the principal ideal* $(\pi^{-k}\langle f, f\rangle^{-1}D(m, f, g))=\mathfrak{B}/\mathfrak{A}$ *with mutually prime integral ideals* $\mathfrak{A}$ *and* $\mathfrak{B}$ *of* K. *Then we have* $\mathfrak{A}_K^{\rho}\mathfrak{B}_K=(C)_K$. *Here, for any integral ideal* $\mathfrak{M}$ *of* K, *we decompose* $\mathfrak{M}=\mathfrak{M}_F\mathfrak{M}_K$ *with the smallest integral ideal* $\mathfrak{M}_F$ *of* F *dividing* $\mathfrak{M}$ *and the remaining* K-*ideal* $\mathfrak{M}_K$ (for details of this definition, see § 1).

Let us give some remarks:

(1) All the prime divisors of $\mathfrak{A}$ are "congruence divisors" of f except for trivial factors. This fact is a direct consequence of Shimura's proof of his algebraicity theorem in [**4**] and was indicated by Doi and Hida;

(2) When the conductor N is a prime, we can easily see that the prime divisors of $(C)_F$ are the factors of N or N^e-1 for the positive integer $e=2m+2-k-l$. Thus in this case, the K-part $(C)_K$ is roughly equal to the whole ideal $(C)=(N\times a(N)^{\rho}\times(a(N)-b(N)^{\rho}N^{k-1-m}))$ as mentioned above in Theorem 1;

(3) The property similar to the second assertion of Theorem 1 holds under some restrictions even if g is an Eisenstein series (see § 1, Proposition 3).

In § 2, we discuss some numerical examples.

§ 1. Proof of Theorem 1.

We keep the notation and the assumptions in the introduction throughout this section. We define complex numbers α_p, α'_p, β_p, $\beta'_p\in \boldsymbol{C}$ for rational primes p by

$$1-a(p)x+\chi(p)p^{k-1}x^2=(1-\alpha_p x)(1-\alpha'_p x),$$

and

$$1-b(p)x+\bar{\chi}(p)p^{l-1}x^2=(1-\beta_p x)(1-\beta'_p x),$$

where x is an indeterminate. Then we know (cf. [**4**, Lemma 1])

$$D(s, f, g)=\prod_p [X_p(s)Y_p(s)^{-1}],$$

where p runs over all rational primes,

$$X_p(s)=1-\alpha_p\alpha'_p\beta_p\beta'_p p^{-2s},$$

and

$$Y_p(s)=(1-\alpha_p\beta_p p^{-s})(1-\alpha_p\beta'_p p^{-s})(1-\alpha'_p\beta_p p^{-s})(1-\alpha'_p\beta'_p p^{-s}).$$

Both the conductors of f and g being N, for every prime divisor p of N, we may put

$$\alpha_p=a(p), \qquad \alpha'_p=0,$$

and

$$\beta_p = b(p)\,, \qquad \beta'_p = 0\,,$$

therefore we have

$$X_p(s) = 1\,,$$

and

$$Y_p(s) = 1 - a(p)b(p)p^{-s}\,.$$

Let us further put

$$D_N(s, f, g) = \{\prod_{p \mid N} Y_p(s)\} \times D(s, f, g)\,.$$

Then we have

$$D_N(s, f, g) = \sum_{(n, N)=1} a(n)b(n)n^{-s}\,, \tag{1.1}$$

and

$$D_N(s, f^\rho, g^\rho) = \sum_{(n, N)=1} a(n)^\rho b(n)^\rho n^{-s} \tag{1.2}$$

for $f^\rho(z) = \sum_{n=1}^{\infty} a(n)^\rho e(nz)$ and $g^\rho(z) = \sum_{n=1}^{\infty} b(n)^\rho e(nz)$. Since we know

$$a(n)^\rho = \bar{\chi}(n)a(n)$$

and

$$b(n)^\rho = \chi(n)b(n)$$

for all integers n prime to N, (1.1) and (1.2) imply that

$$D_N(s, f, g) = D_N(s, f^\rho, g^\rho)\,. \tag{1.3}$$

For every prime divisor p of N, we have $a(p)a(p)^\rho = p^{k-\delta(p)}$ if $p \in A$ and otherwise, $a(p) = 0$ (see Asai [1] or Doi-Miyake [2]). Therefore we see that

$$1 - a(p)b(p)p^{-s} = \begin{cases} \{a(p)^\rho - b(p)p^{k-\delta(p)-s}\}/a(p)^\rho & \text{if } p \in A\,, \\ 1 & \text{if } p \notin A\,, \end{cases}$$

and

$$1 - a(p)^\rho b(p)^\rho p^{-s} = \begin{cases} \{a(p) - b(p)^\rho p^{k-\delta(p)-s}\}/a(p) & \text{if } p \in A\,, \\ 1 & \text{if } p \notin A\,. \end{cases}$$

It follows from the identity $\langle f, f \rangle = \langle f^\rho, f^\rho \rangle$ that

$$\begin{aligned} &\pi^{-k}\langle f, f\rangle^{-1} D(m, f, g)/[N \times \prod_{p \in A} \{a(p)^\rho (a(p) - b(p)^\rho p^{k-\delta(p)-m})\}] \\ &= \pi^{-k}\langle f^\rho, f^\rho\rangle^{-1} D(m, f^\rho, g^\rho)/[N \times \prod_{p \in A} \{a(p)(a(p)^\rho - b(p)p^{k-\delta(p)-m})\}] \end{aligned} \tag{1.4}$$

for an integer m with $(1/2)(k+l-2) < m < k$. On the other hand, [4, Theorem 3] shows

$$(\pi^{-k}\langle f, f\rangle^{-1} D(m, f, g))^\rho = \pi^{-k}\langle f^\rho, f^\rho\rangle^{-1} D(m, f^\rho, g^\rho)\,. \tag{1.5}$$

Consequently $\pi^{-k}\langle f, f\rangle^{-1}D(m, f, g)/[N\times\prod_{p\in A}\{a(p)^{\rho}(a(p)-b(p)^{\rho}p^{k-\delta(p)-m})\}]$ is real and therefore, belongs to F.

Now, for any integral ideal $\mathfrak{M}$ of K, write $\mathfrak{M}=\prod\mathfrak{P}^{\alpha(\mathfrak{P})}$ with prime ideals $\mathfrak{P}$ and non-negative integers $\alpha(\mathfrak{P})$. For a prime ideal $\mathfrak{p}$ of F, we define a non-negative integer $\beta(\mathfrak{p})$ by

$$\beta(\mathfrak{p})=\begin{cases}\left[\dfrac{\alpha(\mathfrak{P})}{2}\right] & \text{if } \mathfrak{p} \text{ is ramified as } \mathfrak{p}=\mathfrak{P}^2 \text{ in } K,\\ \alpha(\mathfrak{P}) & \text{if } \mathfrak{p} \text{ remains prime as } \mathfrak{p}=\mathfrak{P} \text{ in } K,\\ \operatorname{Min}\{\alpha(\mathfrak{P}), \alpha(\mathfrak{P}^{\rho})\} & \text{if } \mathfrak{p} \text{ is split as } \mathfrak{p}=\mathfrak{P}\mathfrak{P}^{\rho} \text{ in } K,\end{cases}$$

where $[r]$ indicates the largest integer not exceeding r. Then we put $\mathfrak{M}_F=\prod\mathfrak{p}^{\beta(\mathfrak{p})}$ and $\mathfrak{M}_K=\mathfrak{M}/\mathfrak{M}_F$. In short, the ideal $\mathfrak{M}_F$ is the smallest integral ideal of F dividing $\mathfrak{M}$ as mentioned in the introduction. Now we are going to prove the second assertion of Theorem 1 in a slightly general setting.

LEMMA 2. *Let a be a nonzero element of K and c an algebraic integer of K. Write the principal ideal $(a)=\mathfrak{B}/\mathfrak{A}$ with mutually prime integral ideals $\mathfrak{A}$ and $\mathfrak{B}$ of K. Assume that a/c belongs to F. Then we have $\mathfrak{A}_K^{\rho}\mathfrak{B}_K=(c)_K$.*

PROOF. From the assumption, $\mathfrak{B}/\{\mathfrak{A}(c)\}=(\mathfrak{B}_F/\{\mathfrak{A}_F(c)_F\})\times(\mathfrak{B}_K/\{\mathfrak{A}_K(c)_K\})$ is an ideal of F; therefore, $\mathfrak{B}_K/\{\mathfrak{A}_K(c)_K\}$ must be an ideal of F. Now we suppose that a positive power $\mathfrak{P}^e$ of a prime ideal $\mathfrak{P}$ of K divides $\mathfrak{B}_K$. First we consider the case $\mathfrak{P}\neq\mathfrak{P}^{\rho}$. Since $\mathfrak{B}_K/\{\mathfrak{A}_K(c)_K\}$ is an ideal of F, we have $\mathfrak{A}_K^{\rho}\mathfrak{B}_K(c)_K^{\rho}=\mathfrak{A}_K\mathfrak{B}_K^{\rho}(c)_K$. From the definition of the K-part $\mathfrak{B}_K$, $\mathfrak{P}$ is prime to $\mathfrak{B}_K^{\rho}$ and also $\mathfrak{P}$ is prime to $\mathfrak{A}_K$. Therefore $\mathfrak{P}^e$ divides $(c)_K$. Next suppose $\mathfrak{P}=\mathfrak{P}^{\rho}$. Then $e=1$. Assume $\mathfrak{P}\nmid(c)_K$. Then, $\mathfrak{P}$ divides the F-ideal $\mathfrak{B}_K/\{\mathfrak{A}_K(c)_K\}$ with exponent 1, a contradiction; therefore, $\mathfrak{P}$ divides $(c)_K$. Thus we know that $\mathfrak{B}_K|(c)_K$. Put $(c)_K=\mathfrak{B}_K\mathfrak{D}$ with an integral ideal $\mathfrak{D}$ of K. Since $\mathfrak{A}_K\mathfrak{D}=(\mathfrak{B}_K/\{\mathfrak{A}_K(c)_K\})^{-1}$ is still an ideal of F, we see that if $\mathfrak{P}^e$ divides $\mathfrak{A}_K$, then similarly as above, $(\mathfrak{P}^e)^{\rho}$ must divide $\mathfrak{D}$, and therefore, $\mathfrak{A}_K^{\rho}|\mathfrak{D}$. We may put $(c)_K=\mathfrak{A}_K^{\rho}\mathfrak{B}_K\mathfrak{E}$ with an integral ideal $\mathfrak{E}$ of K. Since $\mathfrak{B}_K/\{\mathfrak{A}_K(c)_K\}$ is an ideal of F, we know that $\mathfrak{E}$ is an ideal of F. On the other hand, since $\mathfrak{E}$ divides the K-part $(c)_K$, $\mathfrak{E}$ coincides with $\mathfrak{E}_K$. Consequently we conclude $\mathfrak{E}=1$ and $\mathfrak{A}_K^{\rho}\mathfrak{B}_K=(c)_K$.

We take $\pi^{-k}\langle f, f\rangle^{-1}D(m, f, g)$ and C in Theorem 1 as a and c in Lemma 2, respectively. Then the second assertion of Theorem 1 follows from the first assertion and Lemma 2.

We note here that if $m<k-1$ or all primes p of A satisfy Condition ($\mathrm{C_a}$), then $C'=\prod_{p\in A}[a(p)^{\rho}\{a(p)-b(p)^{\rho}p^{k-\delta(p)-m}\}]$ is integral and therefore, we can similarly prove the assertions of Theorem 1 by replacing C by C'.

The second assertion of Theorem 1 also holds with some modification even when we take an Eisenstein series in place of the cusp form g in Theorem 1.

However, the analogue of the first assertion is not necessarily valid in this case (see below Example 3). Let us explain this in detail. Let l be a positive integer and let ψ_1 and ψ_2 Dirichlet characters defined modulo N_1 and N_2, respectively. Put $\tilde{\chi}=\psi_1\psi_2$ and $N=N_1N_2$. Assume that $\tilde{\chi}(-1)=(-1)^l$ and that one of the following conditions is satisfied:

(i) If $l=2$ and both ψ_1 and ψ_2 are the identities, then $N_1=1$ and N_2 (>1) is square-free; or,

(ii) Both ψ_1 and ψ_2 are primitive.

Moreover we put

$$b_0=\begin{cases} 0 & \text{if } l\neq 1 \text{ and } \psi_1 \text{ is not the identity, or} \\ & \quad l=1 \text{ and neither } \psi_1 \text{ nor } \psi_2 \text{ is the identity,} \\ -\dfrac{1}{24}\prod_{p\mid N}(1-p) & \text{if } l=2 \text{ and both } \psi_1 \text{ and } \psi_2 \text{ are the identities,} \\ -\dfrac{1}{2l}B_{l,\tilde{\chi}} & \text{otherwise,} \end{cases}$$

where $B_{l,\tilde{\chi}}$ is the l-th generalized Bernoulli number belonging to the character $\tilde{\chi}$. Now we define the Eisenstein series with characters ψ_1 and ψ_2 by

$$E(z;\psi_1,\psi_2)=b_0+\sum_{n=1}^{\infty}\{\sum_{\substack{dd'=n\\ d>0}}\psi_1(d')\psi_2(d)d^{l-1}\}e(nz).$$

Then $E(z;\psi_1,\psi_2)$ is a holomorphic modular form of weight l, level N and the character $\tilde{\chi}$ (see Hecke [**3**, Satz 44], and also [**2**, Theorem 4.7.1]). Now we take a primitive cusp form f of conductor N, character χ and weight k as in Theorem 1. Since for every positive integer n prime to N, we have

$$(\sum_{\substack{dd'=n\\ d>0}}\psi_2(d')\psi_1(d)d^{l-1})^{\rho}=\chi(n)\sum_{\substack{dd'=n\\ d>0}}\psi_1(d')\psi_2(d)d^{l-1},$$

the similar argument as in the proof of Theorem 1 shows that

$$D_N(s,f,E(z;\psi_1,\psi_2))=D_N(s,f^{\rho},E(z;\psi_2,\psi_1)^{\rho}), \tag{1.3$'$}$$

and

$$\pi^{-k}\langle f,f\rangle^{-1}D(m,f,E(z;\psi_1,\psi_2))\times N\times\prod_{p\in A}\{a(p)(a(p)^{\rho}-b(p)p^{k-\delta(p)-m})\} \tag{1.4$'$}$$
$$=\pi^{-k}\langle f^{\rho},f^{\rho}\rangle^{-1}D(m,f^{\rho},E(z;\psi_2,\psi_1)^{\rho})\times N\times\prod_{p\in A}\{a(p)^{\rho}(a(p)-b'(p)^{\rho}p^{k-\delta(p)-m})\},$$

where

$$b(p)=\psi_1(p)+\psi_2(p)p^{l-1},$$

and

$$b'(p)=\psi_2(p)+\psi_1(p)p^{l-1}.$$

Consequently we obtain

PROPOSITION 3. *Write the principal ideal* $(\pi^{-k}\langle f, f\rangle^{-1}D(m, f, E(z; \psi_1, \psi_2)))=\mathfrak{B}/\mathfrak{A}$ *with mutually prime integral ideals* $\mathfrak{A}$ *and* $\mathfrak{B}$ *of* K *and also write the principal ideal* $(\pi^{-k}\langle f, f\rangle^{-1}D(m, f, E(z; \psi_2, \psi_1)))=\mathfrak{D}/\mathfrak{C}$ *with mutually prime integral ideals* $\mathfrak{C}$ *and* $\mathfrak{D}$ *of* K. *If a prime divisor* $\mathfrak{P}$ *of the principal ideal* $(N\times\prod_{p\in A}[a(p)^{\rho}\{a(p)-b'(p)^{\rho}p^{k-\delta(p)-m}\}])$ *is prime to both* $\mathfrak{C}^{\rho}$ *and the principal ideal* $(N\times\prod_{p\in A}[a(p)\{a(p)^{\rho}-b(p)p^{k-\delta(p)-m}\}])$, *then* $\mathfrak{P}$ *divides* $\mathfrak{B}$.

§ 2. Numerical examples.

Under the same notation and the assumptions as in the previous sections, we define an element $S(m)=S(m, f, g)$ of K by

$$S(m)=\pi^{-k}\langle f, f\rangle^{-1}D(m, f, g)/\gamma,$$

where

$$\gamma=\frac{\Gamma(2m+2-k-l)}{\Gamma(m)\Gamma(m+1-l)}\cdot\frac{(-1)^{k-1-m}\cdot 4^{k-1}\cdot N}{3}\times\prod_{p\mid N}(1+p^{-1}),$$

the product being taken over all prime divisors p of N. This modification of our number $\pi^{-k}\langle f, f\rangle^{-1}D(m, f, g)$ is just for convenience of our numerical computation of these numbers and does not affect the assertions of Theorem 1. Thus our theorem can be stated for our number $S(m, f, g)$ instead of $\pi^{-k}\langle f, f\rangle^{-1}D(m, f, g)$ (see § 1, Lemma 2). The number $S(m)$ can be computed by the method of Shimura ([**4**, Example p. 801]), and we write the principal ideal $(S(m))=\mathfrak{B}/\mathfrak{A}$ with mutually prime integral ideals $\mathfrak{A}$ and $\mathfrak{B}$ of K as in Theorem 1. We give here some numerical examples. In the prime factorization of our numerical data, we put * for large factors which we do not know whether they are primes or not. For any modular form $h(z)=\sum_{n=0}^{\infty}c(n)e(nz)$, we denote by $\boldsymbol{Q}(h)$ the field generated over $\boldsymbol{Q}$ by $c(n)$ for all n. Now we take $N=13$ and $\chi=\bar{\chi}=\left(\frac{13}{-}\right)$:

EXAMPLE 1. Let $k=6$ and $l=4$. We take $f\in S_6(\Gamma_0(13), \chi)$ and $g\in S_4(\Gamma_0(13), \bar{\chi})$. Then we have $\dim S_6(\Gamma_0(13), \chi)=6$, $\dim S_4(\Gamma_0(13), \bar{\chi})=2$, $\boldsymbol{Q}(g)=\boldsymbol{Q}(\sqrt{-1})$ and $\boldsymbol{Q}(f)=\boldsymbol{Q}(\alpha)$ with a root α of the equation:

$$\phi(x)=x^6+161x^4+5856x^2+18864=0.$$

Moreover we obtain the following numbers:

$$S(5)=[11\alpha^5+2(1+12\sqrt{-1})\alpha^4+(1423+18\sqrt{-1})\alpha^3+46(4+45\sqrt{-1})\alpha^2$$
$$+12(2831+126\sqrt{-1})\alpha+24(64+603\sqrt{-1})]/[2\cdot 7\cdot\phi'(\alpha)],$$

$$N_{K/Q}(\text{Numerator of } S(5))=2^{54}\cdot 3^{26}\cdot 13^{16}\cdot \mathit{233}\cdot \mathit{12281}\cdot \mathit{18181},$$

$$N_{K/Q}(a(13)-b(13)^{\rho})=2^{14}\cdot 3^{6}\cdot 13^{14}\cdot \mathit{233}\cdot \mathit{12281}\cdot \mathit{18181}\,,$$

$$N_{Q(\alpha)/Q}(\phi'(\alpha))=-2^{22}\cdot 3^{10}\cdot 23^{2}\cdot 37^{2}\cdot 113^{2}\cdot 131\cdot 163^{2}\,,$$

where

$$\phi'(x)=\frac{d\phi}{dx}(x)\,,$$

$$a(13)=(5\alpha^{5}-42\alpha^{4}+721\alpha^{3}-5682\alpha^{2}+27276\alpha-93960)/144\,,$$

and

$$b(13)=13(2-3\sqrt{-1})\,.$$

Therefore, $\mathfrak{B}_K$ coincides with the ideal $(a(13)^{\rho}\{a(13)-b(13)^{\rho}\})_K$ up to the prime divisors of 2, 3 and 13. In this case, no prime divisors outside the ideal $(a(13)^{\rho}\{a(13)-b(13)^{\rho}\})_K$ appear in $\mathfrak{B}$.

EXAMPLE 2. Next we take $k=8$ and $l=4$, and g is as in Example 1. We take $f\in S_8(\Gamma_0(13), \chi)$. Then we have $\dim S_8(\Gamma_0(13), \chi)=6$, and $\boldsymbol{Q}(f)=\boldsymbol{Q}(\alpha)$ with a root α of the equation:

$$\phi(x)=x^{6}+449x^{4}+37224x^{2}+205776=0\,.$$

We obtain that

(i) $$S(6)=-[32\alpha^{5}-(5-174\sqrt{-1})\alpha^{4}+5(2015-18\sqrt{-1})\alpha^{3}-5(227-6669\sqrt{-1})\alpha^{2}$$
$$+3(132196-2520\sqrt{-1})\alpha+12(2495+37683\sqrt{-1})]/[3\cdot 5\cdot 7\cdot \phi'(\alpha)]\,,$$

$$N_{K/Q}(\text{Numerator of } S(6))=2^{56}\cdot 3^{24}\cdot \mathit{5}^{25}\cdot 13^{16}\cdot \mathit{457}\cdot \mathit{5441}^{2}\cdot \mathit{9202421}\,,$$

$$N_{K/Q}(a(13)-b(13)^{\rho}\cdot 13)=2^{14}\cdot 3^{4}\cdot \mathit{5}^{3}\cdot 13^{26}\cdot \mathit{457}\cdot \mathit{9202421}\,,$$

$$N_{Q(\alpha)/Q}(\phi'(\alpha))=-2^{26}\cdot 3^{6}\cdot 5^{4}\cdot 41^{2}\cdot 1429\cdot 25104281^{2}\,,$$

where

$$a(13)=(65\alpha^{5}-78\alpha^{4}+26845\alpha^{3}-15990\alpha^{2}+1696500\alpha+511368)/480\,,$$

and

$$b(13)=13(2-3\sqrt{-1})\,.$$

In this case, $\mathfrak{B}_F$ is non-trivial and has a factor prime to the principal ideal $(a(13)-b(13)^{\rho}\cdot 13)$; namely, a prime factor of *5441* divides $\mathfrak{B}_F$. Note that the degree of this factor in F over $\boldsymbol{Q}$ is 1. The similar assertion holds for the prime factors of $\mathfrak{B}_K$ except for some small primes. These phenomena occur persistently in the limit of our calculation we have already done.

(ii) $$S(7)=[119\alpha^{5}-(1-357\sqrt{-1})\alpha^{4}+(38053-27\sqrt{-1})\alpha^{3}-2(169-35469\sqrt{-1})\alpha^{2}$$
$$+12(120419-129\sqrt{-1})\alpha+24(331+31185\sqrt{-1})]/[7\cdot 17\cdot \phi'(\alpha)]\,,$$

$$N_{K/Q}(\text{Numerator of } S(7))=\mathit{2}^{57}\cdot 3^{18}\cdot 5^{10}\cdot 13^{28}\cdot \mathit{139}^{2}\cdot \mathit{205350452847487 13}^{*},$$

$$N_{K/Q}(a(13)-b(13)^{\rho})=2^{19}\cdot 3^{4}\cdot 5^{4}\cdot 13^{18}\cdot \mathit{205350452847487 13}^{*}.$$

In this case, $\mathfrak{B}_F$ has a prime factor of *139* which is prime to the principal ideal $(a(13)-b(13)^{\rho})$ and has the degree 1 in F over $\boldsymbol{Q}$.

EXAMPLE 3. Now we take Eisenstein series E_1 and E_2 of weight 2; namely we put $E_1=E\left(z;\left(\frac{13}{-}\right),\text{id.}\right)$ and $E_2=E\left(z;\text{id.},\left(\frac{13}{-}\right)\right)$. We take $f\in S_8(\Gamma_0(13),\chi)$. Then we have $\boldsymbol{Q}(E_1)=\boldsymbol{Q}(E_2)=\boldsymbol{Q}$ and $\boldsymbol{Q}(f)=\boldsymbol{Q}(\alpha)$ with α as in Example 2. We obtain that

$$S(6,f,E_1)=-[238\alpha^5+475\alpha^4+79244\alpha^3+100817\alpha^2+2407488\alpha+21588]/[3\cdot 7\cdot 17\cdot\phi'(\alpha)],$$

$$S(6,f,E_2)=-[237\alpha^5-495\alpha^4+78667\alpha^3-108677\alpha^2+2392716\alpha+38172]/[3\cdot 7\cdot 17\cdot\phi'(\alpha)],$$

$$N_{Q(\alpha)/Q}(\text{Numerator of } S(6,f,E_1))=-2^{22}\cdot 3^{8}\cdot 5^{6}\cdot 13^{11}\cdot 103\cdot \mathit{109}\cdot 2411\cdot \mathit{2593}\cdot \mathit{1678613},$$

$$N_{Q(\alpha)/Q}(a(13)-b'(13)^{\rho}\cdot 13)=2^{5}\cdot 3^{3}\cdot 5^{3}\cdot 13^{6}\cdot \mathit{109}\cdot \mathit{2593}\cdot \mathit{1678613},$$

$$N_{Q(\alpha)/Q}(\text{Numerator of } S(6,f,E_2))=-2^{22}\cdot 3^{7}\cdot 5^{6}\cdot 13^{16}\cdot 103\cdot \mathit{1861}\cdot \mathit{2087}\cdot 2411,$$

$$N_{Q(\alpha)/Q}(a(13)-b''(13)^{\rho}\cdot 13)=2^{5}\cdot 3^{2}\cdot 5^{3}\cdot 13^{11}\cdot \mathit{1861}\cdot \mathit{2087},$$

where

$$a(13)=(65\alpha^5-78\alpha^4+26845\alpha^3-15990\alpha^2+1696500\alpha+511368)/480,$$

$$b'(13)=1,$$

and

$$b''(13)=13.$$

Let $S(6,f,E_1)=\mathfrak{B}/\mathfrak{A}$ with mutually prime integral ideals $\mathfrak{A}$ and $\mathfrak{B}$ of $\boldsymbol{Q}(\alpha)$ and let $S(6,f,E_2)=\mathfrak{D}/\mathfrak{C}$ with mutually prime integral ideals $\mathfrak{C}$ and $\mathfrak{D}$ of $\boldsymbol{Q}(\alpha)$. Then we observe that $\mathfrak{B}$ and the principal ideal $(a(13)-b'(13)^{\rho}\cdot 13)$ have prime divisors of *109*, *2593* and *1678613* in common and that $\mathfrak{D}$ and the principal ideal $(a(13)-b''(13)^{\rho}\cdot 13)$ have prime divisors of *1861* and *2087* in common. The prime factors of *103* and *2411* in $\mathfrak{B}$ are prime to the principal ideal $(a(13)-b'(13)^{\rho}\cdot 13)$, but they are not real. Thus the analogue of the first assertion of Theorem 1 fails to hold in this case.

We list some other examples below in the case $N=5$, $\chi=\bar{\chi}=\left(\frac{5}{-}\right)$, $8\leqq k\leqq 16$ and $l=6$. We write

$$T=T(m)=N_{K/\boldsymbol{Q}}(\text{Numerator of } S(m)),$$

and

$$L=L(m)=N_{K/\boldsymbol{Q}}(a(5)-b(5)^{\rho}\cdot 5^{k-1-m}).$$

We give the table of $\dim S_k(\Gamma_0(5), \chi)$:

k	6	8	10	12	14	16
$\dim S_k(\Gamma_0(5), \chi)$	2	2	4	4	6	6

Table (I): The defining polynomial $\phi(x)$ for $\boldsymbol{Q}(f)$ and the discriminant of $\phi(x)$.

k	$\phi(x)$	Discriminant of $\phi(x)$
6	x^2+44	$-2^4\cdot 11$
8	x^2+116	$-2^4\cdot 29$
10	$x^4+1708x^2+1216$	$2^{18}\cdot 3^4\cdot 5^4\cdot 19\cdot 809^2$
12	$x^4+4132x^2+2496256$	$2^{20}\cdot 3^4\cdot 5^4\cdot 7^2\cdot 11^2\cdot 179^2\cdot 199$
14	$x^6+41052x^4+440779968x^2+617678127104$	$-2^{52}\cdot 3^{20}\cdot 5^{12}\cdot 269^2\cdot 521\cdot 7541^2 \times 10577429^2$
16	$x^6+117588x^4+2455515648x^2+4160982695936$	$-2^{70}\cdot 3^{12}\cdot 5^{12}\cdot 11\cdot 29^4\cdot 31\cdot 863^2 \times 1061^2\cdot 53497637^2$

Table (II): The denominators of $S(m)$.

k	m	
8	7	$3\cdot\phi'(\alpha)$
10	8	$2^3\cdot 7\cdot\phi'(\alpha)$
	9	$-13\cdot\phi'(\alpha)$
12	9	$2^2\cdot 3^2\cdot 5\cdot\phi'(\alpha)$
	10	$-2\cdot 5\cdot 13\cdot\phi'(\alpha)$
	11	$31\cdot\phi'(\alpha)$
14	10	$2^2\cdot 3^2\cdot 5\cdot 11\cdot\phi'(\alpha)$
	11	$-2\cdot 3\cdot 11\cdot 13\cdot\phi'(\alpha)$

k	m	
14	12	$2^2\cdot 3\cdot 31\cdot\phi'(\alpha)$
	13	$-13\cdot 313\cdot\phi'(\alpha)$
16	11	$5\cdot 7\cdot 11\cdot 13\cdot\phi'(\alpha)$
	12	$-2^2\cdot 7\cdot 13^2\cdot\phi'(\alpha)$
	13	$7\cdot 13\cdot 31\cdot\phi'(\alpha)$
	14	$-2\cdot 7\cdot 13\cdot 313\cdot\phi'(\alpha)$
	15	$71\cdot 521\cdot\phi'(\alpha)$

Here α is a root of $\phi(x)$ and $\phi'(x)=\frac{d\phi}{dx}(x)$.

Table (III): $T(m)$ and $L(m)$.

k	m	
8	7	$T=3^3\cdot5^4\cdot11$
		$L=2^6\cdot3^3\cdot5^8\cdot11$
10	8	$T=2^{40}\cdot3^4\cdot5^{18}\cdot7^8\cdot6011$
		$L=2^{14}\cdot3^4\cdot5^{22}\cdot6011$
	9	$T=2^{24}\cdot3^4\cdot5^{31}\cdot379\cdot39979$
		$L=2^{14}\cdot3^4\cdot5^{17}\cdot379\cdot39979$
12	9	$T=2^{42}\cdot3^{19}\cdot5^{18}\cdot7^4\cdot11^4\cdot31\cdot47$
		$L=2^{14}\cdot3^5\cdot5^{30}\cdot31\cdot47$
	10	$T=2^{38}\cdot3^5\cdot5^{31}\cdot7^6\cdot31\cdot109^2\cdot153877$
		$L=2^{14}\cdot3^5\cdot5^{25}\cdot31\cdot153877$
	11	$T=2^{32}\cdot3^5\cdot5^{32}\cdot7^4\cdot31^2\cdot389\cdot643\cdot3391$
		$L=2^{14}\cdot3^7\cdot5^{18}\cdot7^2\cdot389\cdot643\cdot3391$
14	10	$T=2^{118}\cdot3^{71}\cdot5^{54}\cdot11^6\cdot23^2\cdot269^4\cdot683\cdot5791$
		$L=2^{26}\cdot3^{11}\cdot5^{50}\cdot683\cdot5791$
	11	$T=2^{108}\cdot3^{40}\cdot5^{69}\cdot11^4\cdot17^2\cdot19^2\cdot23^3\cdot269^4\cdot5903^2\cdot18802789043$
		$L=2^{30}\cdot3^8\cdot5^{43}\cdot23\cdot18802789043$
	12	$T=2^{114}\cdot3^{49}\cdot5^{66}\cdot7^4\cdot47\cdot251^2\cdot269^4\cdot619\cdot2833^2\cdot4874017157$
		$L=2^{26}\cdot3^9\cdot5^{38}\cdot7^2\cdot47\cdot619\cdot4874017157$
	13	$T=2^{96}\cdot3^{43}\cdot5^{82}\cdot7^6\cdot11^2\cdot269^4\cdot353747^2\cdot168405448423318469277268 7^*$
		$L=2^{26}\cdot3^9\cdot5^{26}\cdot1684054484233184692772687^*$

k	m	
16	11	$T=2^{106}\cdot 3^{24}\cdot 5^{58}\cdot 11^{11}\cdot 13^{4}\cdot 29^{8}\cdot 79^{2}\cdot 409^{2}\cdot 863^{4}\cdot 991\cdot 5701^{2}\cdot 28549$
		$L=2^{22}\cdot 3^{6}\cdot 5^{64}\cdot 11\cdot 991\cdot 28549$
	12	$T=2^{134}\cdot 3^{18}\cdot 5^{74}\cdot 11^{2}\cdot 13^{4}\cdot 29^{8}\cdot 37^{2}\cdot 163\cdot 257\cdot 739^{2}\cdot 863^{4}\cdot 3929^{2}\cdot 38669 \times 107603$
		$L=2^{22}\cdot 3^{6}\cdot 5^{56}\cdot 163\cdot 257\cdot 38669\cdot 107603$
	13	$T=2^{106}\cdot 3^{22}\cdot 5^{68}\cdot 7^{14}\cdot 11^{2}\cdot 13^{4}\cdot 29^{8}\cdot 863^{4}\cdot 920193557^{2} \times 18409196539129609^{*}$
		$L=2^{22}\cdot 3^{10}\cdot 5^{48}\cdot 7^{2}\cdot 18409196539129609^{*}$
	14	$T=2^{126}\cdot 3^{22}\cdot 5^{86}\cdot 7^{6}\cdot 11^{3}\cdot 29^{8}\cdot 163\cdot 223\cdot 863^{4}\cdot 77628664507^{2} \times 7239374722421197 5379^{*}$
		$L=2^{22}\cdot 3^{6}\cdot 5^{40}\cdot 11\cdot 163\cdot 223\cdot 72393747224211975379^{*}$
	15	$T=2^{120}\cdot 3^{26}\cdot 5^{82}\cdot 7^{4}\cdot 11^{2}\cdot 29^{8}\cdot 863^{4}\cdot 1259\cdot 5009\cdot 14831\cdot 24379\cdot 98299 \times 18511104979^{2}\cdot 261306370933$
		$L=2^{22}\cdot 3^{6}\cdot 5^{28}\cdot 11^{2}\cdot 1259\cdot 5009\cdot 14831\cdot 24379\cdot 98299\cdot 261306370933$

References

[1] T. Asai, On the Fourier coefficients of automorphic forms at various cusps and some applications to Rankin's convolution, J. Math. Soc. Japan, **28** (1976), 48-61.

[2] K. Doi and T. Miyake, Automorphic forms and number theory (in Japanese), Kinokuniya Shoten, Tokyo, 1976.

[3] E. Hecke, Über Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung II, Math. Ann., **114** (1937), 316-351.

[4] G. Shimura, The special values of the zeta functions associated with cusp forms, Comm. Pure Appl. Math., **29** (1976), 783-804.

Yoshitaka MAEDA

Department of Mathematics
Hokkaido University
Sapporo 060, Japan