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On the sample continuity of $\mathscr{S}'$-processes

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1. Introduction and results.

Let E be a nuclear Fréchet space and E' the topological dual space (of which the Schwartz space $\mathscr{S}'$ of tempered distributions is a typical one). We denote by $\langle x, \xi\rangle$, $x\in E'$, $\xi\in E$ the canonical bilinear form on $E'\times E$. Let $\boldsymbol{X}=\{X_t; t\in[0, \infty)\}$ be a stochastic process defined on a complete probability space $(\Omega, \mathfrak{F}, P)$ with values in E'. In the previous paper [4] the author showed that $X_1=\{X_t; t\in[0, 1]\}$ has a strongly continuous version if for each $\xi\in E$, the process $\langle X_t, \xi\rangle$ has a continuous version and satisfies the moment condition

$$\int_\Omega \sup_{t\in Q}|\langle X_t, \xi\rangle|^\rho dP<+\infty\,, \tag{1.1}$$

where $\rho>0$ and Q is a countable dense subset of $[0, 1]$.

In this paper, we will prove the similar results without assuming the moment condition. The results are stated as follows:

THEOREM 1. *Let E be a nuclear Fréchet space and $\boldsymbol{X}$ an E'-valued stochastic process such that for each ξ in E the real stochastic process $X_\xi=\{\langle X_t, \xi\rangle; t\in[0, \infty)\}$ has a continuous version. Then $\boldsymbol{X}$ has a strongly continuous version.*

THEOREM 2. *Let E be a nuclear Fréchet space and $\boldsymbol{X}$ an E'-valued stochastic process such that for each ξ in E the real stochastic process X_ξ has a version which is right continuous and has left-hand limits. Then $\boldsymbol{X}$ has a version which is right continuous and has left-hand limits in the strong topology of E'.*

The proof of Theorem 1 will be given in Section 2. The proof of Theorem 2 is quite similar to that of Theorem 1, so that we will omit it. As applications of Theorem 1, we will give a characterization of the existence of a continuous version with respect to a certain norm and a generalized Kolmogorov's criterion for continuity in Section 3.

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2. Proof.

Let $\|\cdot\|_1 \leqq \|\cdot\|_2 \leqq \cdots \leqq \|\cdot\|_p \leqq \cdots$ be an increasing sequence of Hilbertian semi-norms defining the topology of E, E_p the completion by $\|\cdot\|_p$, E'_p the topological dual space of E_p and $\|\cdot\|_{-p}$ the dual norm of E'_p. $\boldsymbol{R}^n$ and T_+ denote an n-dimensional Euclidean space and a closed interval $[0, T]$ respectively.

It is enough to prove that for any fixed $T>0$, $\{X_t ; t \in T_+\}$ has a strongly continuous version. Let D be a countable dense subset of T_+. First we prove

LEMMA 1. *For any $\varepsilon>0$ there exist a natural number p and a $\delta>0$ such that*

$$\int_\Omega \sup_{t\in D} |1-e^{i\langle X_t, \xi\rangle}| \, dP \leqq \varepsilon + 2\frac{\|\xi\|_p^2}{\delta^2} \tag{2.1}$$

for every ξ in E.

Before we show the lemma, we will introduce the following;

$$M(\xi)=\int_\Omega \frac{\sup_{t\in D}|\langle X_t, \xi\rangle|}{1+\sup_{t\in D}|\langle X_t, \xi\rangle|} dP, \qquad \xi \in E .$$

Then $M(\xi)$ has the following properties.

1) $M(\xi) \geqq 0$ and $M(-\xi)=M(\xi)$.

2) $M(\xi+\eta) \leqq M(\xi)+M(\eta)$ for any ξ, η in E.

3) $M(\xi)$ is a lower semi-continuous function on E.

4) $\lim_{n\to\infty} M(\xi/n)=0$.

1) and 2) are trivial, so we prove 3) and 4). If $\xi_n \to \xi$ in E, we have

$$\begin{aligned}
\liminf_{n\to\infty} M(\xi_n) &\geqq \int_\Omega \liminf_{n\to\infty} \frac{\sup_{t\in D}|\langle X_t, \xi_n\rangle|}{1+\sup_{t\in D}|\langle X_t, \xi_n\rangle|} dP \\
&\geqq \int_\Omega \frac{\liminf_{n\to\infty}\sup_{t\in D}|\langle X_t, \xi_n\rangle|}{1+\liminf_{n\to\infty}\sup_{t\in D}|\langle X_t, \xi_n\rangle|} dP \\
&\geqq \int_\Omega \frac{\sup_{t\in D}(\liminf_{n\to\infty}|\langle X_t, \xi_n\rangle|)}{1+\sup_{t\in D}(\liminf_{n\to\infty}|\langle X_t, \xi_n\rangle|)} dP \\
&= M(\xi),
\end{aligned}$$

so that 3) is proved. Since X_ξ has a continuous version, $\sup_{t\in D}|\langle X_t, \xi\rangle| < +\infty$ almost surely, so that by Lebesgue's bounded convergence theorem 4) is proved.

Now Lemma 1.2.3 (page 386) of D. Xia [5] tells us that the properties 1), 2), 3) and 4) imply that $M(\xi)$ is continuous at O in E. For any $\varepsilon>0$ there exists a $\delta_1>0$ such that

$$|1-e^{is}|\leqq\frac{\varepsilon}{2} \qquad \text{if } |s|\leqq\delta_1 .$$

Put $\delta_2=\min\left\{\delta_1, \frac{-1+\sqrt{1+\varepsilon}}{2}\right\}$. Since $M(\xi)$ is continuous at O in E there exist a natural number p and a $\delta>0$ such that $M(\xi)\leqq(\delta_2)^2$ if $\|\xi\|_p<\delta$, so that

$$P(\omega;\sup_{t\in D}|\langle X_t, \xi\rangle|\geqq\delta_2)\leqq\frac{1+\delta_2}{\delta_2}M(\xi)$$

$$\leqq\delta_2(1+\delta_2)\leqq\frac{\varepsilon}{4}.$$

Therefore if $\|\xi\|_p<\delta$ we get

$$\int_\Omega \sup_{t\in D}|1-e^{i<X_t,\xi>}|\,dP$$

$$\leqq\int_{\tilde{\Omega}} \sup_{t\in D}|1-e^{i<X_t,\xi>}|\,dP+2P(\Omega\setminus\tilde{\Omega})$$

$$\leqq\frac{\varepsilon}{2}+2\frac{\varepsilon}{4}=\varepsilon ,$$

where $\tilde{\Omega}=\{\omega;\sup_{t\in D}|\langle X_t, \xi\rangle|<\delta_2\}$.

On the other hand it always holds that $\sup_{t\in D}|1-e^{i<X_t,\xi>}|\leqq2$, so that if $\|\xi\|_p\geqq\delta$ we have

$$\int_\Omega \sup_{t\in D}|1-e^{i<X_t,\xi>}|\,dP\leqq2\frac{\|\xi\|_p^2}{\delta^2}.$$

Thus the proof of Lemma 1 is completed.

Following the idea of K. Itô [2], we will proceed with our argument. Since E is separable, there exists a countable dense subset $F=\{\xi_1, \xi_2, \cdots\}$ of E. For each natural number n we choose a complete orthonormal system $\{e_j^n\}$ of E_n by the Schmidt orthogonalization of F. Then it is evident that

$$\xi_k=\sum_{j=1}^{m(n,k)} a_j^n(k)e_j^n+\theta_k^n , \qquad \text{where } m(n, k)\leqq k \text{ and } \|\theta_k^n\|_n=0. \tag{2.2}$$

For a given $\varepsilon>0$ let p be the natural number determined by Lemma 1. Since E is nuclear, there exists a natural number $q>p$ such that

$$\sum_{j=1}^{\infty}\|e_j^q\|_p^2<+\infty . \tag{2.3}$$

Then we have

LEMMA 2.

$$P(\omega;\sup_{t\in D}\|X_t\|_{-q}<+\infty)\geqq 1-2\frac{\sqrt{e}}{\sqrt{e}-1}\varepsilon. \tag{2.4}$$

According to the estimation of A. Badrikian [1], for $C>0$ we have

$$\begin{aligned}
(2.5)\quad & P(\omega;\sup_{t\in D}\sum_{j=1}^{\infty}\langle X_t, e_j^q\rangle^2>C^2)\\
&=\lim_{n\to\infty}P(\omega;\sup_{t\in D}\sum_{j=1}^{n}\langle X_t, e_j^q\rangle^2>C^2)\\
&\leqq\lim_{n\to\infty}\frac{\sqrt{e}}{\sqrt{e}-1}\int_\Omega(1-\exp(-\sup_{t\in D}\sum_{j=1}^{n}\langle X_t, e_j^q\rangle^2/2C^2))dP\\
&=\lim_{n\to\infty}\frac{\sqrt{e}}{\sqrt{e}-1}\int_\Omega\sup_{t\in D}(1-\exp(-\sum_{j=1}^{n}\langle X_t, e_j^q\rangle^2/2C^2))dP\\
&=\lim_{n\to\infty}\frac{\sqrt{e}}{\sqrt{e}-1}\int_\Omega\sup_{t\in D}\Big\{\int_{R^n}(1-\exp(i\sum_{j=1}^{n}y_j\langle X_t, e_j^q\rangle))\\
&\qquad\frac{C^n}{(\sqrt{2\pi})^n}\exp(-\sum_{j=1}^{n}C^2y_j^2/2)dy_1dy_2\cdots dy_n\Big\}dP\\
&\leqq\lim_{n\to\infty}\frac{\sqrt{e}}{\sqrt{e}-1}\int_{R^n}\Big(\int_\Omega\sup_{t\in D}|1-\exp(i\sum_{j=1}^{n}y_j\langle X_t, e_j^q\rangle)|\,dP\Big)\\
&\qquad\frac{C^n}{(\sqrt{2\pi})^n}\exp(-\sum_{j=1}^{n}C^2y_j^2/2)dy_1dy_2\cdots dy_n.
\end{aligned}$$

By (2.1) of Lemma 1 we have

(2.6) the last term of the above inequalities

$$\begin{aligned}
&\leqq\lim_{n\to\infty}\frac{\sqrt{e}}{\sqrt{e}-1}\int_{R^n}\Big(\varepsilon+\frac{2}{\delta^2}\|\sum_{j=1}^{n}y_je_j^q\|_p^2\Big)\frac{C^n}{(\sqrt{2\pi})^n}\exp(-\sum_{j=1}^{n}C^2y_j^2/2)dy_1dy_2\cdots dy_n\\
&=\frac{\sqrt{e}}{\sqrt{e}-1}\Big(\varepsilon+\frac{2}{\delta^2}(\sum_{j=1}^{\infty}\|e_j^q\|_p^2/C^2)\Big).
\end{aligned}$$

It follows from (2.3) and the estimations (2.5) and (2.6) that

$$P(\omega;\sup_{t\in D}\sum_{j=1}^{\infty}\langle X_t, e_j^q\rangle^2<+\infty)\geqq 1-\frac{\sqrt{e}}{\sqrt{e}-1}\varepsilon. \tag{2.7}$$

On the other hand, it follows from (2.2) that $\|\theta_j^q\|_p=0$, $j=1, 2, \cdots$, for $\|\cdot\|_p\leqq\|\cdot\|_q$. Then by changing e_j^q for θ_j^q in the estimations (2.5) and (2.6), we have

(2.8)
$$P(\omega\,;\sup_{t\in D}\sum_{j=1}^{\infty}\langle X_t,\ \theta_j^q\rangle^2>0)$$
$$=\lim_{m\to\infty}P\Big(\omega\,;\sup_{t\in D}\sum_{j=1}^{\infty}\langle X_t,\ \theta_j^q\rangle^2>\frac{1}{m}\Big)$$
$$\leqq\lim_{m\to\infty}\lim_{n\to\infty}\frac{\sqrt{e}}{\sqrt{e}-1}\Big(\varepsilon+\frac{2m}{\delta^2}(\sum_{j=1}^{n}\|\theta_j^q\|_p^2)\Big)$$
$$=\frac{\sqrt{e}}{\sqrt{e}-1}\,\varepsilon\,.$$

Put $\Gamma=\{\omega\,;\sup_{t\in D}\sum_{j=1}^{\infty}\langle X_t,\ e_j^q\rangle^2<+\infty\}\cap\{\omega\,;\sup_{t\in D}\sum_{j=1}^{\infty}\langle X_t,\ \theta_j^q\rangle^2=0\}$, then from (2.7) and (2.8) we get

(2.9)
$$P(\Gamma)\geqq 1-2\frac{\sqrt{e}}{\sqrt{e}-1}\,\varepsilon\,.$$

Now, since F is dense in E and $X_t(\omega)\in E'$ for every $\omega\in\Omega$ and $t\in D$, then for each ξ in E satisfying $\|\xi\|_q\leqq 1$ there exists a sequence $\{\xi_{k_\nu}\}$ of elements of F such that

(2.10)
$$\langle X_t(\omega),\ \xi\rangle=\lim_{\nu\to\infty}\langle X_t(\omega),\ \xi_{k_\nu}\rangle\qquad\text{for each }\omega\in\Omega\text{ and }t\in D\,,$$

(2.11)
$$\lim_{\nu\to\infty}\|\xi-\xi_{k_\nu}\|_q=0\,.$$

If $\omega\in\Gamma$, by (2.2), (2.10) and (2.11) we have for each $t\in D$,

$$|\langle X_t(\omega),\ \xi\rangle|=\lim_{\nu\to\infty}|\sum_{j=1}^{m(q,\,k_\nu)}a_j^q(k_\nu)\langle X_t(\omega),\ e_j^q\rangle+\langle X_t(\omega),\ \theta_{k_\nu}^q\rangle|$$
$$\leqq\lim_{\nu\to\infty}(\sum_{j=1}^{m(q,\,k_\nu)}a_j^q(k_\nu)^2)^{1/2}(\sum_{j=1}^{\infty}\langle X_t(\omega),\ e_j^q\rangle^2)^{1/2}$$
$$=\lim_{\nu\to\infty}\|\xi_{k_\nu}\|_q(\sum_{j=1}^{\infty}\langle X_t(\omega),\ e_j^q\rangle^2)^{1/2}$$
$$\leqq(\sum_{j=1}^{\infty}\langle X_t(\omega),\ e_j^q\rangle^2)^{1/2}\,.$$

Therefore

$$\sup_{\|\xi\|_q\leqq 1}\sup_{t\in D}|\langle X_t(\omega),\ \xi\rangle|\leqq(\sup_{t\in D}\sum_{j=1}^{\infty}\langle X_t(\omega),\ e_j^q\rangle^2)^{1/2}<+\infty\,.$$

Thus the proof of Lemma 2 is completed.

Let $\{\varepsilon_N\}$ be a sequence of positive numbers decreasing to 0. Then by (2.4) of Lemma 2 there exists an increasing sequence of natural numbers $\{p_N\}$ such that $P(\omega; \sup_{t\in D}\|X_t\|_{-p_N}<+\infty)\geqq 1-\varepsilon_N$. Then we have

$$P(\bigcup_{N=1}^{\infty}\{\omega; \sup_{t\in D}\|X_t\|_{-p_N}<+\infty\})=1. \tag{2.12}$$

Now let us proceed to construct the strongly continuous version. Our method is similar to the proof of Theorem 2 in [3]. Put $\Lambda_N=\{\omega; \sup_{t\in D}\|X_t\|_{-p_N}<+\infty\}$, then $\Lambda_N\subset\Lambda_{N+1}$, $N=1, 2, \cdots$. Set $\Omega_1=\Lambda_1$, $\Omega_2=\Lambda_2-\Lambda_1, \cdots, \Omega_N=\Lambda_N-\Lambda_{N-1}, \cdots$. For each ξ in E we denote by $\{X_\xi(t, \omega); t\in[0, \infty)\}$ the continuous version of X_ξ. By the nuclearity of E, for each natural number N there exists a natural number $q_N>p_N$ such that $\sum_{j=1}^{\infty}\|e_j^{q_N}\|_{p_N}^2<+\infty$. Then if $\omega\in\Omega_N$ we have

$$\sum_{j=1}^{\infty}\sup_{t\in T_+}(X_{e_j^{q_N}}(t, \omega))^2\leqq\sum_{j=1}^{\infty}(\sup_{t\in D}\|X_t(\omega)\|_{-p_N}^2)\|e_j^{q_N}\|_{p_N}^2<+\infty \tag{2.13}$$

and

$$\sum_{j=1}^{\infty}\sup_{t\in T_+}(X_{\theta_j^{q_N}}(t, \omega))^2\leqq\sum_{j=1}^{\infty}(\sup_{t\in D}\|X_t(\omega)\|_{-p_N}^2)\|\theta_j^{q_N}\|_{p_N}^2=0. \tag{2.14}$$

Then by repeating the argument deriving (2.4) from (2.9), (2.13) and (2.14) tells us that for each $t\in T_+$,

$$\|X_t(\omega)\|_{-q_N}<+\infty \qquad \text{almost surely in } \Omega_N,$$

so that

$$\|X_t(\omega)\|_{-q_N}^2=\sum_{j=1}^{\infty}\langle X_t(\omega), e_j^{q_N}\rangle^2 \qquad \text{almost surely in } \Omega_N. \tag{2.15}$$

Let $\{x_j^{q_N}\}$ be a sequence of elements in E'_{q_N} such that $\langle x_j^{q_N}, e_i^{q_N}\rangle=\delta_{ji}$, where $\delta_{ji}=1$ if $j=i$ and $\delta_{ji}=0$ if $j\neq i$. By (2.13), for $\omega\in\Omega_N$ we can construct a $\|\cdot\|_{-q_N}$-continuous path $\{z_t^N(\omega); t\in T_+\}$ as follows:

$$z_t^N(\omega)=\sum_{j=1}^{\infty}X_{e_j^{q_N}}(t, \omega)x_j^{q_N},$$

where the right hand limit means $\|\cdot\|_{-q_N}$-convergence.

Set

$$Z_t(\omega)=\begin{cases} z_t^N(\omega) & \text{if } \omega\in\Omega_N, \\ 0 & \text{if } \omega\notin\bigcup_{N=1}^{\infty}\Omega_N. \end{cases}$$

Then $\{Z_t; t\in T_+\}$ becomes a strongly continuous version of $\{X_t; t\in T_+\}$ by (2.15). This completes the proof.

3. Application.

Important corollaries follow from our proof of the theorem. Let $X_T=\{X_t; t\in T_+\}$ be an E'-valued stochastic process. For each ξ in E we denote by $X_T(\xi)$ the real stochastic process $\{\langle X_t, \xi\rangle; t\in T_+\}$. We will say that X_T is p-continuous if there exists a countable dense subset I of T_+ and for any $\varepsilon>0$ and $\lambda>0$ there exists a $\delta>0$ such that $P(\omega; \sup_{t\in I}|\langle X_t, \xi\rangle|>\varepsilon)\leqq\lambda$ if $\|\xi\|_p\leqq\delta$.

For example, the previous condition (1.1) implies X_1 is p-continuous for some natural number p.

Now we have

COROLLARY 1. *Let E be a nuclear Fréchet space and X_T an E'-valued p-continuous stochastic process such that for each ξ in E the real stochastic process $X_T(\xi)$ has a continuous version. Then there exists a natural number $k>p$ such that X_T has a $\|\cdot\|_{-k}$-continuous version.*

COROLLARY 2. *Let E be a nuclear Fréchet space and X_T an E'-valued p-continuous stochastic process such that for each ξ in E the real stochastic process $X_T(\xi)$ has a version which is right continuous and has left-hand limits. Then there exists a natural number $k>p$ such that X_T has a version which is right continuous and has left-hand limits in the $\|\cdot\|_{-k}$-topology.*

SKETCH OF THE PROOF OF COROLLARIES. In these cases $M(\xi)$ defined in the previous section becomes continuous in the $\|\cdot\|_p$-topology at O in E. Letting $\varepsilon\downarrow 0$ in (2.4) of Lemma 2, we get

$$P(\omega; \sup_{t\in D}\|X_t\|_{-q}<+\infty)=1.$$

The rest of the proof is quite similar to that in Section 2.

As the corollary to Theorem 1 we have

COROLLARY 3 (generalized Kolmogorov's criterion for continuity). *Let E be a nuclear Fréchet space and for each ξ in E let $V_t(\xi)$ be a non-negative, non-decreasing and continuous function of t. Let $\boldsymbol{X}$ be an E'-valued stochastic process such that for each $T>0$ and each ξ in E,*

$$\int_\Omega|\langle X_t, \xi\rangle-\langle X_s, \xi\rangle|^{\alpha_{\xi,T}}dP\leqq\gamma_{\xi,T}(V_t(\xi)-V_s(\xi))^{\beta_{\xi,T}+1}, \quad 0\leqq s\leqq t\leqq T,$$

where $\alpha_{\xi,T}$, $\beta_{\xi,T}$ and $\gamma_{\xi,T}$ are strictly positive numbers. Then $\boldsymbol{X}$ has a strongly continuous version.

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