Isotropic minimal immersions of spheres into spheres

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1. Introduction.

It is an interesting and important problem to investigate the existence and the rigidity of a minimal immersion of a Riemannian manifold into a unit sphere. In the case of an n-dimensional sphere $S^n$, associated with each positive integer $s$, there exists an isometric minimal immersion $\phi_{n,s}: S_{k(s)}^n \rightarrow S_1^m(s)$, where $S_l$ denotes an l-dimensional sphere with constant sectional curvature $c$ and $k(s)$ and $m(s)$ are given as follows:

$$k(s)=\frac{n}{s(s+n-1)},$$
$$m(s)=(2s+n-1) \left(\frac{(s+n-2)!}{s!(n-1)!}\right)-1.$$  

$\phi_{n,s}$ is given by s-th eigenfunctions of the Laplacian $\Delta$ on $S^n$ (T. Takahashi [9]). These immersions $\phi_{n,s}$ are called “standard minimal immersions” (cf. § 2).

It will be convenient to say that a minimal immersion $\varphi: S^n \rightarrow S_1 \subset R^{l+1}$ is full if $\varphi(S^n)$ is not contained in a hyperplane of $R^{l+1}$ and that two such immersions $\varphi_1, \varphi_2$ are equivalent if there exists an isometry $\rho$ of $S_1$ such that $\varphi_2=\rho^*\varphi_1$.

For the rigidity of the immersion $\phi_{n,s}$, do Carmo and Wallach [4] showed the following result.

**Theorem ([4]).** In the case of $s=1, 2,$ and $3$, the immersion $\phi_{n,s}$ is rigid. Namely any isometric minimal immersion $\varphi$ of $S_{k(s)}^n$ into $S_1$ is equivalent to $\phi_{n,s}$. However when $n \geq 3$ and $s \geq 4$ the immersion $\phi_{n,s}$ is not rigid. That is, the set of equivalence classes of isometric minimal immersions of $S_{k(s)}^n$ into $S_1$ can be smoothly parametrized by a compact convex body $L \subset W$ in a vector space $W$, with $\dim W=N(n, s) \geq 18$.

In this note we consider characterizations of the standard minimal immersions in such a broad class of minimal immersions. First we characterize it by making use of the notion of isotropic immersions introduced by B. O’Neill [5]. We say that an $R^l$-valued symmetric multi-linear form $B$ on $R^n$ is isotropic if $\|B(u, u, \cdots, u)\|=\text{constant}$ for any unit vectors $u$ in $R^n$ (cf. § 2). Then we have the following result.
THEOREM A. Let $\varphi : S_{n(s)}^{s} \rightarrow S_{l}^{1}$ be an isometric minimal immersion. Assume that $\varphi$ is full and that $n \geq 3$ and $s \geq 4$. If the degree of $\varphi \geq \left[ \frac{s}{2} \right]$, where $\left[ \frac{s}{2} \right]$ is the largest integer less than or equal to $s/2$, and the $j$-th fundamental form $B_j$ is isotropic for $2 \leq j \leq \left[ \frac{s}{2} \right]$, then we have $l=m(s)$ and $\varphi$ is equivalent to the standard minimal immersion $\varphi_{n,s}$.

We shall refer to the notions of higher fundamental forms and the degree of the immersion in §2.

Next we characterize the standard minimal immersion using the concept of a helical geodesic immersion. Let $\varphi : M \rightarrow \overline{M}$ be an isometric immersion of a connected complete Riemannian manifold $M$ into a Riemannian manifold $\overline{M}$. If for each geodesic $\gamma$ of $M$ the curve $\varphi \cdot \gamma$ in $\overline{M}$ has constant curvatures of osculating order $d$ which are independent of $\gamma$, then $\varphi$ is called a helical geodesic immersion of order $d$ (K. Sakamoto [8]). It is known that a strongly harmonic manifold admits a helical geodesic minimal immersion into a sphere (A. Besse [2]). Sakamoto [8] stated that the study of helical geodesic immersions will be useful for the study of the conjecture that the harmonic manifolds are locally symmetric. In this paper we show the following result.

THEOREM B. Let $\varphi : S_{n(s)}^{s} \rightarrow S_{l}^{1}$ be a helical geodesic minimal immersion. Assume that $\varphi$ is full. Then $\varphi$ is equivalent to the standard minimal immersion $\varphi_{n,s}$; in particular the order of the helical geodesic immersion $\varphi$ is equal to $s$ and $l=m(s)$.

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2. The standard minimal immersions and their properties.

2.1. The standard minimal immersions. Let $M=G/K$ be an $n$-dimensional compact homogeneous Riemannian group and $V^s$ the $s$-th eigenspace of the Laplacian $\Delta_M$ corresponding to the $s$-th eigenvalue $\lambda$. We define an inner product $\langle \cdot , \cdot \rangle$ in $V^s$ by

$$\langle f, h \rangle = \int_M f \cdot h \, d\mu, \quad f, h \in V^s.$$ 

For simplicity, we normalize the canonical measure $d\mu$ of $(M, g)$ in such a way that $\int_M d\mu = \dim V^s = m(s) + 1$. Let $\{f_0, f_1, \cdots, f_{m(s)}\}$ be an orthonormal basis for $V^s$ and define a map $\varphi : M \rightarrow \mathbb{R}^{m(s)+1}$ by $\varphi(p) = (f_0(p), \cdots, f_{m(s)}(p))$, $p \in M$. The transitive action $G$ on $M$ induces a natural action on $V^s$ by $(g \cdot f)(p) = f(g^{-1}p)$, $g \in G$, $p \in M$. It is easily seen that $\sum_{i=0}^{m(s)} f_i(p) = 1$ for all $p \in M$, i.e., $\varphi(M) \subset S_{l}^{1}$. 

Isotropic minimal immersions

The irreducibility of the linear isotropy action of \(K\) and the \(G\)-invariance of the metric \(g\) imply that \(\phi\) is an isometric immersion of \((M, cg)\) into \(S^{m(S)}\). By a theorem of Takahashi \([9]\), \(\phi\) is then a minimal immersion of \((M, cg)\) into \(S^{m(S)}\) and \(c=\lambda_s/n\). We shall call this isometric minimal immersion \(\phi\) of \((M, (\lambda_s/n)g)\) into \(S^{m(S)}\) the \(s\)-th standard minimal immersion of \(M\).

The standard minimal immersion can be described in other words as follows. Take an orthonormal basis \(\{f_0, f_1, \cdots, f_{m(S)}\}\) of \(V^s\) such that \(e_0=\phi(eK)\) \(=(f_0(eK), \cdots, f_{m(S)}(eK))\), where \(e\) is the identity element of \(G\). Let \(A\) be an isometry of \(V^s\) into \(R^{m(S)+1}\) such that \(A(f_j)=e_j, j=0, 1, \cdots, m(s)\). Let \(G\) act on \(R^{m(S)+1}\) so that \(A\) is a \(G\)-isomorphism. Then by a simple computation we get \(\phi(gK)=A(g\cdot f_0), g\in G\). Since \(A\) is an isometry, we can consider \(\phi\) as an isometric minimal immersion of \((M, (\lambda_s/n)g)\) into a unit hypersphere in \(V^s\) defined by \(\phi(gK)=g\cdot f_0, g\in G\).

Let \(\varphi : M \rightarrow \overline{M}\) be an isometric immersion of a Riemannian homogeneous space \(M=G/K\) into a Riemannian manifold of constant sectional curvature \(\overline{M}\). We say that \(\varphi\) is equivariant if there exists a continuous homomorphism \(\rho\) of \(G\) into the isometry group \(I(\overline{M})\) of \(\overline{M}\) such that

\[\varphi(g \cdot p) = \rho(g) \varphi(p) \quad p \in M, g \in G.\]

It is easily seen that the standard minimal immersion is naturally equivariant.

2.2. Higher fundamental forms and degrees of isometric immersions. In this part, we define the higher fundamental forms and the degree of an isometric immersion (Wallach \([10]\)). Let \(\overline{M}\) be a Riemannian manifold of constant curvature. Let \(\varphi : M \rightarrow \overline{M}\) be an isometric immersion of a Riemannian manifold \(M\) into \(\overline{M}\). Let \(B_a\) be the second fundamental form of \(\varphi\) at \(p \in M\) and \(O_p^2\) be the linear span of the image of \(B_a\) in the normal space \(N_p(M)\) of the immersion \(\varphi\) at \(p \in M\). We call \(\varphi_a T_pM + O_p^2\) the second osculating space at \(p \in M\). We say that \(p \in M\) is degree 2 regular if \(O_p^2\) is of maximal dimension. Let \(R_2 \subset M\) be the set of all degree 2 regular points of \(M\). Then \(R_2\) is open in \(M\). Let \(N_a\) be the normal projection in \(N_p(M)\) relative to \(N_p(M) = O_p^2 + (O_p^2)^\perp\) (we write \(v \rightarrow v^{N_2} \in (O_p^2)^\perp\)). We define \(B_a(u_1, u_2, u_3) = (\hat{\nabla}_{u_1}(B_a(u_3, u_2)))^{N_2}\) for \(u_1, u_2, u_3 \in T_pM\) arbitrarily extended to the vector fields on \(M\), where \(\hat{\nabla}\) denotes the Riemannian connection on \(\overline{M}\). \(B_a\) is well-defined and defines a symmetric tensor field on \(R_2\). Let \(O_p^a\) be the linear span of the image of \(B_a\). We call \(B_a\) the third fundamental form of \(\varphi\) at \(p\) and \(\varphi_a T_pM + O_p^2 + O_p^a\) the third osculating space. We call a point \(p \in R_2\) degree 3 regular if \(O_p^a\) is maximal. We define \(B_j, O_j^a\) for \(j=2, 3, \cdots\) by recursion as above on the space \(R_{j-1}\) of all degree \(j-1\) regular points of \(M\). We call \(B_j\) the \(j\)-th fundamental form of \(\varphi\) and \(\varphi_a T_pM + O_p^2 + \cdots + O_p^a\) the \(j\)-th osculating space. Clearly the above process must eventually stop since \(\dim(\varphi_a T_pM + O_p^2 + O_p^3 + \cdots + O_p^a) \leq \dim T_p\overline{M}\).
Let $d$ be the first integer $\geqq 2$ such that $B_d \not\equiv 0$ but $B_{d+1} \equiv 0$. Then we call $d$ the degree of $\varphi$ and the set of all $d$-regular points will be called the set of all completely regular points of $M$, denoted $M' = R_d$. In particular, when $\varphi$ is totally geodesic, i.e., $B_1 \equiv 0$, we say that $\varphi$ has degree 1.

**Lemma 2.1** (Wallach [10]).

1. $B_j : T_pM \times T_pM \times \cdots \times T_pM \to O_p^j$ is an $O_p^j$-valued symmetric $j$-linear form on $T_pM$ for $p \in R_{j-1}$. Then $B_j$ induces a linear map $S^j(T_pM) \to O_p^j$, where $S^j(T_pM)$ denotes the $j$-fold symmetric power of $T_pM$.

2. Let $e_1, \cdots, e_n$ be an orthonormal basis of $T_pM$. Set $r_p = \sum_{i=1}^n e_i^2 \in S^2(T_pM)$. If $\varphi : M \to \overline{M}$ is minimal, then

$$\ker B_j \supset r_p \cdot S^{j-2}(T_pM), \quad j \geqq 2.$$ 

2.3. Higher fundamental forms of the standard minimal immersions. Let $\psi : M \to S^n$ be the standard minimal immersion of a compact homogeneous space $M = G/K$ defined in 2.1. Since $\psi$ is equivariant, the set of all completely regular points of $M$ coincides with $M$. Moreover the following properties hold.

**Lemma 2.2.**

1. $B_j$ is $G$-invariant and commutes with $\rho(g)$,

$$B_j(g \cdot u_1, \cdots, g \cdot u_j) = \rho(g) B_j(u_1, \cdots, u_j)$$

$$\rho(g)O_p^j = O_{g \cdot p}^j$$

$$N_j \cdot \rho(g) = \rho(g) N_j, \quad g \in G.$$ 

In particular, $B_j : S^j(T_{eK}M) \to O_{eK}^j(M)$ is a $K$-homomorphism.

2. $V^s$ admits an orthogonal direct sum decomposition

$$V^s = R \cdot \varphi(eK) + \psi_* T_{eK}M + O_{eK}^2 + \cdots + O_{eK}^d,$$ 

where $d$ is the degree of $\psi$.

**Remark 2.3.** When $M = G/K$ is a compact rank 1 symmetric space, $K$ acts transitively on the unit sphere of $T_{eK}M$. Then by Lemma 2.2 (1),

$$\|B_j(k \cdot u, \cdots, k \cdot u)\| = \|\rho(k)B_j(u, \cdots, u)\| = \|B_j(u, \cdots, u)\| \quad k \in K.$$ 

Thus $B_j$ is isotropic at $eK$ and again by Lemma 2.2 (1) $B_j$ is constant isotropic on $M$.

**Remark 2.4.** The degrees of the standard minimal immersions of a compact rank 1 symmetric space $M$ into spheres are computed.

1. When $M = S^n$, the degree of the $s$-th standard minimal immersion $\psi_s$ is $s$.

2. When $M$ is a complex projective space $P_n(C)$, a quaternion projective space $P_n(H)$, or a Cayley projective plane $P_2(Cay)$, the degree of $\psi_s$ is $2s$.

Do Carmo and Wallach [4] showed the above result in the case of a sphere and K. Mashimo [6, 7] calculated the degree for the other cases.
3. Higher fundamental forms of isotropic minimal immersions.

Let \( \varphi_{n,s}: M_{k(s)}^{n} \rightarrow S_{1}^{m(s)} \) be the standard minimal immersion of an \( n \)-dimensional compact rank 1 symmetric space \( M \) into a unit sphere corresponding to the \( s \)-th eigenvalue, where \( M_{k(s)}^{n} \) has the induced Riemannian metric by \( \varphi_{n,s} \). Let \( \varphi: M_{k(s)}^{n} \rightarrow S_{1}^{l} \) be another minimal immersion corresponding to the same eigenvalue. We compare the higher fundamental forms of \( \varphi \) with those of \( \varphi_{n,s} \) when the higher fundamental forms of \( \varphi \) are isotropic. Namely we show the following.

**Proposition 3.1.** We denote by \( B_{j} \) and \( \hat{B}_{j} \) the \( j \)-th fundamental forms of \( \varphi \) and \( \varphi_{n,s} \) respectively. Let \( i \) be an integer such that \( 2 \leq i \leq \min \{ \text{degree of } \varphi, \text{degree of } \varphi_{n,s} \} \). If \( B_{k} \) is isotropic for \( 2 \leq k \leq i \) at every degree \( k-1 \) regular point \( p \in R_{k-1} \), with respect to \( \varphi \), then we have

\[
\langle B_{k}(u_{1}, \cdots, u_{k}), B_{k}(v_{1}, \cdots, v_{k}) \rangle = \langle \hat{B}_{k}(u_{1}, \cdots, u_{k}), \hat{B}_{k}(v_{1}, \cdots, v_{k}) \rangle,
\]

\( 2 \leq k \leq i \), \( u_{1}, \cdots, u_{k}, v_{1}, \cdots, v_{k} \in T_{p}M \) at every point \( p \in R_{k-1} \). In particular, the set of all degree \( k \) regular points with respect to \( \varphi \) coincides with \( M \) for \( 2 \leq k \leq i \).

As preliminaries we state two well-known lemmas.

**Lemma 3.2.** Let \( B \) be an \( R^{k} \)-valued symmetric \( j \)-linear form on \( R^{n} \). \( B \) is \( \lambda \)-isotropic, i.e., \( \| B(x, \cdots, x) \| = \lambda \) for any unit vector \( x \in R^{n} \), if and only if

\[
S_{2j}\{ \langle B(u_{1}, \cdots, u_{j}), B(u_{j+1}, \cdots u_{2j}) \rangle \} = \lambda^{2}S_{2j}\{ \langle u_{1}, u_{2} \rangle \cdots \langle u_{2j-1}, u_{2j} \rangle \}
\]

for \( u_{1}, \cdots, u_{2j} \in R^{n} \),

where \( S_{2j} \) denotes the symmetrizer of order \( 2j \).

Next we recall the equations of Gauss and Ricci. We prepare notations. Let \( \varphi: M \rightarrow S_{1}^{l} \) be an isometric immersion. We denote by \( \overline{\nabla} \) and \( \nabla \) the covariant differentiations on \( S_{1}^{l} \) and \( M \) respectively. \( \overline{\nabla} \) denotes the covariant differentiation with respect to the induced connection in the normal bundle. We define the covariant differentiation \( \overline{\nabla} \) on \( T(M) \oplus N(M) \) as follows: For any \( N(M) \)-valued tensor field \( S \) of type \((0, k)\), we define

\[
\langle \overline{\nabla}_{X}S(Y_{1}, \cdots, Y_{k}) \rangle = \nabla_{X}(S(Y_{1}, \cdots, Y_{k})) - \sum_{i=1}^{k} S(Y_{1}, \cdots \nabla_{X}Y_{i}, \cdots, Y_{k})
\]

and \( \overline{\nabla}S \) is also defined by \( \langle \overline{\nabla}S(X, Y_{1}, \cdots, Y_{k}) \rangle = \langle \overline{\nabla}_{X}S(Y_{1}, \cdots, Y_{k}) \rangle \).

**Lemma 3.3.**

(1) **Gauss equation**:

\[
\langle R(X, Y)Z, W \rangle = \langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle + \langle B_{g}(X, W), B_{g}(Y, Z) \rangle - \langle B_{g}(X, Z), B_{g}(Y, W) \rangle,
\]

where \( R \) denotes the curvature tensor with respect to \( \overline{\nabla} \).

(2) **Ricci equation**: 
\[ \langle R^{\perp}(X, Y)\xi, \eta \rangle = \langle [H_{\xi}, H_{\eta}](X), Y \rangle, \]

where \( R^{\perp} \) denotes the curvature tensor with respect to \( \nabla^{\perp} \) and \( H_{\xi} \) denotes the second fundamental tensor corresponding to the normal vector field \( \xi \). \( H_{\xi} \) is related to \( B_{2} \) as \( \langle H_{\xi}X, Y \rangle = \langle B_{2}(X, Y), \xi \rangle \).

(3) Ricci formula:
\[
\overline{\nabla}^{2}S(U, V, X_{1}, \cdots, X_{k}) - \overline{\nabla}^{2}S(V, U, X_{1}, \cdots, X_{k}) = \overline{R}(U, V)S(X_{1}, \cdots, X_{k}) = R^{\perp}(U, V)(S(X_{1}, \cdots, X_{k})) - \sum_{i=1}^{k}S(X_{1}, \cdots, R(U, V)X_{i}, \cdots, X_{k}).
\]

We prove Proposition 3.1 inductively. First we start under the assumption that the second fundamental form \( B_{2} \) of \( \varphi \) is \( \lambda \)-isotropic. We recall \( B_{2} \) is constant isotropic (Remark 2.3). Using Lemma 3.2 and Gauss equation we obtain
\[
(3.1) \quad 3\langle B_{2}(u, v), B_{2}(x, y) \rangle = \lambda^{2}\{\langle u, v \rangle \langle x, y \rangle + \langle u, x \rangle \langle y, v \rangle + \langle u, y \rangle \langle v, x \rangle \}
- \langle R(u, y)v, x \rangle - \langle R(u, x)v, y \rangle + \langle u, x \rangle \langle y, v \rangle
- \langle u, v \rangle \langle x, y \rangle + \langle u, y \rangle \langle x, v \rangle - \langle u, v \rangle \langle x, y \rangle.
\]

Since \( \varphi \) is minimal,
\[
0 = 3\left( \sum_{i=1}^{n} B_{2}(e_{i}, e_{i}), \sum_{j=1}^{n} B_{2}(e_{j}, e_{j}) \right) = \lambda^{2}n(n+2) + 2(\tau - n(n-1)),
\]
where \( \tau \) is the scalar curvature of \( M \). Then we have
\[
\lambda^{2} = 2(n(n-1)-\tau)/n(n+2).
\]

Therefore the right-hand-side of (3.1) for \( \varphi \) coincides with that of \( \psi_{n,s} \). Thus we have \( \langle B_{2}(u, v), B_{2}(x, y) \rangle = \langle \hat{B}_{2}(u, v), \hat{B}_{2}(x, y) \rangle \), which implies that the dimension of \( O_{p}^{2} \) of \( \varphi \) is equal to that of \( \psi_{n,s} \) at every point \( p \in M \). Therefore every point of \( M \) is degree 2 regular for the immersion \( \varphi \).

Next step we shall show that \( \langle \overline{\nabla}B_{2}(x, y, z), B_{2}(u, v) \rangle = \langle \overline{\nabla}\hat{B}_{2}(x, y, z), \hat{B}_{2}(u, v) \rangle = 0 \) on \( M \). Since \( \psi_{n,s} \) is equivariant, it is sufficient to prove this at the origin \( o=\exp X \) of \( M \). For an arbitrary vector \( x \in T_{o}M \), we denote by \( \gamma \) the geodesic of \( M \) such that \( \gamma(0) = o \) and \( \gamma(0) = x \). Let \( (G, K) \) be a symmetric pair corresponding to \( M \) and \( g=\mathfrak{t}+\mathfrak{m} \) the canonical decomposition. Then the geodesic \( \gamma \) is described in such a way that \( \gamma(t) = \exp tX \cdot o \), where \( \exp tX \) denotes a one-parameter subgroup of \( G \) and \( X \) is a vector in \( \mathfrak{m} \) corresponding to \( x \). Moreover \( (\exp tX) \cdot y \) is a parallel vector field along \( \gamma \), which we denote by \( Y \). Similarly we set \( Z=(\exp tX) \cdot z, U=(\exp tX) \cdot u \), etc. Then
\[
\langle \dot{B}_2(Y, Z), \dot{B}_2(U, V) \rangle = \langle \dot{B}_2((\exp tX)y, (\exp tX)z), \dot{B}_2((\exp tX)u, (\exp tX)v) \rangle \\
= \langle \rho(\exp tX)\dot{B}_2(y, z), \rho(\exp tX)\dot{B}_2(u, v) \rangle \\
= \langle \dot{B}_2(y, z), \dot{B}_2(u, v) \rangle.
\]
Therefore \(\langle \dot{B}_2(Y, Z), \dot{B}_2(U, V) \rangle = \text{constant} \) along \(\gamma\). Since 
\[
\langle \dot{B}_2(Y, Z), \dot{B}_2(U, V) \rangle = \langle \dot{B}_2(Y, Z), \dot{B}_2(U, V) \rangle = \text{constant} \)
along \(\gamma\), we have 
\[
\langle \nabla \dot{B}_2(x, y, z), \dot{B}_2(u, v) \rangle + \langle \dot{B}_2(y, z), \nabla \dot{B}_2(x, u, v) \rangle = 0.
\]
Since the above equation holds for any vectors \(x, y, z, u\) and \(v\), and \(\nabla \dot{B}_2\) and \(\dot{B}_2\) are symmetric, 
\[
\langle \nabla \dot{B}_2(x, y, z), B_2(u, v) \rangle = -\langle \dot{B}_2(y, z), \nabla \dot{B}_2(x, u, v) \rangle = \langle \nabla \dot{B}_2(u, y, z), B_2(x, v) \rangle = -\langle \dot{B}_2(u, z), \nabla \dot{B}_2(y, x, v) \rangle = \langle \nabla \dot{B}_2(v, u, z), B_2(y, x) \rangle = -\langle \dot{B}_2(u, v), \nabla \dot{B}_2(x, y, z) \rangle.
\]
Therefore we get 
\[
\langle \nabla \dot{B}_2(x, y, z), B_2(u, v) \rangle = 0.
\]
Similarly we have \(\langle \nabla \dot{B}_2(x, y, z), \dot{B}_2(u, v) \rangle = 0\). By the definition of the third fundamental form, we obtain \(B_3 = \nabla \dot{B}_2\) and \(\dot{B}_3 = \nabla \dot{B}_2\).

Next we shall show that 
\[
(3.2) \quad \langle B_3(X, Z, W), B_3(Y, U, V) \rangle - \langle B_3(Y, Z, W), B_3(X, U, V) \rangle \\
= \langle R^+(X, Y)B_2(Z, W) - B_2(R(X, Y)Z, W) \rangle \\
- B_2(Z, R(X, Y)W), B_2(U, V) \rangle
\]
and the same equation holds for the third fundamental form \(\dot{B}_3\) of \(\varphi_{n, s}\). Since, 
for any vector fields \(Y, Z, W, U\) and \(V\), \(\langle \nabla B_2(Y, Z, W), B_2(U, V) \rangle = 0\), differ-
entiating it with respect to \(X\), we have 
\[
\langle \nabla \nabla B_2(X, Y, Z, W), B_2(U, V) \rangle = -\langle \nabla B_2(Y, Z, W), \nabla B_2(X, U, V) \rangle \\
= -\langle B_3(Y, Z, W), B_3(X, U, V) \rangle.
\]
This, together with Lemma 3.3 (3), gives (3.2). For \(\dot{B}_3\), the situation is quite similar. By Lemma 3.3 (2), 
\[
\langle R^+(X, Y)B_3(Z, W), B_3(U, V) \rangle = \langle [H_{B_3(Z, W)}, H_{B_3(U, V)}](X), Y \rangle.
\]
Moreover 
\[
\langle H_{B_3(Z, W)}(X, Y), B_3(Z, W) \rangle = \langle \dot{B}_3(Z, W), \dot{B}_3(Z, W) \rangle = \langle H_{B_3(Z, W)}(X, Y) \rangle.
\]
Thus we have
\[ \langle R^\perp(X, Y)B_3(Z, W) - B_2(R(X, Y)Z, W) - B_2(Z, R(X, Y)W), B_2(U, V)\rangle \]
\[ = \langle R^\perp(X, Y)B_3(Z, W) - B_2(R(X, Y)Z, W) - B_2(Z, R(X, Y)W), B_2(U, V)\rangle. \]

Now we shall prove Proposition 3.1 for \( B_3 \). We recall the third fundamental form \( \hat{B}_3 \) of \( \varphi_{n,s} \) is constant isotropic. Namely there exists a constant \( \lambda_0 \) such that for any unit tangent vector \( u \) of \( M, \| \hat{B}_3(u, u) \| = \lambda_0 \). On the other hand we assume that \( B_3 \) of \( \varphi \) is \( \lambda \)-isotropic and \( \lambda \) is not necessarily constant on \( M \).

By Lemma 3.2,
\[
\mathcal{S}_s\{\langle B_3(u_1, u_2, u_3), B_3(u_4, u_5, u_6)\rangle\} = \lambda^2 \mathcal{S}_s\{\langle u_1, u_2\rangle\langle u_3, u_4\rangle\langle u_5, u_6\rangle\}
\]
for \( u_1, u_2, \ldots, u_6 \in T_pM \) at an arbitrary point \( p \in M \). This, together with (3.2), yields
\[
(3.3) \quad 6! \langle B_3(u_1, u_2, u_3), B_3(u_4, u_5, u_6)\rangle
\]
\[ = \lambda^2 \mathcal{S}_s\{\langle u_1, u_2\rangle\langle u_3, u_4\rangle\langle u_5, u_6\rangle\} + \langle \lambda^2 \mathcal{S}_s\{\langle u_1, u_2\rangle\langle u_3, u_4\rangle\langle u_5, u_6\rangle\} + \cdots. \]

Similarly
\[
(3.4) \quad 6! \langle \hat{B}_3(u_1, u_2, u_3), \hat{B}_3(u_4, u_5, u_6)\rangle
\]
\[ = \lambda_0^2 \mathcal{S}_s\{\langle u_1, u_2\rangle\langle u_3, u_4\rangle\langle u_5, u_6\rangle\} + \langle \lambda_0^2 \mathcal{S}_s\{\langle u_1, u_2\rangle\langle u_3, u_4\rangle\langle u_5, u_6\rangle\} + \cdots. \]

Since the immersion \( \varphi \) is minimal, for an orthonormal basis \( \{e_1, \ldots, e_n\} \) we get
\[
0 = (6!) \sum_i \langle B_3(e_i, e_i, e_i), B_3(e_i, e_i, e_i)\rangle
\]
\[ = \lambda^2 48(n^3 + 6n^2 + 8n) + (\text{term not containing } \lambda). \]

Similarly \( 0 = \lambda_0^2 48(n^3 + 6n^2 + 8n) + (\text{the same term as above}) \). We remark that terms besides the first term of the right-hand-side of (3.3) are equal to those of the right-hand-side of (3.4). Therefore we have \( \lambda = \lambda_0 \) and again by comparing (3.3) and (3.4) we get \( \langle B_3(u_1, u_2, u_3), B_3(u_4, u_5, u_6)\rangle = \langle \hat{B}_3(u_1, u_2, u_3), \hat{B}_3(u_4, u_5, u_6)\rangle \). Also this implies the dimension of \( O^3_p \) of \( \varphi \) is equal to that of \( \varphi_{n,s} \) at every point \( p \in M \). Therefore every point of \( M \) is degree 3 regular for the immersion \( \varphi \).

Now we will apply the mathematical induction. For this we set the assumptions of the induction as follows:

1. We assume that \( j \geq 3 \) and that every point of \( M \) is degree \( j \) regular for the immersion \( \varphi \). Moreover we assume that
\[
\langle B_k(u_1, \ldots, u_k), B_k(v_1, \ldots, v_k)\rangle = \langle \hat{B}_k(u_1, \ldots, u_k), \hat{B}_k(v_1, \ldots, v_k)\rangle
\]
for \( 2 \leq k \leq j \) at every point of \( M \).
This implies that the vector bundle over $M$ which is given by restricting the tangent bundle of $S_1^l$ to $M$ admits the following orthogonal decomposition;

$$TS_1^l|_M = TM + O^2 + O^3 + \cdots + O^j + Q_j,$$

where $O^k, 2 \leq k \leq j$, denotes the vector bundle whose fibre at $p$ consists of $O_p^k$, and $Q_j$ denotes the vector bundle whose fibre at $p$ consists of the orthogonal complement of $\sum_{k=2}^{j} O_p^k$ in $N_p(M)$.

(2) Next we assume that $\Delta B_{k-1}(u_1, \cdots, u_k)$ has the components only in $O^{k-2}$ and $O^k$ for $3 \leq k \leq j$. By definition $B_k(u_1, \cdots, u_k)$ is the component of $\Delta B_{k-1}(u_1, \cdots, u_k)$ in $O^k$. On the other hand we define a tensor field $D_k$ as follows; $D_k(u_1, \cdots, u_k)$ is the component of $\Delta B_{k-1}(u_1, \cdots, u_k)$ in $O^{k-2}$. Similarly we denote by $\bar{D}_k$ the corresponding tensor field for $\psi_{n,s}$. Moreover we assume that $\langle D_k(u_1, \cdots, u_k), D_k(v_1, \cdots, v_k) \rangle = \langle \bar{D}_k(u_1, \cdots, u_k), \bar{D}_k(v_1, \cdots, v_k) \rangle$ for $3 \leq k \leq j$.

(3) Finally we assume that $B_{j+1}$ is $\lambda$-isotropic.

Under these assumptions, we shall show that the same statements hold for $B_{j+1}$. We proceed on the same way as the process from $B_2$ to $B_3$.

Step 1.

$$\langle \nabla B_{j}(u_0, u_1, \cdots, u_j), B_{j}(v_1, \cdots, v_j) \rangle + \langle B_{j}(u_1, \cdots, u_j), \nabla B_{j}(u_0, v_1, \cdots, v_j) \rangle = 0.$$  

This can be proved in the same way as in the case $j=2$ by using the assumption (1), i.e.,

$$\langle B_{j}(u_1, \cdots, u_j), B_{j}(v_1, \cdots, v_j) \rangle = \langle \hat{B}_{j}(u_1, \cdots, u_j), \hat{B}_{j}(v_1, \cdots, v_j) \rangle.$$

Step 2.

$$\langle \nabla B_{j}(x, y, u_2, \cdots, u_j), B_{j}(v_1, \cdots, v_j) \rangle = \langle \nabla B_{j}(y, x, u_2, \cdots, u_j), B_{j}(v_1, \cdots, v_j) \rangle.$$  

By the assumption (2), $\nabla B_{j} = B_{j} + D_{j}$. Since

$$\nabla D_{j}(x, y, u_2, \cdots, u_j) = \nabla^2 D_{j}(x, y, u_2, \cdots, u_j) - D_{j}(\nabla x, U_2, \cdots, U_j) - \cdots - D_{j}(\nabla y, U_2, \cdots, U_j)$$

and

$$D_{j}(Y, U_2, \cdots, U_j) \in O^{j-2}, \quad \nabla D_{j}(x, y, u_2, \cdots, u_j) \in \sum_{k=2}^{j-1} O^k.$$  

Thus we have

$$\langle \nabla B_{j}(x, y, u_2, \cdots, u_j), B_{j}(v_1, \cdots, v_j) \rangle$$

$$= \langle \nabla B_{j-1}(x, y, u_2, \cdots, u_j), B_{j}(v_1, \cdots, v_j) \rangle$$

$$= \langle \nabla B_{j-1}(y, x, u_2, \cdots, u_j) + R^{\perp}(x, y)B_{j-1}(u_2, \cdots, u_j) \rangle.$$
For $j \geq 3$, $H_{B_{j}(v_{1}, \ldots, v_{j})}=0$. Therefore by Ricci equation,
\[
\langle R^\perp(x, y)B_{j-1}(u_{2}, \cdots, u_{j}), B_{j}(v_{1}, \cdots, v_{j})\rangle=0.
\]
Thus the statement of Step 2 holds.

**Step 3.** \(\langle \nabla B_{j}(u_{0}, u_{1}, \cdots, u_{j}), B_{j}(v_{1}, \cdots, v_{j})\rangle=0\), that is, the component of \(\nabla B_{j}\) in \(O^{j}\) vanishes. By Step 2, \(\langle \nabla B_{j}(x, y, u_{2}, \cdots, u_{j}), B_{j}(v_{1}, \cdots, v_{j})\rangle\) is symmetric with respect to \(x, y, u_{2}, \cdots, u_{j}\), which, combined with Step 1, implies the assertion.

The following two facts are easily seen.

**Step 4.**
\[
\langle \nabla B_{j}(u_{0}, u_{1}, \cdots, u_{j}), B_{j-1}(v_{1}, \cdots, v_{j-1})\rangle = -\langle B_{j}(u_{1}, \cdots, u_{j}), B_{j}(u_{0}, v_{1}, \cdots, v_{j-1})\rangle.
\]

**Step 5.** For $2 \leq k \leq j-2$, \(\langle \nabla B_{j}(u_{0}, u_{1}, \cdots, u_{j}), B_{k}(v_{1}, \cdots, v_{k})\rangle=0\). By Step 3, Step 4, and Step 5, we see that \(\nabla B_{j}\) has the components only in \(O^{j-1}\) and \(Q_{j}\). By the definition we set \(B_{j+1}(u_{0}, u_{1}, \cdots, u_{j})=\text{the } Q_{j}\text{-component of } \nabla B_{j}(u_{0}, u_{1}, \cdots, u_{j})\) and \(D_{j+1}(u_{0}, u_{1}, \cdots, u_{j})=\text{the } O^{j-1}\text{-component of } \nabla B_{j}(u_{0}, u_{1}, \cdots, u_{j})\). Similarly we can define \(\bar{B}_{j+1}\) and \(\bar{D}_{j+1}\) for the standard minimal immersion \(\varphi_{n,s}\). By Step 4, we have
\[
\langle D_{j+1}(u_{0}, u_{1}, \cdots, u_{j}), B_{j-1}(v_{1}, \cdots, v_{j-1})\rangle = -\langle B_{j}(u_{1}, \cdots, u_{j}), B_{j}(u_{0}, v_{1}, \cdots, v_{j-1})\rangle.
\]
Similarly
\[
\langle \bar{D}_{j+1}(u_{0}, u_{1}, \cdots, u_{j}), \bar{B}_{j-1}(v_{1}, \cdots, v_{j-1})\rangle = -\langle \bar{B}_{j}(u_{1}, \cdots, u_{j}), \bar{B}_{j}(u_{0}, v_{1}, \cdots, v_{j-1})\rangle.
\]

Here we remark that by the assumptions of the induction there exists a unique linear isometry \(A\) of \(O^{j-1}_{\varphi}\) of \(\varphi\) onto that of \(\varphi_{n,s}\) such that \(AB_{j-1}(u_{1}, \cdots, u_{j-1}) = \bar{B}_{j-1}(u_{1}, \cdots, u_{j-1})\). By the above two equations and the assumptions of the induction, we have
\[
\langle AD_{j+1}(u_{0}, u_{1}, \cdots, u_{j}), \bar{B}_{j-1}(v_{1}, \cdots, v_{j-1})\rangle
= -\langle D_{j+1}(u_{0}, u_{1}, \cdots, u_{j}), B_{j-1}(v_{1}, \cdots, v_{j-1})\rangle
= -\langle \bar{D}_{j+1}(u_{0}, u_{1}, \cdots, u_{j}), \bar{B}_{j-1}(v_{1}, \cdots, v_{j-1})\rangle,
\]
which implies that \(AD_{j+1}(u_{0}, u_{1}, \cdots, u_{j}) = \bar{B}_{j+1}(u_{0}, u_{1}, \cdots, u_{j})\). Thus we get
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Step 6.
\[ \langle D_{j+1}(u_0, \cdots, u_j), D_{j+1}(v_0, \cdots, v_j) \rangle = \langle \hat{D}_{j+1}(u_0, \cdots, u_j), \hat{D}_{j+1}(v_0, \cdots, v_j) \rangle. \]

Step 7.
\[ \langle B_{j+1}(u_0, u_1, \cdots, u_j), B_{j+1}(v_0, v_1, \cdots, v_j) \rangle = \langle B_{j+1}(v_0, u_1, \cdots, u_j), B_{j+1}(u_0, v_1, \cdots, v_j) \rangle = \langle D_{j+1}(v_0, u_1, \cdots, u_j), D_{j+1}(u_0, v_1, \cdots, v_j) \rangle \]
\[ - \sum_{i=1}^{j} \langle B_{j}(u_1, \cdots R(u_{0}, v_{0})u_{i}, \cdots u_{j}), B_{j}(v_1, \cdots, v_{j}) \rangle. \]

Since \( \langle \overline{\nabla}B_{j}(v_{0}, u_{1}, \cdots, u_{j}), B_{j}(v_{1}, \cdots, v_{j}) \rangle = 0 \), we see that
\[ \langle \overline{\nabla}^{2}B_{j}(u_{0}, v_{0}, u_{1}, \cdots, u_{j}), B_{j}(v_{1}, \cdots, v_{j}) \rangle = -\langle \overline{\nabla}B_{j}(v_{0}, u_{1}, \cdots, u_{j}), \overline{\nabla}B_{j}(u_{0}, v_{1}, \cdots, v_{j}) \rangle \]
\[ - \langle B_{j+1}(v_{0}, u_{1}, \cdots, u_{j}), B_{j+1}(u_{0}, v_{1}, \cdots, v_{j}) \rangle. \]

This, together with Lemma 3.3 (3), gives Step 7.

We remark that the right-hand-side of the equation of Step 7 for the immersion \( \varphi \) is quite equal to that of the standard minimal immersion \( \psi_{n,s} \). By the same method as in the case of \( B_{3} \) we get

Step 8.
\[ \langle B_{j+1}(u_0, u_1, \cdots, u_j), B_{j+1}(v_0, v_1, \cdots, v_j) \rangle = \langle \hat{B}_{j+1}(u_0, u_1, \cdots, u_j), \hat{B}_{j+1}(v_0, v_1, \cdots, v_j) \rangle. \]

Thus our proof of Proposition 3.1 finishes.

In the remainder of this section we show the following property about \( \hat{D}_{j} \) of the standard minimal immersions of spheres.

**Proposition 3.4.** Let \( \psi_{n,s} : S_{k(s)}^{n} \to S_{1}^{m(s)} \) be the standard minimal immersion of a sphere into a unit sphere \((n \geq 2)\) and for this immersion we use the notations \( \hat{B}_{j} \) and \( \hat{D}_{j} \) defined formerly. Then for an arbitrary unit tangent vector \( x \),
\[ \hat{D}_{j+1}(x, \cdots, x) = -(\lambda_{j}^{2} / \lambda_{j-1}^{2}) \hat{B}_{j-1}(x, \cdots, x) \quad (j \geq 3), \]
where
\[ \lambda_{j} = \| \hat{B}_{j}(x, \cdots, x) \| \quad \text{and} \quad \lambda_{j-1} = \| \hat{B}_{j-1}(x, \cdots, x) \|. \]

Before the proof, we prepare the following algebraic lemma.

**Lemma 3.5.** Let \( F \) be a symmetric \( k \)-linear form on \( \mathbb{R}^{n} \) \((k \geq 2)\). Suppose \( F(v, \cdots, v)/\|v\|^{k} \) is constant and \( \sum_{i=1}^{k} F(v_{1}, \cdots, v_{k-2}, e_{i}, e_{i}) = 0 \) for any \( v_{1}, \cdots, v_{k-2}, \)
where \( \{e_1, \ldots, e_n\} \) is an orthonormal basis of \( \mathbb{R}^n \). Then \( F \equiv 0 \).

**Proof.** We set \( \lambda = F(v, \ldots, v) \) for a unit vector \( v \in \mathbb{R}^n \). If \( k \) is odd, we have

\[
\lambda = F(-v, \ldots, -v) = (-1)^k F(v, \ldots, v) = -F(v, \ldots, v) = -\lambda,
\]

so that \( \lambda = 0 \). Thus we get \( F \equiv 0 \). If \( k \) is even,

\[
F(u_1, \ldots, u_{2j}) = \frac{1}{(2j)!} \lambda S_{2j}\{\langle u_1, u_2 \rangle \cdots \langle u_{2j-1}, u_{2j} \rangle\}.
\]

Then for a fixed unit vector \( u \),

\[
0 = \sum_{i=1}^{n} F(u, \ldots, u, e_i, e_i) = K \lambda
\]

for some \( K > 0 \). Therefore \( \lambda = 0 \) and we have \( F \equiv 0 \).

**Proof of Proposition 3.4.** When \( n = 2 \), we can easily prove Proposition 3.4. So we assume that \( n \geq 3 \). Since \( \phi_{n,s} \) is an equivariant immersion, it is sufficient to show the relation at the origin \( eK \) of \( S^n_{eK} \). Since \( \hat{D}_{j+1}(x, \ldots, x) \in O(eK)^j \), it is sufficient to show that

\[
\langle \hat{D}_{j+1}(x, \ldots, x), \hat{B}_{j-1}(v_1, \cdots, v_{j-1}) \rangle = -(\lambda_j^2/\lambda_{j-1}^2) \langle \hat{B}_{j-1}(x, \cdots, x), \hat{B}_{j-1}(v_1, \cdots, v_{j-1}) \rangle
\]

for any \( v_1, \ldots, v_{j-1} \). We recall that

\[
\langle \hat{D}_{j+1}(x, \ldots, x), \hat{B}_{j-1}(v_1, \cdots, v_{j-1}) \rangle = -\langle \hat{B}_{j}(x, \cdots, x), \hat{B}_{j}(x, v_1, v_{j-1}) \rangle.
\]

Therefore we shall show that

\[
\langle \hat{B}_{j}(x, \cdots, x), \hat{B}_{j}(x, v_1, \cdots, v_{j-1}) \rangle
\]

\[
= (\lambda_j^2/\lambda_{j-1}^2) \langle \hat{B}_{j-1}(x, \cdots, x), \hat{B}_{j-1}(v_1, \cdots, v_{j-1}) \rangle.
\]

We set

\[
F_k(v_1, \ldots, v_k) = \langle \hat{B}_{j}(x, \cdots, x), \hat{B}_{j}(x, v_1, \cdots, v_k) \rangle
\]

\[
-(\lambda_j^2/\lambda_{j-1}^2) \langle \hat{B}_{j-1}(x, \cdots, x), \hat{B}_{j-1}(v_1, \cdots, v_k) \rangle
\]

for \( 1 \leq k \leq j-1 \). Then \( F_k \) is a symmetric \( k \)-linear form on the tangent space \( T_{eK}S^n \). We define the subspace \( V \) of \( T_{eK}S^n \) by \( V = \{v \in T_{eK}S^n ; \langle v, x \rangle = 0\} \) and we use an orthonormal basis \( \{e_1, \ldots, e_n\} \) of \( T_{eK}S^n \) such that \( e_1 = x \), and \( e_2, \ldots, e_n \in V \). We see that there exists a subgroup \( K' \) of \( K = SO(n) \) such that \( k \cdot x = x \) for any \( k \in K' \) and \( K' \) acts transitively on the unit sphere of \( V \). So for \( k \in K' \) and a unit vector \( v \in V \) we have

\[
F_k(k \cdot v, \ldots, k \cdot v)
\]

\[
= \langle \hat{B}_{j}(x, \cdots, x), \hat{B}_{j}(x, \cdots, x, k \cdot v, \cdots, k \cdot v) \rangle
\]

\[
-(\lambda_j^2/\lambda_{j-1}^2) \langle \hat{B}_{j-1}(x, \cdots, x), \hat{B}_{j-1}(x, \cdots, x, k \cdot v, \cdots, k \cdot v) \rangle.
\]
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$$=\langle \hat{B}_j(k \cdot x, \cdots, k \cdot x), \hat{B}_j(k \cdot x, \cdots, k \cdot x, k \cdot v, \cdots, k \cdot v) \rangle$$

$$-(\lambda_j^2/\lambda_{j-1}^2)\langle B_{j-1}(k \cdot x, \cdots k \cdot x), B_{j-1}(k \cdot x, \cdots k \cdot x, k \cdot v, \cdots k \cdot v) \rangle$$

$$=\langle \rho(k)\hat{B}_j(x, \cdots, x), \rho(k)\hat{B}_j(x, \cdots, x, v, \cdots, v) \rangle$$

$$-(\lambda_j^2/\lambda_{j-1}^2)\langle \rho(k)B_{j-1}(x, \cdots x), \rho(k)B_{j-1}(x, \cdots x, v, v) \rangle$$

$$=F_i(v, v, v)$$

Therefore $F_i(v, \cdots, v)/\|v\|^i$ is constant on $V$. Since $\hat{B}_j$ and $\hat{B}_{j-1}$ are isotropic, $F_1(x)=\lambda_j^2-(\lambda_j^2/\lambda_{j-1}^2)\lambda_{j-1}^2=0$. By Lemma 3.2, it is easy to see that $\langle \hat{B}_j(x, \cdots, x), \hat{B}_{j-1}(x, \cdots, x, v) \rangle=0$ for $v \in V$, which implies that $F_i(v)=0$ for $v \in V$. Therefore we have $F_i \equiv 0$ on $T_{eK}S^n$.

Next we shall show that $F_2 \equiv 0$. Since $\psi_{n,s}$ is minimal, we get

$$\sum_{i=1}^n F_2(e_i, e_i) = \langle \hat{B}_j(x, \cdots, x), \sum_{i=1}^n B_{j}(x, v_{1}, v_{j-1}) \rangle$$

$$=\langle \hat{B}_{j-1}(x, \cdots, x, v), \sum_{i=1}^n B_{j-1}(x, x, v) \rangle$$

Since $F_2(e_1, e_1) = F_2(x, x) = 0$, we have $\sum_{i=2}^n F_2(e_i, e_i)=0$. Using Lemma 3.3, we obtain $F_2 \equiv 0$ on $V$. This, together with $F_1 \equiv 0$, implies $F_2 \equiv 0$ on $T_{eK}S^n$.

Now we apply a mathematical induction. We assume that $F_i \equiv 0$ on $T_{eK}S^n$ for $1 \leq i \leq k-1$, where $k \geq 3$. Since $\sum_{i=1}^n F_k(v_1, \cdots, v_{k-2}, e_i, e_i)=0$ and $F_k(v_1, \cdots, v_{k-2}, x, x) = F_{k-2}(v_1, \cdots, v_{k-2})=0$, we have $\sum_{i=1}^n F_k(v_1, \cdots, v_{k-2}, e_i, e_i)=0$. Again by Lemma 3.5, $F_k \equiv 0$ on $V$. Combining this with $F_i \equiv 0$ for $1 \leq i \leq k-1$, we have $F_k \equiv 0$ on $T_{eK}S^n$. In particular $F_{j-1} \equiv 0$ on $T_{eK}S^n$. Then

$$\langle \hat{B}_j(x, \cdots, x, v_1, v_{j-1}) \hat{B}_{j-1}(x, \cdots, x, v_{j-1}) \rangle$$

Thus Proposition 3.4 is proved.

4. On the rigidity of isotropic minimal immersions.

In this section we show two theorems.

Theorem 4.1. Let $\phi_{n,s}: M_{k(s)}^n \rightarrow S_1^m(s)$ be the standard minimal immersion of an $n$-dimensional compact rank 1 symmetric space into a unit sphere corresponding to the $s$-th eigenvalue. Let $\phi: M_{k(s)}^n \rightarrow S_1^m(s)$ be another minimal immersion corresponding to the same eigenvalue and assume that $\phi$ is full. We assume that
in the case of $M=S^n$ the degree of $\phi \geq \lceil s/2 \rceil$ and the $j$-th fundamental form $B_j$ of $\phi$ is isotropic at every degree $j-1$ regular point $p \in R_{j-1}$ for $2 \leq j \leq \lceil s/2 \rceil$ and that in the case of $M=P_n(C), P_n(H)$, or $P_s(Cay)$ the degree of $\phi \geq s$ and the $j$-th fundamental form $B_j$ of $\phi$ is isotropic at every degree $j-1$ regular point $p \in R_{j-1}$ for $2 \leq j \leq s$. Then we see that $\phi$ is equivalent to the standard minimal immersion $\phi_{n,s}$ and in particular $l=m(s)$.

Furthermore, we get the following result on the non-rigidity for $S^n$.

**Theorem 4.2.** For $n \geq 3$ and $s \geq 6$, there are many inequivalent minimal immersions of $S^m_{n,(a)}$ into a unit sphere such that $B_k$ is isotropic on $S^m_{n,(a)}$ for $2 \leq k \leq \lceil s/2 \rceil-1$.

We apply the method of do Carmo and Wallach (4). We formulate the rigidity problem following do Carmo and Wallach.

**Proposition 4.3.** Let $\varphi : M^m_{n,(a)} \to S^l_1$ be a minimal immersion. Then there exists a symmetric positive semi-definite linear map $A$ of $R^{m(n)+1}$ such that $\varphi$ is equivalent to $A^*\varphi_{n,s}$. Furthermore $\varphi$ is equivalent to $\varphi_{n,s}$ if and only if the associated symmetric linear mapping $A$ of $\varphi$ is equivalent to the identity map.

**Proposition 4.4.** Let $A$ be a symmetric positive semi-definite linear map of $R^{m(n)+1}$. We assume that $A^*\varphi_{n,s}$ is a minimal immersion of $M^m_{n,(a)}$ into a unit sphere and we denote by $B_j$ and $\dot{B}_j$ the $j$-th fundamental form of $A^*\varphi_{n,s}$ and that of $\varphi_{n,s}$ respectively. (When there is no danger of confusion, we use $\varphi$ instead of $\varphi_{n,s}$.) Then we have $B_k(u_1, u_2) = A\dot{B}_k(u_1, u_2)$ at any point $p \in M$. Furthermore if $B_k$ is isotropic on $M$ for $2 \leq k \leq j$, then $B_k(u_1, \ldots, u_k) = A\dot{B}_k(u_1, \ldots, u_k)$ for $3 \leq k \leq j+1$. Under the same assumption, for the orthogonal decomposition $R^{m(n)+1} = R \cdot \varphi(p) + \varphi_{*}T_pM + O_{p}^{2} + \cdots + O_{p}^{d}$ at $p$ with respect to $\varphi$, where $d$ is the degree of $\varphi$, $A$ is an isometric linear mapping on the subspace

$$R \cdot \varphi(p) + \varphi_{*}T_pM + O_{p}^{2} + \cdots + O_{p}^{d}$$

at any point $p \in M$.

We remark that we identify (as usual) $T_p S^n_{n,(a)}$ with the subspace of $R^{m(n)+1}$. Under the identification, the above statements make sense.

**Proof of Proposition 4.4.** By the argument in the proof of Proposition 3.1, $R^{m(n)+1}$ admits the following orthogonal decomposition with respect to the immersion $A^*\varphi$:

$$R^{m(n)+1} = R \cdot A^*\varphi(p) + (A^*\varphi)_{*}T_pM + O_{p}^{2} + \cdots + \tilde{O}_{p}^{l} + (\tilde{Q})_{p}.$$

Since $A^*\varphi$ is an isometric immersion of $M$ into $S^n_{n,(a)}$, it is easily seen that $A$ is an isometric linear map from $R \cdot \varphi(p)$ to $R \cdot A^*\varphi(p)$ and also from $\varphi_{*}T_pM$ to $(A^*\varphi)_{*}T_pM$. We denote by $\hat{\nabla}$ and $\nabla$ the Riemannian connections on $S^n_{n,(a)}$ and $R^{m(n)+1}$ respectively. Then we see that $\hat{\nabla}_XY = \nabla_XY + \langle X, Y \rangle \varphi$ at $p \in S^n_{n,(a)}$ for any vector fields $X, Y$ on $S^n_{n,(a)}$. For an arbitrary unit vector $x$ in $T_pM$, we set $\gamma(t)$ to be the geodesic such that $\gamma(0) = p$ and $\gamma'(0) = x$. We denote by $\sigma$ the curve in $S^n_{n,(a)}$ defined by $\sigma = \varphi \circ \gamma$ and naturally consider $\sigma$ also as a curve in
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$R^{m(x)+1}$. Then we have

$$B_{2}(x, x)=\nabla_{(A\circ\sigma)_{*}}(A\circ\sigma)=\nabla_{(A\circ\sigma)_{*}}(A\circ\sigma)+A\circ\phi(p)$$

$$=\frac{d^2}{dt^2}A\circ\sigma(t)|_{t=0}+A\circ\phi(p)$$

$$=A\frac{d^2}{dt^2}\sigma(t)|_{t=0}+A\circ\phi(p)$$

$$=A(\frac{d^2}{dt^2}\sigma(t)|_{t=0}+\phi(p))$$

$$=A(\tilde{\nabla}_{\dot{\sigma}}\dot{\sigma})$$

$$=AB_{2}(x, x)$$

Since $B_{2}(x, x)$ and $\tilde{O}^{2}$ span linearly $O^{2}$ and $O^{2}$ respectively, we have $AB_{2}(u_{1}, u_{2})=B_{2}(u_{1}, u_{2})$ for $u_{1}, u_{2}\in T_{p}M$. If the second fundamental form $B_{2}$ of $A\circ\phi$ is isotropic on $M$, then by Proposition 3.1 we get $\langle B_{2}(u_{1}, u_{2}), B_{2}(v_{1}, v_{2})\rangle =\langle A\tilde{B}_{2}(u_{1}, u_{2}), A\tilde{B}_{2}(v_{1}, v_{2})\rangle$. Therefore $A$ is an isometric linear mapping from $O^{2}$ to $\tilde{O}^{2}$.

Next we prove Proposition 4.4 for the third fundamental form $B_{3}$. Since $B_{3}$ is isotropic, by the proof of Proposition 3.1, $B_{3}(x, x, x)=\tilde{\nabla}B_{3}(x, x, x)$. Then

$$B_{3}(x, x, x)=\tilde{\nabla}_{\dot{\gamma}}(B_{3}(\dot{\gamma}, \dot{\gamma}))$$

$$=\nabla_{(A\circ\psi)_{*}x}(B_{3}(\dot{\gamma}, \dot{\gamma}))+A(\psi_{*}H_{B_{3}(x,x)}x)$$

$$=\frac{d}{dt}(B_{3}(\dot{\gamma}, \dot{\gamma}))+A(\psi_{*}H_{B_{3}(x,x)}x)$$

$$=A(\frac{d}{dt}B_{3}(\dot{\gamma}, \dot{\gamma})+\psi_{*}H_{B_{3}(x,x)}x)$$

$$=A(\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma})$$

This implies that $B_{3}(u_{1}, u_{2}, u_{3})=AB_{3}(u_{1}, u_{2}, u_{3})$. If $B_{3}$ is isotropic, by Proposition 3.1, $A$ is an isometric linear mapping from $O^{3}$ to $\tilde{O}^{3}$. We assume that $B_{i}(u_{1}, \cdots, u_{i})=AB_{i}(u_{1}, \cdots, u_{i})$ for $2\leq i\leq k$ ($k\geq 3$) at every point of $M$ and that $A: O^{i}\to \tilde{O}^{i}$ is an isometric linear mapping for all $i$. Then by the argument of the proof of Proposition 3.1, $B_{k+1}(x, \cdots, x)+D_{k+1}(x, \cdots, x)$

$$=\tilde{\nabla}_{\dot{\gamma}}(B_{k}(\dot{\gamma}, \cdots, \dot{\gamma}))$$

$$=\nabla_{(A\circ\psi)_{*}x}(B_{k}(\dot{\gamma}, \cdots, \dot{\gamma}))+A(\psi_{*}H_{B_{k}(x,x)}x)$$
Here again recalling the proof of Proposition 3.1, we have
\[
\langle AD_{k+1}(x, \cdots, x), B_{k-1}(v_1, v_{k-1}) \rangle
= \langle A\mathring{D}_{k+1}(x, \cdots, x), AB_{k-1}(v_1, \cdots, v_{k-1}) \rangle
= \langle D_{k+1}(x, \cdots, x), B_{k-1}(v_1, v_{k-1}) \rangle.
\]
Therefore \( AD_{k+1}(x, \cdots, x) = D_{k+1}(x, \cdots, x) \). Thus we get \( B_{k+1}(x, \cdots, x) = AB_{k+1}(x, \cdots, x) \).
This implies that \( B_{k+1}(u_1, \cdots, u_{k+1}) = AB_{k+1}(u_1, \cdots, u_{k+1}) \).
If \( B_{k+1} \) is isotropic on \( M \), \( A \) is an isometric linear mapping from \( O^{k+1} \) to \( \tilde{O}^{k+1} \).
Thus Proposition 4.4 is proved.

Now we identify the space of all symmetric linear mappings of \( R^{m(s)+1} \) with \( S(R^{m(s)+1}) \), the symmetric square of \( R^{m(s)+1} \) as follows: if \( u, v \in R^{m(s)+1} \), \( u \cdot v \in S(R^{m(s)+1}) \) (the symmetric product of two vectors will be denoted by \( u \cdot v \)), and if \( t \in R^{m(s)+1} \), we set \( u \cdot v(t) = \frac{1}{2} \langle u, t \rangle v + \langle v, t \rangle u \), where \( \langle , \rangle \) is the inner product on \( R^{m(s)+1} \). Under this identification, the inner product on \( S(R^{m(s)+1}) \) is given by \( (A, B) = \text{tr} AB \). We note that if \( A \in S(R^{m(s)+1}) \) and \( u, v \in R^{m(s)+1} \), then \( \langle Au, v \rangle = \langle A, u \cdot v \rangle \).

For the orthogonal decomposition \( R^{m(s)+1} = R \cdot \phi(p) + \psi T_p M + O_p^2 + \cdots + O_p^d \) with respect to the standard minimal immersion \( \phi_{n,s} \), let \( S(R \cdot \phi(p) + \phi T_p M + O_p^2 + \cdots + O_p^d) \) be the symmetric square of \( R \cdot \phi(p) + \phi T_p M + O_p^2 + \cdots + O_p^d \). And let \( W^{(j)} \) be the subspace of \( S(R^{m(s)+1}) \) spanned by \( \bigcup_{p \in M} S(R \cdot \phi(p) + \phi T_p M + O_p^2 + \cdots + O_p^d) \). Let \( W_0 \) and \( W_1 \) be the subspace of \( S(R^{m(s)+1}) \) spanned by \( \bigcup_{p \in M} \phi(p) \cdot \phi(p) \) and \( \bigcup_{p \in M} S(\phi T_p M) \), respectively. Let \( W_j \) be the subspace of \( S(R^{m(s)+1}) \) spanned by \( \bigcup_{p \in M} S(O_p^j) \) for any \( j \).

**Lemma 4.5.** (1) If \( A \in S(R^{m(s)+1}) \) and \( A \geq 0 \) (i.e., \( A \) is positive semi-definite), then \( A \cdot \phi_{n,s} \) is a minimal immersion of \( M_{k(s)} \) in \( S^{m(s)} \) such that the \( k \)-th funda-
mental form $B_k$ of $A\ast \phi_{n,s}$ is isotropic on $M$ for $2 \leq k \leq j \leq$ the degree of $\phi_{n,s}$ if and only if $A^2-I \in (W^{(j)})^\perp$, where $(W^{(j)})^\perp$ denotes the orthogonal complement of $W^{(j)}$ in $S^2(R^{m(s)+1})$ and $I$ denotes the identity transformation of $R^{m(s)+1}$.

(2) $W_0+W_1+ \cdots + W_j = W^{(j)}$.

(3) $W_0 \subset W_1$.

PROOF. (1) It is known that $A\ast \phi_{n,s}$ is a minimal immersion if and only if $A^2-I \in W_1^\perp$, where $W_1^\perp$ denotes the orthogonal complement of $W_1$ in $S^2(R^{m(s)+1})$ (Wallach [10]). If $A\ast \phi_{n,s}$ is a minimal immersion such that $B_k$ is isotropic for $2 \leq k \leq j$, then, by Proposition 4.4, $A$ is an isometric linear mapping on the subspace $R \cdot \psi(p) + \phi_{n,s}T_pM + O_p^2 + \cdots + O_p^j$ of $R^{m(s)+1}$ at every point $p \in M$. Therefore $\langle Au, Av \rangle = \langle u, v \rangle$ for any two vectors $u, v \in R \cdot \psi(p) + \phi_{n,s}T_pM + O_p^2 + \cdots + O_p^j$. This implies that $A^2-I$ is orthogonal to $S^2(R \cdot \psi(p) + \phi_{n,s}T_pM + O_p^2 + \cdots + O_p^j)$. Since $p$ is arbitrary in $M$, $A^2-I$ is orthogonal to $W^{(j)}$.

Conversely if $A^2-I \in (W^{(j)})^\perp$, $A\ast \phi_{n,s}$ is a minimal immersion of $M_{k(s)}$ in $S^2(R)$.

Furthermore by Proposition 4.4 we have

\[ 0 = \langle A^2-I, \dot{B}_2(u_1, u_2) \cdot \dot{B}_2(v_1, v_2) \rangle = \langle AB_2(u_1, u_2), AB_2(v_1, v_2) \rangle - \langle B_2(u_1, u_2), B_2(v_1, v_2) \rangle = \langle AB_2(u_1, u_2), AB_2(v_1, v_2) \rangle - \langle B_2(u_1, u_2), B_2(v_1, v_2) \rangle \]

at every point $p$, where $B_2$ as usual denotes the second fundamental form of $A\ast \phi_{n,s}$. Thus $B_2$ is isotropic. Repeating this process, we can prove Lemma 4.3 (1).

(2) It is trivial that $W_0+W_1+ \cdots + W_j \subset W^{(j)}$. We shall prove $W_0+W_1+ \cdots + W_j \subset W^{(j)}$. We assume that $C \in S^2(R^{m(s)+1})$ is orthogonal to $W_0+W_1+ \cdots + W_j$. Let $t > 0$ be such that $I+tC \geq 0$ and let $A$ be the positive square root of $I+tC$. Then $A^2-I = tC$ is orthogonal to $W_1$. Therefore $A\ast \phi_{n,s}$ is a minimal immersion of $M_{k(s)}$ to $S^2(R)$. Since $A^2-I \in W_2^\perp$, by the argument in (1) the second fundamental form $B_2$ of $A\ast \phi_{n,s}$ is isotropic on $M$. Then Proposition 4.4 implies that $B_2 = AB_2$. Similarly, since $A^2-I \in W_3^\perp$, $B_3$ is isotropic on $M$. Repeating this process, we see that the $k$-th fundamental form $B_k$ of $A\ast \phi_{n,s}$ is isotropic for $2 \leq k \leq j$. Then (1) implies that $A^2-I = tC \perp W^{(j)}$. Thus (2) is proved.

(3) It is proved in Wallach [10].

Lemma 4.6. If the degree of the standard minimal immersion $\phi_{n,s}$ is $d \geq 2$, then $W^{(j)} = S^2(R^{m(s)+1})$ for $j = \lfloor d/2 \rfloor$.

PROOF. We use again the method of Wallach [10]. We show that if $x \in T_pM$ and if $\sigma : (-\varepsilon, \varepsilon) \to M$ is the geodesic through $p$ in $M$ with tangent vector $x$, then $\sigma', \sigma' \sigma(0), \sigma' \sigma(0), \cdots, \sigma' \sigma^{(d)}(0) \in W^{(j)}$, where we identify $\sigma(t)$ with the curve $\phi_{n,s}(t)$ in $R^{m(s)+1}$ and $\sigma^{(k)}(t)$ denotes $\frac{d^k}{dt^k} \sigma(t)$ in $R^{m(s)+1}$. We
shall show only that \( p \cdot \sigma^{(d)}(0) \in W^{(f)} \). We assume that \( d = \text{odd} \), i.e., \( d = 2j + 1 \). When \( d = \text{even} \), the proof is quite similar. We easily see that \( \sigma^{(f)}(t) \in R \cdot \phi(\sigma(t)) \) 
+ \( \phi_{s}T_{\sigma(t)}M + O_{d}(\sigma(t)) + \cdots + O_{d}(\sigma(t)) \), which implies \( \sigma^{(f)}(t) \cdot \sigma^{(f)}(t) \in W^{(f)} \). First we have \( (\sigma^{(f)} \cdot \sigma^{(f)})'(0) = 2\sigma^{(f)}(0) \cdot \sigma^{(f+1)}(0) \in W^{(f)} \). Similarly

\[
(\sigma^{(f-1)} \cdot \sigma^{(f)})'_{(0)}(0) = \sigma^{(f)}(0) \cdot \sigma^{(f)}(0) + 2\sigma^{(f)}(0) \cdot \sigma^{(f+1)}(0) + \sigma^{(f+1)}(0) \cdot \sigma^{(f+2)}(0) 

= 3\sigma^{(f)}(0) \cdot \sigma^{(f)}(0) + \sigma^{(f+1)}(0) \cdot \sigma^{(f+2)}(0) \in W^{(f)}
\]

Therefore we get \( \sigma^{(f)}(0) \cdot \sigma^{(f+1)}(0) \in W^{(f)} \). Similarly

\[
(\sigma^{(f-1)} \cdot \sigma^{(f)})'_{(0)}(0) = \sigma^{(f)}(0) \cdot \sigma^{(f)}(0) + 3\sigma^{(f)}(0) \cdot \sigma^{(f+1)}(0) 

+ 3\sigma^{(f+1)}(0) \cdot \sigma^{(f+2)}(0) + \sigma^{(f+2)}(0) \cdot \sigma^{(f+3)}(0) \in W^{(f)}
\]

These, together with the former results, imply \( \sigma^{(f-1)}(0) \cdot \sigma^{(f+1)}(0) \in W^{(f)} \). Repeating these calculations, we see that \( p \cdot \sigma^{(f+1)}(0) = p \cdot \sigma^{(f)}(0) \in W^{(f)} \). By the same method we can prove that \( p \cdot \sigma^{(k)}(0) \in W^{(f)} \) for \( 0 \leq k \leq d \). If \( C \in S^{(n)}(R^{m(s)+1}) \) and \( C \) is orthogonal to \( W^{(f)} \), then \( 0 = \langle C, p \cdot \sigma^{(k)}(0) \rangle = \langle Cp, \sigma^{(k)}(0) \rangle \) for \( 0 \leq k \leq d \). The \( O_{p}^{k} \)-component of \( \sigma^{(k)}(0) \) is just equal to \( B_{k}(x, \cdots, x) \) and \( O_{p}^{k} \) is linearly spanned by \( B_{k}(x, \cdots, x) \), \( x \in T_{p}M \). Using \( \langle Cp, \sigma^{(k)}(0) \rangle = 0 \) for \( 0 \leq k \leq d \) and for an arbitrary vector \( u \in R \cdot \psi(p) + \psi_{*}T_{p}M + O_{p}^{2} + \cdots + O_{p}^{d} = R^{m(s)+1} \), we can prove inductively that \( \langle Cp, u \rangle = 0 \). Thus we get \( Cp = 0 \). As the standard minimal immersion \( \phi_{n,s} \) is full, we have \( C = 0 \). Thus \ref{lem:4.6} is proved.

**Proof of Theorem 4.1.** If \( A \cdot \phi_{n,s} \) is a minimal immersion and the \( k \)-th fundamental form \( B_{k} \) of \( A \cdot \phi_{n,s} \) is isotropic on \( M \) for \( 2 \leq k \leq \left\lceil d/2 \right\rceil \) ( \( d \) is the degree of \( \phi_{n,s} \)), then by \ref{lem:4.5} \( A^{2} - I \in (W^{(d/2)})^{-1} \). \ref{lem:4.6} implies that \( A^{2} - I = 0 \) and then \( A = I \). By Remark 2.4, \ref{thm:4.1} is proved.

To prove \ref{thm:4.2}, we review do Carmo and Wallach’s results. For the remainder of this section we assume that \( M = S_{n}^{m(s)} = G/K \), where \( G = \text{SO}(n+1) \) and \( K = \text{SO}(n) \).

**Lemma 4.7** (Do Carmo and Wallach [4]). Let \( \phi_{n,s} : S_{n}^{m(s)} \to S_{n}^{m(s)+1} \subset V^{s} \) be the standard minimal immersion described in the second way (cf. §2). If \( V^{s} \) is orthogonally decomposed as \( V^{s} = R \cdot \phi(eK) + \phi_{s}T_{eK}M + O_{eK}^{2} + \cdots + O_{eK}^{d} = R^{m(s)+1} \) associated with \( \phi_{n,s} \), then \( O_{eK}^{j} \) is the \( \text{SO}(n) \)-module of spherical harmonics of order \( j \) on the \((n-1)\)-unit sphere.

From now on we denote the above decomposition by \( V^{s} = V_{0} + V_{1} + \cdots + V_{s} \), where \( V_{i} \) is the \( K (= \text{SO}(n)) \)-module of spherical harmonics of order \( i \) on \( S_{n-1}^{m(s)} \).

Now we prepare some results about representation theory of \( \text{SO}(n+1) \). We first give the classification of representations of \( \text{SO}(n+1) = G \). Let \( T \subset G \) be the subgroup of matrices of the form;
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\[
\begin{pmatrix}
A_1 & 0 \\
A_2 & \ddots \\
0 & \ddots \\
& \ddots \\
0 & & \ddots & 0 \\
& & & A_p \\
& & & & 1
\end{pmatrix}
\]

if \( n=2p \)

or

\[
\begin{pmatrix}
A_1 & 0 \\
A_2 & \ddots \\
& \ddots \\
& & \ddots & 0 \\
& & & & A_p \\
& & & & & 1
\end{pmatrix}
\]

if \( n=2p-1 \)

where

\[
A_i=\begin{pmatrix}
\cos t_i & \sin t_i \\
-\sin t_i & \cos t_i
\end{pmatrix}
\]

Let \( \mathfrak{g} \) be the linear Lie algebra of \( G \) and \( \mathfrak{h} \) be the Lie algebra of \( T \) in \( \mathfrak{g} \). Let \( E_{ij} \) be the \((n+1)\times(n+1)\)-matrix whose \((i, j)\)-entry is 1 and all other entries are zero. Set \( h_i=E_{2i-1,2i}-E_{2i,2i-1}, i=1, \ldots, p \). Then \( h_1, \ldots, h_p \) is a basis of \( \mathfrak{h} \). Let \( \mathfrak{h}^* \) be the (real) dual of \( \mathfrak{h} \), and let \( \lambda_1, \ldots, \lambda_p \) be the dual basis for \( h_1, \ldots, h_p \). Order the elements of \( \mathfrak{h}^* \) lexicographically relative to \( \lambda_1, \ldots, \lambda_p \). Then the highest weights of (complex) irreducible finite dimensional representations of \( G \) are of the form \( \lambda_{(m)}=\sum_i m_i \lambda_i \), where \( m_1, \ldots, m_p \) are integers satisfying

\[
\begin{align*}
\{ & m_1 \geq m_2 \geq \cdots \geq m_p \geq 0 \quad n=2p, \\
& m_1 \geq m_2 \geq \cdots \geq |m_p| \quad n=2p-1. 
\end{align*}
\]

We shall denote the representation of \( SO(n+1) \) with highest weight \( \lambda_{(m)} \) by \( _nV^{(m)} \).

**Theorem 4.8** (The Branching Theorem [3]). Notations being as above, we have as a \( K \)-module,

\[
_nV^{(m)}=\sum_{(m')^{n-1}}V^{(m')}\]

where the summation is taken over all integers \( m'_1, \ldots, m'_p \) such that

\[
\begin{align*}
& m_1 \geq m_2 \geq \cdots \geq m_p \geq 0 \quad n=2p, \\
& m_1 \geq m_2 \geq \cdots \geq |m_p| \quad n=2p-1.
\end{align*}
\]

Now we state the formula which gives the \( SO(n+1) \)-module decomposition of \( ^l_nV^{(l,0,\cdots,0)} \), where \( ^l_nV^{(l,0,\cdots,0)} \) is the symmetric product of \( _nV^{(l,0,\cdots,0)} \) and is naturally an \( SO(n+1) \)-module.

**Theorem 4.9** (Do Carmo and Wallach [4]). If \( l\geq 1 \) and \( n\geq 3 \), then, as an
SO(n+1)-module,
\[
S^n_{\mathbb{R}}(V^{(i, \ldots, 0)}) = \sum_{j=0}^{\lfloor l/2 \rfloor} \mathbb{R}V^{(2l-2j, 2j, \ldots, 0)} + \mathbb{R}V^{(l-1, \ldots, 0)}.
\]

If \( n=2 \), as an SO(3)-module
\[
S^n_{\mathbb{R}}(V^{(i, \ldots, 0)}) = \mathbb{R}V^{(i, \ldots, 0)} + \mathbb{R}V^{(i-1, \ldots, 0)}.
\]

Next we state the Frobenius reciprocity. We first need the notion of an induced representation. Let \( G \) be a compact topological group and let \( K \) be a closed subgroup. Let \( V \) be a finite dimensional \( K \)-module over \( C \). Let \( \Gamma(V) \) be the vector space of all continuous functions \( f : G \to V \) such that \( f(xk) = k^{-1}f(x) \) for all \( x \in G, k \in K \). Let \( G \) act on \( \Gamma(V) \) by \( L_xf(y) = f(x^{-1}y) \), \( x, y \in G \). Then \( \Gamma(V) \) is a \( G \)-module which is called the \( G \)-module induced by \( V \).

**Lemma 4.10** (Frobenius Reciprocity). Let \( U \) be an arbitrary finite dimensional \( G \)-module over \( C \). Then \( \text{Hom}_G(U, \Gamma(V)) \) is canonically isomorphic to \( \text{Hom}_K(U, V) \).

We return to the standard minimal immersion \( \phi_{n,s} \) of \( S^K(V) \) in \( S^K(V) \). We shall complexify the real representations. Let \( (V^s)^C \) be the complexification of \( V^s \). It is well-known that \( (V^s)^C \) is an irreducible \( G \)-module over \( C \) with highest weight \( s\lambda_i \). We naturally extend \( G \)-invariant inner product \( , \) of \( V^s \) to the \( G \)-invariant Hermitian inner product \( , \) of \( (V^s)^C \). By Lemma 4.1, \( (V^s)^C \) is decomposed, as a \( K \)-module, as \( (V^s)^C = (V_0)^C + (V_1)^C + \cdots + (V_i)^C \), where \( (V_i)^C \) is a complexification of \( V_i \). It is also well-known that \( (V_i)^C \) is an irreducible \( K \)-module over \( C \) with highest weight \( i\lambda_i \). Next we complexify \( S^k(V^s) \) and denote it by \( S^k((V^s)^C) \). \( S^k((V^s)^C) \) is naturally isomorphic to \( S^k((V^s)^C) \). The Hermitian inner product \( , \) of \( S^k((V^s)^C) \) extended naturally from the inner product of \( S^k(V^s) \) coincides with the Hermitian product of \( S^k((V^s)^C) \) induced from the Hermitian product of \( (V^s)^C \). And it is \( G \)-invariant. Here we recall \( W_i, i=0,1, \ldots, s \), and in this case they are described as follows;

\[
W_0 = \left\{ \bigcup_{p \in M} \phi(p) \cdot \phi(p) \right\}_R = \{ G \cdot \phi(eK) \}_R = \{ G \cdot S^k(V_0) \}_R
\]
\[
W_1 = \left\{ \bigcup_{p \in M} S^k(T_pM) \right\}_R = \{ G \cdot S^k(T_{eK}M) \}_R = \{ G \cdot S^k(V_1) \}_R
\]
and
\[
W_i = \left\{ \bigcup_{p \in M} S^k(O_{p}) \right\}_R = \{ G \cdot S^k(O_{eK}) \}_R = \{ G \cdot S^k(V_i) \}_R \quad i \geq 2,
\]
where \( \{ G \cdot S^k(V_0) \}_R \) denotes the linear span of the orbit of \( S^k(V_0) \) in \( S^k(V^s) \). Therefore \( W_i \) is a \( G \)-submodule of \( S^k(V^s) \). Similarly we complexify \( W_j \) and denote it by \( W_j^c \). Naturally \( W_j^c \) is a \( G \)-submodule of \( S^k(V^s)^C \). We also remark that \( W_j^c \) is isomorphic to \( \{ G \cdot S^k(V_j^c) \}_C \). From now on for simplicity we denote the complexification of a real \( G \)-module (or \( K \)-module) by the same notation.
used for the real object. For example we denote $S^i(V^s)$, $W_j$, $V_i$, \ldots, etc. instead of $S^i(V',W_j',V_i', \ldots$.

**Lemma 4.11.** For a fixed positive integer $j$, let $U$ be the sum of those $G$-submodules of $S^i(V^s)$ not containing, as $K$-submodules, $n^{-1}V^{(2i-k,0,0,0,0, \ldots,0)}$ if $n \geq 4$ and $iV^{(1,0,0, \ldots,0)}$ if $n=3$, where $i$ and $k$ are integers satisfying $0 \leq i \leq j$ and $0 \leq k \leq [j/2]$. Then $U$ is orthogonal to $W_1 + \cdots + W_j$ in $S^i(V^s)$.

**Proof.** We shall show that $U$ is orthogonal to $W_i$, $i=1, \ldots, j$. We denote by $I(S^i(V_i))$ the $G$-module induced by the $K$-module $S^i(V_i)$ over $C$. Let $W=\{u \in S^i(V^s); (u, S^i(V_i))=0\}$, where $(\ , \ )$ is the Hermitian inner product in $S^i(V^s)$. Then as a $K$-module, $S^i(V^s)$ admits the orthogonal direct sum decomposition $S^i(V^s)=S^i(V_i)+W$. Let $P: S^i(V^s) \to S^i(V_i)$ be the corresponding projection. We claim that $W_i$ is contained in $I(S^i(V_i))$ as a $G$-submodule. To see this, define, for each $u \in S^i(V^s)$, a map $f_u: G \to S^i(V_i)$ by $f_u(g)=P(g^{-1}u)$. It is easily verified that $f_u \in I(S^i(V_i))$. Next define a map $\alpha: S^i(V^s) \to I(S^i(V_i))$ by $\alpha(u)=f_u$. Since $\alpha(g^{-1}u)(g)=P(g^{-1}(g^{-1}u))=P((g_0^{-1}g)^{-1}u)=(L_{g_0}(\alpha(u))(g_0))$, we conclude that $\alpha \in \text{Hom}_G(S^i(V^s), I(S^i(V_i)))$. If $u \in S^i(V^s)$ and $\alpha(u)=0$, then for any $g \in G$, $0=\alpha(u)(g)=P(g^{-1}u)$. Thus $0=(g^{-1}u, S^i(V_i))=(u, g \cdot S^i(V_i))$. We thus see that $\ker \alpha=W_i$. Therefore $\alpha: W_i \to I(S^i(V_i))$ is a $G$-module isomorphism, which proves our claim. Now consider the $G$-module $U$. Using the above fact and Frobenius Reciprocity (Lemma 4.10), we obtain $\dim_C \text{Hom}_G(U, W_i) \leq \dim_C \text{Hom}_G(U, I(S^i(V_i)))=\dim_C \text{Hom}_G(U, S^i(V_i))$. Since $V_i$ is isomorphic to $n^{-1}V^{(1,0,0,0, \ldots,0)}$ as a $K$-module, Theorem 4.9 implies that $U$ does not contain a $K$-submodule of $S^i(V_i)$. Therefore $\dim_C \text{Hom}_G(U, W_i)=0$. It follows that $U$ is orthogonal to $W_i$. Thus Lemma 4.11 is proved.

**Proof of Theorem 4.2.** By Theorem 4.9, there exists the $G$-submodule $n^{-1}V^{(2i-k,0,0,0,0, \ldots,0)}$ of $S^i(V^s)$. Theorem 4.8 implies that $n^{-1}V^{(2i-k,0,0,0,0, \ldots,0)}$ does not contain, as $K$-submodule, $n^{-1}V^{(2i-k,0,0,0,0, \ldots,0)}$ for $0 \leq i \leq [s/2]-1$ and $0 \leq k \leq [i/2]$ if $n \geq 4$ and $iV^{(1,0,0,0,0, \ldots,0)}$ for $0 \leq i \leq [s/2]-1$ if $n=3$. Thus by Lemma 4.11 the dimension of the orthogonal complement of $W_1 + \cdots + W_{[s/2]-1}$ in $S^i(V^s)$ is positive. This, together with Lemma 4.5, gives Theorem 4.2.

5. Helical geodesic minimal immersions.

First we give the definition of a helical geodesic immersion following Sakamoto ([8]). Let $\gamma: I \to M$ be a $C^\infty$-curve parametrized by the arc-length $s$. Let $\gamma^{(1)}=\gamma$ be the unit tangent vector and put $\kappa_s=\|\nabla_s \gamma\|$. If $\kappa_s$ vanishes on $I$, then $\gamma$ is said to be of order 1. If $\kappa_s$ is not identically zero, then we define $\gamma^{(2)}$ by $\nabla_\gamma \gamma^{(1)}=\kappa_s \gamma^{(2)}$ on the set $I_2=\{s \in I; \kappa_s(s) \neq 0\}$. Put $\kappa_2=\|\nabla_\gamma \gamma^{(2)} + \kappa_s \gamma^{(1)}\|$. If $\kappa_2 \equiv 0$ on $I_2$, then $\gamma$ is said to be of order 2. If $\kappa_s$ is not identically zero on $I_2$, then we define $\gamma^{(3)}$ by $\nabla_\gamma \gamma^{(2)}=-\kappa_s \gamma^{(1)} + \kappa_2 \gamma^{(3)}$. Inductively we put $\kappa_{d+1}=\|\nabla_\gamma \gamma^{(d)} + \kappa_d \gamma^{(d-1)}\|$ and if $\kappa_{d+1} \equiv 0$ on $I_d$, then $\gamma$ is said to be of order $d$. 

\[4.8\] 
\[4.9\] 
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\[4.13\] 
\[4.14\] 
\[4.15\] 
\[4.16\] 
\[4.17\] 
\[4.18\] 
\[4.19\] 
\[4.20\]
DEFINITION. Let \( \varphi : M \to \overline{M} \) be an isometric immersion of a connected complete Riemannian manifold \( M \) into a Riemannian manifold \( \overline{M} \) and \( \sigma : I \to M \) be an arbitrary geodesic in \( M \) parametrized by the arc-length. If the curve \( \gamma = \varphi \circ \sigma \) in \( \overline{M} \) is of order \( d \) and has constant curvatures \( \kappa_2, \cdots, \kappa_d \) which do not depend on \( \sigma \), then \( \varphi \) is called a helical geodesic immersion of order \( d \).

REMARK. It is known that a strongly harmonic manifold admits a helical geodesic minimal immersion into a sphere (Besse [2]). In particular the standard minimal immersions of compact rank one symmetric spaces into spheres are helical geodesic and minimal.

PROPOSITION 5.1. Let \( \varphi : S^{n}_{k(s)} \to S^{l}_{j} \) be a minimal immersion. \( \varphi \) is a helical geodesic immersion if and only if the \( j \)-th fundamental form \( B_j \) is isotropic for \( 2 \leq j \leq (\text{degree of } \varphi) \). In particular if \( \varphi \) is a helical geodesic minimal immersion of a sphere into a unit sphere, the order of \( \varphi \) is equal to the degree of \( \varphi \).

PROOF. Suppose that \( \varphi \) is a helical geodesic minimal immersion of order \( d \) of \( S^{n}_{k(s)} \) into \( S^{l}_{j} \). We use the same notations of covariant differentiations on \( S^{n}_{k(s)} \), \( S^{l}_{j} \), the normal bundle of \( \varphi \) etc. as in §3. Let \( \sigma : I \to S^{n}_{k(s)} \) be an arbitrary geodesic parametrized by the arc-length \( s \). We put \( X(s) = \delta(s) \). We denote by \( \kappa_3, \cdots, \kappa_d \) the curvatures of \( \gamma = \varphi \circ \sigma \) in \( S^{l}_{j} \). We shall compute the Frenet frame \( \{ \gamma^{(1)}, \gamma^{(2)}, \cdots, \gamma^{(d)} \} \). Since \( \tilde{\nabla}_{X}\gamma^{(1)} = B_2(X, X) \), we have \( \gamma^{(1)} = \kappa_2^{-1}B_2(X, X) \). Moreover \( \Vert B_2(X, X) \Vert = \kappa_2 \) for any unit vector \( X \) and then the second fundamental form \( B_2 \) of \( \varphi \) is isotropic on \( S^{n}_{k(s)} \). Next

\[
\tilde{\nabla}_{X}\gamma^{(2)} = \kappa_3^{-1}\tilde{\nabla}_{X}(B_2(X, X)) = \kappa_3^{-1}\{-H_{B_2(X, X)}X + \nabla B_2(X, X, X) \}.
\]

Since \( B_2 \) is isotropic, \( H_{B_2(X, X)}X = (\kappa_2)X \) and by the proof of Proposition 3.1 we have \( \nabla B_2(X, X) = B_3(X, X, X) \). It follows that \( \tilde{\nabla}_{X}\gamma^{(2)} = -\kappa_3X + \kappa_3^{-1}B_3(X, X, X) \). Thus \( \gamma^{(2)} = (\kappa_2\kappa_3)^{-1}B_3(X, X, X) \). Similarly we have \( \Vert B_3(X, X, X) \Vert = \kappa_2\kappa_3 \) for any unit tangent vector \( X \) and then \( B_3 \) is isotropic on \( S^{n}_{k(s)} \). Here we apply a mathematical induction. Let \( j \) be a fixed natural number satisfying \( 3 \leq j \leq d \). We assume that \( \gamma^{(k)} \) is described as \( \gamma^{(k)} = (\kappa_2 \cdots \kappa_k)^{-1}B_k(X, \cdots, X) \) and \( B_k \) is isotropic on \( S^{n}_{k(s)} \) for \( 2 \leq k \leq j \). Under these assumptions, we have

\[
\tilde{\nabla}_{X}\gamma^{(j)} = (\kappa_2 \cdots \kappa_j)^{-1}\tilde{\nabla}_{X}(B_j(X, \cdots, X)) = (\kappa_2 \cdots \kappa_j)^{-1}\{-H_{B_j(X, \cdots, X)}X + \nabla B_j(X, \cdots, X) \} \quad (j \geq 3).
\]

Since \( B_j \) is isotropic, the argument in the proof of Proposition 3.1 implies that

\[
\nabla B_j(X, \cdots, X) = B_{j+1}(X, \cdots, X) + D_{j+1}(X, \cdots, X).
\]

Noticing that we can prove Proposition 3.4 under the only condition that \( B_s \) is isotropic for \( 2 \leq k \leq j \), we have

\[
D_{j+1}(X, \cdots, X) = -(\lambda_j^2/\lambda_{j-1}^2)B_{j-1}(X, \cdots, X),
\]
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where \( \lambda_j = \|B_j(X, \cdots, X)\| \) and \( \lambda_{j-1} = \|B_{j-1}(X, \cdots, X)\| \). So we have \( D_j + (X, \cdots, X) = -\kappa_j B_{j-1}(X, \cdots, X) \). It follows that

\[
\tilde{\gamma}_X \gamma^{(j)} = -\kappa_j \gamma^{(j-1)} + (\kappa_2 \cdots \kappa_j)^{-1} B_{j+1}(X, \cdots, X).
\]

Thus we obtain \( \gamma^{(j+1)} = (\kappa_2 \cdots \kappa_{j+1})^{-1} B_{j+1}(X, \cdots, X) \) and that \( B_{j+1} \) is isotropic. By

the above argument we see that the Frenet frame \( \{ \gamma^{(1)}, \gamma^{(2)}, \cdots, \gamma^{(d)} \} \) of \( \gamma \) is given by

\[
\gamma^{(j)} = (\kappa_2 \cdots \kappa_j)^{-1} B_j(X, \cdots, X) \quad \text{for} \quad 2 \leq j \leq d.
\]

Therefore we see that the order of \( \varphi \) is equal to the degree of \( \varphi \) and that \( B_j \) is isotropic for \( 2 \leq j \leq d \). Noticing that if the \( j \)-th fundamental form \( B_j \) of a minimal immersion \( \varphi \) is isotropic, then it is constant isotropic (§ 3), we can prove the converse similarly.

By Proposition 5.1, we easily get

COROLLARY 5.2. Let \( \varphi_{n,s} : S^n_{k(s)} \to S^1 \) be the \( s \)-th standard minimal immersion \((n \geq 2)\). Then \( \varphi_{n,s} \) is a helical geodesic immersion of order \( s \).

THEOREM 5.3. Let \( \varphi : S^n_{k(s)} \to S^1 \) be a helical geodesic minimal immersion. Assume that \( \varphi \) is full. Then \( \varphi \) is equivalent to the standard minimal immersion \( \varphi_{n,s} \) and in particular the order of the helical geodesic immersion \( \varphi \) is \( s \) and \( l = m(s) \).

We prepare some lemmas before the proof of Theorem 5.3. Let \( X \in T_{x_0}M - \{0\} \) and \( \gamma : s \to \exp_{x_0} \frac{s}{\|X\|} X \) be the geodesic. Let \( \{Y_i\}_{i=1, \ldots, n} \) be Jacobi fields along \( \gamma \) such that \( Y_i(0) = 0 \) for every \( i \) and \( \{Y'_i(0)\}_{i=1, \ldots, n} \) forms an orthonormal basis of the orthogonal complement of \( X \) in \( T_{x_0}M \). Then we define \( \theta : TM \to R \) by

\[
\begin{cases}
\theta(0) = 1 \\
\theta(X) = \|X\|^{-n+1} \det (Y_1(\|X\|), \cdots, Y_n(\|X\|)),
\end{cases}
\]

where the determinant should be understood with respect to the parallel frame field of \( \{Y'_i(0)\} \). It is known that \( S^n \) is a globally harmonic manifold, i.e., there exists a \( C^\infty \)-function \( \Theta : R_n \to R \) such that \( \theta(X) = \Theta(\|X\|) \) for every \( x \in S^n \) and every \( X \in T_x S^n \).

LEMMA 5.4 (Berger-Gauduchon-Mazet [1] p. 134). Let \( f \) be a \( C^\infty \)-function on \( S^n \) of the form \( f(x) = F(\delta(x, x_0)) \) (i.e., which depends only on the distance to \( x_0 \)), where \( \delta \) denotes the distance function. Then we have

\[
\Delta f = -\frac{d^2F}{ds^2} \left( \frac{\theta'_{x_0}}{\theta_{x_0}} + \frac{n-1}{s} \right) \frac{dF}{ds},
\]

where \( \Delta \) denotes the Laplacian and \( \theta'_{x_0} \) is the radial derivative of \( \theta_{x_0} \) in \( T_{x_0}S^n \).

LEMMA 5.5 (Sakamoto [8]). Let \( \varphi : M \to S^1 \) be a helical geodesic immersion. Then there exists a \( C^\infty \)-function \( F : R_n \to R \) such that the Euclidean inner product
of position vectors $\varphi(x)$ and $\varphi(y)$ is given by $\langle \varphi(x), \varphi(y) \rangle = F(\delta(x, y))$.

**Lemma 5.6** (Besse [2] p. 177, p. 178). Let $\varphi : M \rightarrow S^1$ be a helical geodesic immersion of order $d$ and let $\sigma : I \rightarrow M$ be a geodesic parametrized by the arc-length. Then the curvatures $\kappa_2, \cdots, \kappa_d$ of the curve $\gamma = \varphi \circ \sigma$ in $S^1$ are completely determined by $F^{(k)}(0), k = 1, 2, \cdots$, where $F$ is the function introduced in Lemma 5.5. In particular the order of $\varphi$ is determined by $F^{(k)}(0), k = 1, 2, \cdots$.

**Proof of Theorem 5.3.** We denote by $F$ and $\mathring{F}$ the functions introduced in Lemma 5.5 associated with the helical geodesic minimal immersion $\varphi$ and the standard minimal immersion $\psi_{n,s}$ respectively. For a fixed point $x_0 \in S^n$ we define the functions $f$ and $\mathring{f}$ on $S^n$ by

$$f(x) = \langle \varphi(x), \varphi(x_0) \rangle = F(\delta(x, x_0))$$

$$\mathring{f}(x) = \langle \psi(x), \psi(x_0) \rangle = \mathring{F}(\delta(x, x_0)),$$

where $\langle , \rangle$ denotes the Euclidean inner product. By a well-known theorem (Takahashi [9]), $f(x)$ and $\mathring{f}(x)$ are eigenfunctions of the Laplacian on $S_{k(S)}^n$ with eigenvalue $n$. By Lemma 5.4, we have

$$- \frac{d^2 F}{ds^2} + \frac{dF}{ds} \left( \frac{\theta_{x_0}'}{\theta_{x_0}} + \frac{n-1}{s} \right) = nF$$

and

$$- \frac{d^2 \mathring{F}}{ds^2} + \frac{d\mathring{F}}{ds} \left( \frac{\theta_{x_0}'}{\theta_{x_0}} + \frac{n-1}{s} \right) = n\mathring{F}.$$ 

Since $F(0) = \mathring{F}(0) = 1$ and $F'(0) = \mathring{F}'(0) = 0$, we obtain $F \equiv \mathring{F}$. Then Lemma 5.6 implies that the order of $\varphi$ is equal to the order of $\psi_{n,s} = s$. By Proposition 5.1, the $j$-th fundamental form $B_j$ of $\varphi$ is isotropic for $2 \leq j \leq s$. This, together with Theorem 4.1, gives Theorem 5.3.

**Corollary 5.7.** Let $\varphi : S_{k_{10}}^n \rightarrow S^1$ be a minimal immersion. If the $j$-th fundamental form $B_j$ is isotropic for $2 \leq j \leq (\text{degree of } \varphi)$, then $\varphi$ is equivalent to the standard minimal immersion $\psi_{n,s}$.

**Proof.** By Proposition 5.1, $\varphi$ is a helical geodesic immersion. By Theorem 5.3 we see that $\varphi$ is equivalent to $\psi_{n,s}$.

**References**


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