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Boundaries of the Teichmüller spaces of finitely generated Fuchsian groups of the second kind

By Hiro-o YAMAMOTO

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The Teichmüller space $T(\Gamma)$ of a Fuchsian group Γ can be embedded, as a bounded domain, into the Banach space $B(\Gamma)$ of bounded quadratic differentials for Γ. Hence the boundary $\partial T(\Gamma)$ of $T(\Gamma)$ can be defined naturally. The boundary $\partial T(\Gamma)$ was investigated by Bers [**Ber**], Maskit [$\mathbf{M}_3$], Abikoff [**A**] and others. However, most of them were done under the restriction that Γ is a finitely generated Fuchsian group of the first kind, that is, $T(\Gamma)$ is finitely dimensional. In this paper we investigate the boundaries of Teichmüller spaces of finitely generated Fuchsian group of the second kind. In this case the dimension of $T(\Gamma)$ is infinite. For each $\phi\in T(\Gamma)\cup\partial T(\Gamma)$ the meromorphic homeomorphism W_ϕ of the lower half plane L can be defined, and induces an isomorphism $\chi_\phi : \Gamma\rightarrow W_\phi \Gamma W_\phi^{-1}$. A point $\phi\in\partial T(\Gamma)$ is called a cusp if a hyperbolic element is mapped to a parabolic one under χ_ϕ. The existence of cusps was proved by Bers [**Ber**] if Γ is a finitely generated Fuchsian group of the first kind. In section 2 we prove the existence of cusps even if Γ is a finitely generated Fuchsian group of the second kind. As usual we can prove the above statement by obtaining cusps as limits of sequences in $T(\Gamma)$ obtained by squeezing deformations, the definition of which and necessary preliminaries are exhibited in § 1. In § 3 we prove that if $\phi\in\partial T(\Gamma)$ is not a cusp, then $\chi_\phi(\Gamma)$ is a quasi-Fuchsian group, which never exists on the boundaries of the Teichmüller spaces of finitely generated Fuchsian groups of the first kind. By using the method in the proof of this theorem, we also give an alternative proof of the existence of cusps and of the estimate of outradii of the Teichmüller spaces of parabolic or finite cyclic groups, which are due to Sekigawa [$\mathbf{Se}_1$], [$\mathbf{Se}_2$]. Finally, in § 4 we prove the existence of geometrically infinite cusps.

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§1. Preliminaries.

A Moebius transformation g is a conformal automorphism of the Riemann sphere $\hat{C}=C\cup\{\infty\}$, which is of the form $g(z)=(az+b)/(cz+d)$, where a, b, c and d are complex numbers satisfying the equation $ad-bc=1$. In this paper g is sometimes denoted by $(a, b; c, d)$. Let G be a discrete group of Moebius transformations. The ordinary set $\Omega(G)$ of G is the maximal subset of $\hat{C}$ where G acts discontinuously. The set $\Lambda(G)=\hat{C}-\Omega(G)$ is called the limit set of G. If $\Lambda(G)$ consists of finitely many points, then G is said to be elementary. If $\Lambda(G)$ contains infinitely many points, then G is said to be Kleinian. A quasi-Fuchsian group is a Kleinian group keeping an oriented quasi-circle invariant. A totally degenerate group is a Kleinian group whose ordinary set is connected and simply connected.

For a Kleinian group G the quotient space $S=\Omega(G)/G$ has a natural complex structure so that the canonical projection π is holomorphic. Let $\Omega'(G)$ be $\Omega(G)$ with elliptic fixed points of G removed. A connected subsurface M of S is of type (g, m, n) if $M'=M\cap(\Omega'(G)/G)$ is a compact Riemann surface of genus g with m discs and n points removed. If π is locally ν_j-to-1 in a punctured neighborhood of a point $P_j\in M-M'$ in M', then M is said to have the signature $(g, m, n; \nu_1, \cdots, \nu_n)$. Throughout this paper, we assume that $\nu_1\leqq\cdots\leqq\nu_n$. Note that $2\leqq\nu_1$ and $\nu_n\leqq\infty$. If $M-M'$ consists of n_1 points, then $\nu_j<\infty$ for $j=1, \cdots, n_1$ and $\nu_j=\infty$ for $j=n_1+1, \cdots, n$.

From now on Γ denotes a finitely generated Fuchsian group of the second kind keeping the upper half plane U invariant, that is, $\gamma(U)=U$ for each $\gamma\in\Gamma$ and $(\boldsymbol{R}\cup\{\infty\})\cap\Omega(\Gamma)\neq\emptyset$.

A measurable function μ defined on $\hat{C}$ whose L^∞-norm is less than one is called a Beltrami differentials for Γ if $\mu\circ\gamma(z)\overline{\gamma'(z)}/\gamma'(z)=\mu(z)$ for almost all $z\in\hat{C}$ and all $\gamma\in\Gamma$. Denote by $M(\Gamma)$ the set of all Beltrami differentials for Γ which vanishes in the lower half plane L.

Now we review well known results on Teichmüller spaces (see [**Ber**]). For each $\mu\in M(\Gamma)$, there exists a unique quasi-conformal automorphism w^μ of $\hat{C}$ keeping the three points 0, 1 and ∞ invariant with the complex dilatation μ. Set $\phi^\mu=[w^\mu|L]$, where $w^\mu|L$ is the restriction of w^μ to L and $[f]=(f''/f')'-(1/2)(f''/f')^2$ is the Schwarzian derivative of f. Then ϕ^μ belongs to the Banach space $B(\Gamma)$ of bounded quadratic differentials for Γ, that is, the holomorphic function ϕ^μ defined on L satisfies the equation $\phi^\mu\circ\gamma(z)\gamma'(z)^2=\phi^\mu(z)$ for each $\gamma\in\Gamma$ and each $z=x+\sqrt{-1}y$, and the norm $\|\phi^\mu\|=\sup_{z\in L}y^2|\phi^\mu(z)|$ is finite. The mapping $\Phi: M(\Gamma)\ni\mu\mapsto\phi^\mu\in B(\Gamma)$ is continuous and open. The Teichmüller space $T(\Gamma)$ of Γ is the image of $M(\Gamma)$ under Φ. The space $T(\Gamma)$ is a bounded domain of $B(\Gamma)$, and the boundary of $T(\Gamma)$ is denoted by $\partial T(\Gamma)$. For each $\phi\in B(\Gamma)$ the

Schwarzian differential equation $2\eta''+\phi\eta=0$ has two linearly independent solutions, which are determined uniquely under the normalized conditions $\eta_1'=\eta_2=0$ and $\eta_1=\eta_2'=1$ at $z=-\sqrt{-1}$. If ϕ is in the closure $\mathrm{Cl}\,T(\Gamma)$ $(=T(\Gamma)\cup\partial T(\Gamma))$ of $T(\Gamma)$, then $W_\phi=\eta_1/\eta_2$ is meromorphic and univalent, and the isomorphism $\chi_\phi:\gamma\mapsto W_\phi\circ\gamma\circ W_\phi^{-1}$ takes Γ into the Kleinian group $\chi_\phi(\Gamma)=W_\phi\Gamma W_\phi^{-1}$. A point $\phi\in\partial T(\Gamma)$ is called a cusp if there exists a hyperbolic $\gamma\in\Gamma$ such that $\chi_\phi(\gamma)$ is parabolic. For a cusp ϕ, the Kleinian group $\chi_\phi(\gamma)$ is also called a cusp. Let E be a non-empty subset of $\Omega(G)$ invariant under a Kleinian or elementary group G. A set $R\subset E$ is called a fundamental set for G in E if $g(R)\cap R=\emptyset$ for each $g\in G-\{\mathrm{id.}\}$ and if $\bigcup_{g\in G} g(R)=E$. A connected set $P\subset R$ is called a cusped region if $\pi(P)$ is a doubly connected neighborhood of a puncture of $\Omega(G)/G$.

LEMMA 1.1. *There is a subset S of L included compactly in $\Omega(\Gamma)$ with $\|\phi\| = \sup_{z\in S} y^2|\phi(z)|$ for each $\phi\in B(\Gamma)$.*

PROOF. Let R be a fundamental set for Γ in L. Note that $\|\phi\|=\sup_{z\in R} y^2|\phi(z)|$. Without loss of generality, we may assume that R consists of one cusped region $P=\{z=x+\sqrt{-1}\,y\,;\ 0\leqq x<1,\ y<-2\}$ ([**Sh**]) and a set R_0 included compactly in $\Omega(\Gamma)$. In P, ϕ can be expanded in the form $\sum_{j=1}^{\infty} a_j\exp(-2\pi\sqrt{-1}\,jz)$. Set $\lambda_1(z)=y^2|\exp(-2\pi\sqrt{-1}\,z)|=y^2|\exp(2\pi y)|$. Then it holds that $\lambda_1(z)\leqq\lambda_1(-2)$ for each $y\leqq-2$. Since $\lambda_2(z)=\sum_{j=1}^{\infty} a_j\exp(-2\pi\sqrt{-1}(j-1)z)$ is holomorphic, we can find a point $z_1=x_1-2\sqrt{-1}$ with $|\lambda_2(z)|\leqq|\lambda_2(z_1)|$ for each $z\in P$. Suming up the above, we have $\sup_{z\in P} y^2|\phi(z)|=(-2)^2|\phi(z_1)|$. Clearly the set $R-P$ has the desired property.

For each $\mu\in M(\Gamma)$ there exists a unique quasi-conformal automorphism F^μ of $\hat{C}$ satisfying the equation $F^\mu_{\bar z}=\mu F^\mu_z$ and keeping the three points $-\sqrt{-1}$, $-2\sqrt{-1}$ and $-3\sqrt{-1}$ invariant. As is well known, $W_{\Phi(\mu)}\circ(F^\mu)^{-1}$ is a Moebius transformation. Let $\{\mu_t\}_{t\in R}$ be a family of Beltrami differentials for Γ. Then for the sake of simplicity, we write F_t for F^{μ_t}. The automorphism F_t takes Γ into a Kleinian group $F_t\Gamma F_t^{-1}$ and there is a quasi-conformal homeomorphism f_t of S such that the following diagram

$$\begin{array}{ccc}
\Omega(\Gamma) & \xrightarrow{\quad F_t \quad} & \Omega(F_t\Gamma F_t^{-1}) \\
\downarrow & & \downarrow \\
S=\Omega(\Gamma)/\Gamma & \xrightarrow{\quad f_t \quad} & S_t=\Omega(F_t\Gamma F_t^{-1})/F_t\Gamma F_t^{-1}
\end{array}$$

is commutative, where the vertical arrows represent the canonical projections. We can define the Poincaré metric $\rho_t(z)|dz|$ with the negative constant curvature on $\Omega(F_t\Gamma F_t^{-1})$ since the boundary $\Lambda(F_t\Gamma F_t^{-1})$ of $\Omega(F_t\Gamma F_t^{-1})$ consists of more than two points. So S_t has the natural hyperbolic metric $\bar{\rho}_t(\zeta)|d\zeta|$, where $\zeta=\pi(z)$ is a local parameter and $\bar{\rho}_t(\zeta)|d\zeta|=\rho_t(z)|dz|$. The length of a curve σ on S_t measured by $\bar{\rho}_t(\zeta)|d\zeta|$ is denoted by $l(\sigma, S_t)$.

A sequence $\{g_n=(a_n, b_n; c_n, d_n)\}_{n=1}^{\infty}$ of Moebius transformations is said to converge to a Moebius transformation $g=(a, b; c, d)$ if there exists a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$, where ε_n is either 1 or -1, such that $\varepsilon_n a_n$, $\varepsilon_n b_n$, $\varepsilon_n c_n$ and $\varepsilon_n d_n$ converge to a, b, c and d, respectively. A sequence $\{G_n\}_{n=1}^{\infty}$ of Kleinian groups, where G_n is generated by $g_{1,n}, \cdots, g_{k,n}$, is said to converge to a Kleinian group G generated by $g_1, \cdots, g_k$ if $g_{j,n}$ converges to g_j, $j=1, \cdots, k$.

LEMMA 1.2. *Let* $\{\Phi(\mu_j)\}_{j=1}^{\infty}\subset T(\Gamma)$ *be a sequence converging to a point* $\phi\in T(\Gamma)\cup\partial T(\Gamma)$. *Then there exists a Moebius transformation* g *such that the Kleinian group* $F_j\Gamma F_j^{-1}$ *converges to* $g^{-1}\chi_{\phi}(\Gamma)g$.

PROOF. Since $[F_j|L]=[W_{\Phi(\mu_j)}]$, there exists a Moebius transformation g_j which is identical with $W_{\Phi(\mu_j)}\circ F_j^{-1}$ in $F_j(L)$. Since the sequences $\{g_j(-\sqrt{-1})\}_{n=1}^{\infty}=\{W_{\Phi(\mu_j)}(-\sqrt{-1})\}_{n=1}^{\infty}$, $\{g_j(-2\sqrt{-1})\}_{n=1}^{\infty}=\{W_{\Phi(\mu_j)}(-2\sqrt{-1})\}_{n=1}^{\infty}$ and $\{g_j(-3\sqrt{-1})\}_{n=1}^{\infty}=\{W_{\Phi(\mu_j)}(-3\sqrt{-1})\}_{n=1}^{\infty}$ converge to points which are different from one another, $\{g_j\}_{j=1}^{\infty}$ converges to a Moebius transformation. Therefore so does $\{F_j|L\}_{j=1}^{\infty}=\{g_j^{-1}\circ W_{\Phi(\mu_j)}\}_{j=1}^{\infty}$. Now the proof of Lemma 1.2 is clear.

A set of simple loops $\{\alpha_i\}_{i=1}^{q}$ on a Riemann surface M is said to be homotopically independent if the following holds:

(i) $\alpha_i\cap\alpha_j=\emptyset$, $1\leqq i<j\leqq q$,

(ii) α_i is not freely homotopic to α_j, $1\leqq i<j\leqq q$, and

(iii) α_i bounds neither a disc nor a punctured disc, $1\leqq i\leqq q$.

Let $\{\alpha_i\}_{i=1}^{q}$ be a homotopically independent set of loops on $(U\cap\Omega'(\Gamma))/\Gamma$. For every i, we can find a doubly connected domain D_i of $(U\cap\Omega'(\Gamma))/\Gamma$ containing α_i with $\mathrm{Cl}\,D_i$ is compact in $(U\cap\Omega'(\Gamma))/\Gamma$ and $\mathrm{Cl}\,D_i\cap\mathrm{Cl}\,D_j=\emptyset$, $1\leqq i<j\leqq q$. Let $\{\mu_t\}_{t\in[0,1)}$ be a set of Beltrami differentials for Γ satisfying

(i) μ_t vanishes in $\hat{C}-\pi^{-1}\left(\bigcup_{i=1}^{q} D_i\right)$,

(ii) there exists a simple loop $\bar{\alpha}_i\subset D_i$ freely homotopic to α_i with $\lim_{t\to1} l(f_t(\bar{\alpha}_i), S_t)=0$,

(iii) on any compact subset of $\hat{C}-\pi^{-1}\left(\bigcup_{i=1}^{q}\alpha_i\right)$, μ_t converges to a measurable function whose L^{∞}-norm is less than one as $t\to1$, and

(iv) each component of $f_t(D_i-\bar{\alpha}_i)$ is conformally equivalent to the annulus $1-t<|z|<2$.

Then we say that Γ is squeezed with respect to $\{\alpha_i\}_{i=1}^{q}$ if t tends to 1.

We note that $\{\bar{\alpha}_i\}_{i=1}^{q}$ is homotopically independent and that, though neither $\{D_i\}_{i=1}^{q}$ nor $\{\bar{\alpha}_i\}_{i=1}^{q}$ can be determined uniquely by $\{\alpha_i\}_{i=1}^{q}$, this is not essential for our later argument. By abuse of language, we write merely α_i for $\bar{\alpha}_i$.

The existence of squeezing deformations was shown in [$\mathbf{Y}_1$].

§ 2. Geometrically finite cusps.

The purpose of this section is to investigate properties of groups obtained by squeezing deformations.

Kleinian groups can be naturally regarded as discontinuous groups acting in the Euclidean upper half space $\{(z, t); z\in\hat{C}, t>0\}=\{(x_1, x_2, x_3)\in\boldsymbol{R}^3; x_3>0\}$. A Kleinian group G is said to be geometrically finite if G has a Dirichlet fundamental region in the upper half space surrounded by a finite number of hyperbolic planes, that is, Euclidean half planes or hemispheres orthogonal to C ([$\mathbf{G}_2$, p. 245]). Quasi-Fuchsian groups and elementary groups are geometrically finite, and totally degenerate groups are not ([$\mathbf{G}_1$]).

Let δ^* be the union of all components δ's of $\Omega(\Gamma)-\pi^{-1}\left(\bigcup_{i=1}^{q}\alpha_i\right)$ such that δ/Γ_δ has the signature $(0, 1, 2; 2, 2)$, where Γ_δ is the stabilizer subgroup $\{\gamma\in\Gamma; \gamma(\delta)=\delta\}$ of δ in Γ. Denote $\Omega(\Gamma)-\pi^{-1}\left(\bigcup_{i=1}^{q}\alpha_i\right)-\delta^*$ by $\hat{\delta}$.

THEOREM 2.1. *If a finitely generated Fuchsian group Γ of the second kind is squeezed with respect to a homotopically independent set $\{\alpha_i\}_{i=1}^{q}$ of loops on $(\Omega'(\Gamma)\cap U)/\Gamma$, then the following holds.*

(i) *The point $\Phi(\mu_t)$ converges to a cusp $\psi\in\partial T(\Gamma)$.*

(ii) *The Kleinian group $\chi_\psi(\Gamma)$ is a geometrically finite Kleinian group with an invariant component Δ_0.*

(iii) *There exists a homeomorphism F of $\hat{\delta}$ onto $\Omega(\chi_\psi(\Gamma))$ such that if for a component δ of $\hat{\delta}$, the Riemann surface δ/Γ_δ has the signature $(g, m, n; \nu_1, \cdots, \nu_n)$, then $F(\delta)/\chi_\psi(\Gamma)_{F(\delta)}$ has the signature $(g, 0, m+n; \nu_1, \cdots, \nu_{m+n})$, where $\nu_{n+1}=\cdots=\nu_{m+n}=\infty$.*

The proof of Theorem 2.1 is divided into several lemmas. First, we will show the existence of a cusp ψ to which a sequence $\{\Phi(\mu_{t_j})\}_{j=1}^{\infty}\subset\{\Phi(\mu_t)\}_{t\in[0,1)}$ converges. Secondly, we will show the Kleinian group $\hat{\chi}_\psi(\Gamma)=\lim_{j\to\infty}F_{t_j}\Gamma F_{t_j}^{-1}$ satisfies (ii) and (iii), which means that $\chi_\psi(\Gamma)$ also does by Lemma 1.2. Finally, the uniqueness of the limit $\lim_{t\to1}\Phi(\mu_t)$ will be shown.

Let δ_0 be the component of $\hat{\delta}$ including L.

LEMMA 2.2. *There exists a sequence $\{F_{t_j}\}_{j=1}^{\infty}\subset\{F_t\}_{t\in[0,1)}$ such that $F_{t_j}|\delta_0$ converges to a homeomorphism F_{δ_0} of δ_0, and the convergence is uniform on every compact subset of δ_0.*

PROOF. Let $\{E_k\}_{k=1}^{\infty}$ be a sequence of compact subset of δ_0 with $E_k\subset E_{k+1}^{\circ}$

and $\bigcup_{k=1}^{\infty} E_k^\circ = \delta_0$, where E_k° is the interior of E_k. By the definition of squeezing deformations, $F_t | E_2^\circ$ is a K_2-quasi-conformal homeomorphism keeping the three points $-\sqrt{-1}$, $-2\sqrt{-1}$ and $-3\sqrt{-1}$ invariant, where $K_2 = (1 + \operatorname{ess\,sup}_{z \in E_2^\circ} |\mu(z)|)/(1 - \operatorname{ess\,sup}_{z \in E_2^\circ} |\mu(z)|)$. So the family $\{F_t | E_2^\circ\}_{t \in [0,1)}$ of K_2-quasi-conformal homeomorphism of E_2° is normal ([**LV**, p. 73]). Therefore we can find a sequence $\{F_{2,j}\}_{j=1}^{\infty}$ in $\{F_t\}_{t \in [0,1)}$ such that $F_{2,j} | E_1^\circ$ converges to a K_2-quasiconformal homeomorphism of E_2° and the convergence is uniform on E_1 ([**LV**, p. 74]). By the same reasoning as above we can find a subsequence $\{F_{k+1,j}\}_{j=1}^{\infty}$ of $\{F_{k,j}\}_{j=1}^{\infty}$ such that $F_{k+1,j} | E_{k+1}^\circ$ converges to a K_{k+1}-quasi-conformal homeomorphism of E_{k+1}° and the convergence is uniform on E_k. The mapping $F_{\delta_0} = \lim_{j \to \infty} F_{j,j} | E_j^\circ$ is a homeomorphism of δ_0. Set $F_{t_j} | E_j^\circ = F_{j,j} | E_j^\circ$. Then $\{F_{t_j}\}_{j=1}^{\infty} \subset \{F_t\}_{t \in [0,1)}$ is the desired sequence.

Denote by $\tau(\alpha_i)$ the set consisting of elements of Γ which keep a component of $\pi^{-1}(\alpha_i)$.

LEMMA 2.3. *There exists a sequence $\{\phi_j\}_{j=1}^{\infty} \subset \{\Phi(\mu_t)\}_{t \in [0,1)}$ converging to a cusp ψ.*

PROOF. Set $\phi_j = [F_{t_j} | L] = \Phi(\mu_{t_j}) \in \{\Phi(\mu_t)\}_{t \in [0,1)}$, where $\{F_{t_j}\}_{j=1}^{\infty}$ is the sequence obtained in Lemma 2.2. Since by Lemma 2.2 the sequence $\{F_{t_j} | L\}_{j=1}^{\infty}$ of meromorphic homeomorphism of L converges to F_{δ_0} uniformly on the closure of the set S obtained in Lemma 1.1, so do $\{(F_{t_j} | L)'\}_{j=1}^{\infty}$, $\{(F_{t_j} | L)''\}_{j=1}^{\infty}$ and $\{(F_{t_j} | L)'''\}_{j=1}^{\infty}$. Therefore ϕ_j converges to $\psi = [F_{\delta_0} | L]$. Let γ be a hyperbolic element of $\tau(\alpha_i)$. Then $\hat{\chi}_\psi(\gamma)$ is parabolic ([**Y**$_1$]). Therefore we see that $\psi \in \partial T(\Gamma)$ is a cusp.

Since $F_{\delta_0}(\delta_0) \subset \Omega(\lim_{j \to \infty} F_{t_j} \Gamma F_{t_j}^{-1})$, we can define the isomorphism $\hat{\chi}_\psi$ of Γ mapping γ to $\lim_{j \to \infty} F_{t_j} \circ \gamma \circ F_{t_j}^{-1}$.

LEMMA 2.4. *The homeomorphism F_{δ_0} obtained in Lemma 2.2 maps δ_0 onto an invariant component Δ_0 of $\hat{\chi}_\psi(\Gamma)$.*

PROOF. It is clear that $F_{\delta_0}(\delta_0) \subset \Omega(\hat{\chi}_\psi(\Gamma))$. Let Δ_0 be the component of $\hat{\chi}_\psi(\Gamma)$ including $F_{\delta_0}(\delta_0)$. Since $F_{\delta_0}(\delta_0)$ is invariant under $\hat{\chi}_\psi(\Gamma)$, so is Δ_0. By Ahlfors' finiteness theorem ([**Ah**]) and the definition of squeezing deformations, both $\Delta_0'/\hat{\chi}_\psi(\Gamma)$ and $F_{\delta_0}(\delta_0)'/\hat{\chi}_\psi(\Gamma)$ are compact Riemann surfaces with finitely many points removed. So the set $X = \Delta_0'/\hat{\chi}_\psi(\Gamma) - F_{\delta_0}(\delta_0)'/\hat{\chi}_\psi(\Gamma)$ either is empty or consists of finitely many points. Assume that X is not empty. Let σ be a simple closed loop which bounds a disc on $\Delta_0'/\hat{\chi}_\psi(\Gamma)$ containing exactly one point of X. Then a component $\tilde{\sigma}$ of $\pi^{-1}(\sigma)$ is invariant only under the trivial element in $\hat{\chi}_\psi(\Gamma)$. On the other hand, by the definition of the squeezing deformations $F_{\delta_0}^{-1}(\tilde{\sigma})$ is invariant under a non-trivial element γ so that $\tilde{\sigma}$ is invariant under a non-trivial element $\hat{\chi}_\psi(\gamma)$. This contradiction means that X is empty, and hence

$F_{\delta_0}(\delta_0)=\Delta_0$.

Here we recall some classical results on Fuchsian groups. Let δ be a component of $\Omega(\Gamma)/\Gamma-\pi^{-1}\left(\bigcup_{i=1}^{q}\alpha_i\right)$. Then a stabilizer subgroup Γ_δ of δ in Γ is elementary if and only if δ/Γ_δ has the signature (0, 1, 2; 2, 2) ([**F**, Chapter VI]). In this case, Γ_δ is generated by two elliptic elements e_1 and e_2 with the relations $e_1^2=e_2^2=\mathrm{id.}$, and Γ_δ consists of elements $e_1^m\circ(e_2\circ e_1)^n$, $n\in\boldsymbol{Z}$, $m=0, 1$. Therefore any hyperbolic element $\gamma\in\Gamma_\delta$ is of the form $(e_2\circ e_1)^n$, $n\in\boldsymbol{Z}-\{0\}$. Note that $\pi(\delta)$ is surrounded by one loop $\alpha^*\in\{\alpha_i\,;\,i=1,\cdots,q\}$, which divides $\Omega(\Gamma)/\Gamma$ into the component δ/Γ_δ and another one. Let $\bar{\alpha}$ be a component of $\pi^{-1}(\alpha^*)\cap\partial\delta$. Then, since $\bar{\alpha}$ is invariant under $e_2\circ e_1$, $\partial\delta\cap\Omega(\Gamma)$ consists of $\bar{\alpha}=(e_2\circ e_1)^n(\bar{\alpha})$ and $e_1(\bar{\alpha})=e_1\circ(e_2\circ e_1)^n(\bar{\alpha})$, $n\in\boldsymbol{Z}$. Note that both $\bar{\alpha}$ and $e_1(\bar{\alpha})$ are kept invariant under all hyperbolic elements.

LEMMA 2.5. *There exists a subsequence, again denoted by* $\{F_{t_j}\}_{j=1}^{\infty}$, *of* $\{F_{t_j}\}_{j=1}^{\infty}$ *obtained in Lemma* 2.2 *such that* F_{t_j} *converges to a homeomorphism* F *of* $\hat{\delta}$ *which maps each component of* $\hat{\delta}$ *onto a component of* $\hat{\chi}_\phi(\Gamma)$.

PROOF. Let $\delta_0,\cdots,\delta_p$ be a complete non-conjugate list of components of $\hat{\delta}$. Set $\varepsilon_m=\delta_m\cap(\hat{C}-\pi^{-1}(D_i))$, $m=1,\cdots,p$. Since $F_{t_j}|\varepsilon_m$ is meromorphic and does not take the value $-\sqrt{-1}$, $-2\sqrt{-1}$ and $-3\sqrt{-1}$, the sequence $\{F_{t_j}|\varepsilon_m\}_{j=1}^{\infty}$ is a normal family and a subsequence, again denoted by $\{F_{t_j}|\varepsilon_m\}_{j=1}^{\infty}$, of $\{F_{t_j}|\varepsilon_m\}_{j=1}^{\infty}$ converges to a mapping F_{ε_m} of ε_m which is either a meromorphic homeomorphism of ε_m or a constant map. If $F_{\varepsilon_m}(\varepsilon_m)$ is a point ζ, then ζ is kept invariant under $\hat{\chi}_\phi(\Gamma_{\varepsilon_m})$. Therefore any pair of non-elliptic elements g_1 and g_2 of the discrete group $\hat{\chi}_\phi(\Gamma_{\varepsilon_m})$ are commutative ([**L**, p. 94]), and so are $\hat{\chi}_\phi^{-1}(g_1)$ and $\hat{\chi}_\phi^{-1}(g_2)$. On the other hand, there exists a pair of hyperbolic elements in Γ_{ε_m} which are not commutative each other since Γ_{ε_m} is non-elementary. This contradiction implies that $F_{t_j}|\varepsilon_m$ converges to a meromorphic homeomorphism of ε_m. By the same reasoning as in the proof of Lemma 2.4, we can find a subsequence, again denoted by $\{F_{t_j}\}_{j=1}^{\infty}$, of $\{F_{t_j}\}_{j=1}^{\infty}$ such that $F_{t_j}|\bigcup_{k=0}^{p}\delta_k$ converges to a homeomorphism of $\bigcup_{k=0}^{p}\delta_k$ which maps δ_k onto a component of $\hat{\chi}_\phi(\Gamma)$. Since $(F_{t_j}\circ\gamma\circ F_{t_j}^{-1})(F_{t_j}(z))=F_{t_j}(\gamma(z))$ for each j, each $\gamma\in\Gamma$, and each $z\in\bigcup_{k=0}^{p}\delta_k$, $F_{t_j}|\hat{\delta}$ converges to a homeomorphism of $\hat{\delta}$, which has the desired property.

LEMMA 2.6. *Let* δ *be a component of* δ^* *and* $\{F_{t_j}\}_{j=1}^{\infty}$ *the sequence obtained in Lemma* 2.5. *Then* $F_{t_j}(\delta)$ *converges to the fixed point of* $\hat{\chi}_\phi(\gamma)$, *where* γ *is a hyperbolic element of the elementary group* Stab δ.

PROOF. Let ε be any compact subset of δ. Then, since $\{F_{t_j}|\varepsilon\}_{j=1}^{\infty}$ is a normal family ([**LV**, p. 73]), a subsequence, again denoted by $\{F_{t_j}|\varepsilon\}_{j=1}^{\infty}$, converges to either a quasi-conformal homeomorphism of ε, or a mapping of ε onto two points or a constant map ([**LV**, p. 74]). If $\{F_{t_j}|\varepsilon\}_{j=1}^{\infty}$ converges to a quasi-conformal homeomorphism, then by the same reasoning as in the proof of Lemma 2.4 and by the definition of squeezing deformations, the component $\lim_{j\to\infty} F_{t_j}(\delta)/\hat{\chi}_\phi(\mathrm{Stab}\,\delta)$ of $\Omega(\hat{\chi}_\phi(\Gamma))/\hat{\chi}_\phi(\Gamma)$ has the signature $(0, 0, 3; 2, 2, \infty)$. This contradiction implies that the set $\lim_{j\to\infty} F_{t_j}(\delta)$ consists of at most two points. Assume the existence of a point in $\bar{\delta}=\lim_{j\to\infty} F_{t_j}(\delta)$ which is distinct from the fixed point ξ of $\hat{\chi}_\phi(e_2\circ e_1)$. Then, since $\bar{\delta}$ is invariant under $\hat{\chi}_\phi(\Gamma)$, $\bar{\delta}$ includes infinitely many points. Because of this contradiction, $\bar{\delta}$ is identical with the point ξ.

LEMMA 2.7. *Let $g\in\hat{\chi}_\phi(\Gamma)$ be loxodromic. Then there is a simple arc σ in $F(\hat{\delta}-L)\cup\{z\in\hat{C}\,;\,\hat{g}(z)=z$ for some $\hat{g}\in\hat{\chi}_\phi(\bigcup_{i=1}^{q}\tau(\alpha_i))\}\cup\{\xi(g), \xi'(g)\}$ joining $\xi(g)$ to $\xi'(g)$ such that $(\sigma-\{\xi(g), \xi'(g)\})\cap\Lambda(\hat{\chi}_\phi(\Gamma))$ is discrete on $\sigma-\{\xi(g), \xi'(g)\}$, where $\xi(g)$ and $\xi'(g)$ are the attracting and repelling fixed points of g, respectively.*

PROOF. If g belongs to the stabilizer subgroup of a component Δ of $F(\hat{\delta}-L)$, then we can easily find an arc in Δ with the desired property.

In the other case, the element $\gamma=\hat{\chi}_\phi^{-1}(g)$ does not keep any component of $\hat{\delta}-L$ invariant. Let z be a point of $U\cap\hat{\delta}$, σ' a simple arc joining z to $\gamma(z)$ and $\delta_1, \cdots, \delta_r$ components of $\hat{\delta}\cup\delta^*$ such that $\bigcup_{k=1}^{r}\mathrm{Cl}\,\delta_k\supset\sigma'$ and such that $\delta_k\cap\sigma'$ is non-empty and connected, $k=1, \cdots, r$. Let γ_k be a hyperbolic element of $\mathrm{Stab}(\mathrm{Cl}\,\delta_k\cap\mathrm{Cl}\,\delta_{k+1})$. Since by Lemmas 2.5 and 2.6 the arc $\sigma''=F(\sigma'\cap(\hat{\delta}\cup\delta^*))\cup(\bigcup_{k=1}^{r-1}\lim_{j\to\infty}\xi(F_{t_j}\circ\gamma_k\circ F_{t_j}^{-1}))$ is a simple one included in $F(\hat{\delta}-L)\cup\{z\in\hat{C}\,;\,g(z)=z$ for some $\hat{g}\in\hat{\chi}_\phi(\bigcup_{i=1}^{q}\tau(\alpha_i))\}$, so is $(\bigcup_{k=-\infty}^{\infty}g^k(\sigma''))\cup\{\xi(g), \xi'(g)\}$, which has the desired properties. Using this lemma, we prove the following, which together with the definition of squeezing deformations completes the proof of (iii) in Theorem 2.1.

LEMMA 2.8. *The homeomorphism F obtained in Lemma 2.5 maps $\hat{\delta}$ onto $\Omega(\hat{\chi}_\phi(\Gamma))$.*

PROOF. Assume that our assertion is false. Then Lemma 2.5 implies the existence of another component Δ of $\hat{\chi}_\phi(\Gamma)$ than the invariant component Δ_0 bounded by a quasi-circle ([**M**$_4$]) with $\Delta\cap F(\hat{\delta})=\emptyset$. Let $g\in\hat{\chi}_\phi(\Gamma)$ be a loxodromic element keeping Δ invariant. Then there exists a simple arc σ' in $\mathrm{Cl}\,\Delta\subset\hat{C}-F(L)$ joining the attracting fixed point ξ of g to the repelling one ξ'. Lemma 2.7 shows the existence of another simple arc σ'' than σ' in $\hat{C}-\Delta-F(L)$ joining ξ to ξ' such that $(\sigma'-\{\xi, \xi'\})\cap\Lambda(\chi_\phi(\Gamma))$ is discrete on $\sigma'-\{\xi, \xi'\}$. So both regions E_1 and E_2 surrounded by the simple loop $\sigma=\sigma'\cup\sigma''$ contain a point of $\Lambda(\hat{\chi}_\phi(\Gamma))$. Since the closed set $\mathrm{Cl}\,F(L)$ is invariant under $\hat{\chi}_\phi(\Gamma)$, $\mathrm{Cl}\,F(L)$ includes $\Lambda(\hat{\chi}_\phi(\Gamma))$ ([**L**, p. 105]). Therefore the set E_m contains a point z_m of $F(L)$, $m=1, 2$. Join z_1 to z_2 by an arc σ^* in $F(L)$. Since σ^* is included in $F(L)$, so is a point $z^*\in\sigma^*\cap\sigma$. On the other hand, the point z^* is not included in $F(L)$ since

$\sigma \cap F(L) = \emptyset$. This contradiction completes the proof of our lemma.

In the next lemma we determine parabolic elements of $\hat{\chi}_\phi(\Gamma)$.

LEMMA 2.9. *Let g be a parabolic element of $\hat{\chi}_\phi(\Gamma)$. Then $\hat{\chi}_\phi^{-1}(g)$ either is parabolic or is in $\bigcup_{i=1}^{q} \tau(\alpha_i)$.*

PROOF. First we consider the case where g is in the stabilizer subgroup $\hat{\chi}_\phi(\Gamma)_\Delta$ of a component $\Delta\ (\neq \Delta_0)$ in $\hat{\chi}_\phi(\Gamma)$. Since $\hat{\chi}_\phi(\Gamma)$ is a Kleinian group with an invariant component Δ_0, $\hat{\chi}_\phi(\Gamma)_\Delta$ is quasi-Fuchsian ([**M**$_4$]). Therefore we can find a disc invariant under g, the image of which under the natural projection π is a punctured disc D^* on $\Omega(\hat{\chi}_\phi(\Gamma))/\hat{\chi}_\phi(\Gamma)$. Lemma 2.8 and the definition of squeezing deformations show that $\pi \circ F \circ \pi^{-1}(D^*)$ is either a punctured disc on $\Omega(\Gamma)/\Gamma$ or a doubly connected domain surrounded by some α_i and another simple loop. This implies the desired conclusion.

Next, consider the other case where, apart from Δ_0, no component of $\hat{\chi}_\phi(\Gamma)$ is kept invariant by g. Assume that our assertion is not true. Then there exists a hyperbolic $\gamma \in \Gamma - \bigcup_{i=1}^{q} \tau(\alpha_i)$ such that $g = \hat{\chi}_\phi(\gamma)$ is parabolic. Let σ be a simple loop in $\Omega(\Gamma)$ separating one of the fixed point of γ from the other such that, for each component δ of $\hat{\delta} \cup \delta^*$, $\sigma \cap \delta$ is either connected or empty. We may assume that $\sigma \cap \pi^{-1}(\bigcup_{i=1}^{q} \alpha_i)$ consists of finitely many points since $\sigma \subset \Omega(\Gamma)$. So $F(\sigma \cap (\hat{\delta} \cup \delta^*))$ and the fixed points of finitely many parabolic elements $g_1, \cdots, g_k$ of $\hat{\chi}_\phi(\bigcup_{i=1}^{q} \tau(\alpha_i))$ make a simple loop σ^* by Lemmas 2.5 and 2.6. Since the fixed point ξ of g is on $\sigma^* \cap \{\xi_1, \cdots, \xi_k\}$, where ξ_m is the fixed point of the parabolic g_m, ξ is identical with some ξ_m. Therefore g and g_m are commutative, and so are the hyperbolic $\hat{\chi}_\phi^{-1}(g) \notin \bigcup_{i=1}^{q} \tau(\alpha_i)$ and the hyperbolic $\hat{\chi}_\phi^{-1}(g_m) \in \bigcup_{i=1}^{q} \tau(\alpha_i)$. This contradiction completes the proof of our lemma.

Let H be a subgroup of a Kleinian group G. A Jordan domain on $\hat{C}$ is called a precisely invariant disc for H in G, if $h(E) = E$ for $h \in H$ and $g(E) \cap E = \emptyset$ for each $g \in G - H$ and if $(\mathrm{Cl}\, E - \Lambda(H)) \subset \Omega(G)$.

LEMMA 2.10. *For each maximal parabolic cyclic subgroup H of $\hat{\chi}_\phi(\Gamma)$, there exist two precisely invariant discs for H in $\hat{\chi}_\phi(\Gamma)$ mutually disjoint to each other.*

PROOF. Let h be an element of H. Then by Lemma 2.9 $\hat{\chi}_\phi^{-1}(h)$ either is parabolic or is in $\bigcup_{i=1}^{q} \tau(\alpha_i)$. If $\hat{\chi}_\phi^{-1}(h)$ is parabolic, then there exist two precisely invariant discs E_1 and E_2 for $\hat{\chi}_\phi^{-1}(H)$ in the quasi-Fuchsian group Γ contained in $\hat{\delta}$ which are mutually disjoint to each other. So $F(E_1)$ and $F(E_2)$ are the desired discs. If $\hat{\chi}_\phi^{-1}(h) \in \bigcup_{i=1}^{q} \tau(\alpha_i)$, then by the definition of squeezing deformations $F(E_1)$

and $F(E_2)$ are the desired discs, where E_1 and E_2 are two components of $(\bigcup_{i=1}^{q} \pi^{-1}(D_i-\alpha_i))\cap\hat{\delta}$ invariant under $\hat{\chi}_{\phi}^{-1}(h)$.

Using this lemma, we will prove the following, which together with Lemma 2.4 completes the proof of (ii) in Theorem 2.1.

LEMMA 2.11. *The Kleinian group $\hat{\chi}_{\phi}(\Gamma)$ is geometrically finite.*

PROOF. Since $\hat{\chi}_{\phi}(\Gamma)$ has an invariant component, $\hat{\chi}_{\phi}(\Gamma)$ is formed from $G_1, \cdots, G_r$ and from $g_1, \cdots, g_s$ by finitely many applications of Maskit's combination theorems, where G_m is quasi-Fuchsian, totally degenerate or elementary, and g_m is a Moebius transformation. For terminologies and a proof, see [**M**$_1$], [**M**$_2$], [**M**$_3$], [**M**$_4$]. Assume that some G_m is totally degenerate. If G_m contains a parabolic element h, then there exists exactly one precisely invariant disc for the maximal subgroup H containing h in G_m. Since $\Omega(G_m)\supset\Omega(\hat{\chi}_{\phi}(\Gamma))$, there exists at most one precisely invariant disc for H in $\hat{\chi}_{\phi}(\Gamma)$, which contradicts Lemma 2.10. If G_m does not contain a parabolic element, then G_m contains purely loxodromic subgroup G'_m which is isomorphic to the fundamental group of a compact Riemann surface. On the other hand, since $\chi_{\phi}^{-1}(G'_m)$ is a purely hyperbolic subgroup of a Fuchsian group of the second kind, $\chi_{\phi}^{-1}(G'_m)$ is free, and so is G'_m. This is a contradiction. Now we have seen that no G_m is totally degenerate. Therefore $\hat{\chi}_{\phi}(\Gamma)$ is geometrically finite ([**M**$_5$], [**Y**$_2$]).

To complete the proof of Theorem 2.1, we need only to prove the following.

LEMMA 2.12. *The limit $\lim_{t\to 1}\Phi(\mu_t)$ exists.*

PROOF. Let $\{\Phi(\mu_{t_j})\}_{j=1}^{\infty}$ and $\{\Phi(\mu_{t_j}^{*})\}_{j=1}^{\infty}$ be sequences in $\{\Phi(\mu_t)\}_{t\in[0,1)}$ converging to ϕ and ϕ^*, respectively. Since both $\hat{\chi}_{\phi}(\Gamma)$ and $\hat{\chi}_{\phi^*}(\Gamma)$ are geometrically finite, the conformal homeomorphism $F^*\circ F^{-1}$ of $F(\hat{\delta})=\Omega(\hat{\chi}_{\phi}(\Gamma))$ onto $F^*(\hat{\delta})$ $=\Omega(\hat{\chi}_{\phi^*}(\Gamma))$ is the restriction of a Moebius transformation to $\Omega(\hat{\chi}_{\phi}(\Gamma))$ ([**M**$_5$]). Therefore it holds that $\phi^*=[F^*|L]=[F|L]=\phi$.

§ 3. Quasi-Fuchsian groups on the boundaries of Teichmüller spaces.

The main purpose of this section is to prove the following.

THEOREM 3.1. *Let Γ be a finitely generated Fuchsian group of the second kind. If $\phi\in\partial T(\Gamma)$ is not a cusp, then $\chi_{\phi}(\Gamma)$ is quasi-Fuchsian, so that $\chi_{\phi}(\Gamma)$ is geometrically finite.*

Recall the following theorem on Kleinian groups on the boundaries of Teichmüller spaces of finitely generated Fuchsian groups of the first kind. Then we recognize a clear difference between the following theorem and ours.

THEOREM (Bers [**Ber**] and Greenberg [**G**$_1$]). *Let Γ^* be a finitely generated Fuchsian group of the first kind. If $\phi\in\partial T(\Gamma^*)$ is not a cusp, then $\chi_{\phi}(\Gamma^*)$ is totally degenerate, so that $\chi_{\phi}(\Gamma^*)$ is geometrically infinite.*

Now we prove Theorem 3.1 by showing the existence of a quasi-Fuchsian group of the first kind including $\chi_{\phi}(\Gamma)=W_{\phi}\Gamma W_{\phi}^{-1}$ as a subgroup, where W_{ϕ} is the meromorphic homeomorphism of the lower half plane L defined in § 1. For the sake of simplicity, we assume that the Riemann surface L/Γ is of type (g, l, n).

First we construct a group G^* including $\chi_{\phi}(\Gamma)$. Let ω be a fundamental polygon for Γ in L. Here we may assume that the boundary of ω is connected and consists of finitely many circular arcs called sides which are pairwise identified by elements $\gamma_1, \cdots, \gamma_l$ of Γ (which generate Γ), and γ_1 corresponds to the single boundary component of $\Omega(\Gamma)/\Gamma$ (cf. [$\mathbf{G}_2$] 1.2). We denote by s and s' the sides of ω identified by γ_1, namely $\gamma_1(s)=s'$. Let V_0 be a point on $W_{\phi}(s)$. Let $C_1, \cdots, C_{2r}$ be circles in $W_{\phi}(\omega)\cup W_{\phi}(s)\cup W_{\phi}(s')$ satisfying the following:

(i) C_1 meets $\mathrm{Cl}(\hat{C}-W_{\phi}(\omega))$ only at V_0, and C_{2r} does $\mathrm{Cl}(\hat{C}-W_{\phi}(\omega))$ only at the point $V_{2r}=W_{\phi}\circ\gamma\circ W_{\phi}^{-1}(V_0)$,

(ii) C_j is tangent to C_{j+1}, $1\leqq j\leqq 2r-1$, and

(iii) $C_j\cap C_k=\emptyset$, $k-j\geqq 2$, $1\leqq j<k\leqq 2r$.

Set $V_j=C_j\cap C_{j+1}$, $j=1, \cdots, 2r-1$. Let g_{l+j} be the parabolic transformation with the fixed point V_{2j-1} which maps V_{2j-2} onto V_{2j}, $1\leqq j\leqq r$. Let $\hat{\Omega}$ be the component of $W_{\phi}(\omega)\cap\left(\bigcap_{i=1}^{2r}\mathrm{ext}\, C_j\right)$ not containing the image of a point near $(\mathrm{Cl}\,\omega)\cap\boldsymbol{R}$ under W_{ϕ}. Let G^* be the group generated by $g_1, \cdots, g_{l+r}$, where $g_j=\chi_{\phi}(\gamma_j)$, $1\leqq j\leqq l$.

A Kleinian group with a simply connected invariant component is called B-group.

LEMMA 3.2. *The Kleinian group G^* is a B-group.*

PROOF. Let $c_1, \cdots, c_{2r}$ be circles in $\mathrm{Cl}\,\omega\cup\{x-\sqrt{-1}y\,;\,x+\sqrt{-1}y\in\omega\}$ satisfying the following:

(i) c_1 (resp. c_{2r}) is tangent to s (resp. s') at the point v_0 (resp. v_{2r}) on $\boldsymbol{R}$, so that c_1 and c_{2r} are orthogonal to $\boldsymbol{R}$,

(ii) $c_j\cap c_k=\emptyset$, $k-j\geqq 2$, $1\leqq j<k\leqq 2r$, and

(iii) c_j is tangent to c_{j+1} at a point on $\boldsymbol{R}$, $1\leqq j\leqq 2r-1$.

Set $v_j=c_j\cap c_{j+1}$, $1\leqq j\leqq 2r-1$. Let γ_{l+j} be the parabolic transformation with the fixed point v_{2j-1} which maps v_{2j-2} onto v_{2j}. Then the group Γ^* generated by $\bigcup_{j=1}^{l+r}\gamma_j$ is a Fuchsian group of the first kind keeping L invariant. Note that $\hat{\omega}=\omega\cap\left(\left(\bigcap_{j=1}^{2r}\mathrm{ext}\, c_j\right)\cup\left(\bigcap_{j=1}^{r}c_{2j-1}\right)\right)$ is a connected fundamental set for Γ^* in L.

Let Ψ_1 be a homeomorphism of the boundary $\partial\hat{\omega}$ of $\hat{\omega}$ onto $\partial\hat{\Omega}$ satisfying the following:

(i) $\Psi_1(s)=W_{\phi}(s)\cap\partial\hat{\Omega}$,

(ii) $\Psi_1(s')=W_{\phi}(s')\cap\partial\hat{\Omega}$,

(iii) $\Psi_1(c_j\cap L)=C_j\cap\partial\hat{\Omega}$, $j=1, \cdots, 2r$, and

(iv) $\Psi_1(z)=W_\phi(z)$ for $z\in\partial\omega-(s\cup s')-\left(\bigcup_{j=1}^{2r}c_j\right)$.

Then we can construct a homeomorphism Ψ_2 of $\mathrm{Cl}\,\hat{\omega}$ onto $\mathrm{Cl}\,\hat{\Omega}$ which is identical with Ψ_1 on $\partial\hat{\omega}$. The correspondence $\gamma_j\mapsto g_j$, $1\leqq j\leqq l+r$, can be naturally extended to an isomorphism $\chi^*:\Gamma^*\to G^*$, which is identical with χ_ϕ on Γ, because Γ^* (resp. G^*) is the free product of Γ (resp. $\chi_\phi(\Gamma)$) and $\langle\gamma_{l+1}\rangle,\cdots$, and $\langle\gamma_{l+r}\rangle$ (resp. $\langle g_{l+1}\rangle,\cdots$, and $\langle g_{l+r}\rangle$), where $\langle\gamma_j\rangle$ is the cyclic group generated by γ_j. Let Ψ_3 be the restriction of Ψ_2 to $\omega^*=\mathrm{Cl}\,\hat{\omega}\cap L$. Now we can construct a homeomorphism Ψ of L onto $\{g(z);\,g\in G^*,\,z\in\Psi_3(\omega^*)\}$ by setting $\Psi(\gamma(z))=\chi^*(\gamma)(\Psi_3(z))$ for $z\in\omega^*$ and $\gamma\in\Gamma^*$. Clearly L is simply connected, so is $\Psi(L)$. Note that G^* is discontinuous in $\Psi(L)$ and that G^* keeps $\Psi(L)$ invariant. To complete the proof of our lemma, we must prove that $\Psi(L)$ is identical with the component Δ_1 of G^* including $\Psi(L)$. By Ahlfors finiteness theorem [**Ah**] and by the construction of G^* both Δ_1'/G^* and $\Psi(L)'/G^*$ are compact Riemann surfaces with finitely many points removed. The rest of the proof of our lemma is the same as that of Lemma 2.4.

LEMMA 3.3. *Let γ be an element of Γ^*. Then, if $\chi^*(\gamma)$ is parabolic, so is γ.*

PROOF. First we consider the case of $\chi^*(\gamma)\in\chi^*(\Gamma)$. Then, since ϕ is not a cusp, $\chi^*(\gamma)=\chi_\phi(\gamma)$ is parabolic if and only if so is γ.

Secondly, assume that $\chi^*(\gamma)$ is an element of the group $\chi^*(\Gamma_1)$, where Γ_1 is the group generated by $\gamma_{l+1},\cdots,\gamma_{l+r}$. Note that Γ_1 (resp. $\chi^*(\Gamma_1)$) is a Kleinian group with an invariant component formed from parabolic cyclic groups via Maskit's combination theorem I, where all amalgamated subgroups are trivial. For terminologies and a proof, see [$\mathbf{M}_1$], [$\mathbf{M}_4$]. So an element γ of Γ_1 (resp. $\chi^*(\gamma)$ of $\chi^*(\Gamma_1)$) is parabolic if and only if γ (resp. $\chi^*(\gamma)$) is conjugate to an element of $\bigcup_{j=1}^{r}\langle\gamma_{l+j}\rangle$ in Γ_1 $\left(\text{resp. }\bigcup_{j=1}^{r}\langle\chi^*(\gamma_{l+j})\rangle\text{ in }\chi^*(\Gamma_1)\right)$. Now it is clear that our assertion is true.

Finally, if $\chi^*(\gamma)$ is in $G^*-\chi^*(\Gamma)-\chi^*(\Gamma_1)$, then $\chi^*(\gamma)$ can be represented as the free product of elements of $\chi^*(\Gamma)$ and $\chi^*(\Gamma_1)$. Recall an elementary fact that γ, γ^{-1} and $g\circ\gamma\circ g^{-1}$ for each $g\in G^*$ are parabolic simultaneously. This permits us to assume that $\chi^*(\gamma)$ is of the form $\bar{\gamma}_k\circ\bar{g}_k\circ\cdots\circ\bar{\gamma}_1\circ\bar{g}_1$, where $\bar{g}_j\in\chi^*(\Gamma_1)-\{\mathrm{id.}\}$ and $\bar{\gamma}_j\in\chi^*(\Gamma)-\{\mathrm{id.}\}$. Let $\sigma\subset\left\{W_\phi(\omega)\bigcap_{j=1}^{2r}\mathrm{ext}\,C_j\right\}\cup\{V_0,V_{2r}\}$ be a Jordan loop separating the set $\bigcup_{j=1}^{2r}\mathrm{int}\,C_j$ from the point ∞. Recall that $\chi^*(\Gamma)$ is the free product of cyclic groups $\langle g_{l+1}\rangle,\cdots,\langle g_{l+r}\rangle$. Since $\bar{g}_1$ is an element of $\chi^*(\Gamma_1)-\{\mathrm{id.}\}$, $\bar{g}_1$ maps $\mathrm{ext}\,\sigma$ onto $\mathrm{int}\,\bar{g}_1(\sigma)$ included in $\mathrm{int}\,\sigma$. Note that $\bar{g}_1(\sigma)\cap\sigma$ is empty or $\{V_0\}$ or $\{V_{2r}\}$. Since $\bar{\gamma}_1$ is an element of $\chi^*(\Gamma)-\{\mathrm{id.}\}$ and since $\mathrm{int}\,\bar{g}_1(\sigma)$ is included in a fundamental domain $W_\phi(\omega)$ for $\chi^*(\Gamma)$ in $W_\phi(L)$, $\bar{\gamma}_1$ maps $\mathrm{int}\,\bar{g}_1(\sigma)$ onto $\bar{\gamma}_1(\mathrm{int}\,g_1(\sigma))$ $=\mathrm{int}\,\bar{\gamma}_1\circ\bar{g}_1(\sigma)$ included in $W_\phi(L)-\mathrm{Cl}\,W_\phi(\omega)$ $(\subset\mathrm{ext}\,\sigma)$. Note that the set $\bar{\gamma}_1\circ\bar{g}_1(\sigma)\cap\sigma$ is empty or $\{V_0\}$ or $\{V_{2r}\}$. Inductively we can show that the transformation

$\chi^*(\gamma)$ maps $\text{ext}\,\sigma$ onto $\text{int}\chi^*(\gamma)(\sigma)$ included in $W_\phi(L)-\text{Cl}\,W_\phi(\omega)$ ($\subset\text{ext}\,\sigma$), and that $\chi^*(\gamma)(\sigma)\cap\sigma$ is empty or $\{V_0\}$ or $\{V_{2r}\}$. On the other hand, since $\chi^*(\gamma)$ is parabolic, $\chi^*(\gamma)(\sigma)\cap\sigma$ is not empty, so that $\chi^*(\gamma)(\sigma)\cap\sigma$ is either $\{V_0\}$ or $\{V_{2r}\}$. Therefore $\bar{\gamma}_j\circ\cdots\circ\bar{g}_1(\sigma)\cap\sigma$ is either $\{V_0\}$ or $\{V_{2r}\}$, $1\leqq j\leqq k$ and $\chi^*(\gamma)$ is one of the following forms or its power: $\chi^*(\gamma_1^{-1})\circ g_{l+r}\circ\cdots\circ g_{l+1}$ or $\chi^*(\gamma_1)\circ g_{l+1}^{-1}\circ\cdots\circ g_{l+r}^{-1}$. In any case, obviously γ is a parabolic element whose fixed point is either v_0 or v_{2r}. Now we complete the proof of our lemma.

The following lemma completes the proof of Theorem 3.1.

LEMMA 3.4. *The B-group G^* is quasi-Fuchsian, and so is $\chi_\phi(\Gamma)$.*

PROOF. By a classification theorem on B-groups ([**M**$_8$]), Lemma 3.3 implies that G^* is quasi-Fuchsian or totally degenerate. The latter cannot occur because G^* has another component containing a point of $W_\phi(L)-\text{Cl}\,\Psi(L)$ than $\Psi(L)$. Therefore G^* is quasi-Fuchsian, and so is the subgroup $\chi_\phi(\Gamma)$ of G^*.

In the above proof, we constructed a Kleinian group containing the original group as a subgroup. In general, this method is sometimes valid for our solving problem on (quasi-) Fuchsian group of the second kind. Here we show two applications of this method.

The first one is a direct answer to the problem stated in Introduction, which is a corollary to Theorem 2.1.

PROPOSITION 3.5. *Let Γ be a finitely generated Fuchsian group of the second kind. Then there exists a cusp on $\partial T(\Gamma)$.*

PROOF. Let Γ^* be a finitely generated Fuchsian group of the first kind including Γ as a subgroup. Then $T(\Gamma^*)\subset T(\Gamma)$. Bers [**Ber**] showed the existence of a sequence $\{\phi_j\}_{j=1}^\infty\subset T(\Gamma^*)$ converging to a cusp $\phi\in\partial T(\Gamma^*)$ ($\subset\text{Cl}\,T(\Gamma)$) such that $W_\phi\circ\gamma\circ W_\phi^{-1}$ is parabolic for a hyperbolic $\gamma\in\Gamma$ ($\subset\Gamma^*$). This means that $\phi\in\partial T(\Gamma)$ is a cusp.

The next application is to give an estimate of the outradii of the Teichmüller spaces of some cyclic groups.

PROPOSITION 3.6 ([**Se**$_1$], [**Se**$_2$]). *Let Γ be the trivial, or an elliptic or a parabolic cyclic group keeping the upper half plane invariant. Then the outradius $o(\Gamma)=\sup_{\phi\in T(\Gamma)}\|\phi\|$ of $T(\Gamma)$ is equal to 3/2.*

PROOF. We prove our proposition only in the case where Γ is generated by a parabolic transformation γ. In the other case, the proof can be given in the same way. For any $\varepsilon>0$, there exists a finitely generated Fuchsian group Γ^* of the first kind which contains a parabolic element γ^* with $o(\Gamma^*)>3/2-\varepsilon$ ([**C**]). Then we have $3/2-\varepsilon<o(\Gamma^*)\leqq o(\langle\gamma^*\rangle)=o(\langle\gamma\rangle)\leqq 3/2$, where $\langle\gamma^*\rangle$ denotes the group generated by γ^*. The equality was pointed out by Sekigawa [**Se**$_1$] and the last inequality is due to Nehari [**N**]. Since ε is arbitrarily small, we have the desired conclusion.

§ 4. Geometrically infinite cusps.

In this section we prove the following.

THEOREM 4.1. *Let ϕ be a cusp obtained in Theorem 2.1 such that $\Omega(G)/G = S_0+S_1+\cdots+S_k$, where $G=\hat{\chi}_\phi(\Gamma)$ and S_0 is the component of $\Omega(G)/G$ including $F(L)/G$. Then for any proper subset $J=\{j_1, \cdots, j_s\}$ of $K=\{1, \cdots, k\}$ such that no S_u is of type (0, 0, 3) for $u\in K-J$, there exists a geometrically infinite cusp $\phi\in\partial T(\Gamma)$ such that $\Omega(\hat{\chi}_\phi(\Gamma))/\hat{\chi}_\phi(\Gamma)=S_0+S_{j_1}+\cdots+S_{j_s}$.*

PROOF. The following proof of Theorem 4.1 is merely a copy of that of Theorem 7 in [**A**]. Therefore we state only an outline of the proof. For details, see [**A**]. For each $r\in K-J$, let Δ_r be a component of $\pi^{-1}(S_r)$. Since Δ_r/G_{Δ_r} is not of type (0, 0, 3), a sequence $\{\nu_{r,n}\}_{n=1}^\infty$ of Beltrami differentials for G_{Δ_r} with the supports on Δ_r such that $F^{\nu_{r,n}}G_{\Delta_r}(F^{\nu_{r,n}})^{-1}$ converges to a totally degenerate group H_r. Let ν_n be the Beltrami differential for G with the support on $\bigcup_{r\in K-J}\bigcup_{g\in G}g(\Delta_r)$ which is identical with $\nu_{n,r}$ on Δ_r. Then, since $F^{\nu_n}\circ F\circ\Gamma\circ(F^{\nu_n}\circ F)^{-1}$ is geometrically finite ([$\mathbf{Y}_2$]), there exists a sequence $\{\mu_{j,n}\}_{j=1}^\infty\subset M(\Gamma)$ with $\lim_{j\to\infty}[F^{\mu_{j,n}}|L]=[F^{\nu_n}\circ F|L]$. Note that $[F^{\mu_{j,n}}|L]\in T(\Gamma)$ and that there exists a hyperbolic $\gamma\in\Gamma$ such that $F^{\nu_n}\circ F\circ\gamma\circ(F^{\nu_n}\circ F)^{-1}$ is parabolic, so that $\phi_n=[F^{\nu_n}\circ F|L]\in\partial T(\Gamma)$ is a cusp. As in the proof of Lemma 2.3, a subsequence, again denoted by $\{\phi_n\}_{n=1}^\infty$, of $\{\phi_n\}_{n=1}^\infty$ converges to a cusp ϕ. For each n and each $r\in K$ we can find a K-quasi-conformal automorphism $F(n, r)$ of $\hat{C}$ with $F^{\nu_n}G_{\Delta_r}(F^{\nu_n})^{-1}=F(n, r)G_{\Delta_r}F(n, r)^{-1}$ ([**A**, Lemma 5]). So we can find a K-quasi-conformal automorphism $F(r)$ of $\hat{C}$ and a subsequence, again denoted by $\{F^{\nu_n}\}_{n=1}^\infty$, of $\{F^{\nu_n}\}_{n=1}^\infty$ such that $\lim_{n\to\infty}F^{\nu_n}G_{\Delta_r}(F^{\nu_n})^{-1}=\lim_{n\to\infty}F(n, r)G_{\Delta_r}F(n, r)^{-1}=F(r)H_rF(r)^{-1}$. Since the Kleinian group $\hat{\chi}_\phi(\Gamma)$ with an invariant component has the totally degenerate group $F(r)H_rF(r)^{-1}$ as a factor subgroup, $\hat{\chi}_\phi(\Gamma)$ is geometrically infinite ([$\mathbf{M}_5$], [$\mathbf{Y}_2$]). Thus we complete the proof of our theorem.

References

[A] W. Abikoff, On boundaries of Teichmüller spaces and on Kleinian groups III, Acta Math., **134** (1975), 211-234.

[Ah] L.V. Ahlfors, Finitely generated Kleinian groups, Amer. J. Math., **86** (1964), 413-429.

[B] A.F. Beardon, The geometry of discrete groups, in "Discrete groups and automorphic functions", 47-72, Academic Press, New York and London, 1977.

[Ber] L. Bers, On boundaries of Teichmüller spaces and on kleinian groups I, Ann. of Math., **91** (1970), 570-600.

[C] T. Chu, On the outradius of finitely dimensional Teichmüller spaces, in "Discontinuous groups and Riemann surfaces", Ann. of Math. Studies, **79**, 99-104, Princeton Univ. Press, 1974.

[F] L.R. Ford, Automorphic functions, Chelsea, New York, 1951.

[G_1] L. Greenberg, Fundamental polyhedra for Kleinian groups, Ann. of Math., **84** (1966), 433-441.
[G_2] L. Greenberg, Finiteness theorems for Fuchsian and Kleinian groups, in "Discrete groups and automorphic functions", 199-257, Academic Press, New York and London, 1977.
[L] J. Lehner, Discontinuous groups and automorphic functions, Amer. Math. Soc., Providence, 1964.
[LV] O. Lehto-K.I. Virtanen, Quasi-conformal mappings in the plane, Springer-Verlag, Berlin, 1973.
[M_1] B. Maskit, On Klein's combination theorem, Trans. Amer. Math. Soc., **130** (1965), 499-509.
[M_2] B. Maskit, On Klein's combination theorem II, Trans. Amer. Math. Soc., **131** (1968), 32-39.
[M_3] B. Maskit, On boundaries of Teichmüller spaces and on kleinian groups II, Ann. of Math., **91** (1970), 607-639.
[M_4] B. Maskit, Decomposition of certain Kleinian groups, Acta Math., **130** (1973), 243-263.
[M_5] B. Maskit, On the classification of Kleinian groups, II—signatures, Acta Math., **138** (1977), 17-42.
[N] Z. Nehari, The Schwarzian derivatives and schlicht functions, Bull. Amer. Math. Soc., **55** (1949), 545-551.
[Se_1] H. Sekigawa, The outradius of the Teichmüller space, Tôhoku Math.J., **30** (1978), 607-612.
[Se_2] H. Sekigawa, Schwarzian derivatives of some conformal mappings, Tôhoku Math. J., **31** (1979), 309-318.
[Sh] H. Shimizu, On discontinuous groups operating on the product of the upper half planes, Ann. of Math., **77** (1963), 33-71.
[Y_1] H. Yamamoto, Squeezing deformations in Schottky spaces, J. Math. Soc. Japan, **31** (1979), 227-243.
[Y_2] H. Yamamoto, Constructibility and geometric finiteness of Kleinian groups, Tôhoku Math. J., **32** (1980), 353-362.

Hiro-o YAMAMOTO
Mathematical Institute
Tôhoku University
Sendai 980, Japan