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Group extensions and Plancherel formulas

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§ 1. Introduction.

The purpose of this paper is to describe the Plancherel formula for some locally compact groups and to investigate the associated objects through the intermediary of a suitable normal subgroup and something related with it.

A. Kleppner and R. L. Lipsman ([12], [13]) discussed this problem under the assumptions that a normal subgroup N of G is "essentially" of type I (cf. Definition 5-1 for detail), the action of G on $\hat{N}$ is smooth, and G is isotropically of type I almost everywhere. We can regard their results as a "little group analysis" in the Plancherel formula context.

In this paper, when N is "essentially" of type I, instead of Kleppner and Lipsman's smoothness (type I'ness) condition, we assume that the action of G on $\hat{N}$ is locally essentially free (Definition 5-4). Whereas it is out of extent of the Mackey theory, we can do the "little group analysis" about the Plancherel objects. We will be mainly interested in the non type I groups as the subjects of this extended analysis.

The (central) decomposition of the Haar weight on $C^*(G)$ into Δ_G-semicharacters is regarded as the Plancherel formula. A measure (class) which gives the central decomposition of the left regular representation of G and so gives the Plancherel formula of G is called Plancherel measure (class). Since the Haar measure of G is Δ_G-relatively invariant with respect to inner automorphism, the above "Plancherel formula" can be regarded as the "global duality" of G.

In order to establish the theory of decompositions of Haar weight, we must make free use of the inductions and the direct integral decompositions of semitraces. We discuss these matters in § 2. The author has received a recent preprint of N. V. Pedersen (On the left regular representations of locally compact groups), after having finished the preparation of this paper, which contains discussions of similar problems but the conclusions are slightly different. In contrast to Pedersen, we used and refined the decomposition of left Hilbert algebra established by C. E. Sutherland [21]. Moreover, in this section we discussed the case of projective semitraces in order to treat the problems in the group extension situation more closely in future.

In § 3, we discuss a delicate problem on central decompositions, and the notion of the Plancherel theory. And we discuss the projective Plancherel theory. In § 4, we discuss the examples of the Plancherel theory for some concrete groups. § 5 is a general theory. We consider the problem to compose the Plancherel theory of G from those of a suitable normal subgroup N and of suitable closed subgroups of G/N. When N is "essentially" of type I, we show that this system works well if we assume that the action of G on $\hat{N}$ is locally essentially free. Next, we consider the case that N is not "essentially" of type I. [**22**], [**21**] and "On the left ⋯" cited above treated the case that N is equal to the kernel of modular function of G (the maximal unimodular subgroup). In this case, the central property of the obtained decomposition of the left regular representation follows from the special choice of N. We discuss the case in which N is not necessarily the maximal unimodular subgroup. We present two examples (discrete nilpotent group and almost connected Lie group) in which we can apply a similar analysis.

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§ 2. Induction and direct integral decomposition of semitraces.

2-1. Induction of semitraces.

The induction of semitraces was studied by N. V. Pedersen. For the notational convenience, we explain here the outline of his method and make some additional remarks.

Let A be a separable C^*-algebra, ϕ a weight on A. We use the following notations. $n_\phi=\{x\in A:\phi(x^*x)<+\infty\}$, $N_\phi=\{x\in A:\phi(x^*x)=0\}$ and $m_\phi=n_\phi^* n_\phi$.

Let G be a separable locally compact group (s. l. c. g.), dg a left Haar measure, Δ_G the modular function. Inner automorphisms of G induce a group $\{\alpha_g: g\in G\}$ of automorphisms of $C^*(G)$. For each element $k\in C_c(G)$ ($C_c(G)$ is the set of continuous functions with compact supports), $\alpha_g k$ is defined as $(\alpha_g k)(g')=\Delta_G(g)k(g^{-1}g'g)$ for $g'\in G$. Let χ be a continuous homomorphism from G to $\boldsymbol{R}^+$.

We write down the definitions of χ-semitrace, χ-semitraceclass representation, χ-semicharacter and χ-semitrace type representation for the convenience of the reader (cf. 2-1 and 2-2 in [**16**]).

DEFINITION 0. (A) A χ-semitrace on G is a lower semicontinuous weight on $C^*(G)$ such that

(1) $\phi(x)=\sup\{\phi(y): y\in \bar{m}_\phi\ 0\leqq y\leqq x\}$,

(2) $\phi(\alpha_g(x))=\chi(g)\phi(x),\ g\in G,\ x\in C^*(G)^+$.

(B) A χ-semitraceclass representation is a pair $(\pi, \tilde{\phi})$ consisting of a representation π of G and a faithful normal semifinite (f. n. s.) weight on $R(\pi)$ (the von Neumann algebra generated by the range of π) such that

(1) $\tilde{\phi}(\pi(g)T\pi(g)^{-1})=\chi(g)\tilde{\phi}(T)$, $g\in G$, $T\in R(\pi)^{+}$,

(2) $\pi(C^{*}(G))\cap m_{\tilde{\phi}}$ is weakly dense in $R(\pi)$.

(C) When $(\pi, \tilde{\phi})$ is a χ-semitraceclass representation, we call π a χ-semitrace type representation.

(D) χ-semitrace is called a χ-semicharacter when the G. N. S. representation associated with it is factorial.

Now, we consider the induction of semitraces. Let N and G be s. l. c. g.'s and $N\lhd G$ (i. e. N is a closed normal subgroup of G). G acts on $C^{*}(N)$ by restricting the action of $\{\alpha_{g}: g\in G\}$. Let η be a continuous homomorphism from G to $\boldsymbol{R}^{+}$, and χ denotes the restriction of η to N. Suppose that a χ-semitrace ϕ of $C^{*}(N)$ is η-relatively invariant under the action of G. Let $A_{\phi}=n_{\phi}\cap n_{\phi}^{*}/N_{\phi}$. A_{ϕ} has a natural left Hilbert algebra structure. Let H_{ϕ} be the Hilbert space completion of n_{ϕ}/N_{ϕ} (A_{ϕ} is dense in H_{ϕ}), U_{ϕ} be the left von Neumann algebra of A_{ϕ} and $\tilde{\phi}$ be the canonical weight on it (cf. [**16**] 2-1).

Let π_{ϕ} be the G. N. S. representation of G on H_{ϕ} given by ϕ. We set $M(G, \phi)$ be the set of Borel functions $k: G\rightarrow U_{\phi}$ such that (i) $k(gn)=\pi_{\phi}(n)^{-1}k(g)$ for all $g\in G$, $n\in N$. (ii) k is norm bounded. (iii) $\dot{g}\rightarrow\|k(g)\|$ is with compact support on G/N. $C_{c}(G)$ is naturally embedded in $M(G, \phi)$. $M(G, \phi)$ has a natural structure of involutive algebra which is an extension of the ordinary structure of $C_{c}(G)$. $\sharp$ denotes the involution of $M(G, \phi)$.

Let $n(G, \phi)\equiv\left\{k\in M(G, \phi): \int_{G/N}\tilde{\phi}(k(g)^{*}k(g))d\dot{g}<+\infty\right\}$, $N(G, \phi)\equiv\left\{k\in M(G, \phi): \int_{G/N}\tilde{\phi}(k(g)^{*}k(g))d\dot{g}=0\right\}$. $H(G, \phi)$ denotes the completion of $n(G, \phi)/N(G, \phi)$ with respect to the inner product naturally introduced in it. Let $A(G, \phi)\equiv(n(G, \phi)\cap n(G, \phi)^{\sharp})/N(G, \phi)$. $A(G, \phi)$ has automatically a left Hilbert algebra structure and is dense in $H(G, \phi)$. Let $U(G, \phi)$ denote the left von Neumann algebra of $A(G, \phi)$ and $\tilde{\bar{\phi}}'$ be the canonical weight on it. The left representation of $A(G, \phi)$ on $H(G, \phi)$ induces a unitary representation of G which is unitarily equivalent to $\bar{\pi}_{\phi}\equiv\mathrm{Ind}_{N\uparrow G}\,\pi_{\phi}$. $\tilde{\bar{\phi}}$ denotes the image of $\tilde{\bar{\phi}}'$ by the spatial transformation from $U(G, \phi)$ to $R(\bar{\pi}_{\phi})$.

LEMMA 2-1 ([**17**] Proposition 2.1.1). *$(\bar{\pi}_{\phi}, \tilde{\bar{\phi}})$ is an $\eta\cdot\Delta_{G/N}$-semitraceclass representation. Let $\bar{\phi}\equiv\tilde{\bar{\phi}}\circ\bar{\pi}_{\phi}|_{C^{*}(G)^{+}}$. Then $\bar{\phi}$ is an $\eta\cdot\Delta_{G/N}$-semitrace on $C^{*}(G)$. The unitary representation of G which we get from $\bar{\phi}$ by the* G. N. S. *construction is equivalent to $\bar{\pi}_{\phi}$. We call $\bar{\phi}$ the semitrace induced from ϕ and denote it* $\mathrm{Ind}_{N\uparrow G}\phi$.

2-2. Direct integral decomposition of semitraces.

Let A be a separable C^{*}-algebra. Let $\{\sigma_{t}: t\in\boldsymbol{R}\}$ be a one parameter $*$-automorphism group of A and ϕ be a weight on A. When ϕ satisfies the K. M. S.

condition with respect to $\{\sigma_t : t\in \boldsymbol{R}\}$ (p. 94 of [21]), we say that ϕ is σ-K. M. S. We assume that each K. M. S. weight is lower semicontinuous.

When ϕ is σ-K. M. S., N_ϕ is a closed two sided ideal and we construct a left Hilbert algebra A_ϕ and a left representation (π_ϕ, H_ϕ). Let M_ϕ be the left von Neumann algebra, $\tilde{\phi}$ be the canonical weight on it and $\tilde{\sigma}$ be the modular automorphism group determined by $\tilde{\phi}$.

LEMMA 2-2 ([21] Theorem 5.3). *Let ϕ be a densely defined σ-K. M. S. weight on A. Suppose that $\pi_\phi \cong \int_\Gamma^\oplus \pi_\gamma d\mu(\gamma)$ is a decomposition of π_ϕ such that the diagonal algebra is contained in the centre $ZR(\pi_\phi)$ of $R(\pi_\phi)$. Then there exists a family of weights $\{\phi_\gamma : \gamma\in\Gamma\}$ on A which satisfies the following conditions.*

(1) *$x\in A^+$, $x\to\phi_\gamma(x)$ is measurable and $\phi(x)=\int_\Gamma \phi_\gamma(x)d\mu(\gamma)$.*

(2) *for μ a.a. γ, ϕ_γ is σ-K. M. S.*

(3) *for μ a.a. γ, ϕ_γ is densely defined.*

(4) *for μ a.a. γ, the representation determined by ϕ_γ is unitarily equivalent to π_γ.*

REMARK 2-3. In [21], ϕ was assumed to be faithful and the decomposition of π_ϕ central. But the statements are valid under weaker conditions here stated.

Let ϕ be a σ-K. M. S. weight on a separable C^*-algebra A. Let G be an s. l. c. g. and χ be a continuous homomorphism from G to $\boldsymbol{R}^+$. Suppose that G acts on A as a continuous $*$-automorphism group $\{\alpha_g : g\in G\}$ and $\phi(\alpha_g(v))=\chi(g)\phi(v)$ for $\forall g\in G$, $\forall v\in A^+$. Then there exists a σ-weakly continuous $*$-automorphism group $\{\tilde{\alpha}_g : g\in G\}$ on $R(\pi_\phi)$ such that $\tilde{\alpha}_g(\pi_\phi(v))=\pi_\phi(\alpha_g(v))$ and $\tilde{\phi}(\tilde{\alpha}_g(T))=\chi(g)\tilde{\phi}(T)$ $\forall T\in R(\pi_\phi)^+$.

Now, suppose that π_ϕ has a direct integral decomposition such that every element of the diagonal algebra is fixed under the action of G. We have,

$$\{\phi, \pi_\phi, M_\phi, \tilde{\phi}, \tilde{\alpha}_g\} \cong \int_\Gamma^\oplus \{\phi_\gamma, \pi_\gamma, M_\gamma, \tilde{\phi}_\gamma, \tilde{\alpha}_g^\gamma\}\, d\mu(\gamma)\,,$$

and for μ a.a. γ, $\{\tilde{\alpha}_g^\gamma : g\in G\}$ is a σ-weakly continuous $*$-automorphism group of M_γ.

THEOREM 2-4. *For μ a.a. γ, $\phi_\gamma(\alpha_g(v))=\chi(g)\phi_\gamma(v)$ for $\forall g\in G$, $\forall v\in A^+$.*

PROOF. We prove that, for μ a.a. γ, the equality $\tilde{\phi}_\gamma(\tilde{\alpha}_g^\gamma(T_\gamma))=\chi(g)\tilde{\phi}_\gamma(T_\gamma)$ holds for $\forall g\in G$, $\forall T_\gamma\in M^+$. Let $T\in M^+$ and $T\cong\int_\Gamma^\oplus T_\gamma d\mu(\gamma)$. We have $\tilde{\phi}(T)=\int_\Gamma^\oplus \tilde{\phi}_\gamma(T_\gamma)d\mu(\gamma)$ and $\tilde{\phi}(\tilde{\alpha}_g(T))=\chi(g)\int_\Gamma \tilde{\phi}_\gamma(T_\gamma)d\mu(\gamma)=\int_\Gamma \chi(g)\tilde{\phi}_\gamma(T_\gamma)d\mu(\gamma)$. For each $g\in G$, we put $\tilde{\phi}^g(T)\equiv\chi(g)\tilde{\phi}(T)$, $\tilde{\phi}_\gamma^{g,1}(T_\gamma)\equiv\chi(g)\tilde{\phi}_\gamma(T_\gamma)$ and $\tilde{\phi}_\gamma^{g,2}(T_\gamma)\equiv\tilde{\phi}_\gamma(\tilde{\alpha}_g^\gamma(T_\gamma))$ for $\forall T\in M^+$, $\forall T_\gamma\in M^+$. Clearly $\tilde{\phi}^g$, $\tilde{\phi}_\gamma^{g,1}$'s are f. n. s. weights, and by the continuity of $\tilde{\alpha}_g^\gamma$, $\phi_\gamma^{g,2}$'s are also f. n. s. We consider the measurability. To this aim we refer measurability and weak measurability of family of weights to Definition

4.1 in [21]. Since $\{\gamma\to\tilde{\phi}_\gamma\}$ is measurable, $\{\gamma\to\tilde{\phi}_\gamma^{g,1}\}$ is measurable. By the property of $\{\alpha_g^\gamma\}$, $\{\gamma\to\phi_\gamma^{g,2}\}$ is weakly measurable. Using Theorem 4.25 in [21], we have for μ a. a. γ,

$$\tilde{\phi}_\gamma(\tilde{\alpha}_g^\gamma(T_\gamma))=\chi(g)\tilde{\phi}_\gamma(T_\gamma) \qquad \text{for} \quad \forall T_\gamma\in M_\gamma^+ .$$

Since G is assumed to be separable, there exists a countable dense set $\{g_j\}$ in G. Let $\mathfrak{N}_j$ be an exceptional null set relative to g_j and $\mathfrak{N}\equiv\bigcup\mathfrak{N}_j$. $\mathfrak{N}$ is also a μ null set. For every $\gamma\in\Gamma\setminus\mathfrak{N}$ and every j we have $\tilde{\phi}_\gamma(\tilde{\alpha}_{g_j}^\gamma(T_\gamma))=\chi(g_j)\tilde{\phi}_\gamma(T_\gamma)$ for every $T_\gamma\in M_\gamma^+$. Since $\tilde{\phi}_\gamma$'s are σ-weakly lower semicontinuous and χ is a continuous homomorphism, this equality holds for every $g\in G$ and every $\gamma\in\Gamma\setminus\mathfrak{N}$.

q. e. d.

Again, let χ be a continuous homomorphism from G to $\boldsymbol{R}^+$. χ gives rise to a continuous $*$-automorphism group $\{\sigma_t^\chi : t\in\boldsymbol{R}\}$ on $C^*(G)$ such that for $k\in C_c(G)$ we have $(\sigma_t^\chi k)(g)=\chi^{it}(g)k(g)$. It is to be remarked that $C_c(G)$ is $\{\sigma_t^\chi\}$ invariant.

LEMMA 2-5 ([2], p. 67–p. 68). *Let ϕ be a χ-semitrace on $C^*(G)$, then ϕ is σ^χ-K. M. S.*

By this lemma, we are able to apply the general theory of direct integral decomposition and Theorem 2-4 to the case of semitraces. Let $\{e\}$ be the trivial normal subgroup of G. $C^*(\{e\})$ has a G invariant identical trace ϕ^i. Let $\phi^G\equiv \mathrm{Ind}_{\{e\}\uparrow G}\phi^i$. ϕ^G is a Δ_G-semitrace of $C^*(G)$. The representation of G corresponding to ϕ^G is the left regular representation λ^G. As $n_{\phi^G}\supset C_c(G)$, ϕ^G is densely defined. We call ϕ^G the Haar weight of G.

Let G and N be s. l. c. g.'s and $N\lhd G$. Let $\tilde{\phi}^N$ be the canonical weight on $M^N\equiv R(\lambda^N)$ corresponding to the Haar weight ϕ^N of N and $\{\tilde{\sigma}_t\}$ be the modular automorphism group on M^N determined by $\tilde{\phi}^N$. Let Δ_N be defined by $\int_N k(g^{-1}ng)dn =\Delta_N(g)\int_N k(n)dn$ for $k\in C_c(N)$, where dn is a Haar measure of N. (The restriction of Δ_N to N is just the modular function of N.)

LEMMA 2-6. *ϕ^N and $\tilde{\phi}^N$ are Δ_N-relatively invariant under the action of G.*

Let $\lambda^N\cong\int_\Gamma^\oplus\lambda_\gamma^N d\mu(\gamma)$ be a direct integral decomposition over some standard measure space (Γ, μ). Suppose that each element of the diagonal algebra is fixed under the action of G. Let $\phi^N=\int_\Gamma\phi_\gamma^N d\mu(\gamma)$ be the corresponding decomposition.

PROPOSITION 2-7. *For μ a. a. γ,*

(1) *ϕ_γ^N is a Δ_N-semitrace of $C^*(N)$, and it is Δ_N-relatively invariant under the action of G.*

(2) *λ_γ^N is a semitrace type representation associated with the semitrace ϕ_γ^N.*

PROOF. We consider $(G, C^*(N))$ and apply Theorem 2-11 which appears after in 2-3 to this situation. q. e. d.

REMARK 2-8. This proposition assures the possibility of inducing ϕ_γ^N to G

for μ a.a. γ.

COROLLARY 2-9. *The Haar weight of s.l.c.g. can be decomposed into Δ_G--semicharacters.*

PROOF. Put $G=N$ in Proposition 2-7. q.e.d.

2-3. The relation between the induction and the decomposition of semitraces.

Let $G \triangleright N$ and ϕ be a χ-semitrace on $C^*(N)$ which is η-relatively invariant under the action of G for a positive real character η of G. We assume first that $n_\phi \supset C_c(N)$. This assumption is unnecessarily strong and we will make it weaker in Corollary 2-12. Let $\pi_\phi \cong \int_\Gamma^\oplus \pi_\gamma d\mu(\gamma)$ and the diagonal algebra be fixed elementwise under the action of G on $R(\pi_\phi)$ determined by α (cf. 2.1). Then, the corresponding f.n.s. weight $\tilde{\phi}$ on $R(\pi_\phi)$ is decomposed accordingly $\left(\tilde{\phi}=\int_\Gamma^\oplus \tilde{\phi}_\gamma d\mu(\gamma)\right)$. We can assume that all $\tilde{\phi}_\gamma$'s are η-relatively invariant. If we put $\phi_\gamma \equiv \tilde{\phi}_\gamma \circ \pi_\gamma |_{C^*(N)^+}$, we have $\phi=\int_\Gamma \phi_\gamma d\mu(\gamma)$. Let $\bar{\phi} \equiv \mathrm{Ind}_{N\uparrow G}\phi$ and $(\pi_{\bar{\phi}}, \tilde{\bar{\phi}})$ be the semitraceclass representation corresponding to $\bar{\phi}$. Due to Mackey, we have $\pi_{\bar{\phi}} \cong \int_\Gamma^\oplus (\mathrm{Ind}_{N\uparrow G}\pi_\gamma) d\mu(\gamma)$.

LEMMA 2-10. *Let G and H be s.l.c.g.'s and $G \triangleright H$. Let π be a unitary representation of H on H_π. Let $\{\tilde{\alpha}_g : g \in G\}$ be a $*$-automorphism group on $R(\pi_\phi)$ such that $(g\circ\pi)(h)=\tilde{\alpha}_g(\pi(h))$ for every $h \in H$. Suppose that $\pi \cong \int_\Gamma^\oplus \pi_\gamma d\mu(\gamma)$ and the diagonal algebra is fixed by $\tilde{\alpha}_g$ $(g \in G)$ elementwise. If $U \equiv \mathrm{Ind}_{H\uparrow G}\pi$, $U \cong \int_\Gamma^\oplus (\mathrm{Ind}_{H\uparrow G}\pi_\gamma) d\mu(\gamma)$. The diagonal algebra of this decomposition is contained in $R(U)$.*

PROOF. This lemma is an easy generalization of Lemma 3-4-3 of Pukanszky [18]. q.e.d.

Let $\bar{\phi}_\gamma \equiv \mathrm{Ind}_{N\uparrow G}\phi_\gamma$, and $\tilde{\bar{\phi}}_\gamma$ be the f.n.s. weight of the von Neumann algebra given by π_γ corresponding to $\bar{\phi}_\gamma$. In this case $R(\pi_\phi)$ is also decomposed by the same diagonal algebra as appeared at the beginning of this section.

THEOREM 2-11. *In the above situation, $\{\gamma \to \tilde{\bar{\phi}}_\gamma\}$ is a measurable family of f.n.s. weights and $\tilde{\bar{\phi}}=\int_\Gamma^\oplus \tilde{\bar{\phi}}_\gamma d\mu(\gamma)$. From this we have $\bar{\phi}=\int_\Gamma \bar{\phi}_\gamma d\mu(\gamma)$.*

PROOF. We must treat null sets carefully. Since N is separable, there exists a countable dense subset $\{\xi_j\}$ of $C_c(N)$ containing an approximate unit in $C_c(N)$. Let P_1 be the $\boldsymbol{Q}(i)$ (rational complex numbers) coefficient $*$-subalgebra of $C_c(N)$ generated by $\{\xi_j\}$. Since P_1 is countable, by removing some null set, we can assume that, for every $\gamma \in \Gamma$, P_1 is contained in n_{ϕ_γ}. Since n_{ϕ_γ} is hereditary, n_{ϕ_γ} contains the $*$-subalgebra P_2 generated by P_1.

For $k \in C_c(G)$, let $F(k)(g)(n) \equiv k(gn)$ $(\forall g \in G, \forall n \in N)$. $F(k)(g)$ is considered to be a map from G to $C_c(N)$. Let $h\in P_2$. Then $(F(k)(g)\cdot h)(n)=\int_N k(gn')h(n'^{-1}n)dn'$ for every $n\in N$. Let $(k\circ h)(g)\equiv(F(k)(g)\cdot h)(e)$. Clearly $k\circ h$ belongs to $C_c(G)$. By the left invariance of dn, $F(k)\cdot h=F(k\circ h)$. Since $h\in n_{\phi_\gamma}$, $k\circ h\in n_{\bar{\phi}_\gamma}$. Analogously $k\circ h\in n_{\bar{\phi}}$. Let $\{k_i: i=1, 2, \cdots\}$ be a countable dense subset of $C_c(G)$ containing an approximate unit of $C_c(G)$. Let $q_1\equiv\{k\circ h: k\in\{k_i\}, h\in P_1\}$. $q_1\subset C_c(G)$ and q_1 contains an approximate unit of $C_c(G)$, and $q_1\subset n_{\bar{\phi}}\cap n_{\bar{\phi}_\gamma}$ for every $\gamma\in\Gamma$.

Let $k\in C_c(G)$ and $h\in C_c(N)$. We have

$$\begin{aligned}F((k\circ h)^*)(g)&=\Delta_{G/N}(\dot{g})\alpha_g{}^{-1}(F(k)(g^{-1})\circ h^*)\\&=\alpha_g{}^{-1}(h^*)\Delta_{G/N}(g)\alpha_g{}^{-1}(F(k)(g^{-1})^*)\\&=F(k^*)(g)\alpha_g{}^{-1}(h^*)\,.\end{aligned}$$

Since P_1 is $*$-closed, $h\in P_1$ implies $h^*\in P_1$. And since n_{ϕ_γ} is G invariant, we have $(k\circ h)^*\subset C_c(G)\cap n_{\bar{\phi}_\gamma}\cap n_{\bar{\phi}}$. Let q_2 be the $\boldsymbol{Q}(i)$ coefficient $*$-subalgebra generated by q_1. We have $q_2\subset C_c(G)\cap n_{\bar{\phi}_\gamma}\cap n_{\bar{\phi}}$ and q_2 has an approximate unit. Let $\tilde{\chi}\equiv\Delta_{G/N}\cdot\eta$. For every $\gamma\in\Gamma$, $\bar{\phi}$ and $\bar{\phi}_\gamma$'s are $\sigma^{\tilde{\chi}}$-K. M. S. Let q_3 be the $\boldsymbol{Q}(i)$ coefficient $*$-subalgebra generated by $\{\sigma_t(v): t\in\boldsymbol{Q}, v\in q_2{}^*q_2\}$. q_3 is countable, dense, $*$-closed and $\{\sigma_t: t\in\boldsymbol{Q}\}$ invariant. Since $C_c(G)$, $m_{\bar{\phi}}$ and $m_{\bar{\phi}_\gamma}$'s are $\sigma^{\tilde{\chi}}$ invariant, $q_3\subset C_c(G)\cap m_{\bar{\phi}}\cap m_{\bar{\phi}_\gamma}$. Let q_4 be the $*$-subalgebra generated by q_3. By the "Polarization identity", if a weight is finite on q_3^+, it is finite on q_4^+.

$\tilde{\sigma}^{\tilde{\chi}}$, $\tilde{\sigma}_\gamma^{\tilde{\chi}}$'s denote the modular automorphism group on von Neumann algebras corresponding to $\sigma^{\tilde{\chi}}$. Let $u_3\equiv\pi_{\bar{\phi}}(q_3)$, $u_4\equiv\pi_{\bar{\phi}}(q_4)$, $u_3^\gamma\equiv\pi_{\bar{\phi}_\gamma}(q_3)$ and $u_4^\gamma\equiv\pi_{\bar{\phi}_\gamma}(q_4)$. u_3 and u_3^γ's are countable. These subsets satisfy the same property for $(\bar{\bar{\phi}}, \tilde{\sigma}^{\tilde{\chi}})$, $(\bar{\bar{\phi}}_\gamma, \tilde{\sigma}_\gamma^{\tilde{\chi}})$'s as in C^*-situation. $R(\pi_{\bar{\phi}})\cong\int_\Gamma^\oplus M_\gamma d\mu(\gamma)$. We can assume that for every $\gamma\in\Gamma$, $M_\gamma=R(\mathrm{Ind}_{N\uparrow G}\pi_\gamma)$. Let $\bar{\bar{\phi}}=\int_\Gamma\bar{\bar{\phi}}_\gamma' d\mu(\gamma)$ be the corresponding decomposition. $\bar{\bar{\phi}}_\gamma$'s are f. n. s. and we can assume that all $\bar{\bar{\phi}}_\gamma$'s are $\tilde{\chi}$-relatively invariant under the action of G. By removing some null set we can assume that, for every $\gamma\in\Gamma$, $u_3^\gamma\subset m_{\bar{\bar{\phi}}_\gamma'}$, and this implies that, for every $\gamma\in\Gamma$, $u_4\subset m_{\bar{\bar{\phi}}_\gamma'}$.

By the construction of induced semitrace we can identify the representation spaces of $(\mathrm{Ind}_{N\uparrow G}\pi_\gamma)$'s and $\pi_{\bar{\phi}}$ with the completion of left Hilbert algebras which give induced semitraces. Since $H\pi_{\bar{\phi}}=L^2(G/N)\otimes H\pi_{\bar{\phi}}$, and $H\pi_\phi\cong\int_\Gamma^\oplus H(\gamma)d\mu(\gamma)$, we have $H\pi_{\bar{\phi}}\cong\int_\Gamma^\oplus(L^2(G/N)\otimes H(\gamma))d\mu(\gamma)$. Let $k\in C_c(G)$ take finite values for induced semitraces. By the above consideration we can remove some μ null set N_k from Γ such that for $\gamma\in\Gamma\setminus N_k$ we have $\|F(k)^\gamma(g)\|^2=\int_{G/N}\|F(k)(g)^\gamma\|^2dg$, where $F(k)\cong\int_\Gamma^\oplus F(k)^\gamma d\mu(\gamma)$ and $F(k)(g)\cong\int_\Gamma^\oplus F(k)(g)^\gamma d\mu(\gamma)$ (as the decompositions of left Hilbert algebras). Since u_3^γ is countable, we can assume that for every $\gamma\in\Gamma$,

$\tilde{\bar{\phi}}'_\gamma$ and $\tilde{\bar{\phi}}_\gamma$ are equal for the elements of u_3^γ of the form $\pi_{\tilde{\phi}_\gamma}(k^{**}k)$. By the "Polarization identity", $\tilde{\bar{\phi}}'_\gamma$ and $\tilde{\bar{\phi}}_\gamma$ are equal on u_3^γ, and so are on u_4^γ. u_5^γ denotes the $*$-subalgebra generated by $\{\sigma_{\gamma,t}(T_\gamma): t\in \boldsymbol{R},\ T_\gamma\in u_4^\gamma\}$. We show that $\tilde{\bar{\phi}}_\gamma=\tilde{\bar{\phi}}'_\gamma$ holds on u_5^γ. Since u_4^γ is $*$-closed and every automorphism preserves $*$, it is sufficient to prove it for monomials. We define two functions on $\boldsymbol{R}^n$ by

$$c(t_1, \cdots, t_n)\equiv\tilde{\bar{\phi}}_\gamma(\tilde{\sigma}^{\tilde{\chi}}_{\gamma,t_1}(T^1_\gamma)\cdots\tilde{\sigma}^{\tilde{\chi}}_{\gamma,t_n}(T^n_\gamma))$$
$$c'(t_1, \cdots, t_n)\equiv\tilde{\bar{\phi}}'_\gamma(\tilde{\sigma}^{\tilde{\chi}}_{\gamma,t_1}(T^1_\gamma)\cdots\tilde{\sigma}^{\tilde{\chi}}_{\gamma,t_n}(T^n_\gamma))$$

where $(t_1, \cdots, t_n)\in\boldsymbol{R}^n$ and $T^i_\gamma\in u_4$. For $n=1$, the validity of $c=c'$ comes from the invariance of weights under modular automorphism groups. Suppose that $n\geqq 2$. Since $\tilde{\bar{\phi}}_\gamma$ and $\tilde{\bar{\phi}}'_\gamma$ are $\tilde{\sigma}^{\tilde{\chi}}_\gamma$-K.M.S. and all T^i_γ's are contained in $n_{\tilde{\bar{\phi}}_\gamma}\cap n_{\tilde{\bar{\phi}}_\gamma}{}^*\cap n_{\tilde{\bar{\phi}}'_\gamma}\cap n_{\tilde{\bar{\phi}}'_\gamma}{}^*$, $t_1\to c(t_1, \cdots, t_n)$, $t_n\to c(t_1, \cdots, t_n)$, $t_1\to c'(t_1, \cdots, t_n)$ and $t_n\to c'(t_1, \cdots, t_n)$ are continuous. For $2\leqq i\leqq n-1$, $t_i\to c(t_1, \cdots, t_i, \cdots, t_n)$ and $t_i\to c'(t_1, \cdots, t_i, \cdots, t_n)$ are continuous because $x\to\tilde{\bar{\phi}}_\gamma(yxz)(\tilde{\bar{\phi}}'_\gamma(yxz))$ $(y, z\in n_{\tilde{\bar{\phi}}_\gamma}\cap n_{\tilde{\bar{\phi}}'_\gamma}\cap n_{\tilde{\bar{\phi}}_\gamma}{}^*\cap n_{\tilde{\bar{\phi}}'_\gamma}{}^*)$ are matrix elements of σ-weakly continuous representations of M_γ. Therefore $(t_1, \cdots, t_n)\to c(t_1, \cdots, t_n)$ $(c'(t_1, \cdots, t_n))$ is separately continuous. Since by the construction of u_4^γ $c=c'$ on $\boldsymbol{Q}^n$, we have $c=c'$ on $\boldsymbol{R}^n$. This shows that $\tilde{\bar{\phi}}_\gamma|_{u_5^\gamma}=\tilde{\bar{\phi}}'_\gamma|_{u_5^\gamma}$.

Since u_5^γ is a $*$-closed σ-weakly dense subalgebra of $m_{\tilde{\bar{\phi}}_\gamma}\cap m_{\tilde{\bar{\phi}}'_\gamma}$ invariant under the modular automorphism group, by virtue of Theorem 5.9 of G.K. Pedersen and M. Takesaki [**15**] we have $\tilde{\bar{\phi}}_\gamma=\tilde{\bar{\phi}}'_\gamma$. Therefore $\{\gamma\to\tilde{\bar{\phi}}_\gamma\}$ is a measurable family of f.n.s. weights and $\tilde{\bar{\phi}}=\int_\Gamma^\oplus\tilde{\bar{\phi}}_\gamma d\mu(\gamma)$. This shows that $\bar{\phi}=\int_\Gamma\bar{\phi}_\gamma d\mu(\gamma)$. q.e.d.

COROLLARY 2-12. *For the proof of Theorem* 2-11, *it was necessary that* m_ϕ *had a good countable subset. Under the condition stated in the first part of this section, we can derive the same conclusion for the direct integral components of Haar weights.*

We will use this stronger result in § 5.

2-4. Projective semitraces.

Like ordinary semitraces, we can define the notion of a projective semitrace and develope a theory of induction and direct integral decomposition. For this purpose we need the notion of twisted C^*-algebra defined in [**13**]. The arguments of [**13**] p. 104–p. 106 are valid without any assumption of type I'ness or unimodularity. We will employ the same notation. Let G be an s.l.c.g. and ω be a multiplier on G. i.e. a function on $G\times G$ with value in $\boldsymbol{T}$ which is Borel measurable and satisfies cocycle condition (cf. p. 215 in [**11**]). There exists a conditional expectation P from $C^*(G(\omega))$ to $C^*(G, \omega)$. Then weights on $C^*(G, \omega)$ can be regarded as weights on $C^*(G(\omega))$. This correspondence preserves lower semicontinuity and densely defindness. Let χ be a continuous homomorphism from G to $\boldsymbol{R}^+$. χ can be lifted on $G(\omega)$.

DEFINITION 2-13. A lower semicontinuous weights ϕ on $C^*(G, \omega)$ is called an

ω-χ-semitrace if $\phi\circ P$ is a χ-semitrace on $C^*(G(\omega))$.

The following propositions hold. The proof of them are carried out by reduction to the ordinary case.

PROPOSITION 2-14. *Let G, N be s.l.c.g.'s, $G\rhd N$, ω be a multiplier on G and $\omega'\equiv\omega|_{N\times N}$. Let η be a continuous homomorphism from G to $\boldsymbol{R}^+$, and $\chi\equiv\eta|_N$. Let ϕ be an ω'-χ-semitrace on $C^*(N, \omega')$ which is η-relatively invariant under the action of $G(\omega)$. Then we can construct a canonical ω-$\eta\cdot\Delta_{G/N}$-semitrace on $C^*(G, \omega)$. $\mathrm{Ind}^{\omega}_{N\uparrow G}\phi$ denotes this semitrace.*

PROPOSITION 2-15. *A densely defined ω-χ-semitrace on $C^*(G, \omega)$ can be decomposed into densely defined ω-χ-semitraces.*

PROPOSITION 2-16. *In the situation analogous to Theorem 2-11, ϕ can be decomposed into ω-semitraces which are η-relatively invariant under the action of G.*

PROPOSITION 2-17. *The stage theorem holds for the induction of projective semitraces.*

PROPOSITION 2-18. *In the projective semitrace context, the processes of induction and direct integral decomposition commute.*

§ 3. Plancherel theory.

3-1. A remark on the central decomposition.

Let A be a separable C^*-algebra and π be a representation of A. $\pi\cong\int_{\Gamma}^{\oplus}\pi_\gamma d\mu(\gamma)$ is called a coarse decomposition when the diagonal algebra is contained in $R(\pi)$. When we want to carry out the central decomposition of π, we often decompose π coarsely first and carry out the central decomposition of each component. That is to say $\pi\cong\int_{\Gamma}^{\oplus}\pi_\gamma d\mu(\gamma)$ (coarse) and $\pi_\gamma\cong\int_{\widehat{A}}^{\oplus}\pi_{\gamma,\delta}d\mu_\gamma(\delta)$ (central). But $\left\{\gamma\to\int_{\widehat{A}}^{\oplus}\pi_{\gamma,\delta}d\mu_\gamma(\delta)\right\}$ may not be measurable, so we cannot understand the central decomposition of π as a double integral directly. But by [4] and [8], the next lemma holds.

LEMMA 3-1 ([4], [8]). *Let $\pi\cong\int_{\Gamma}^{\oplus}\pi_\gamma d\mu(\gamma)$ be coarse and for μ a.a. γ $\pi_\gamma\cong\int_{\widehat{A}}^{\oplus}\pi_{\gamma,\delta}d\mu_\gamma(\delta)$ be central. Then for μ a.a. γ there exists a family of Borel measures $\{\mu'_\gamma\}$ on $\widehat{A}$ satisfies the following* (1) *and* (2). *Put $\pi'_\gamma=\int_{\widehat{A}}^{\oplus}\pi_{\gamma,\delta}d\mu'_\gamma(\delta)$ for μ a.a. γ. Then,* (1) *each π'_γ is unitarily equivalent to π_γ, and $\{\gamma\to\pi'_\gamma\}$ is measurable,* (2) *π is unitarily equivalent to $\int_{\Gamma}^{\oplus}\pi'_\gamma d\mu(\gamma)$.*

In the above situation we write

$$\pi\cong\int_{\Gamma}\int_{\widehat{A}}^{\oplus}\pi_{\gamma,\delta}d\mu_\gamma(\delta)d\mu(\gamma)(m)\,.$$

REMARK 3-2. Let G be an s.l.c.g. and ϕ be a densely defined χ-semitrace on $C^*(G)$. When we have $\phi=\int_{\Gamma}\phi_{\gamma}d\mu(\gamma)$ and $\phi_{\gamma}=\int_{\Delta_{\gamma}}\phi_{\gamma,\delta}d\mu_{\gamma}(\delta)$, then we can write as follows.

$$\phi=\int_{\Gamma}\left\{\int_{\Delta_{\gamma}}\phi_{\gamma,\delta}d\mu_{\gamma}(\delta)\right\}d\mu(\gamma)\quad\left(=\int_{\Gamma}\int_{\Delta_{\gamma}}\phi_{\gamma,\delta}d\mu_{\gamma}(\delta)d\mu(\gamma)\text{ for notational convenience}\right).$$

This is the greatest merit of considering the decomposition of semitraces rather than the decomposition of representations.

3-2. Plancherel theory.

By using the notion of semitraces, we can have a Plancherel theory for general separable locally compact groups. Let λ^G be the left regular representation of G and ϕ^G be the Haar weight on $C^*(G)$. Then by [**3**], there exists a standard Borel measure μ^G on $\widehat{G}$ unique up to measure classes and a measurable field of factor representations $\{\gamma\to\lambda_{\gamma}^G\}$ such that we have $\lambda^G\cong\int_{\widehat{G}}^{\oplus}\lambda_{\gamma}^G d\mu^G(\gamma)$, where the diagonal algebra is the centre of $R(\lambda^G)$. This algebra also gives the decomposition of the canonical f.n.s. weight $\tilde{\phi}^G$ on $R(\lambda^G)$ corresponding to ϕ^G. That is to say, by removing some null set from $\widehat{G}$ if necessary, $\tilde{\phi}^G=\int_{\widehat{G}}^{\oplus}\tilde{\phi}_{\gamma}^G d\mu^G(\gamma)$. If we put $\phi_{\gamma}^G\equiv\tilde{\phi}_{\gamma}^G\circ\lambda_{\gamma}^G|_{C^*(G)^+}$, we have $\phi^G=\int_{\widehat{G}}\phi_{\gamma}^G d\mu(\gamma)$. This is a decomposition of the Haar weight ϕ^G into Δ_G-semicharacters.

DEFINITION 3-3 (§6 in [**20**]). We call $\phi^G=\int_{\widehat{G}}\phi_{\gamma}^G d\mu(\gamma)$ the Plancherel formula of G, μ^G a Plancherel measure and its measure class the Plancherel measure class. The study of the central decomposition of λ^G, Plancherel measures and Plancherel formula will be called the Plancherel theory for G.

REMARK 3-4. When G is not of type I, we do not know yet the whole set of $\widehat{G}$. But in many interesting cases, we can determine a smaller part which supports μ^G.

3-3. Projective Plancherel formula.

Let G be an s.l.c.g. and ω be a multiplier on G (cf. 2-4). Let $\phi^{G,\omega}$ be the ω-Δ_G-semitrace on $C^*(G,\omega)$ induced from the trivial trace on $C^*(\{e\})$. We call $\phi^{G,\omega}$ an ω-Haar weight. $\lambda^{G,\omega}$ denotes the ω-regular representation of G and $\flat$ denotes the involution of $L^1(G,\omega)$.

LEMMA 3-5. *$\phi^{G,\omega}$ is a densely defined σ^{Δ_G}-K.M.S. weight. For $k\in L^1(G,\omega)\cap L^2(G)$, $\phi^{G,\omega}(k^{\flat}*_{\omega}k)=(k^{\flat}*_{\omega}k)(e)$. Therefore the elements of $C_c(G(\omega))$ considered as elements of $C^*(G(\omega))$ are contained in $n_{\phi^{G,\omega}}$.*

When G is unimodular this is the Lemma 2-1 of [**13**]. $\widehat{G}^{\omega}$ denotes the set of quasi-equivalence classes of ω factor representations of G. There exists a standard Borel measure $\mu^{G,\omega}$ on $\widehat{G}^{\omega}$ such that $\lambda^{G,\omega}\cong\int_{\widehat{G}^{\omega}}\lambda_{\gamma}^{G,\omega}d\mu^{G,\omega}(\gamma)$ (central) and $\phi^{G,\omega}=\int_{\widehat{G}^{\omega}}\phi_{\gamma}^{G,\omega}d\mu^{G,\omega}(\gamma)$.

DEFINITION 3-6. We call $\phi^{G,\omega}=\int_{\widehat{G}^{\omega}}\phi_{\gamma}^{G,\omega}d\mu^{G,\omega}(\gamma)$ an ω-Plancherel formula of G, (the class of) $\mu^{G,\omega}$ an (the) ω-Plancherel measure (class) of G. The study of the decomposition of $\lambda^{G,\omega}$, the ω-Plancherel formula and the ω-Plancherel measure (class) will be called the ω-Plancherel theory of G.

The projective Plancherel theory is necessary when we study the Plancherel theory on group extension context.

§ 4. Plancherel formula for concrete groups.

4-1. Groups whose regular representations or projective regular representations are factorial.

In this case, the Plancherel measure concentrates on one point. Examples of these groups are ICC groups and $\boldsymbol{Z}^2$ with "irrational" multipliers.

4-2. Solvable Lie group.

EXAMPLE 1. Mautner group.

$G=\boldsymbol{R}\times_s\boldsymbol{C}^2$. The multiplication is given by

$$(x, z_1, z_2)(x', z_1', z_2')=(x+x', e^{ix'}z_1+z_1', e^{2\pi ix'}z_2+z_2').$$

For $(w_1, w_2)\in\boldsymbol{C}^2$, $\chi^{(w_1,w_2)}\in\widehat{\boldsymbol{C}}^2$ is defined by

$$\chi^{(w_1,w_1)}(z_1, z_2)\equiv\exp[i\operatorname{Re}(\bar{w}_1z_1+\bar{w}_2z_2)] \qquad \text{for} \quad z_i\in\boldsymbol{C}.$$

When $w_1\neq 0$ and $w_2\neq 0$, the stabilizer at $\chi^{(w_1,w_2)}$ of the action of $\boldsymbol{R}$ on $\widehat{\boldsymbol{C}}^2$ is trivial. Let $U^{(w_1,w_2)}\equiv\operatorname{Ind}_{C^2\uparrow G}\chi^{(w_1,w_2)}$. If $w_1\neq 0$ and $w_2\neq 0$, $U^{(w_1,w_2)}$ is irreducible. We define r_j, θ_j $(r_j>0, 0\leqq\theta_j<2\pi)$ by $w_j=r_je^{i\theta_j}$ $(j=1, 2)$. Let

$$W^{(r_1,r_2)}\equiv\int_0^{2\pi}\int_0^{2\pi\oplus}U^{(r_1e^{i\theta_1},r_2e^{i\theta_2})}d\theta_1d\theta_2.$$

PROPOSITION 4-1. (1) *When* $r_1\neq 0$ *and* $r_2\neq 0$, $W^{(r_1,r_2)}$ *is a normal representation which generates a hyperfinite* II_∞ *factor.*

(2) $\lambda^G\cong\int_0^\infty\int_0^{\infty\oplus}W^{(r_1,r_2)}dr_1dr_2$ *and this is central.*

(3) $\phi^G=\int_0^\infty\int_0^\infty\phi^{(r_1,r_2)}dr_1dr_2$ *and this is a Plancherel formula of* G. *If* r_1 *and* $r_2\neq 0$, $\phi^{(r_1,r_2)}$ *is the character which gives* $W^{(r_1,r_2)}$. *For* $k\in C_c(G)$ *we have*

$$\phi^{(r_1,r_2)}(k^* * k)=\int_{-\infty}^\infty\int_0^{2\pi}\int_0^{2\pi}\left|\int_C\int_C f(x, z_1, z_2)\chi^{(r_1e^{i\theta_1},r_2e^{i\theta_2})}dz_1dz_2\right|^2 d\theta_1d\theta_2dx.$$

REMARK 4-2. Since $R(W^{(r_1,r_2)})$'s are II_∞ factors, there is no appropriate normalization of traces or the Plancherel measures.

EXAMPLE 2. Dixmier group.

G is diffeomorphic to $\boldsymbol{R}^2\times\boldsymbol{C}^2\times\boldsymbol{R}$, and the multiplication is given by

$$(t, s, z_1, z_2, r)(t', s', z_1', z_2', r')$$
$$=(t+t', s+s', e^{2\pi it'}z_1+z_1', e^{2\pi is'}z_2+z_2', r+r'+t's).$$

For $(w_1, w_2, \tilde{r})\in\boldsymbol{C}^2\times\boldsymbol{R}$, $\chi^{(w_1,w_2,\tilde{r})}\in\widehat{\boldsymbol{C}^2\times\boldsymbol{R}}$ is defined by

$$\chi^{(w_1,w_2,\tilde{r})}(z_1, z_2, r)\equiv\exp[2\pi i\,\mathrm{Re}(\bar{w}_1z_1+\bar{w}_2z_2+\tilde{r}r)].$$

Each orbit on $\widehat{\boldsymbol{C}^2\times\boldsymbol{R}}$ is given for R_1, $R_2\geqq 0$ and $\tilde{r}$ by

$$O(R_1, R_2, \tilde{r})\equiv\{\chi^{(w_1,w_2,\tilde{r})}: |w_1|=R_1, |w_2|=R_2\}.$$

Clearly all orbits are closed. We only consider the case when $R_1>0$, $R_2>0$.

Let $H\equiv\{(m, n, z_1, z_2, r): m, n\in\boldsymbol{Z}, z_1, z_2\in\boldsymbol{C}, r\in\boldsymbol{R}\}$. Then this is the common stabilizer group. $\boldsymbol{Z}^2$ is the common little group.

Suppose that $w_1=R_1$, $w_2=R_2$. We determine the projective extension $\tilde{\chi}^{(R_1,R_2,\tilde{r})}$ of $\chi^{(R_1,R_2,\tilde{r})}$ by the trivial extension. In this case a Mackey obstruction cocycle $\alpha^{\tilde{r}}$ is given by

$$\alpha^{\tilde{r}}((m_1, n_1), (m_2, n_2))\equiv e^{-2\pi i\tilde{r}m_2n_1}\qquad(m_i, n_i\in\boldsymbol{Z}).$$

When $w_1=R_1e^{2\pi it}$ and $w_2=R_2e^{2\pi is}$ ($t\neq 0$ or $s\neq 0$), we define $\tilde{\chi}^{(w_1,w_2,\tilde{r})}$ by $(t, s)\cdot\tilde{\chi}^{(R_1,R_2,\tilde{r})}$. In this case we can take the same Mackey obstruction cocycle $\alpha^{\tilde{r}}$. Moreover, we assume that $\tilde{r}$ is irrational.

Let $\sigma^{\tilde{r}}$ be the $\alpha^{\tilde{r}}$-regular representation of $\boldsymbol{Z}^2$. This is a II_1-normal representation. Let $\hat{\sigma}^{\tilde{r}}$ be the lifting of $\sigma^{\tilde{r}}$ to H. Let $U^{(w_1,w_2,\tilde{r})}\equiv\chi^{(w_1,w_2,\tilde{r})}\otimes\hat{\sigma}^{\tilde{r}}$. This is a normal representation of H. Let $V^{(w_1,w_2,\tilde{r})}\equiv\mathrm{Ind}_{H\uparrow G}U^{(w_1,w_2,\tilde{r})}$. Then $V^{(w_1,w_2,\tilde{r})}$ is a normal representation of G. Let

$$W^{(R_1,R_2,\tilde{r})}\equiv\int_0^1\int_0^{1\oplus}V^{(R_1e^{2\pi i\theta_1},R_2e^{2\pi i\theta_2},\tilde{r})}d\theta_1d\theta_2.$$

$V^{(w_1,w_2,\tilde{r})}$ is quasi equivalent to $W^{(R_1,R_2,\tilde{r})}$ if $|w_i|=R_i$.

PROPOSITION 4-3. (1) *When* $R_1>0$, $R_2>0$ *and* $\tilde{r}$ *is irrational,* $W^{(R_1,R_2,\tilde{r})}$ *is a* II_∞*-normal representation.*

(2) $\lambda^G\cong\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty\oplus}W^{(R_1,R_2,\tilde{r})}dR_1dR_2d\tilde{r}$ *and this is central.*

(3) $\phi^G=\int_{\boldsymbol{R}/\boldsymbol{Q}}\int_{(0,\infty)}\int_{(0,\infty)}\phi^{(R_1,R_2,\tilde{r})}\,dR_1dR_2d\tilde{r}$ *and this is a Plancherel formula of* G. $\phi^{(R_1,R_2,\tilde{r})}$*'s are the characters which give* $W^{(R_1,R_2,\tilde{r})}$*'s and for* $k\in C_c(G)$ *we have*

$$\phi^{(R_1,R_2,\tilde{r})}(k^{*}*k)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^1\int_0^1\left|\int_{-\infty}^{\infty}\int_C\int_C k(t,\ s,\ z_1,\ z_2,\ r)\exp\left[2\pi i\,\mathrm{Re}(R_1e^{i\theta_1}\bar{z}_1)\right]\right.$$

$$\left.\exp\left[2\pi i\,\mathrm{Re}(e^{i\theta_2}\bar{z}_2)\right]e^{2\pi i\tilde{r}r}\,dz_1dz_2dr\right|^2 d\theta_1 d\theta_2\,dsdt\,.$$

REMARK 4-4. In Mautner group and Dixmier group cases, stabilizer groups are constant almost everywhere.

4-3. Heisenberg type groups.

We investigate Heisenberg type groups which S. Kawakami defined in his Master Thesis [**9**].

Let X, Y and Z be separable locally compact abelian groups, and B be a continuous bihomomorphism from $X\times Y$ to Z. $G\equiv X\times_s(Y\times Z)$ and the multiplication is given by

$$(x,\ y,\ z)(x',\ y',\ z')=(x+x',\ y+y',\ z+z'+B(x',\ y))\,.$$

For $\beta\in\hat{Y}$, $\gamma\in\hat{Z}$, χ^{β} and χ^{γ} denote the corresponding character of Y and Z. The action of X on $\widehat{Y\times Z}$ is given by

$$x\cdot(\chi^{\beta}\times\chi^{\gamma})\equiv\chi^{\beta-(\gamma\cdot B(x))}\,,\qquad \text{where}\quad \chi^{\gamma\cdot B(x)}(y)\equiv\chi^{\gamma}(B(x,\ y))\,.$$

The orbit containing (β,γ) $(O(\beta,\gamma))$ is the translation of (β,γ) by the subgroup Y_{γ} of $\hat{Y}$, where $Y_{\gamma}\equiv\{\gamma\cdot B(x): x\in X\}$. $O(\beta,\gamma)$'s are not necessarily locally closed and so G is not necessarily of type I.

We have $\bar{O}(\beta,\gamma)=\beta\cdot\bar{Y}_{\gamma}\times\{\gamma\}$. Since $\beta\cdot\bar{Y}_{\gamma}$ depends only on the $\bar{Y}_{\gamma}$ coset containing it, we can write $\bar{O}(\dot{\beta},\gamma)\equiv O(\beta,\gamma)$. We have $Y\times Z=\bigcup_{\gamma\in\hat{Z},\,\dot{\beta}\in\hat{Y}/\bar{Y}_{\gamma}}\bar{O}(\dot{\beta},\gamma)$, where each $\bar{O}(\dot{\beta},\gamma)$ is G invariant and closed, and each G orbit contained in it is dense. As the stabilizer at (β,γ) depends only on γ, we can write this X_{γ}. Let $G_{\gamma}\equiv X_{\gamma}\times_s(Y\times Z)$. For $\alpha\in\hat{X}_{\gamma}$, χ^{α} denotes the corresponding character of X_{γ}. Let $\chi^{(\gamma,\beta,\alpha)}\equiv\chi^{\alpha}\times\chi^{\beta}\times\chi^{\gamma}$, and $F(\gamma,\dot{\beta},\alpha)\equiv\{\chi^{\alpha}\times\chi^{\beta'}\times\chi^{\gamma}: \beta'\in\beta\cdot\bar{Y}_{\gamma}\}$. $F(\gamma,\dot{\beta},\alpha)$ is a closed subset of $\hat{G}_{\gamma}$ and each G orbit contained in it is dense. We have

$$\bigcup_{\dot{\beta}\in\hat{Y}/\bar{Y}_{\gamma},\,\alpha\in\hat{X}_{\gamma}}F(\gamma,\ \dot{\beta},\ \alpha)=\{\chi^{\delta}\in\hat{G}_{\gamma}: \chi^{\delta}|_{Y\times Z}\in\bar{O}(\dot{\beta},\ \gamma)\}\,.$$

(a) Prim(G).

Since $C^*(G)=X\times_{\alpha}C_0(\widehat{Y\times Z})$ and the primitive ideal space of such a C^*-algebra is already known [**6**], we have only to apply the arguments in [**6**] to our case. Let $F\equiv\bigcup F(\gamma,\dot{\beta},\alpha)$. For $\chi^{(\gamma,\beta,\alpha)}\in F$, let $U^{(\gamma,\beta,\alpha)}\equiv\mathrm{Ind}_{G_{\gamma}\uparrow G}\chi^{(\gamma,\beta,\alpha)}$. Since $U^{(\gamma,\beta,\alpha)}$ is irreducible, $J(\gamma,\beta,\alpha)\equiv\ker(U^{(\gamma,\beta,\alpha)})$ is a primitive ideal of $C^*(G)$.

LEMMA 4-5. *$J(\gamma,\beta,\alpha)=J(\gamma',\beta',\alpha')$ if and only if $\gamma=\gamma'$, $\dot{\beta}=\dot{\beta}'$ and $\alpha=\alpha'$.*

This lemma shows that there exists an injection from $\tilde{F}\equiv\{F(\gamma,\dot{\beta},\alpha): \gamma\in\hat{Z}$, $\dot{\beta}\in\hat{Y}/\bar{Y}_{\gamma},\ \alpha\in\hat{X}_{\gamma}\}$ to Prim(G).

LEMMA 4-6. *For every $J\in$Prim(G), there exists a (γ,β,α) such that $J=$*

$J(\gamma, \beta, \alpha)$.

This lemma is proved in a general situation in [6].

PROPOSITION 4-7 (Theorem 3.1 in [6]). *There exists a bijective correspondense between* $\tilde{F}$ *and* Prim(G).

(b) Some part of normal representations.

We construct a normal representation of G for each element of $\tilde{F}$ defined in (a). Let ${}^{\gamma}\mu^{Y}$ be a Haar measure on Y. Then $\delta_{\gamma}\times{}^{\gamma}\mu^{Y}\times\delta_{\alpha}$ is a G invariant ergodic measure on $F(\gamma, \dot{\beta}, \alpha)$. Let

$$V^{(\gamma,\dot{\beta},\alpha)}\equiv\int_{\dot{Y}_{\gamma}}^{\oplus}(\chi^{\alpha}\times\chi^{\beta+\beta'}\times\chi^{\gamma})d^{\gamma}\mu^{Y}(\beta') \qquad (\beta'\in\dot{\beta}).$$

We can consider $V^{(\gamma,\dot{\beta},\alpha)}$ as a representation of an abelian C^*-algebra $C_0(F(\gamma, \dot{\beta}, \alpha))$.

$$C_0(F(\gamma, \dot{\beta}, \alpha))^{+}\ni k \longrightarrow \int_{F(\gamma,\dot{\beta},\alpha)}(\chi^{\alpha}\times\chi^{\beta+\beta'}\times\chi^{\gamma})(k)d^{\gamma}\mu^{Y}(\beta')$$

gives a lower semicontinuous trace $\tau^{(\gamma,\dot{\beta},\alpha)}$ on $C_0(F(\gamma, \dot{\beta}, \alpha))$.

LEMMA 4-8. *$\tau^{(\gamma,\dot{\beta},\alpha)}$ is semifinite. The f.n.s. trace $\tilde{\tau}^{(\gamma,\dot{\beta},\alpha)}$ on $R(V^{(\gamma,\dot{\beta},\alpha)})$ given by $\tau^{(\gamma,\dot{\beta},\alpha)}$ makes $V^{(\gamma,\dot{\beta},\alpha)}$ a trace class representation and gives a G invariant lower semicontinuous semifinite trace on $C^*(G_{\gamma})$.*

PROOF. Since $\delta_{\alpha}\times\beta\cdot{}^{\gamma}\mu^{Y}\times\delta_{\gamma}$ is a Radon measure on $F(\gamma, \dot{\beta}, \alpha)$, $C_c(F(\gamma, \dot{\beta}, \alpha))$ is contained in $n_{\tau(\gamma,\dot{\beta},\alpha)}$. This shows that $\tau^{(\gamma,\dot{\beta},\alpha)}$ is densely defined and so semifinite. The second statement is a consequence from the general theory.

q. e. d.

Let $W^{(\gamma,\dot{\beta},\alpha)}\equiv\mathrm{Ind}_{G_{\gamma}\uparrow G}V^{(\gamma,\dot{\beta},\alpha)}$. Since the trace corresponding to $W^{(\gamma,\dot{\beta},\alpha)}$ is G invariant and semifinite and G/G_{γ} is abelian, $W^{(\gamma,\dot{\beta},\alpha)}$ is of trace class. And since $\delta_{\alpha}\times\beta\cdot{}^{\gamma}\mu^{Y}\times\delta_{\gamma}$ is G-ergodic, $W^{(\gamma,\dot{\beta},\alpha)}$ is factorial.

PROPOSITION 4-9. (1) *$W^{(\gamma,\dot{\beta},\alpha)}$ is a normal representation of G.*

(2) *The map from $\tilde{F}\ni F^{(\gamma,\dot{\beta},\alpha)}$ to "the quasi equivalence class of $W^{(\gamma,\dot{\beta},\alpha)}$" $\in\hat{G}_{\mathrm{norm}}$ is an injection.*

(3) *For $k\in C_c(G)$ the value of induced character $\bar{\tau}^{(\gamma,\dot{\beta},\alpha)}$ corresponding to $W^{(\gamma,\dot{\beta},\alpha)}$ is given by*

$$\bar{\tau}^{(\gamma,\dot{\beta},\alpha)}(k^{*}*k)=\int_{X/X_{\gamma}}\int_{\dot{Y}_{\gamma}}\left|\int_{X_{\gamma}}\int_{Y}\int_{Z}k(x+x', y', z')\chi^{\alpha}(x')\right.$$

$$\left.\cdot\chi^{\beta+\beta'}(y')\chi^{\gamma}(z')d\eta^{X_{\gamma}}(x')d\eta^{Y}(y')d\eta^{Z}(z')\right|^{2}d^{\gamma}\mu^{Y}(\beta')d\nu^{\gamma}(x),$$

where $\eta^{X_{\gamma}}$, η^{Y} and η^{Z} are Haar measures on X_{γ}, Y and Z, and ν^{γ} is the Haar measure on X/X_{γ} determined by η^{X} and $\eta^{X_{\gamma}}$.

(c) The regular representation and the Plancherel formula.

We fix Haar measures η^{X}, η^{Y} and η^{Z} of X, Y and Z respectively. Let μ^{X}, μ^{Y} and μ^{Z} be the Plancherel measures dual to η^{X}, η^{Y} and η^{Z} respectively. ϕ_{γ} denotes the trace on $C^*(Z)$ corresponding to χ^{γ}.

Let $W_{\gamma}\equiv\mathrm{Ind}_{Z\uparrow G}\chi^{\gamma}$ and $\phi_{\gamma}^{G}\equiv\mathrm{Ind}_{Z\uparrow G}\phi_{\gamma}$. Then we have

$$\lambda^G \cong \int_{\hat{Z}}^{\oplus} W_\gamma d\mu^Z(\gamma) \quad \text{and} \quad \phi^G = \int_{\hat{Z}} \phi_\gamma^G d\mu(\gamma).$$

Next we consider the decompositions of ϕ_γ^G and W_γ. Let η^{X_γ} and ${}^\gamma\mu^Y$ be the Haar measures on X_γ and $\bar{Y}_\gamma$ respectively, ν^γ and ${}^\gamma\nu^Y$ be the Haar measure on X/X_γ and $\hat{Y}/\bar{Y}_\gamma$ such that

$$\eta^X = \int_{X/X_\gamma} \dot{\varepsilon}\cdot\eta^{X_\gamma} d\nu^\gamma(\dot{\varepsilon}) \quad \text{and} \quad \mu^Y = \int_{\hat{Y}/\bar{Y}_\gamma} \dot{\beta}\cdot{}^\gamma\mu^Y d^\gamma\nu^Y(\dot{\beta}).$$

Let ϕ^α and ϕ^β denote the traces corresponding to χ^α and χ^β respectively. We have

$$W_\gamma \cong \int_{\hat{X}_\gamma\times\hat{Y}}^{\oplus} \mathrm{Ind}_{G_\gamma\uparrow G}(\chi^\alpha\times\chi^\beta\times\chi^\gamma) d\mu^{X_\gamma}(\alpha) d\mu^Y(\beta)$$

and

$$\phi^G = \int_{\hat{X}_\gamma\times\hat{Y}} \mathrm{Ind}_{G_\gamma\uparrow G}(\phi^\alpha\otimes\phi^\beta\otimes\phi^\gamma) d\mu^{X_\gamma}(\alpha) d\mu^Y(\beta).$$

Let

$$V^{(\gamma,\dot{\beta},\alpha)} \equiv \int_{\beta\cdot\bar{Y}_\gamma}^{\oplus} (\chi^\alpha\times\chi^{\beta'}\times\chi^\gamma) d(\beta\cdot{}^\gamma\mu^Y)(\beta')$$

and

$$\tau^{(\gamma,\dot{\beta},\alpha)} \equiv \int_{\beta\cdot\bar{Y}_\gamma} (\phi^\alpha\otimes\phi^{\beta'}\otimes\phi^\gamma) d(\beta\cdot{}^\gamma\mu^Y)(\beta').$$

$\tau^{(\gamma,\dot{\beta},\alpha)}$ is the G invariant lower semicontinuous trace on $C^*(G_\gamma)$ corresponding to $V^{(\gamma,\dot{\beta},\alpha)}$. We have

$$\mathrm{Ind}_{Z\uparrow G_\gamma}\chi^\gamma \cong \int_{\hat{X}_\gamma\times\hat{Y}/\bar{Y}_\gamma}^{\oplus} V^{(\gamma,\dot{\beta},\alpha)} d\mu^{X_\gamma}(\alpha) d^\gamma\nu^Y(\beta),$$

and

$$\mathrm{Ind}_{Z\uparrow G_\gamma}\phi_\gamma = \int_{\hat{X}_\gamma\times\hat{Y}/\bar{Y}_\gamma} \tau^{(\gamma,\dot{\beta},\alpha)} d\mu^{X_\gamma}(\alpha) d^\gamma\nu^Y(\beta).$$

Let $W^{(\gamma,\dot{\beta},\alpha)} \equiv \mathrm{Ind}_{G_\gamma\uparrow G} V^{(\gamma,\dot{\beta},\alpha)}$, $\bar{\tau}^{(\gamma,\dot{\beta},\alpha)} \equiv \mathrm{Ind}_{G_\gamma\uparrow G}\tau^{(\gamma,\dot{\beta},\alpha)}$. Then we have

$$W_\gamma \cong \int_{\hat{X}_\gamma\times\hat{Y}/\bar{Y}_\gamma}^{\oplus} W^{(\gamma,\dot{\beta},\alpha)} d\mu^{X_\gamma}(\alpha) d^\gamma\nu^Y(\dot{\beta})$$

and

$$\phi^G = \int_{\hat{X}_\gamma\times\hat{Y}/\bar{Y}_\gamma} \bar{\tau}^{(\gamma,\dot{\beta},\alpha)} d\mu^{X_\gamma}(\alpha) d^\gamma\nu^Y(\dot{\beta}).$$

THEOREM 4-10. *Let $G = X\times_s(Y\times Z)$ be a Heisenberg type group.*

(1) $\lambda^G \cong \int_{\hat{Z}}\int_{\hat{X}_\gamma\times\hat{Y}/\bar{Y}_\gamma}^{\oplus} W^{(\gamma,\dot{\beta},\alpha)} d\mu^{X_\gamma}(\alpha) d^\gamma\nu^Y(\dot{\beta})(m)$ (cf. 3-1).

This is central.

(2) $\phi^G = \int_{\hat{Z}}\int_{\hat{X}_\gamma\times\hat{Y}/\bar{Y}_\gamma} \bar{\tau}^{(\gamma,\dot{\beta},\alpha)} d\mu^{X_\gamma}(\alpha) d^\gamma\nu^Y(\dot{\beta})$.

This is a Plancherel formula for G.

(3) *Let* $\bar{\bar{\tau}}^{(\gamma,\dot{\beta},\alpha)}$ *denotes the f.n.s. trace on* $R(W^{(\gamma,\dot{\beta},\alpha)})$ *corresponding to* $\bar{\tau}^{(\gamma,\dot{\beta},\alpha)}$. *Then for* $k\in(L_1(G)\cap L_2(G))^2$ *we have*

$$k(x,\,y,\,z)=\int_{\hat{Z}}\int_{\hat{X}_\gamma\times\hat{Y}/\hat{Y}_\gamma}\bar{\bar{\tau}}^{(\gamma,\dot{\beta},\alpha)}(W^{(\gamma,\dot{\beta},\alpha)}(k)W^{(\gamma,\dot{\beta},\alpha)}(x,\,y,\,z)^*)$$

$$\cdot\, d\mu^{X_\gamma}(\alpha)d^\gamma\nu^Y(\dot{\beta})d\mu^Z(\gamma)\qquad (\textit{Inversion formula})\ .$$

REMARK 4-11. The argument used here is topological and the ambiguity of null sets doesn't occur.

§5. Plancherel formula for group extensions.

We describe the central decomposition of the left regular representation of an s.l.c.g. G and the Plancherel formula (or measure) of it by assuming the existence of a suitable normal subgroup N.

Kleppner and Lipsman ([**12**], [**13**]) assumed the smoothness of $(G, \hat{N})$ and the isotropic type I'ness, but made no assumption about the constancy of stabilizer groups. In contrast with their assumptions we put neither the assumption of the smoothness nor the isotropic type I'ness, but assume the local constancy of stabilizers.

5-1. Good normal subgroup case.

Let G and N be s.l.c.g.'s and $N\lhd G$. Let μ^N be a Plancherel measure of N.

DEFINITION 5-1. We say that the pair (G, N) has the property (A) if,

(1) The regular representation of N is of type I.

(2) We can take off a G invariant Borel μ^N null set C from $\hat{N}$ such that, for all $\pi\in\hat{N}\setminus C$, G_π is closed.

(3) We can take off a G invariant Borel μ^N null set C' from $\hat{N}$ such that, for all $\pi\in\hat{N}\setminus C'$, the transitive quasi orbit containing π is canonical (i.e. it corresponds to some central decomposition).

LEMMA 5-2. *Let* (G, N) *satisfy* (A) *and* $\pi\in\hat{N}$ *satisfy the conditions* (2) *and* (3) *of* (A). *Suppose that* U *is a factor representation of* G_π *such that* $U|_N\cong\pi\otimes I$. *Then* $W\equiv\mathrm{Ind}_{G_\pi\uparrow G}U$ *is factorial and* $R(W)'$ *and* $R(U)'$ *are algebraically isomorphic.*

PROOF. The condition (3) shows that the diagonal algebra of the decomposition of $W|_N$ in Mackey's subgroup theorem is contained in $R(W)$. q.e.d.

REMARK 5-3. When N is unimodular (2) and (3) are conclusion from (1). When N is of type I, (G, N) satisfies the property (A).

DEFINITION 5-4. Let $\mu^N=\int_X\mu_t^N d\nu(t)$ be a G ergodic decomposition. (G, N) is said to have the property (B) if, for ν a.a. t, there exists a G invariant Borel subset E_t in $\hat{N}$ which supports μ_t^N and the stabilizers on each point of E_t are equal to some closed subgroup G_t.

G acts on $C^*(N)$ and $R(\lambda^N)$. $\{\alpha_g : g\in G\}$ and $\{\tilde{\alpha}_g : g\in G\}$ denote these actions respectively. $\{\tilde{\alpha}_g : g\in G\}$ is implemented by a strongly continuous unitary representation U of G.

Let $\lambda^N \cong \int_{\hat{N}}^{\oplus} (\lambda_\gamma^N \otimes I_\gamma) d\mu^N(\gamma)$ be central and $\phi^N = \int_{\hat{N}} \phi_\gamma^N d\mu^N(\gamma)$ be the Plancherel formula for N. The action of G on $ZR(\lambda^N)$ (the centre of $R(\lambda^N)$) is equivariant to the action of G on $\hat{N}$. $ZR(\lambda^N)^G$ denotes the set of all G invariant elements. We decompose λ^N and ϕ^N using $ZR(\lambda^N)^G$. We have

$$\lambda^N \cong \int_X^{\oplus} \lambda_t^N d\nu^N(t) \quad \text{and} \quad \phi^N = \int_X \phi_t^N d\nu^N(t) \quad \text{and for } \nu^N \text{ a.a. } t\,,$$

$$\lambda_t^N = \int_{\hat{N}}^{\oplus} \lambda_\gamma^N d\mu_t^N(\gamma) \quad \text{and} \quad \phi_t^N = \int_{\hat{N}} \phi_\gamma^N d\mu_t^N(\gamma)\,.$$

By Proposition 2-7, for ν^N a. a. t, ϕ_t^N is Δ_N-relatively invariant under the action of G and is a densely defined semitrace corresponding to λ_t^N.

By using the decomposition of semitraces we can show the unimodularity of little groups in some special situations.

PROPOSITION 5-5. Suppose that G is unimodular, (G, N) has the properties (A) and (B). Then N and G_t's are all unimodular, and for ν^N a. a. t G_t/N is unimodular.

PROOF. Since $N \lhd G$ and $G_t \lhd G$, they are unimodular (cf. § 1 of [**23**]). Let ϕ_t^N be a Δ_N-relatively invariant component of ϕ^N and $\tilde{\phi}_t^N$ be the f. n. s. trace on $R(\lambda_t^N)$ corresponding to ϕ_t^N. We can assume that the corresponding representation λ_t^N is of type I. Then $\tilde{\phi}_t^N$ is a direct integral of canonical traces on type I factors. Therefore the action of G_t (the restriction of the action of G) on each component is necessarily inner and keep the values of traces invariant. From this we have

$$\tilde{\phi}_t^N(T_t) = \tilde{\phi}_t^N(\tilde{\alpha}_g(T_t)) = \Delta_N(g)\tilde{\phi}^N(T_t) \qquad \text{for} \quad \forall T_t \in R(\lambda_t^N)^+ \quad \text{and} \quad \forall g \in G\,.$$

Therefore we have $\Delta_N(g) \equiv 1$ for every $g \in G_t$. Since N and G_t are unimodular this shows that G_t/N is also unimodular for t which satisfies all the good properties. q. e. d.

REMARK 5-6. When N is regularly embedded in G, this proposition follows from Lemma 2-2 of [**13**].

We consider only t which has good properties. Let

$$W_t \equiv \mathrm{Ind}_{N\uparrow G}\, \lambda_t^N\,, \quad \phi_t^G \equiv \mathrm{Ind}_{N\uparrow G}\, \phi_t^N\,.$$

We have $\lambda^G \cong \int_X^{\oplus} (W_t \otimes I_t) d\nu^N(t)$ where I_t is the identity operator on the representation space of W_t, and $\phi^G = \int_X \phi_t^G d\nu^N(t)$. We consider the decomposition of W_t's and ϕ_t^G's. Let $V_t \equiv \mathrm{Ind}_{N\uparrow G_t}\lambda_t^N$ and $\bar{\phi}_t \equiv \mathrm{Ind}_{N\uparrow G_t}\phi_t^N$. Then $\bar{\phi}_t$ is a Δ_{G_t}-relatively invariant semitrace on $C^*(G_t)$ and V_t corresponds to $\bar{\phi}_t$. We consider the central decomposition of V_t's and $\bar{\phi}_t$'s. By the definition of λ_t^N we have $V_t \cong \int_{\hat{N}}^{\oplus}(\mathrm{Ind}_{N\uparrow G_t}\lambda_\gamma^N)d\mu_t^N(\gamma)$. Since G_t fixes each element of the diagonal algebra of the decomposition of λ_t^N, by Lemma 2-10, the diagonal algebra of the decomposition of V_t is contained in $R(V_t)$. From this fact we can write $\bar{\phi}_t = \int_{\hat{N}}(\mathrm{Ind}_{N\uparrow G_t}\phi_\gamma^N)d\mu_t^N(\gamma)$.

Let $\mathrm{Ind}_{N\uparrow G_t}\lambda_\gamma^N = \int_{\widehat{G}_t}^{\oplus} V_{\gamma,\delta}d\bar{\mu}_\gamma(\delta)$ be central and $\mathrm{Ind}_{N\uparrow G_t}\phi_\gamma^N = \int_{\widehat{G}_t}\bar{\phi}_{\gamma,\delta}d\bar{\mu}_\gamma(\delta)$ be the corresponding decomposition. $\bar{\phi}_t = \int_{\hat{N}}\int_{\widehat{G}_t}\bar{\phi}_{\gamma,\delta}d\bar{\mu}_\gamma(\delta)d\mu_t^N(\gamma)$ is the decomposition of $\bar{\phi}_t$ into semicharacters. Since each λ_γ^N is an irreducible representation of N, we can construct an $\alpha(\gamma)$-representation $\bar{\lambda}_\gamma^N$ of G_t such that $\bar{\lambda}_\gamma^N|_N = \lambda_\gamma^N$, where $\alpha(\gamma)$ is an N invariant Mackey obstruction cocycle at λ_γ^N. $\lambda^{\overline{\alpha(\gamma)}}$ denotes the $\overline{\alpha(\gamma)}$-regular representation of G_t/N. Let Δ be the quasi equivalence classes of $\overline{\alpha(\gamma)}$-factor representations of G_t/N. Let $\lambda^{\overline{\alpha(\gamma)}} \cong \int_{\Delta}\sigma_{\gamma,\varepsilon}d\mu^{\overline{\alpha(\gamma)}}(\varepsilon)$ be central. Then we have $V_\gamma \cong \lambda_\gamma^N \otimes \lambda^{\overline{\alpha(\gamma)}} \cong \int_{\Delta}^{\oplus}(\lambda_\gamma^N \otimes \hat{\sigma}_{\gamma,\varepsilon})d\mu^{\overline{\alpha(\gamma)}}(\varepsilon)$, where $\mu^{\overline{\alpha(\gamma)}}$ is an $\overline{\alpha(\gamma)}$-Plancherel measure of G_t/N.

LEMMA 5-7. *$V_\gamma \cong \int_{\Delta}^{\oplus}(\lambda_\gamma^N \otimes \hat{\sigma}_{\gamma,\varepsilon})d\mu^{\overline{\alpha(\gamma)}}(\varepsilon)$ is central.*

PROOF. Canonical argument shows that $R(V)'$ is spatially isomorphic to $I \otimes R(\hat{\lambda}^{\overline{\alpha(\gamma)}})'$. Therefore we have

$$Z(R(V_\gamma)) = Z(R(V_\gamma))' \cong Z(I \otimes R(\hat{\lambda}^{\overline{\alpha(\gamma)}})') = I \otimes Z(R(\hat{\lambda}^{\overline{\alpha(\gamma)}}))\,. \qquad \text{q. e. d.}$$

We summarize these arguments.

LEMMA 5-8. (1) *$V_t \cong \int_{\hat{N}}\int_{\Delta}^{\oplus}(\lambda_\gamma^N \otimes \hat{\sigma}_{\gamma,\varepsilon})d\mu^{\overline{\alpha(\gamma)}}(\varepsilon)d\mu_t^N(\gamma)(m)$ (cf. 3-1), and this is central.*

(2) *$\bar{\phi}_t = \int_{\hat{N}}\int_{\widehat{G}_t}\bar{\phi}_{\gamma,\delta}d\bar{\mu}_\gamma(\delta)d\mu_t^N(\gamma)$ and this is the decomposition into semicharacters.*

(3) *$d\mu^{\overline{\alpha(\gamma)}}d\mu_t^N$ and $d\mu_\gamma d\mu_t^N$ describe the canonical measure $\bar{\mu}_t$ of V_t and $\bar{\phi}_t$ on $\widehat{G}_t$.*

Since G keeps V_t invariant, G keeps $\bar{\mu}_t$ quasi invariant. Then we consider the ergodic decomposition of standard measure $\bar{\mu}_t$ under the action of G $\left(\text{i.e. } \bar{\mu}_t = \int_Y \bar{\mu}_{t,\tau}d\bar{\nu}^t(\tau)\right)$. The decomposition of V_t and $\bar{\phi}_t$ by the fixed point subalgebra $Z(R(V_t))^G$ are

$$V_t \cong \int_Y^{\oplus} V_{t,\tau}d\bar{\nu}^t(\tau) \quad \text{and} \quad \bar{\phi}_t = \int_Y \bar{\phi}_{t,\tau}d\bar{\nu}^t(\tau)\,, \qquad \text{where for } \bar{\nu}^t \text{ a. a. } \tau$$

$$V_{t,\tau}\cong\int_{\hat{G}_t}^{\oplus}V_{t,\varepsilon}d\bar{\mu}_{t,\tau}(\varepsilon) \quad \text{and} \quad \bar{\phi}_{t,\tau}=\int_{\hat{G}_t}\bar{\phi}_{t,\varepsilon}d\bar{\mu}_{t,\tau}(\varepsilon).$$

Let $W_{t,\tau}\equiv\mathrm{Ind}_{G_t\uparrow G}V_{t,\tau}$. Then we have $W_t\cong\int_Y^{\oplus}W_{t,\tau}d\bar{\nu}^t(\tau)$. By Lemma 2-10, the diagonal algebra is contained in $R(W_t)$. Therefore we have $\phi_t^G=\int_Y\phi_{t,\tau}^G d\bar{\nu}^t(\tau)$ where, for $\bar{\nu}^t$ a.a. τ, $\phi_{t,\tau}^G=\mathrm{Ind}_{G_t\uparrow G}\bar{\phi}_{t,\tau}$.

PROPOSITION 5-9. *The diagonal algebra of* $W_t\cong\int_Y^{\oplus}W_{t,\tau}d\bar{\nu}^t(\tau)$ *is equal to the centre of* $R(W_t)$. *And* $\phi_t^G=\int_Y\phi_{t,\tau}^G d\bar{\nu}^t(\tau)$ *is a decomposition into semicharacters.*

PROOF. It is sufficient to show that for $\bar{\nu}^t$ a.a. τ $W_{t,\tau}$ is factorial. We have $W_{t,\tau}\cong\int_{\hat{G}_t}^{\oplus}(\mathrm{Ind}_{G_t\uparrow G}V_{t,\varepsilon})d\mu_{t,\tau}(\varepsilon)$. We show that for $\bar{\nu}^t$ a.a. τ for $\bar{\mu}_{t,\tau}$ a.a. $(\mathrm{Ind}_{G_t\uparrow G}V_{t,\varepsilon})$'s are factorial. A good $V_{t,\varepsilon}$ has the form $\bar{\lambda}_\gamma^N\otimes\hat{\sigma}_{\gamma,\varepsilon}$. Then $V_{t,\varepsilon}$ is a factor representation such that $V_{t,\varepsilon}|_N=\lambda_\gamma^N\otimes I$. Since N has the property (A) of Definition 5-1 and since we can assume that λ_γ^N satisfies the property (2) and (3) of Definition 5-1, by Lemma 5-2, $\mathrm{Ind}_{G_t\uparrow G}V_{t,\varepsilon}$ is a factor representation whose commuting algebra is algebraically isomorphic to $R(V_{t,\varepsilon})'$. Since we can assume that $V_{t,g\cdot\varepsilon}$ is quasi equivalent ($\sim$) to $g\cdot V_{t,\varepsilon}$, we have

$$\mathrm{Ind}_{G_t\uparrow G}V_{t,g\cdot\varepsilon}\sim(\mathrm{Ind}_{G_t\uparrow G}(g\cdot V_{t,\varepsilon})\cong g\cdot\mathrm{Ind}_{G_t\uparrow G}V_{t,\varepsilon}\cong\mathrm{Ind}_{G_t\uparrow G}V_{t,\varepsilon}.$$

Further $\infty\cdot W_{t,\tau}\cong\int_{\hat{G}_t}^{\oplus}\infty\cdot(\mathrm{Ind}_{G_t\uparrow G}V_{t,\varepsilon})d\bar{\mu}_{t,\tau}(\varepsilon)$, and if $\varepsilon'=g\cdot\varepsilon$ for some $g\in G$ we have that $\infty\cdot\mathrm{Ind}_{G_t\uparrow G}V_{t,\varepsilon}$ is unitarily equivalent to $\infty\cdot\mathrm{Ind}_{G_t\uparrow G}V_{t,\varepsilon'}$. Since $\bar{\mu}_{t,\tau}$ is G ergodic, by using the argument of Proposition 1 in [**5**], $\infty\cdot W_{t,\tau}$ is factorial and so $W_{t,\tau}$ is factorial. q.e.d.

As a conclusion we have the following main theorem.

THEOREM 5-10 (Plancherel formula for group extensions).
Let (G, N) *satisfy* (A) *and* (B).

(1) $\lambda^G\cong\int_X\int_Y^{\oplus}W_{t,\tau}d\bar{\nu}^t(\tau)d\nu^N(t)(m)$ (cf. 3-1).

And this is central.

(2) $\phi^G=\int_X\int_Y\phi_{t,\tau}^G d\bar{\nu}^t(\tau)d\nu^N(t)$.

And this is a Plancherel formula of G.

(3) $d\bar{\nu}^t d\nu^N$ *is a Plancherel measure of G in some sense.*

REMARK 5-11. (1) The measure space (X, ν^N) and $(Y, \bar{\nu}^t)$, the representations $(W_{t,\tau})$'s and the semicharacters $(\phi_{t,\tau}^G)$'s are determined by the informations about the Plancherel objects of N, stabilizers and little groups and the action of G on them.

(2) When N is abelian, and if there exists a continuous cross section from G_t/N to G_t and the Mackey obstruction cocycle can be taken continuously as in

the case of Dixmier group, we can describe $\bar{\phi}_{\gamma,\delta}$'s in terms of G_t/N, $\alpha(\gamma)$ and N.

5-2. Bad normal subgroup case.

When N doesn't have type I regular representation, we cannot say anything about "the induction" of the Plancherel theory in general. Therefore we will add several assumptions on N and (G, N). Let μ^N be a Plancherel measure of N and $\mu^N=\int_X \mu_t^N d\nu^N(t)$ be an ergodic decomposition under the action of G.

DEFINITION 5-12. (G, N) satisfies the property (C) if

(1) For μ^N a.a. $\pi\in\hat{N}$, G_π is closed.

(2) For μ^N a.a. $\pi\in\hat{N}$, the transitive quasi orbit containing π is canonical.

(3) (G, N) has the property (B).

(4) For μ^N a.a. $\pi\in\hat{N}$, the canonical measure of $U_\pi\equiv\mathrm{Ind}_{N\uparrow G_\pi}\pi$ is atomic.

REMARK 5-13. (1) When μ^N concentrates on $\widehat{N}_{\mathrm{norm}}$ and $\widehat{N}_{\mathrm{norm}}\cong\mathrm{Prim}(N)$, then we have (1) and (2).

(2) When for μ^N a.a. π $G_\pi=N$, then (1) and (4) are automatically satisfied.

Let $U_\pi\cong\sum_\varepsilon^\oplus U_{\pi,\varepsilon}$ be central.

LEMMA 5-14. *Let* $W_{\pi,\varepsilon}\equiv\mathrm{Ind}_{G_\pi\uparrow G}U_{\pi,\varepsilon}$. *Then* $W_{\pi,\varepsilon}$ *is a factor representation of* G *such that* $R(W_{\pi,\varepsilon})'$ *is algebraically isomorphic to* $R(U_{\pi,\varepsilon})'$.

PROOF. By the subgroup theorem, $U_\pi|_N\cong\pi\otimes I$. Since $U_{\pi,\varepsilon}$'s are subrepresentations of U_π, $U_{\pi,\varepsilon}|_N$'s are also subrepresentations of $U_\pi|_N\cong\pi\otimes I$. Since $\pi\otimes I$ is factorial, $U_{\pi,\varepsilon}|_N$'s are quasi equivalent to π. By the subgroup theorem, $(\mathrm{Ind}_{G_\pi\uparrow G}\ U_{\pi,\varepsilon})|_N=\int_{G/G_\pi}^\oplus s\cdot(U_{\pi,\varepsilon}|_N\otimes I)d\dot{s}$ and this is central by (2) of Definition 5-12. This shows that $R(W_{\pi,\varepsilon})'\cong R(U_{\pi,\varepsilon})'$ and $W_{\pi,\varepsilon}$ is factorial. q.e.d.

Let U_t, $\bar{\phi}_t$ be the same as in the good normal subgroup case. Then $U_t\cong\int_{\hat{N}}^\oplus\sum_\varepsilon U_{\pi,\varepsilon}d\mu_t^N(\pi)(m)$ is central and $\bar{\phi}_t=\int_{\hat{N}}\sum_\varepsilon\phi_{\pi,\varepsilon}d\mu_t^N(\pi)$ is the decomposition into semicharacters. Let $\sum_\varepsilon\mu_t^N=\int_Y\bar{\mu}_\delta^t d\bar{\nu}^t(\delta)$ be an ergodic decomposition under G. Let

$$W_t\equiv\mathrm{Ind}_{G_t\uparrow G}U_t\,,\quad \phi_t^G\equiv\mathrm{Ind}_{G_t\uparrow G}\phi_t\,,\quad W_\delta\equiv\int_{\widehat{G}_t}^\oplus\mathrm{Ind}_{G_t\uparrow G}U_{\pi,\varepsilon}d\bar{\mu}_\delta^t((\pi,\varepsilon))$$

and

$$\phi_\delta^G\equiv\int_{\widehat{G}_t}\mathrm{Ind}_{G_t\uparrow G}\phi_{\pi,\varepsilon}d\bar{\mu}_\delta^t((\pi,\varepsilon))\qquad(\text{for good }\delta).$$

In conclusion, we have the following theorem.

THEOREM 5-15. *Suppose that* (G, N) *has the property* (C).

(1) $\lambda^G\cong\int_X\int_Y^\oplus W_\delta d\bar{\nu}^t(\delta)d\nu^N(t)(m)$ (cf. 3-1).

And this is central.

(2) $\phi^G=\int_X\int_Y \phi_\delta^G d\bar{\nu}^t(\delta)d\nu^N(t)$.

And this is a Plancherel formula for G.

We present some examples in which we have the property (C).

EXAMPLE 1. Let G be an almost connected Lie group such that G/G_0 (G_0 is the connected component of G) is abelian. We consider G_0 as N. We show that this (G, N) has the property (C). By Theorem 1 of Pukanszky [18], μ^N is concentrated in $\widehat{N}_{\mathrm{norm}}$ and $\widehat{N}_{\mathrm{norm}}\cong\mathrm{Prim}(N)$. Therefore (G, N) satisfies (1) and (2) of Definition 5-12. Since G/N is finite, $(G, \widehat{N}_{\mathrm{norm}})$ is smooth and since G/N is abelian, (G, N) satisfies (3).

Let (M, G, θ, β) be a twisted W^*-covariant system (M is a von Neumann algebra, G is a locally compact group, β is an M^U valued cocycle and θ is a β action of G on M, and they satisfy the axiom of p. 168 in [20]). Suppose that M is a factor and G is a discrete abelian group. Let (U_θ, π_θ) be the canonical covariant representation of (M, G, θ, β) on $H_M\otimes L^2(G)$ which gives rise to $L_{\theta,\beta}(G, M)$ (twisted W^*-crossed product).

LEMMA 5-16. *In the above situation, there exists a subgroup H of G and a map W from H to M^U such that we have $W_gW_{g'}=W_{gg'}\cdot\beta(g, g')$ for every g and g' in H and T is contained in $Z(L_{\theta,\beta}(G, M))$ if and only if T is of the form $\sum_{g\in H} c_gU_\theta(g)\pi_\theta(W_g)^*$, where c_g's are scalars.*

PROOF. Each element of $L_{\theta,\beta}(G, M)$ is of the form $T=\sum_{g'\in G} U_\theta(g')\pi_\theta(A_{g'})$. T is in $Z(L_{\theta,\beta}(G, M))$ if and only if $U_\theta(g)T=TU_\theta(g)$ for every $g\in G$ and $\pi_\theta(B)T=T\pi_\theta(B)$ for every $B\in M$. From the first equality we have

$$\sum_{g'\in G} U_\theta(g)U_\theta(g')\pi_\theta(A_{g'})=\sum_{g'\in G} U_\theta(g')\pi_\theta(A_{g'})U_\theta(g).$$

And using the properties of U_θ and π_θ, we have

$$\sum_{g'\in G} U_\theta(gg')\pi_\theta(\beta(g, g')A_{g'})=\sum_{g'\in G} U_\theta(gg')\pi_\theta(\beta(g', g)\theta_g^{-1}(A_{g'})).$$

From this we have

(i) $\beta(g, g')A_{g'}=\beta(g', g)\theta_g^{-1}(A_{g'})$ for every g and g' in G

From the second equality we have

$$\begin{aligned}\sum_{g'\in G} U_\theta(g')\pi_\theta(A_{g'}B)&=\sum_{g'\in G} \pi_\theta(B)U_\theta(g')\pi_\theta(A_{g'})\\&=\sum_{g'\in G} U_\theta(g')\pi_\theta(\theta_{g'}^{-1}(B)A_{g'}).\end{aligned}$$

This shows that

(ii) $A_{g'}B=\theta_{g'}^{-1}(B)A_{g'}$ for every B in M.

If $\theta_{g'}^{-1}$ is outer, by Corollary 1.2 in [10], $A_{g'}=0$. We can assume that $\theta_{g'}^{-1}$ is inner. This shows that $U_\theta(g')$ is in $\pi_\theta(M)$. Then we can put $W_{g'} \equiv \pi_\theta^{-1}(U_\theta(g'))$. We have

(iii) $\theta_{g'}^{-1}(B) = W_{g'}^* B W_{g'}$ for every B in M.

Let $H \equiv \{g \in G : \theta_g \text{ is inner}\}$. Then H is clearly a subgroup of G. If g and g' are in H, we have $W_g W_{g'} = W_{gg'} \beta(g, g')$ because π_θ is an isomorphism. If we put (iii) into (ii),

$$A_{g'} B = W_{g'}^* B W_{g'} A_{g'} \quad \text{and so} \quad (W_{g'} A_{g'}) B = B (W_{g'} A_{g'}) \quad \text{for every } B \in M.$$

$W_{g'} A_{g'}$ is contained in $Z(M)$. Since M is a factor, there exists a scalar $c_{g'}$ such that $W_{g'} A_{g'} = c_{g'} I$. Then $A_{g'} = c_{g'} W_{g'}^*$.

Conversely, such an $A_{g'}$ satisfies (i). In fact we have $U_\theta(gg')^* = \pi_\theta(\beta(g, g'))$ $U_\theta(g')^* U_\theta(g)^*$ and $U_\theta(g'g)^* = \pi_\theta(\beta(g', g)) U_\theta(g)^* U_\theta(g')^*$ for every g and g' in G. From this $\pi_\theta(\beta(g, g')) U_\theta(g')^* = \pi_\theta(\beta(g', g)) U_\theta(g)^* U_\theta(g')^* U_\theta(g)$. Then we have $\pi_\theta(\beta(g, g') W_{g'}) = \pi_\theta(\beta(g', g) \theta_g^{-1}(W_{g'})^*)$ for every g in G and g' in H. Since π_θ is injective, $\beta(g, g') W_{g'} = \beta(g', g) \theta_g^{-1}(W_{g'})$ holds. q.e.d.

COROLLARY 5-17. *When G is moreover finite, the dimension of $Z(L_{\theta,\beta}(G, M))$ is finite and not greater than the order of G.*

This corollary shows that (G, N) satisfies (4).

EXAMPLE 2. We consider another case in which G acts on $\widehat{N}_{\text{norm}}$ freely. Let $[X_1, \cdots, X_5]$ denotes a basis of a nilpotent Lie algebra $\mathfrak{g}$. We write only non zero brackets.

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \quad [X_1, X_4] = X_5, \quad [X_2, X_3] = X_5.$$

Let $\mathfrak{N} = \sum_{i=2}^{5} \boldsymbol{R} X_i$. Then $\mathfrak{N}$ is an ideal of $\mathfrak{g}$. Let $\tilde{G}$ be the simply connected Lie group corresponding to $\mathfrak{g}$, and $\tilde{N}$ be the connected subgroup determined by $\mathfrak{N}$. G is diffeomorphic to $\boldsymbol{R}^5$ by using the 2'nd canonical system of coordinates with respect to $\{X_1, \cdots, X_5\}$. Let $G \equiv \{(12n_1, n_2, n_3, n_4, n_5) : n_i \in \boldsymbol{Z}\}$ and $N \equiv \{(0, n_2, n_3, n_4, n_5) : n_i \in \boldsymbol{Z}\}$. Then G and N are discrete subgroups of G and $N \lhd G$. N is a direct product of the discrete Heisenberg group and $\boldsymbol{Z}$ (integers), and so N is not of type I and moreover does not have type I regular representation.

Since G is unimodular, the Plancherel measure class of N is supported by $\widehat{N}_{\text{norm}}$. We can compute $\widehat{N}_{\text{norm}}$ and $\text{Prim}(N)$, and this calculation shows that $\widehat{N}_{\text{norm}} \cong \text{Prim}(N)$. The Plancherel measure class is supported by the set of quasi equivalence classes of infinite dimensional normal representations, where the action of G is free. Therefore we conclude that this (G, N) has the property (C).

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