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The axiom of n-planes and convexity in Riemannian manifolds

Dedicated to Professor I. Mogi on his 60th birthday

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1. Introduction.

A characterization of a space of constant curvature is an interesting problem in Riemannian geometry. It has been done by the various methods as seen in [2], [4] and [10]. In particular we are interested in the axiom of n-planes which is stated as follows; a Riemannian manifold M of dimension $m \geqq 3$ is said to satisfy the axiom of n-planes if for each p in M and any n-dimensional subspace T'_p of the tangent space T_pM, there is an n-dimensional totally geodesic submanifold N containing p such that the tangent space of N at p is T'_p, where n is a fixed integer $2 \leqq n < m$. É. Cartan [4] proved that if M satisfies the axiom of n-planes for some n, then M is a space of constant curvature.

Historically, E. Beltrami [1] proved that a space of constant curvature M satisfies the axiom of 2-planes, and the converse was proved by F. Schur [12]. É. Cartan also indicated in [3] that Schur's theorem had been proved by L. Schlaefli [11] in combination with F. Klein [9].

The purpose of the present paper is to exhibit this axiom in terms of convex analysis, i.e., convex combinations and convex hulls.

Let M be a Riemannian manifold without boundary. For a point p in M let $B_r(p)$ denote "the strongly convex (open) ball" with center p and radius r, i.e., every ball which is contained in $B_r(p)$ is convex where the term, *convex*, is used in the following sense. A set $D \subset M$ is convex iff x, y in D implies that there is a unique (distance minimizing geodesic) segment $T(x, y)$ and it is contained in D. From [6] and [7] we know that for each p in M there is an $r>0$ such that $B_r(p)$ is strongly convex. Since the constancy of curvature is a local property, we may direct our attention to the interior of a strongly convex ball.

If U is a subset of $B_r(p)$, then we consider the smallest convex set which contains U. We call it the *convex hull* of U and denote it by HU. Clearly $HU \subset B_r(p)$.

For a set U in M, CU is by definition the set of all points each of which belongs to some segment which joins two points of U, and we put $C^kU:=C(C^{k-1}U)$ inductively, $k=1, 2, 3, \cdots$, $C^0U:=U$. Clearly $HU=\bigcup_{k=0}^{\infty}C^kU$ holds for any $U\subset B_r(p)$. We may think that C^k corresponds to convex combinations in the linear space.

It is the nature of a space of constant curvature that the convex hull of sufficiently close $n+1$ points $x_0, x_1, \cdots, x_n$ can be obtained by $C^k\{x_0, x_1, \cdots, x_n\}$, where the integer k satisfies $2^{k-1}\leqq n<2^k$. And if M satisfies the axiom of n-planes with $2\leqq n<\dim M$, then the set of $n+1$ points $x_0, x_1, \cdots, x_n$, which are sufficiently close to each other, has the property that $C^k\{x_0, x_1, \cdots, x_n\}=H\{x_0, x_1, \cdots, x_n\}$, where the integer k satisfies $2^{k-1}\leqq n<2^k$.

However it is not easy to verify the converse. This is because $C^k\{\ \}$ does not in general carry the structure of a smooth submanifold, and because the dimension of $H\{\ \}$ is in general greater than n.

Thus our main result is

THEOREM 1. *Let* $\dim M$ *be greater than* 3. *If for each point* p *in* M *there exists a convex neighborhood* V *of* p *in* M *such that* $H\{x_0, x_1, x_2, x_3\}=C^2\{x_0, x_1, x_2, x_3\}$ *for any points* x_0, x_1, x_2, x_3 *in* V, *then* M *is a space of constant curvature.*

The author does not know whether the above theorem for convex combinations of three points is true. On this problem the following holds.

THEOREM 2. *Let* $\dim M$ *be greater than* 2. *If for each point* p *in* M *there exists a convex neighborhood* V *of* p *in* M *such that* $H\{x_0, x_1, x_2\}=C^2\{x_0, x_1, m(x_1, x_2)\}\cup C^2\{x_0, x_2, m(x_1, x_2)\}$ *for any points* x_0, x_1, x_2 *in* V, *where* $m(x_1, x_2)$ *is the midpoint of the segment* $T(x_1, x_2)$ *which joins* x_1 *and* x_2, *then* M *is a space of constant curvature.*

In the proofs of our theorems we shall need to estimate the dimensions (defined in [8] p. 24) of convex hulls. For this purpose we will often use the a-measure $m_a(X)$, $0\leqq a<\infty$, of a (separable) metric space X which is defined in [8] p. 102 as follows. Given $\varepsilon>0$, let $m_a^{\varepsilon}(X):=\inf\sum_{i=1}^{\infty}[\delta(A_i)]^a$, where $X=\bigcup_{i=1}^{\infty}A_i$ is any decomposition of X in a countable number of subsets such that for every i the diameter $\delta(A_i)$ of A_i is less than ε, and the superscript a denotes the exponentiation. Let $m_a(X):=\sup_{\varepsilon>0}m_a^{\varepsilon}(X)$.

Concerning this measure it is well known ([8] p. 104) that if X is a metric space such that $m_{n+1}(X)=0$, $0\leqq n<\infty$, then $\dim X\leqq n$, and this fact is used in the proof of Lemma 2 in § 2.

In § 2 we shall give lemmas which are used in the proofs of our theorems and we will prove theorems in § 3. In § 4 we give remarks of the theorems.

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2. Lemmas.

In [5] Cheeger-Gromoll showed that if S is a connected locally convex set in M, then there is a smooth totally geodesic imbedded submanifold N of M such that $N\subset S\subset \bar{N}$, where $\bar{N}$ is the closure of N.

This fact and the axiom of 2-planes furnish the following.

LEMMA 1. *Let $m:=\dim M$ be greater than 2. If for each point p in M and for some n, $2\leqq n<m$, there is a convex neighborhood V of p in M such that $\dim H\{x_0, x_1, \cdots, x_n\}\leqq n$ for any points $x_0, x_1, \cdots, x_n$ in V, then M is a space of constant curvature.*

PROOF. We first claim that $\dim H\{x_0, x_1, \cdots, x_k\}\leqq k$ holds for every k, $2\leqq k\leqq n$, and for any points $x_0, x_1, \cdots, x_k$ in V. Suppose $\dim H\{x_0, x_1, \cdots, x_{n-1}\}>n-1$ for some points $x_0, x_1, \cdots, x_{n-1}$ in V, i.e., $\dim H\{x_0, x_1, \cdots, x_{n-1}\}=n$. Then there exists a smooth totally geodesic n-dimensional imbedded submanifold N such that $N\subset H\{x_0, x_1, \cdots, x_{n-1}\}\subset \bar{N}$. Take a point q in N and a normal vector v of N at q such that $\exp_N v\in V$. Then $\dim H\{x_0, x_1, \cdots, x_{n-1}, \exp_N v\}>n$, a contradiction. Thus we obtain $\dim H\{x_0, x_1, \cdots, x_{n-1}\}\leqq n-1$ for any points $x_0, x_1, \cdots, x_{n-1}$ in V. By the same argument inductively we have our claim. In particular, $\dim H\{x_0, x_1, x_2\}\leqq 2$ for any points x_0, x_1, x_2 in V.

Now we show that M satisfies the axiom of 2-planes. Let T'_p be an arbitrary 2-dimensional subspace of T_pM and let v_1 and v_2 be vectors in T'_p such that $x_1:=\exp_p v_1$ and $x_2:=\exp_p v_2$ belong to V, and p, x_1 and x_2 are non-collinear. Take q in the interior of the segment $T(x_1, x_2)$ joining x_1 and x_2, and take x_0 in V on the other side of q with respect to p on the extension of $T(p, q)$. Let N_0 be the set of all points each of which belongs to a certain segment from x_0 to a point of $T(x_1, x_2)$. Then N_0 is a smooth surface except at x_0. We need to prove that $T_pN_0=T'_p$ and $N_0-\{x_0\}$ is totally geodesic in M. Let N be a smooth totally geodesic submanifold in M such that $N\subset H\{x_0, x_1, x_2\}\subset \bar{N}$. Since $p\in N_0\subset H\{x_0, x_1, x_2\}$ and $\dim H\{x_0, x_1, x_2\}\leqq 2$, it follows that $N_0\subset N$ and $\dim N_0=\dim N=2$. Thus $T_pN_0=T'_p$ and $N_0-\{x_0\}$ is totally geodesic in M.

We know from this lemma that in order to prove our theorems we have only to estimate the dimension of $H\{\ \}$. We then need the following lemma.

LEMMA 2. *Let T_1 and T_2 be two segments contained in a convex set V in M. Let A be the set of all points each of which belongs to some segment joining a point of T_1 and a point of T_2. Then $\dim A\leqq 3$ and A is closed.*

PROOF. Let $x(\tau)$, $0\leqq\tau\leqq\alpha$, and $y(\nu)$, $0\leqq\nu\leqq\beta$, represent segments T_1 and T_2 respectively, and let W_q, for each q in V, be a subset of T_qM where $\exp_q|W_q$ is diffeomorphic onto V. Define a map G of $[0, 1]\times[0, \alpha]\times[0, \beta]$ into T_pM by $G(\mu, \tau, \nu):=(\exp_p|W_p)^{-1}[\exp_{x(\tau)}\{\mu(\exp_{x(\tau)}|W_{x(\tau)})^{-1}(y(\nu))\}]$ for $(\mu, \tau, \nu)\in[0, 1]\times[0, \alpha]\times[0, \beta]$, where p is a fixed point in V. Then G is differentiable, and hence G is Lipschitz continuous. Therefore it follows from the definition of 4-measure that the 4-measure of the image of G is zero since the 4-measure of $[0, 1]\times[0, \alpha]\times[0, \beta]$ is zero. Note that A is the image of $\exp_p\circ G$ and that the property of having at most dimension n is topologically invariant. Thus we conclude $\dim A\leqq 3$ by the fact in § 1.

Closedness of A is evident.

LEMMA 3. *Let p be a fixed point in M. For an arbitrary $\alpha>0$, there exists an $r>0$ such that for any points x, y and z in $B_r(p)$,*

$$\mu(1-\alpha)yz\leqq w_y(\beta\mu)w_z(\gamma\mu)\leqq\mu(1+\alpha)yz$$

for any $\mu\in[0, 1]$, where yz is the distance between y and z, and $w_y(\tau)$, $0\leqq\tau\leqq\beta$, and $w_z(\nu)$, $0\leqq\nu\leqq\gamma$, represent segments $T(x, y)$ and $T(x, z)$ respectively.

PROOF. By a straightforward generalization of Proposition 9.10 in [7] p. 54 we obtain that for given $0<\varepsilon<1$ there is an $r>0$ such that for any non-collinear points x, y and z in $B_r(p)$

$$1-\varepsilon<\|(\exp_x|B_r)^{-1}(y)-(\exp_x|B_r)^{-1}(z)\|_x/yz<1+\varepsilon\,,$$

where $\|\ \|_x$ is the norm in T_xM, and B_r is the r-ball in T_xM centered at the origin.

We then have

$$1-\varepsilon<\|(\exp_x|B_r)^{-1}(w_y(\beta\mu))-(\exp_x|B_r)^{-1}(w_z(\gamma\mu))\|_x/w_y(\beta\mu)w_z(\gamma\mu)<1+\varepsilon$$

for $\mu\neq 0$. Therefore

$$(1-\varepsilon)/(1+\varepsilon)<(1/\mu)(w_y(\beta\mu)w_z(\gamma\mu)/yz)<(1+\varepsilon)/(1-\varepsilon)\,.$$

If we choose an $\varepsilon>0$ which satisfies

$$1-\alpha<(1-\varepsilon)/(1+\varepsilon)<(1+\varepsilon)/(1-\varepsilon)<1+\alpha\,,$$

then it follows that $\mu(1-\alpha)yz\leqq w_y(\beta\mu)w_z(\gamma\mu)\leqq\mu(1+\alpha)yz$ for any $\mu\in[0, 1]$.

3. Proofs of Theorems.

3.1. PROOF OF THEOREM 1. We denote six segments each of which joins x_i and x_j, $0\leqq i<j\leqq 3$ by T_k, $k=1, 2, \cdots, 6$. Then from the assumption $H\{x_0, x_1, x_2, x_3\}=\bigcup_{1\leqq i\leqq j\leqq 6}\{x\in V;\ x$ belongs to some segment which connects a point of T_i and a point of $T_j\}$. Therefore it follows from Lemma 2 and the sum theorem ([8] p. 30), i. e., a separable metric space which is the countable sum of closed subsets of dimension $\leqq n$ has dimension $\leqq n$, that $\dim H\{x_0, x_1, x_2, x_3\}\leqq 3$. Hence we obtain our theorem by Lemma 1.

3.2. PROOF OF THEOREM 2. Let $\alpha>0$ satisfy that $6((1+\alpha)/2)^3<1$. And for this α we choose an $r>0$ such that $B_r(p)\subset V$ satisfies the conclusion of Lemma 3 and the $4r$-ball with center p is strongly convex.

By Lemma 1 it suffices to show that $\dim H\{x_0, x_1, x_2\}\leqq 2$. If $\dim C^2\{y_0, y_1, y_2\}\leqq 2$ for any points y_0, y_1 and y_2 in $B_r(p)$, then $\dim H\{x_0, x_1, x_2\}\leqq 2$ because of the sum theorem.

In fact, $\dim C^2\{y_0, y_1, y_2\}\leqq 2$ is established as follows. From the definition of $C^2\{y_0, y_1, y_2\}$ the diameter of $C^2\{y_0, y_1, y_2\}$ is not greater than $y_0y_1+y_1y_2+y_2y_0$, because $C^2\{y_0, y_1, y_2\}$ is contained in $B_{(y_0y_1+y_1y_2+y_2y_0)/2}(y_0)$. Since $H\{y_0, y_1, y_2\}\supset C^2\{y_0, y_1, y_2\}$, it holds that

$$C^2\{y_0, y_1, y_2\}\subset C^2\{y_0, y_1, m(y_1, y_2)\}\cup C^2\{y_0, y_2, m(y_1, y_2)\}.$$

Hence if we put $y_0':=m(y_1, y_2)$, $y_1':=m(y_0, y_2)$, $y_2':=m(y_0, y_1)$ and $y':=m(y_0, y_0')$, then

$$C^2\{y_0, y_1, y_2\}\subset C^2\{y_0, y_1', y'\}\cup C^2\{y_0, y', y_2'\}\cup C^2\{y_1', y_2, y_0'\}$$
$$\cup C^2\{y_1', y_0', y'\}\cup C^2\{y', y_0', y_2'\}\cup C^2\{y_2', y_0', y_1\},$$

and the diameter of each $C^2\{\ \}$ on the right hand side are not greater than $((1+\alpha)/2)(y_0y_1+y_1y_2+y_2y_0)$ (by Lemma 3). If we repeat this $(n-1)$ times for each $C^2\{\ \}$ of the right hand side, then we obtain 6^n $C^2\{\ \}$'s and their diameters are not greater than $((1+\alpha)/2)^n(y_0y_1+y_1y_2+y_2y_0)$. Hence for given $\varepsilon>0$ there is an n_0 such that $n\geqq n_0$ implies $((1+\alpha)/2)^n(y_0y_1+y_1y_2+y_2y_0)<\varepsilon$. Since $m_3^{\varepsilon}(C^2\{y_0, y_1, y_2\})\leqq\sum[\delta(C^2\{\ \})]^3\leqq 6^n[((1+\alpha)/2)^n(y_0y_1+y_1y_2+y_2y_0)]^3$ for $n\geqq n_0$, we get $m_3(C^2\{y_0, y_1, y_2\})=0$. By the fact introduced in §1, $\dim C^2\{y_0, y_1, y_2\}\leqq 2$. The proof is complete.

4. Remarks.

If we try to describe Theorem 1 with only convex combinations we have Corollary 1. This is because $C^{k+1}U=C^kU$ for every subset U of $B_r(p)$ in M means $HU=C^kU$.

COROLLARY 1. *Let* $\dim M$ *be greater than* 3. *If for each point* p *in* M *there is a convex neighborhood* V *of* p *in* M *such that* $C^3\{x_0, x_1, x_2, x_3\}=C^2\{x_0, x_1, x_2, x_3\}$ *for any points* x_0, x_1, x_2, x_3 *in* V, *then* M *is a space of constant curvature.*

The following corollary is evident by the fact that $H\{y_0, y_1, y_2\}\supset C^2\{y_0, y_1, y_2\}$ for any y_0, y_1 and y_2 in $B_r(p)\subset M$. Moreover it is directly proved by the same way as in the proof of Theorem 2.

COROLLARY 2. *Let* $\dim M$ *be greater than* 2. *If for each point* p *in* M *there is a convex neighborhood* V *of* p *in* M *such that* $H\{x_0, x_1, x_2\}=H\{x_0, x_1, m(x_1, x_2)\}\cup H\{x_0, x_2, m(x_1, x_2)\}$ *for any points* x_0, x_1 *and* x_2 *in* V, *then* M *is a* *space of constant curvature.*

It is natural to ask whether $H\{x_0, x_1, x_2\}$ in the assumption of Theorem 2 could be replaced by $C^2\{x_0, x_1, x_2\}$. On this question we show the following.

THEOREM 3. *Let* $\dim M$ *be greater than* 2. *If for each* p *in* M *there is a convex neighborhood* V *of* p *in* M *such that* $C^2\{x_0, x_1, x_2\}=C^2\{x_0, x_1, x\}\cup C^2\{x_0, x_2, x\}$ *for any points* x_0, x_1 *and* x_2 *in* V *and for any point* x *in the segment* $T(x_1, x_2)$, *then* M *is a space of constant curvature.*

If it is possible to replace x in the assumption with $m(x_1, x_2)$, then this theorem is stronger than Theorem 2. However the author does not know the possibility.

We prepare a lemma.

LEMMA 4. *Let* M *satisfy the assumption in Theorem* 3. *Let* $x(\tau)$, $0\leqq\tau\leqq\alpha$, *and* $y(\nu)$, $0\leqq\nu\leqq\beta$, *represent segments* $T_1=T(y_0, y_1)$ *and* $T_2=T(y_0, y_2)$ *respectively in* $B_r(p)\subset V$ *and let* $t:=\max_{\mu\in[0,1]} x(\alpha\mu)y(\beta\mu)$. *If the* $3r$*-ball with center* p *is strongly convex, then* $C^2\{y_0, y_1, y_2\}$ *is contained in the union of the closed* t*-neighborhood of* T_1 *and the closed* t*-neighborhood of* T_2 *in* M.

PROOF OF LEMMA 4. Choose a partition $0=\mu_0<\mu_1<\cdots<\mu_n=1$ of $[0, 1]$ such that $\alpha(\mu_i-\mu_{i-1})<t$ and $\beta(\mu_i-\mu_{i-1})<t$ for $1\leqq i\leqq n$. Then $C^2\{x(\alpha\mu_{i-1}), x(\alpha\mu_i), y(\beta\mu_{i-1})\}\subset H\{x(\alpha\mu_{i-1}), x(\alpha\mu_i), y(\beta\mu_{i-1})\}\subset\overline{B_t(x(\alpha\mu_{i-1}))}$ and $C^2\{x(\alpha\mu_i), y(\beta\mu_{i-1}), y(\beta\mu_i)\}\subset\overline{B_t(y(\beta\mu_i))}$ for every $1\leqq i\leqq n$, because for every $1\leqq i\leqq n$ $\overline{B_t(x(\alpha\mu_{i-1}))}$ and $\overline{B_t(y(\beta\mu_i))}$ are contained in $B_{3r}(p)$ and hence are convex. Since $C^2\{y_0, y_1, y_2\}\subset\bigcup_{i=1}^{n} C^2\{x(\alpha\mu_{i-1}), x(\alpha\mu_i), y(\beta\mu_{i-1})\}\cup C^2\{x(\alpha\mu_i), y(\beta\mu_{i-1}), y(\beta\mu_i)\}$, $C^2\{y_0, y_1, y_2\}$ is contained in the union of the closed t-neighborhood of T_1 and the closed t-neighborhood of T_2 in M.

PROOF OF THEOREM 3. Let x_0, x_1 and x_2 be any points in $B_r(p)\subset V$ where r is a positive such that the $3r$-ball with center p is strongly convex. Let S be the set of all points each of which belongs to the segment $T(x_0, x)$ for some x in $T(x_1, x_2)$. $S\subset C^2\{x_0, x_1, x_2\}$ is clear. We claim $S=C^2\{x_0, x_1, x_2\}$. In fact, suppose there exists a point $z\in C^2\{x_0, x_1, x_2\}-S$. Let s be the distance between z and S. Since S is closed we have $s>0$. Choose a partition $x_1=z_1, z_2, \cdots, z_n=x_2$ of $T(x_1, x_2)$ in this order such that if $z_i(\tau)$, $0\leqq\tau\leqq\alpha_i$, represents the segment $T(x_0, z_i)$ for each $1\leqq i\leqq n$, and if we put $t_i=\max_{\mu\in[0,1]} z_i(\alpha_i\mu)z_{i+1}(\alpha_{i+1}\mu)$ for each $1\leqq i\leqq n-1$, then $t_i<s$ for all $1\leqq i\leqq n-1$. By Lemma 4 and the assumption $C^2\{x_0, x_1, x_2\}$ is contained in the open s-neighborhood of S in M, a contradiction.

Next we assert $H\{x_0, x_1, x_2\}=C^2\{x_0, x_1, x_2\}$. Let z and y be any points of $C^2\{x_0, x_1, x_2\}$. Then by the above argument there are points z' and y' in $T(x_1, x_2)$ such that $z\in T(x_0, z')$ and $y\in T(x_0, y')$. Since $C^2\{x_0, x_1, x_2\}=C^2\{x_0, x_1, z'\}\cup C^2\{x_0, z', y'\}\cup C^2\{x_0, x_2, y'\}$, where we assume without loss of generality that

x_1, z', y' and x_2 are in this order on $T(x_1, x_2)$, and since $T(z, y)$ is contained in $C^2\{x_0, z', y'\}$, $T(z, y)$ is contained in $C^2\{x_0, x_1, x_2\}$, which implies the convexity of $C^2\{x_0, x_1, x_2\}$.

Thus $H\{x_0, x_1, x_2\}=C^2\{x_0, x_1, x_2\}=S$. Then we conclude $\dim H\{x_0, x_1, x_2\} \leqq 2$, and hence we obtain our theorem by Lemma 1.

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