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A characterization of Azumaya coalgebras over a commutative ring

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§1. Introduction.

Throughout this paper R is a commutative ring with 1, and (C, Δ, ε) is a coalgebra over R, where Δ is the comultiplication of C and ε is the counit of C. As usual we denote $\Delta(c)=\sum c_{(1)}\otimes c_{(2)}$ for each $c\in C$. Furthermore we will set $C^*=\mathrm{Hom}_R(C, R)$, and for each $c^*\in C^*$ and $c\in C$, we denote by $\langle c^*, c\rangle$ the element of R to which c is mapped by c^* in stead of $c^*(c)$. As is well known C^* is an R-algebra whose multiplication is defined by $\langle c^*\cdot d^*, c\rangle=\sum\langle c^*, c_{(1)}\rangle\langle d^*, c_{(2)}\rangle$ (namely, $(c^*\cdot d^*)(c)=\sum c^*(c_{(1)})d^*(c_{(2)})$ by the ordinary description of homomorphisms) for any c^*, $d^*\in C^*$ and $c\in C$. On the other hand, C is a two-sided C^*-module by $c^*\cdot c=\sum c_{(1)}\langle c^*, c_{(2)}\rangle$ and $c\cdot c^*=\sum\langle c^*, c_{(1)}\rangle c_{(2)}$ for any $c^*\in C^*$ and $c\in C$. Then it is easily seen that the C^*-C^*-module structure of $\mathrm{Hom}_R(C, R)$ induced from the C^*-C^*-module structure of C is the same as that induced from the ring structure of $\mathrm{Hom}_R(C, R)=C^*$. In what follows throughout, all $\otimes$ will be $\otimes_R$ and Hom will mean Hom_R.

In this paper we will show that in the case where C is R-finitely generated projective and faithful, C^* is an R-Azumaya algebra if and only if there exist C^*-C^*-isomorphisms Ψ of $C\otimes C$ to $C\otimes_{C^*}C\otimes C$ and μ of $C^*\otimes I$ to C, where $I=\{c\in C\,|\,\sum c_{(1)}\otimes c_{(2)}=\sum c_{(2)}\otimes c_{(1)}\}$, such that $\Psi(c\otimes d)=\sum c\otimes d_{(1)}\otimes d_{(2)}$ and $\mu(c^*\otimes a)=c^*\cdot a$ $(=a\cdot c^*)$ for c, $d\in C$, $c^*\in C^*$ and $a\in I$

§2. Let A, B and S be (not necessarily commutative) rings with identities. We denote as usual ${}_AM_B$ (resp. $M_{A\text{-}B}$) in the case where M is a left A-module as well as a right B-module (resp. a right A-module as well as a right B-module) such that $(am)b=a(mb)$ (resp. $(ma)b=(mb)a$) for all $m\in M$, $a\in A$ and $b\in B$. For any ${}_AP_A$ and ${}_AM_B$, ${}_AN_B$, we will set, respectively,

$$P^A=\{x\in P\,|\,ax=xa \text{ for all } a\in A\},$$

$$\mathrm{Hom}({}_AM_B,\ {}_AN_B)=\{A\text{-}B\text{-homomorphism of } M \text{ to } N\}.$$

Then it is clear that $\mathrm{Hom}({}_AM_B,\ {}_AN_B)=[\mathrm{Hom}(M_B, N_B)]^A=[\mathrm{Hom}({}_AM, {}_AN)]^B$. The similar symbols will be used for $M_{A\text{-}B}$ and $N_{A\text{-}B}$. The following lemmas are well known.

LEMMA 1. *For $M_{A\text{-}S}$, ${}_AN_B$ and $P_{S\text{-}B}$, there exists an S-S-isomorphism*

$$\varphi: \mathrm{Hom}((M\otimes_A N)_B, P_B) \longrightarrow \mathrm{Hom}(M_A, \mathrm{Hom}(N_B, P_B)_A)$$

such that $[\varphi(h)(m)](n)=h(m\otimes n)$ *for* $h\in\mathrm{Hom}((M\otimes_A N)_B, P_B)$ *and* $m\in M$, $n\in N$. φ *induces* $\mathrm{Hom}((M\otimes_A N)_{B\text{-}S}, P_{B\text{-}S})\cong\mathrm{Hom}(M_{A\text{-}S}, \mathrm{Hom}(N_B, P_B)_{A\text{-}S})$.

LEMMA 2. *Under the situation described by P_A, ${}_BM_A$, ${}_BN$, if P is A-finitely generated projective, there exists an isomorphism*

$$\psi: P\otimes_A\mathrm{Hom}({}_BM, {}_BN) \longrightarrow \mathrm{Hom}({}_B\mathrm{Hom}(P_A, M_A), {}_BN)$$

such that $\psi(p\otimes f)(g)=f(g(p))$ *for* $f\in\mathrm{Hom}({}_BM, {}_BN)$, $g\in\mathrm{Hom}(P_A, M_A)$ *and* $p\in P$. *Furthermore, let* $\{p_i^*, p_i\}$ *be a dual basis for* P_A, *namely,* $p=\sum p_i p_i^*(p)$ *for each* $p\in P$. *Then the inverse of* ψ *is given by* $\psi^{-1}(\alpha)=\sum p_i\otimes\alpha_i$ *with* $\alpha_i(m)=\alpha(m^{(L)}\circ p_i^*)$, *where* $m^{(L)}\in\mathrm{Hom}(A_A, M_A)$ *such that* $m^{(L)}(a)=ma$ *for all* $a\in A$.

Next, let M be a module over a commutative ring R, and set $M^*=\mathrm{Hom}(M, R)$. There are R-homomorphisms

$$\theta: M \longrightarrow \mathrm{Hom}(\mathrm{Hom}(M, R), R)\,,$$

$$\sigma: \mathrm{Hom}(M, M) \longrightarrow \mathrm{Hom}(M^*, M^*)$$

such that $\theta(m)(m^*)=m^*(m)$ and $[\sigma(f)(m^*)](m)=m^*(f(m))$, respectively, for any $m\in M$, $m^*\in M^*$ and $f\in\mathrm{Hom}(M, M)$. M is said to be torsionless, or reflexive, if θ is a monomorphism, or an isomorphism, respectively. It is also clear that σ is a ring homomorphism.

LEMMA 3. *Let M, M^* and σ be as above. Then*

(1) *If M is torsionless, we have* $\mathrm{Ker}\,\sigma=0$.

(2) *If M is reflexive, then σ is an isomorphism.*

PROOF. (1). Let $f\in\mathrm{Ker}\,\sigma$. Then for any $m\in M$ and $m^*\in M^*$, we have

$$0=[\sigma(f)(m^*)](m)=m^*(f(m))=\theta(f(m))(m^*)\,.$$

Hence $\theta(f(m))=0$. But $\mathrm{Ker}\,\theta=0$ by assumption. Hence $f(m)=0$ for all $m\in M$. (2). Suppose that θ is an isomorphism. Then we have an isomorphism

$$\tau: \mathrm{Hom}(M^{**}, M^{**}) \longrightarrow \mathrm{Hom}(M, M)\ (=\mathrm{Hom}(M, \theta^{-1})\cdot\mathrm{Hom}(\theta, M^{**}))\,.$$

On the other hand we have a ring homomorphism

$$\sigma^*: \mathrm{Hom}(M^*, M^*) \longrightarrow \mathrm{Hom}(M^{**}, M^{**})$$

defined by the same way as σ, namely, $[\sigma^*(h)(m^{**})](m^*)=m^{**}(h(m^*))$ for any $h\in\mathrm{Hom}(M^*, M^*)$, $m^*\in M^*$ and $m^{**}\in M^{**}$. Then by direct computations, we see that $\sigma\circ(\tau\circ\sigma^*)=$identity, and $(\tau\circ\sigma^*)\circ\sigma=$identity. For example, pick any

$h\in\mathrm{Hom}(M^*, M^*)$ and $m\in M$, and set $n=[\tau\circ\sigma^*(h)](m)$. Set $\theta^*=\mathrm{Hom}(\theta, M^{**})$. Then, $n=\theta^{-1}[(\theta^*\circ\sigma^*(h))(m)]$, and for any $m^*\in M^*$, we have $m^*(n)=\theta(n)(m^*)=\theta(m)(h(m^*))=h(m^*)(m)$. Then,

$$[(\sigma\circ\tau\circ\sigma^*(h))(m^*)](m)=m^*(\tau\circ\sigma^*(h)(m))=m^*(n)=h(m^*)(m).$$

This means that $\sigma\circ\tau\circ\sigma^*(h)=h$. Thus we have $\sigma\circ(\tau\circ\sigma^*)=$identity. The other equality is also evident.

Finally, we will introduce a theorem by K. Hirata [4]. For ${}_AM$ and ${}_AN$, set $S=\mathrm{Hom}({}_AM, {}_AM)$ and $T=\mathrm{Hom}({}_AN, {}_AN)$. Then we obtain the situations ${}_AM_S$ and ${}_AN_T$ and an A-T-homomorphism

$$\iota: N \longrightarrow \mathrm{Hom}(\mathrm{Hom}({}_AN, {}_AM)_S, M_S)$$

such that $\iota(n)(f)=nf\ (=f(n))$ for $n\in N$ and $f\in\mathrm{Hom}({}_AN, {}_AM)$. Then,

LEMMA 4. *${}_AN\langle\oplus_A(M\oplus M\oplus\cdots\oplus M)$ if and only if ι is an isomorphism and $\mathrm{Hom}({}_AN, {}_AM)$ is S-finitely generated projective.*

PROOF. See Theorem 1.2 [4].

§3. Now regard $C\otimes C$ as C^*-C^*-module by $c^*(c\otimes d)d^*=c\cdot d^*\otimes c^*\cdot d$ for $c, d\in C$ and $c^*, d^*\in C^*$. Then $C\otimes C$ becomes a left $C^*\otimes C^{*0}$-module, where C^{*0} is the opposite ring of C^*. Set $A=C^*\otimes C^{*0}$, $N=C\otimes C$, $M=C$ and $S=\mathrm{Hom}({}_AC, {}_AC)=\mathrm{Hom}({}_{C^*}C_{C^*}, {}_{C^*}C_{C^*})$, and apply Lemma 3. First of all we have

(1) $$\iota: C\otimes C \longrightarrow \mathrm{Hom}(\mathrm{Hom}({}_{C^*}C\otimes C_{C^*}, {}_{C^*}C_{C^*})_S, C_S)$$

such that $\iota(c\otimes d)(\alpha)=\alpha(c\otimes d)$ for each $\alpha\in\mathrm{Hom}({}_{C^*}C\otimes C_{C^*}, {}_{C^*}C_{C^*})$ and $c, d\in C$. Next note that there is a C^*-C^*-map

(2) $$\gamma: C^* \longrightarrow \mathrm{Hom}({}_{C^*}C, {}_{C^*}C)$$

such that $\gamma(c^*)=c^{*(R)}$, where $c^{*(R)}$ means the right multiplication of C by c^* $(\in C^*)$. Then by Lemma 1 we have

(3) $$\mathrm{Hom}(C_{C^*}, C^*_{C^*}) \underset{\gamma^*}{\longrightarrow} \mathrm{Hom}(C_{C^*}, \mathrm{Hom}({}_{C^*}C, {}_{C^*}C)_{C^*}) \quad (\gamma_*=\mathrm{Hom}(1_C, \gamma))$$
$$=\mathrm{Hom}(C_{R\text{-}C^*}, \mathrm{Hom}({}_{C^*}C, {}_{C^*}C)_{R\text{-}C^*})\cong\mathrm{Hom}({}_{C^*}C\otimes C_{C^*}, {}_{C^*}C_{C^*}).$$

Now suppose that C_R and C_{C^*} are finitely generated projective. Then by Lemma 2 and (3), we have

(4) $$C\otimes C \longrightarrow \mathrm{Hom}(\mathrm{Hom}({}_{C^*}C\otimes C_{C^*}, {}_{C^*}C_{C^*})_S, C_S)$$
$$\subset\mathrm{Hom}(\mathrm{Hom}({}_{C^*}C\otimes C_{C^*}, {}_{C^*}C_{C^*}), C) \underset{(\gamma_*)^*}{\longrightarrow} \mathrm{Hom}(\mathrm{Hom}(C_{C^*}, C^*_{C^*}), C)$$
$$\cong C\otimes_{C^*}\mathrm{Hom}(C^*, C)=C\otimes_{C^*}\mathrm{Hom}(\mathrm{Hom}(C, R), C)$$
$$\cong C\otimes_{C^*}(C\otimes\mathrm{Hom}(R, C))\cong C\otimes_{C^*}C\otimes C.$$

We will calculate the composition of the above maps concretely. To begin with it is easy to see that the composition of the maps in (3) is given by

$$(5)\qquad \varphi: \mathrm{Hom}(C_{C^*}, C^*_{C^*}) \longrightarrow \mathrm{Hom}({}_{C^*}C\otimes C_{C^*}, {}_{C^*}C_{C^*})$$

such that $\varphi(g)(c\otimes d)=d\cdot g(c)$ for each $g\in\mathrm{Hom}(C_{C^*}, C^*_{C^*})$ and $c, d\in C$. Next suppose that C_{C^*} and C_R are finitely generated projective, and let $\{f_j, c_j\}$ and $\{d_i^*, d_i\}$ be dual bases for C_{C^*} and C_R, respectively. Then by Lemma 2, we have isomorphisms

$$(6)\qquad \nu: \mathrm{Hom}(\mathrm{Hom}(C_{C^*}, C^*_{C^*}), C) \longrightarrow C\otimes_{C^*}\mathrm{Hom}(C^*, C)$$

such that $\nu(\alpha)=\sum c_j\otimes\alpha_j$ for each $\alpha\in\mathrm{Hom}(\mathrm{Hom}(C_{C^*}, C^*_{C^*}), C)$ with $\alpha_j(c^*)=\alpha(c^{*(L)}\circ f_j)$ for each $c^*\in C^*$, and

$$(7)\qquad \tau: \mathrm{Hom}(C^*, C)=\mathrm{Hom}(\mathrm{Hom}(C, R), C) \longrightarrow C\otimes\mathrm{Hom}(R, C)\cong C\otimes C$$

such that $\tau(h)=\sum d_i\otimes h(d_i^*)$ for each $h\in\mathrm{Hom}(C^*, C)$. On the other hand, since S is an R-algebra, we have an inclusion map

$$(8)\qquad i: \mathrm{Hom}(\mathrm{Hom}({}_{C^*}C\otimes C_{C^*}, {}_{C^*}C_{C^*})_S, C_S)\subset\mathrm{Hom}(\mathrm{Hom}({}_{C^*}C\otimes C_{C^*}, {}_{C^*}C_{C^*}), C).$$

Finally set $\Psi=(1_C\otimes\tau)\circ\nu\circ\mathrm{Hom}(\varphi, C)\circ i\circ\iota$, which is exactly (4);

$$(9)\qquad \Psi: C\otimes C \longrightarrow \mathrm{Hom}(\mathrm{Hom}({}_{C^*}C\otimes C_{C^*}, {}_{C^*}C_{C^*}), C) \longrightarrow$$
$$\mathrm{Hom}(\mathrm{Hom}(C_{C^*}, C^*_{C^*}), C) \longrightarrow C\otimes_{C^*}\mathrm{Hom}(C^*, C) \longrightarrow C\otimes_{C^*}C\otimes C.$$

LEMMA 5. *Let C_{C^*} and C_R be finitely generated projective, and Ψ as above. Then $\Psi(c\otimes d)=\sum c\otimes d_{(1)}\otimes d_{(2)}$ for any $c, d\in C$.*

PROOF. Set $[\mathrm{Hom}(\varphi, 1_C)\circ i\circ\iota](c\otimes d)=\alpha$. For each $g\in\mathrm{Hom}(C_{C^*}, C^*_{C^*})$,

$$\alpha(g)=[\mathrm{Hom}(\varphi, 1_C)(\iota(c\otimes d))](g)=\iota(c\otimes d)(\varphi(g))=\varphi(g)(c\otimes d)=d\cdot g(c).$$

On the other hand by (6), $\nu(\alpha)=\sum c_j\otimes G_j$, where

$$G_j(c^*)=\alpha(c^{*(L)}\circ f_j)=d\cdot(c^{*(L)}\circ f_j)(c)=d\cdot(c^*\cdot f_j(c))$$
$$=\sum\langle c^*\cdot f_j(c), d_{(1)}\rangle d_{(2)}=\sum\langle c^*, d_{(1)}\rangle\langle f_j(c), d_{(2)}\rangle d_{(3)}$$

for each $c^*\in C^*$. Then,

$$\Psi(c\otimes d)=(1_C\otimes\tau)\nu(\alpha)=\sum c_j\otimes\tau(G_j)=\sum c_j\otimes d_i\otimes G_j(d_i^*)$$
$$=\sum c_j\otimes d_i\otimes\langle d_i^*, d_{(1)}\rangle\langle f_j(c), d_{(2)}\rangle d_{(3)}$$
$$=\sum c_j\otimes d_i\langle d_i^*, d_{(1)}\rangle\otimes\langle f_j(c), d_{(2)}\rangle d_{(3)}$$
$$=\sum c_j\otimes d_{(1)}\otimes\langle f_j(c), d_{(2)}\rangle d_{(3)}=\sum c_j\otimes d_{(1)}\langle f_j(c), d_{(2)}\rangle\otimes d_{(3)}$$
$$=\sum c_j\otimes f_j(c)\cdot d_{(1)}\otimes d_{(2)}=\sum c_j\cdot f_j(c)\otimes d_{(1)}\otimes d_{(2)}=\sum c\otimes d_{(1)}\otimes d_{(2)}.$$

The author gives his hearty thanks to the referee for various kind of advices. In particular, he showed the author the other method of calculation of the map Ψ, which looked more beautiful and co-algebra theoretical. But the author dared to stick to his original method. The referee also showed him the next equality, which was yielded in the process of the calculation of Ψ. Here we will show it by the other proof.

LEMMA 6. *Suppose that C is R-projective. Then for any $c, d\in C$, we have an equality $\sum c\otimes d_{(1)}\otimes d_{(2)}=\sum c_{(2)}\otimes d\otimes c_{(1)}$ in $C\otimes_{C^*}C\otimes C$. In particular, if C_R and C_{C^*} are finitely generated projective, we have*

$$\Psi(c\otimes d)=\sum c\otimes d_{(1)}\otimes d_{(2)}=\sum c_{(2)}\otimes d\otimes c_{(1)}\,.$$

PROOF. Let $\{c_\nu^*, c_\nu\}$ be a dual basis for C, and let c, d be any elements of C. For each ν we have in $C\otimes_{C^*}C$

$$\sum\langle c_\nu^*, c_{(1)}\rangle c_{(2)}\otimes d=c\cdot c_\nu^*\otimes d=c\otimes c_\nu^*\cdot d=\sum c\otimes d_{(1)}\langle c_\nu^*, d_{(2)}\rangle\,.$$

Then in $C\otimes_{C^*}C\otimes C$, we have $\sum\langle c_\nu^*, c_{(1)}\rangle c_{(2)}\otimes d\otimes c_\nu=\sum c\otimes d_{(1)}\langle c_\nu^*, d_{(2)}\rangle\otimes c_\nu$. Then,

$$\sum c_{(2)}\otimes d\otimes c_{(1)}=\sum c_{(2)}\otimes d\otimes\sum_\nu\langle c_\nu^*, c_{(1)}\rangle c_\nu=\sum c\otimes d_{(1)}\otimes\sum\langle c_\nu^*, d_{(2)}\rangle c_\nu$$
$$=c\otimes d_{(1)}\otimes d_{(2)}\,.$$

REMARK. Here we will introduce the referee's method of the calculation of Ψ very briefly. In the case where C is torsionless, there exists an isomorphism

$$\rho: \mathrm{Hom}(C\otimes_{C^*}C, R)\longrightarrow\mathrm{Hom}({}_{C^*}C\otimes C_{C^*}, {}_{C^*}C_{C^*})$$

such that $\rho(f)(c\otimes d)=\sum c_{(1)}f(c_{(2)}\otimes d)=\sum f(c\otimes d_{(1)})d_{(2)}$ for any $f\in(C\otimes_{C^*}C)^*$. The inverse map of ρ is given by $\rho^{-1}(\alpha)=\varepsilon\circ\alpha$ for any $\alpha\in\mathrm{Hom}({}_{C^*}C\otimes C_{C^*}, {}_{C^*}C_{C^*})$. Then under the same conditions as Lemma 5, the composition of the following maps

$$C\otimes C\xrightarrow[\iota]{}\mathrm{Hom}({}_S\mathrm{Hom}({}_{C^*}C\otimes C_{C^*}, {}_{C^*}C_{C^*}), {}_SC)\longrightarrow\mathrm{Hom}({}_S\mathrm{Hom}(C\otimes_{C^*}C, R), {}_SC)$$
$$\xrightarrow[i]{}\mathrm{Hom}(\mathrm{Hom}(C\otimes_{C^*}C, R), C)=(C\otimes_{C^*}C)\otimes C$$

is exactly $\Psi(c\otimes d)=\sum c\otimes d_{(1)}\otimes d_{(2)}=\sum c_{(2)}\otimes d\otimes c_{(1)}$ for any $c, d\in C$.

Now set $I=\{c\in C\,|\,\sum c_{(1)}\otimes c_{(2)}=\sum c_{(2)}\otimes c_{(1)}\}$ and $C^{C^*}=\{c\in C\,|\,c^*\cdot c=c\cdot c^*$ for all $c^*\in C^*\}$. Then we have

LEMMA 7. *$I\subseteq C^{C^*}$. If C is R-projective, we have $I=C^{C^*}$.*

PROOF. If $\sum c_{(1)}\otimes c_{(2)}=\sum c_{(2)}\otimes c_{(1)}$, then $c^*\cdot c=\sum c_{(1)}\langle c^*, c_{(2)}\rangle=\sum c_{(2)}\langle c^*, c_{(1)}\rangle=c\cdot c^*$. Thus we have $I\subseteq C^{C^*}$. Let $\{c_\nu^*, c_\nu\}$ be a dual basis for C_R, and suppose $c\in C^{C^*}$. Then $\sum c_{(1)}\langle c_\nu^*, c_{(2)}\rangle=\sum c_{(2)}\langle c_\nu^*, c_{(1)}\rangle$ for each ν. Hence $\sum c_{(1)}\otimes c_{(2)}=\sum c_\nu\langle c_\nu^*, c_{(1)}\rangle\otimes c_{(2)}=\sum c_\nu\otimes\langle c_\nu^*, c_{(1)}\rangle c_{(2)}=\sum c_\nu\otimes\langle c_\nu^*, c_{(2)}\rangle c_{(1)}=\sum c_\nu\langle c_\nu^*, c_{(2)}\rangle\otimes c_{(1)}=\sum c_{(2)}\otimes c_{(1)}$. Thus $c\in I$, and we see $C^{C^*}\subseteq I$. Therefore we have $I=C^{C^*}$.

Next suppose that C is R-faithful. Then we see that C^* is a faithful R-algebra. Because, if $r\varepsilon=0$ for some $r\in R$, $rc=r\sum c_{(1)}\langle\varepsilon, c_{(2)}\rangle=\sum c_{(1)}\langle r\varepsilon, c_{(2)}\rangle=0$. This means that $rC=0$.

Now suppose that C is R-finitely generated projective and R-faithful. In this case we see that $C\cong C^{**}$ and $(C\otimes C)^*\cong C^*\otimes C^*$ canonically, and C^* is R-finitely generated projective and R-faithful. Hence C^* is R-Azumaya if and only if $C^*\otimes C^*<\!\oplus(C^*\oplus C^*\oplus\cdots\oplus C^*)$ as C^*-C^*-module by Corollary 1.1 or Corollary 1.2 [**5**]. But this is the case if and only if $C\otimes C<\!\oplus(C\oplus C\oplus\cdots\oplus C)$ as C^*-C^*-module (or as C-C-comodule, since $C\cong C^{**}$ and $(C\otimes C)^*\cong C^*\otimes C^*$ as C^*-C^*-module. Therefore we have

LEMMA 8. *If C is R-finitely generated projective and R-faithful, then the following conditions are equivalent:*

(i) *C^* is an R-Azumaya algebra.*

(ii) *$C\otimes C<\!\oplus(C\oplus C\oplus\cdots\oplus C)$ as C^*-C^*-module.*

(iii) *The map ι of $C\otimes C$ to $\mathrm{Hom}({}_S\mathrm{Hom}({}_{C^*}C\otimes C_{C^*}, {}_{C^*}C_{C^*}), {}_SC)$ is an isomorphism and $\mathrm{Hom}({}_{C^*}C\otimes C_{C^*}, {}_{C^*}C_{C^*})$ is S-finitely generated projective, where $S=\mathrm{Hom}({}_{C^*}C_{C^*}, {}_{C^*}C_{C^*})$.*

THEOREM 1. *Suppose that C is R-finitely generated projective and R-faithful. Then C^* is an Azumaya R-algebra, if and only if the following two maps are isomorphisms;*

$$\Psi: C\otimes C \longrightarrow C\otimes_{C^*}C\otimes C \qquad \Psi(c\otimes d)=\sum c\otimes d_{(1)}\otimes d_{(2)}\ (=\sum c_{(2)}\otimes d\otimes c_{(1)})$$

$$\mu: C^*\otimes I \longrightarrow C \qquad \mu(c^*\otimes a)=c^*\cdot a\ (=a\cdot c^*)$$

where c, $d\in C$, $a\in I$ and $c^*\in C^*$.

PROOF. Suppose that C^* is an Azumaya R-algebra. Then applying Corollary 3.6 [**1**] to a C^*-C^*-module C, we see that $C^*\otimes I=C^*\otimes C^{C^*}\cong C$. Thus μ is an isomorphism. On the other hand, since C^* is an R-progenerator, R is an R-direct summand of C^* (see Corollary 4.2 [**1**]). Hence $I\cong R\otimes I<\!\oplus C^*\otimes I\cong C$. Thus I is also R-finitely generated projective. But $[C^*_{\mathfrak{m}}: R_{\mathfrak{m}}]=[C_{\mathfrak{m}}: R_{\mathfrak{m}}]$ for each maximal ideal $\mathfrak{m}$ of R. Hence I is rank 1 R-projective, and consequently, $R\cong\mathrm{Hom}(I, I)$. Then we have $\mathrm{Hom}({}_{C^*}C, {}_{C^*}C)=\mathrm{Hom}({}_{C^*}C^*\otimes I, {}_{C^*}C^*\otimes I)=C^*\otimes\mathrm{Hom}(I, I)=C^*\otimes R\cong C^*$ and $S\cong\mathrm{Hom}({}_{C^*}C_{C^*}, {}_{C^*}C_{C^*})\cong C^{*C^*}=R$. Therefore, maps γ, φ and i in (2), (5) and (8), respectively, are isomorphisms, while map ι in (1) is an isomorphism by Lemma 8. Hence Ψ is an isomorphism by Lemma 5. Conversely suppose Ψ and μ are isomorphisms. Then for the same reasons as the proof of 'only if' part, I is rank 1 R-projective, and $S\cong C^{*C^*}$ (=the center of C^*). But for any $c^*\in C^*$, $s\in S$ and $a\in I\ (=C^{C^*})$, we have $c^*\cdot(s^*\cdot a)=a\cdot(c^*\cdot s^*)=a\cdot(s^*\cdot c^*)=(a\cdot s^*)\cdot c^*=(s^*\cdot a)\cdot c^*$. This means that $SI\subset I$, and we see that $S\otimes I\cong SI=I$. Then since I is an R-progenerator, we see that S is also rank 1 R-projective. But C^* is an R-progenerator. Therefore, R is an R-direct summand of S by Corollary 4.2 [**1**]. Hence $R=S$, and i is an isomorphism. On the other hand, γ is also an isomorphism, since μ is an isomorphism. Thus we see that $(1_C\otimes\tau)\circ\nu\circ\mathrm{Hom}(\varphi, 1_C)\circ i$ is an isomorphism. Then since Ψ is an isomorphism, ι is an isomorphism. On the other hand, $\mathrm{Hom}({}_{C^*}C\otimes C_{C^*}, {}_{C^*}C_{C^*})=\mathrm{Hom}(C_{C^*}, C^*_{C^*})$ and they are C^*-finitely generated projective, since C_{C^*} is finitely generated projective. Then, they are S-finitely generated projective, since $S=R$ and C^* is R-finitely generated projective. Therefore, C^* is an Azumaya R-algebra by Lemma 8.

Now for a coalgebra C, let σ, η and $\tilde{\eta}$ be such that

$$\sigma: \mathrm{Hom}(C, C) \longrightarrow \mathrm{Hom}(C^*, C^*) \quad (\langle\sigma(f)(c^*), c\rangle=\langle c^*, f(c)\rangle)$$
$$\tilde{\eta}\,; C^*\otimes C^{*0} \longrightarrow \mathrm{Hom}(C, C) \quad (\tilde{\eta}(c^*\otimes d^{*0})(c)=d^*\cdot c\cdot c^*)$$
$$\eta: C^*\otimes C^{*0} \longrightarrow \mathrm{Hom}(C^*, C^*) \quad (\eta(c^*\otimes d^{*0})(a^*)=c^*\cdot a^*\cdot d^*)$$

for any $f\in\mathrm{Hom}(C, C)$, c^*, d^* and $a^*\in C^*$ and $c\in C$. It is easy to see that $\eta=\sigma\circ\tilde{\eta}$. Now by Lemma 3 we have

PROPOSITION 1. *Let C be an R-faithful coalgebra. Then,*

(1) *If C is R-torsionless, and if C^* is R-Azumaya, then $\tilde{\eta}$ is an isomorphism, and consequently,* $\mathrm{Hom}(C, C)\cong\mathrm{Hom}(C^*, C^*)$.

(2) *If C is R-finitely generated projective, then we have that C^* is R-Azumaya if and only if $\tilde{\eta}$ is an isomorphism.*

PROOF. This is obvious by Lemma 3 and by the well known fact that an R-algebra C^* is R-Azumaya if and only if C^* is an R-progenerator and η is an isomorphism (see Theorem 3.4 [**1**]).

§**4.** Let A be an R-algebra which is R-finitely generated projective. Denote its multiplication map by π (i.e., $\pi(a\otimes b)=ab$, for $a, b\in A$), and let $\{a_i^*, a_i\}$ be a dual basis for A_R. Since A is R-finitely generated projective, we have the following natural isomorphisms

$$\rho: A^*\otimes A^* \longrightarrow (A\otimes A)^* \quad \text{and} \quad \theta: A \longrightarrow A^{**}$$

such that $\rho(a^*\otimes b^*)(a\otimes b)=a^*(a)b^*(b)$, and $\theta(a)(a^*)=a^*(a)$ for any $a^*, b^*\in A^*$ and $a, b\in A$. As usual we set $\theta(a)=a^{**}$ for each $a\in A$. Thus $a^{**}(a^*)=a^*(a)$ for $a\in A$ and $a^*\in A^*$. Next set

(10) $$\sigma: (A\otimes A)^* \longrightarrow A^*\otimes A^* \quad (\sigma(\alpha^*)=\sum\alpha_i^*\otimes a_i^*)$$

with $\alpha_i^*\in A^*$ such that $\alpha_i^*(a)=\alpha^*(a\otimes a_i)$ for each $a\in A$ and $\alpha^*\in(A\otimes A)^*$. Then by direct computation we can easily see that $\rho\circ\sigma=1_{(A\otimes A)^*}$ and $\sigma\circ\rho=1_{A^*\otimes A^*}$. Thus $\sigma=\rho^{-1}$. Then by the same way as Proposition 1.1.2 [**6**], we can make A^* a coalgebra whose comultiplication is $\Delta=\mathrm{Hom}(\pi, 1_R)\circ\sigma$. (10) shows that

$$\Delta(a^*)=\sigma(a^*\circ\pi)=\sum b_i^*\otimes a_i^* \quad (a\in A^*), \text{ with}$$

$$b_i^*(a)=a^*\circ\pi(a\otimes a_i)=a^*(aa_i)=(a_i\cdot a^*)(a) \quad (a\in A).$$

This means that $b_i^*=a_i\cdot a^*$ and $\Delta(a^*)=\sum a_i\cdot a^*\otimes a_i^*$. Now set $C=A^*$. Then C is a coalgebra, and C^* is an algebra with $A\cong C^*$ $(=A^{**})$. But it is easily seen that θ is an algebra isomorphism, and the C^*-C^*-module structure of C as coalgebra coinsides with the A-A-module structure of $C=\mathrm{Hom}(A, R)$ regarding $A=C^*$ by θ. Therefore we have by Theorem 1

PROPOSITION 2. *Let A be an R-algebra such that A is a faithful finitely generated projective R-module, and let $\{a_i^*, a_i\}$ be a dual basis of A over R. Then, A is an Azumaya R-algebra, if and only if there exist following two isomorphisms*

$$\Phi: A^*\otimes A^* \longrightarrow A^*\otimes_A A^*\otimes A^* \quad (\Phi(a^*\otimes b^*)=\sum a^*\otimes a_i\cdot b^*\otimes a_i^*)$$

$$\mu: A\otimes H \longrightarrow A^* \quad (\mu(a\otimes h^*)=a\cdot h^*)$$

for any $a\in A$, a^*, $b^*\in A^*$ and $h^*\in H$, where $H=\{h^*\in A^* \mid h^*(ab)=h^*(ba)$ for any $a, b\in A\}$.

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