

## On the perturbation of linear operators in Banach and Hilbert spaces

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### Introduction.

This paper is concerned with the stability theory for several properties of linear operators in Banach and Hilbert spaces.

Let  $A$  be a linear operator with domain  $D(A)$  and range  $R(A)$  in a Banach space  $X$ . Let  $B$  be a linear operator in  $X$ , with  $D(B) \supset D(A)$ . Assume that

(i) there are constants  $a_0, b_0 \geq 0$  such that for all  $u \in D(A)$ ,

$$(0.1) \quad \|Bu\| \leq a_0\|u\| + b_0\|Au\|.$$

In the perturbation theory it is frequently assumed that

(ii)  $b_0$  is less than one.

In fact, under these conditions the following three facts, for example, are well known:

(P1)  $A+B$  is closed if and only if  $A$  is closed;

(P2) if  $A$  is  $m$ -accretive, with  $D(A)$  dense in  $X$ , and  $B$  is accretive then  $A+B$  is also  $m$ -accretive, i. e., if  $-A$  is the generator of a contraction semigroup on  $X$  then so is  $-(A+B)$ , too;

(P3) if  $A$  is selfadjoint and  $B$  is symmetric then  $A+B$  is also selfadjoint (when  $X$  is a Hilbert space).

The main purpose of this paper is to show that condition (ii) can be replaced by (indeed generalized to)

(iii) for every  $u \in D(A)$  there is  $g \in F(Au)$  such that

$$\operatorname{Re}(Bu, g) \geq -c\|u\|^2 - a\|Au\|\|u\| - b\|Au\|^2,$$

where  $a, b$  ( $b < 1$ ) and  $c$  are nonnegative constants.

The appearance of the duality map  $F$  on  $X$  to its adjoint  $X^*$  may be somewhat unfamiliar in the theory of linear operators. But, we need only elementary properties of the duality map. In this connection, we denote by

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$(w, g)$  the pairing between  $w \in X$  and  $g \in X^*$ :  $(w, g)$  is linear in  $w$  and semilinear in  $g$ . Noting further that if  $g \in F(Au)$  then  $(Au, g) = \|Au\|^2$ , we see by (P1) under conditions (i) and (iii) that  $2A = A + A$  is closed if  $A$  is closed. In this case of  $B = A$ , however, we can not take  $b_0 < 1$  in (0.1). This point seems to be an advantage of condition (iii).

In §1 we consider the stability of closedness and bounded invertibility. The latter property is treated in a slightly generalized form. Namely, we introduce the notion of deficiency of a closed linear operator  $A$ :  $\text{def } A = \text{codim } R(A)$ , and establish a stability theorem for invertible operators with closed ranges. §2 is devoted to the preliminaries to §3.

Now let  $A$  be a closed linear accretive operator in a Banach space. Then by definition  $A + \xi$  is invertible and  $R(A + \xi)$  is closed for  $\xi > 0$ . In §3 we consider the perturbation of deficiencies of closed accretive operators. First we show that  $\text{def } (A + B + \xi) = \text{def } (A + \xi)$  for  $\xi > 0$  if  $A$  and  $B$  satisfy conditions (i) and (iii) above, with some additional ones (Theorem 3.2). The same fact under conditions (i) and (ii) is due to Behncke-Focke [1]. In order to prove the fundamental lemma, we can apply the result prepared in §1. Furthermore, we try to generalize Theorem 3.2 (see Theorems 3.3 and 3.7). The main theorems in Okazawa [18] are special cases of these theorems. Nevertheless, the idea for proofs is similar to that in [18]. It may be a singular perturbation of the case (I) in the sense of Kato [11]. But, we have not yet succeeded in proving this. In other words, we can assert nothing about cores of the unperturbed operator. In the same manner we can generalize Theorems 4.1 and 4.5 in [18] (cf. [1], Theorem 4) although we shall not mention it.

§4 is concerned with the  $m$ -accretiveness of the sum of two linear  $m$ -accretive operators in a reflexive Banach space. We shall generalize and unify the theorems in Okazawa [17] and Sohr [20], [21]. The key lemma has already been noted in Okazawa [16]. We collect in §5 several stability theorems for (essentially) selfadjoint operators in a Hilbert space. They are applied in the last §6 to the selfadjointness problem for Schrödinger operators. We can prove that if  $t > -\frac{N}{4}(N-4)$  then  $A + tB = -\Delta + t|x|^{-2}$  with  $D(A + tB) = D(A) \cap D(B)$  is selfadjoint in  $L^2(\mathbb{R}^N)$ . Using the famous result of Kato [10], we shall reconstruct the proof of the Faris-Lavine theorem (cf. [4] and Reed-Simon [19]).

### §1. Stability of closedness and bounded invertibility.

Let  $X, Y$  be two Banach spaces. Let  $A$  be a linear operator from  $X$  to  $Y$ . Namely,  $A$  is a linear operator with domain  $D(A)$  in  $X$  and range  $R(A)$  in  $Y$ . A linear operator  $B$  from  $X$  to  $Y$  is said to be  $A$ -bounded if  $D(A) \subset D(B)$  and there exist nonnegative constants  $a_0$  and  $b_0$  such that for all  $u \in D(A)$ ,

$$(1.1) \quad \|Bu\| \leq a_0 \|u\| + b_0 \|Au\|.$$

For the notion of  $A$ -boundedness we refer to Kato [8], IV-§ 1.1. Now let  $Y^*$  be the adjoint space of  $Y$ . Then  $F$  denotes the duality map on  $Y$  to  $Y^*$ : for every  $y \in Y$ ,

$$F(y) = \{g \in Y^*; (y, g) = \|y\|^2 = \|g\|^2\}.$$

We start with the following

LEMMA 1.1. Let  $A, B$  be linear operators from  $X$  to  $Y$ . Set  $D(A+B) = D(A) \cap D(B)$ . Assume that for every  $u \in D(A+B)$  there is  $g \in F(Au)$  such that

$$(1.2) \quad \operatorname{Re}(Bu, g) \geq -c\|u\|^2 - a\|Au\|\|u\| - b\|Au\|^2,$$

where  $a, b$  ( $b < 1$ ) and  $c$  are nonnegative constants.

Then  $A$  is  $(A+B)$ -bounded:

$$(1.3) \quad \|Au\| \leq (1-b)^{-1}\|(A+B)u\| + K\|u\|, \quad u \in D(A+B),$$

and hence  $B$  is also  $(A+B)$ -bounded:

$$(1.4) \quad \|Bu\| \leq [(1-b)^{-1} + 1]\|(A+B)u\| + K\|u\|, \quad u \in D(A+B),$$

where  $K = a(1-b)^{-1} + \sqrt{c(1-b)^{-1}}$ . Furthermore, for every  $h \in F((A+B)u)$  we have

$$(1.5) \quad \operatorname{Re}(Bu, h) \geq -b(1-b)^{-1}\|(A+B)u\|^2 - K\|(A+B)u\|\|u\|.$$

PROOF. It follows from (1.2) that

$$\begin{aligned} \|Au\|^2 &= \operatorname{Re}(Au, g) \\ &\leq \operatorname{Re}((A+B)u, g) + c\|u\|^2 + a\|Au\|\|u\| + b\|Au\|^2. \end{aligned}$$

So, we have

$$(1-b)\|Au\|^2 \leq [\|(A+B)u\| + a\|u\|]\|Au\| + c\|u\|^2.$$

Solving this inequality, we obtain (1.3) and (1.4).

Next, let  $h \in F((A+B)u)$ . Then we have

$$\|(A+B)u\|^2 = \operatorname{Re}(Au, h) + \operatorname{Re}(Bu, h)$$

and hence  $\operatorname{Re}(Bu, h) \geq \|(A+B)u\|^2 - \|Au\|\|(A+B)u\|$ . Therefore, (1.5) follows from (1.3). Q. E. D.

REMARK 1.2. Let  $A, B$  be linear operators from  $X$  to  $Y$ . If  $B$  is  $A$ -bounded, then for every  $g \in F(Au)$  the inequality (1.2) (with  $c=0$ ) holds. In fact, since  $\operatorname{Re}(Bu, g) \geq -\|Bu\|\|Au\|$ , it follows from (1.1) that

$$\operatorname{Re}(Bu, g) \geq -a_0\|Au\|\|u\| - b_0\|Au\|^2.$$

THEOREM 1.3. Let  $A, B$  be linear operators from  $X$  to  $Y$ , and  $B$  be  $A$ -

bounded. Assume that for every  $u \in D(A)$  there is  $g \in F(Au)$  such that (1.2) with  $b < 1$  holds. Then  $A+B$  is closed if and only if  $A$  is closed.

PROOF. Let  $A$  be closed. Then the closedness of  $A+B$  follows from (1.3) and (1.1). To prove the converse, it suffices to note that  $\|(A+B)u\| \leq a_0\|u\| + (1+b_0)\|Au\|$  for  $u \in D(A)$ , where  $a_0$  and  $b_0$  are constants in (1.1). Q. E. D.

PROPOSITION 1.4. Let  $A, B$  be linear operators from  $X$  to  $Y$ , and  $B$  be  $A$ -bounded. Assume that for every  $u \in D(A)$  there is  $h \in F(Bu)$  such that

$$(1.6) \quad \operatorname{Re}(Au, h) \geq -c\|u\|^2 - a\|Bu\|\|u\|,$$

where  $c$  and  $a$  are nonnegative constants.

If  $A$  is closed, then  $A+tB$  is closed for all  $t > 0$ .

PROOF. Let  $t > 0$ . Then we see from (1.6) that

$$\operatorname{Re}(Au, g) \geq -tc\|u\|^2 - a\|tBu\|\|u\|,$$

where  $g = th \in F(tBu)$ . Therefore, it follows from Lemma 1.1 ( $b=0$ ) that  $A$  is  $(A+tB)$ -bounded:

$$(1.7) \quad \|Au\| \leq 2\|(A+tB)u\| + (a + \sqrt{tc})\|u\|, \quad u \in D(A).$$

The closedness of  $A+tB$  follows from (1.7) and (1.1). Q. E. D.

REMARK 1.5. In Theorem 1.3 and Proposition 1.4 the term "closed" can be replaced by "closable".

Let  $A$  be a closed linear operator from  $X$  to  $Y$ . Then  $A$  is said to be *semi-Fredholm* if  $R(A)$  is closed and at least one of  $\operatorname{nul} A$  and  $\operatorname{def} A$  is finite. Here  $\operatorname{nul} A$  and  $\operatorname{def} A$  are the nullity and deficiency of  $A$ , respectively:

$$\operatorname{nul} A = \dim N(A), \quad N(A) = \{u \in D(A); Au = 0\},$$

$$\operatorname{def} A = \operatorname{codim} R(A) = \dim Y/R(A).$$

For a semi-Fredholm operator  $A$  the *index* is well-defined as

$$\operatorname{ind} A = \operatorname{nul} A - \operatorname{def} A.$$

We note further that a closed linear operator  $A$  from  $X$  to  $Y$  has closed range if and only if

$$\gamma(A) = \inf \left\{ \frac{\|Au\|}{\operatorname{dist}(u, N(A))}; u \in D(A) \right\} > 0.$$

If  $A$  is invertible, i. e.,  $N(A) = \{0\}$ , then  $\|Au\| \geq \gamma(A)\|u\|$  for  $u \in D(A)$ . For these facts we refer to Kato [8] and Goldberg [5].

PROPOSITION 1.6. Let  $A, B$  be linear operators from  $X$  to  $Y$ , with  $D(A) \subset D(B)$ . Assume that for all  $t \in [0, 1]$ ,  $A+tB$  is closed and invertible, with  $\gamma(A+tB) > 0$ . Then  $\operatorname{ind}(A+tB)$  is constant and hence  $\operatorname{ind}(A+B) = \operatorname{ind} A$ .

PROOF. By assumption there are constants  $c(t), k(t) > 0$  such that for all  $u \in D(A)$ ,

$$\|(A+tB)u\| \leq c(t)\|Au\|, \quad \|Au\| \leq k(t)\|(A+tB)u\|;$$

see Kato [8], IV-§ 1.1. Therefore,  $B$  is  $(A+tB)$ -bounded for  $t \in [0, 1]$ :  $\|Bu\| \leq [c(1)+1]k(t)\|(A+tB)u\|$ ,  $u \in D(A)$ . Now set

$$M = \{t \in [0, 1]; \text{ind}(A+tB) \neq \text{ind } A\}.$$

We want to show that  $M$  is empty. Assuming the contrary, set  $t_0 = \inf M$ . Then it follows from the stability theorem for semi-Fredholm operators that  $t_0 > 0$  and that  $\text{ind}(A+tB)$  is constant near  $t=t_0$  (see Kato [8], Theorem IV-5.22). But, since  $\text{ind}(A+tB) = \text{ind } A$  for  $t < t_0$ , this leads to a contradiction:  $\text{ind}(A+tB) = \text{ind } A$  near  $t=t_0$ .  
Q. E. D.

REMARK 1.7. The above proposition generalizes both Theorem 1 in Wüst [23] and Corollary 1 to Theorem 1 in Behncke-Focke [1].

LEMMA 1.8. Let  $A, B$  be linear operators from  $X$  to  $Y$ , with  $D(A) \subset D(B)$ . Let  $A$  be closed and invertible, with  $\gamma(A) > 0$ . Assume that for every  $u \in D(A)$  there is  $g \in F(Au)$  such that (1.2) holds and  $c\gamma^{-2} + a\gamma^{-1} + b < 1$ , where  $\gamma = \gamma(A)$ . Then  $A+B$  is invertible, with the estimate

$$(1.8) \quad \|Au\| \leq \frac{1}{1 - c\gamma^{-2} - a\gamma^{-1} - b} \|(A+B)u\|, \quad u \in D(A).$$

PROOF. Since  $\|u\| \leq \gamma^{-1}\|Au\|$ , it follows from (1.2) that

$$\text{Re}(Bu, g) \geq -(c\gamma^{-2} + a\gamma^{-1} + b)\|Au\|^2, \quad u \in D(A).$$

So, we see that

$$\begin{aligned} \text{Re}((A+B)u, g) &= \|Au\|^2 + \text{Re}(Bu, g) \\ &\geq (1 - c\gamma^{-2} - a\gamma^{-1} - b)\|Au\|^2, \quad u \in D(A). \end{aligned}$$

Therefore, we obtain (1.8) and  $A+B$  is also invertible.  
Q. E. D.

THEOREM 1.9. Let  $A$  and  $B$  be linear operators from  $X$  to  $Y$ , and  $B$  be  $A$ -bounded. Let  $A$  be closed and invertible, with  $\gamma(A) > 0$ . Assume that for every  $u \in D(A)$  there is  $g \in F(Au)$  such that (1.2) holds. If

$$c[\gamma(A)]^{-2} + a[\gamma(A)]^{-1} + b < 1,$$

then  $A+B$  is closed and invertible, with  $\gamma(A+B) > 0$  and  $\text{ind}(A+B) = \text{ind } A$ .

PROOF. It follows from Theorem 1.3 and Lemma 1.8 that for all  $t \in [0, 1]$ ,  $A+tB$  is closed and invertible, with  $\gamma(A+tB) > 0$ . Therefore, by Proposition 1.6,  $\text{ind}(A+tB)$  is constant.  
Q. E. D.

Finally, let  $B(Y, X)$  be the set of all bounded linear operators on  $Y$  to  $X$ .

COROLLARY 1.10. Let  $A, B$  be linear operators from  $X$  to  $Y$ , and  $B$  be  $A$ -

bounded. Let  $A$  be invertible and  $A^{-1} \in B(Y, X)$  (so that  $A$  is closed). Assume that for every  $u \in D(A)$  there is  $g \in F(Au)$  such that (1.2) holds and  $c\|A^{-1}\|^2 + a\|A^{-1}\| + b < 1$ . Then  $A+B$  is invertible and  $(A+B)^{-1} \in B(Y, X)$  with

$$(1.9) \quad \|(A+B)^{-1}\| \leq [1 - c\|A^{-1}\|^2 - a\|A^{-1}\| - b]^{-1} \|A^{-1}\|.$$

PROOF. Since  $\gamma(A) = \|A^{-1}\|^{-1}$ , the assumption of Theorem 1.9 is satisfied and (1.9) follows from (1.8). Q. E. D.

REMARK 1.11. Theorem 1.3 and Corollary 1.10 are respectively generalizations of Theorems IV-1.1 and IV-1.16 in Kato [8].

## § 2. Preliminaries.

Let  $X$  be a Banach space and  $X^*$  be the adjoint space of  $X$ . Let  $S$  be a linear operator with domain  $D(S)$  and range  $R(S)$  in  $X$ . We denote by  $S^*$  the adjoint operator of  $S$  when  $D(S)$  is dense in  $X$ . Let  $S$  be a closed linear operator in  $X$ . Following Kato [8], we denote by  $\text{def } S$  the deficiency of  $S$ :

$$\text{def } S = \text{codim } R(S) = \dim X/R(S).$$

In this paper we shall assume that  $R(S)$  is closed.

The following lemma is simple but useful (see [8], Problem III-1.42).

LEMMA 2.1. Let  $Y$  and  $Z$  be subspaces of a linear space. If  $\text{codim } Y < \dim Z < \infty$ , then  $\dim(Y \cap Z) > 0$ .

We denote by  $\tilde{A}$  the closure of  $A$  when  $A$  is closable.

PROPOSITION 2.2. Let  $S$  be a densely defined linear operator in  $X$ , with  $D(S^*)$  dense in  $X^*$ . Let  $A$  be a closable linear operator in  $X$ . Let  $tS+A$  be closed for all  $t \in (0, 1]$ . Assume that

(i) there is  $\xi > 0$  such that  $R(\tilde{A} + \xi)$  is closed;

(ii) for every  $v(t) \in R(tS + A + \xi)$  there is  $u(t) \in D(S) \cap D(A)$  and a constant  $M > 0$  such that

$$(2.1) \quad tSu(t) + Au(t) + \xi u(t) = v(t), \quad t \in (0, 1],$$

and

$$(2.2) \quad \|u(t)\| + \|Au(t)\| \leq M\|v(t)\|, \quad t \in (0, 1].$$

If  $\text{def } (tS + A + \xi)$  is constant, then we have

$$\text{def } (\tilde{A} + \xi) \leq \text{def } (tS + A + \xi).$$

PROOF. We may assume that  $\text{def } (tS + A + \xi) = k < \infty$ . Consequently,  $R(tS + A + \xi)$  is closed (see e. g. [5], Corollary IV-1.13). First suppose that  $\text{def } (\tilde{A} + \xi) = l < \infty$  and  $l > k$ . Then there exist two subspaces  $Y(t)$  and  $Y_0$  such that

$$R(tS + A + \xi) \oplus Y(t) = X = R(\tilde{A} + \xi) \oplus Y_0.$$

Since  $\dim Y(t) < \dim Y_0$ , we can find  $v(t) \in R(tS + A + \xi) \cap Y_0$  with  $\|v(t)\| = 1$  (see Lemma 2.1). Since  $Y_0$  is locally compact, there are a sequence  $\{t_n\}$  and  $v \in Y_0$  such that  $t_n \rightarrow +0$  and  $v(t_n) \rightarrow v$  ( $n \rightarrow \infty$ ). We shall show that  $v = 0$  in contradiction to  $\|v(t_n)\| = 1$ . Let  $f \in X^*$  be an annihilator of  $R(\tilde{A} + \xi)$ :  $(u, f) = 0$  for all  $u \in R(\tilde{A} + \xi)$ , with the property  $(v, f) = \|v\|$  and  $\|f\| = \|v\| / \text{dist}(v, R(\tilde{A} + \xi))$  (see e.g. [8], Theorem III-1.22). Let  $u(t)$  be as in condition (ii). Then, since  $D(S^*)$  is dense in  $X^*$ , it follows from (2.2) that  $t_n S u(t_n) \rightarrow 0$  ( $n \rightarrow \infty$ ) weakly. Also, we see from (2.1) that  $(v(t_n), f) = t_n (S u(t_n), f)$ . Going to the limit  $n \rightarrow \infty$ , we obtain  $(v, f) = 0$  and hence  $v = 0$  because of  $(v, f) = \|v\|$ .

Next, suppose that  $\text{def}(\tilde{A} + \xi) = \infty$ . Let  $Y_0$  be a  $(k+1)$ -dimensional subspace such that  $R(\tilde{A} + \xi) \cap Y_0 = \{0\}$ . Arguing as above, we are led to a contradiction.

Q. E. D.

We note that  $w \in R(tS + A + \xi)$  for all  $t \in (0, 1]$  implies  $w \in R(\tilde{A} + \xi)$ . In fact,  $w - (A + \xi)u(t) = tSu(t) \rightarrow 0$  ( $t \rightarrow +0$ ) weakly and  $R(\tilde{A} + \xi)$  is weakly closed.

REMARK 2.3. The above proposition extends Theorem 2 in [1] (see also Remark 3.5 below).

### § 3. Stability of indices of closed accretive operators.

Let  $X$  be a Banach space. A linear operator  $A$  with domain  $D(A)$  and range  $R(A)$  in  $X$  is said to be *accretive* if

$$\|(A + \xi)u\| \geq \xi \|u\| \quad \text{for all } u \in D(A) \text{ and } \xi > 0.$$

It is well known that  $R(A + \xi) = X$  either for every  $\xi > 0$  or for no  $\xi > 0$ ; in the former case we say that  $A$  is *m-accretive*. Accordingly, an *m-accretive* operator is necessarily closed.

Let  $F$  be the duality map on  $X$  to  $X^*$ . Then a linear operator  $A$  in  $X$  is accretive if and only if for every  $u \in D(A)$  there is  $f \in F(u)$  such that  $\text{Re}(Au, f) \geq 0$  (see Kato [9]). In this connection, we note that if  $A$  is *m-accretive* and densely defined then

$$(3.1) \quad \text{Re}(Au, f) \geq 0 \quad \text{for all } f \in F(u);$$

see Tanabe [22], Theorem 2.1.5.

Now let  $A$  be a closed linear accretive operator in  $X$ . Then by definition the nullspace  $N(A + \xi)$  is trivial and  $R(A + \xi)$  is closed for  $\xi > 0$ . Namely,  $A + \xi$  is a semi-Fredholm operator, with

$$(3.2) \quad \text{ind}(A + \xi) = -\text{def}(A + \xi).$$

$A$  is *m-accretive* if and only if  $\text{ind}(A + \xi) = 0$  for some  $\xi > 0$ .

The following lemma, which was first noted by Behncke-Focke [1], is fundamental.

LEMMA 3.1. Let  $A, B$  be linear operators in  $X$ , with  $D(A) \subset D(B)$ . Assume that for all  $t \in [0, 1]$ ,  $A + tB$  is closed and accretive. Then  $\text{ind}(A + tB + \xi)$  is constant. In particular,  $\text{ind}(A + B + \xi) = \text{ind}(A + \xi)$  for  $\xi > 0$ .

PROOF. By assumption  $(A + \xi) + tB$  is closed and invertible, with  $\gamma(A + \xi + tB) \geq \xi$ , for  $t \in [0, 1]$  and  $\xi > 0$ . Therefore, the conclusion follows from Proposition 1.6. Q. E. D.

In view of this lemma and Theorem 1.3, we obtain

THEOREM 3.2. Let  $A$  be a densely defined and closed linear operator in  $X$ . Let  $B$  be a linear accretive operator in  $X$ , with  $D(B) \supset D(A)$ . Let  $A + tB$  be accretive ( $0 \leq t \leq 1$ ). Assume that for every  $u \in D(A)$  there is  $g \in F(Au)$  such that

$$\text{Re}(Bu, g) \geq -c\|u\|^2 - a\|Au\|\|u\| - b\|Au\|^2,$$

where  $a, b$  ( $b < 1$ ) and  $c$  are nonnegative constants.

Then  $A + B$  is closed and  $\text{ind}(A + B + \xi) = \text{ind}(A + \xi)$  for  $\xi > 0$ .

PROOF. It suffices to show that  $B$  is  $A$ -bounded. But, since  $D(B)$  is dense in  $X$ ,  $B$  must be closable (see Lumer-Phillips [13], Lemma 3.3). Applying the closed graph theorem, we can conclude the  $A$ -boundedness of  $B$ . Q. E. D.

When  $X$  is a Hilbert space, it is easy to see that the above theorem is a corollary of Theorem 1.9. But, the details may be omitted.

THEOREM 3.3. Let  $A$  be a linear accretive operator in  $X$ . Let  $S$  be a densely defined and closed linear accretive operator in  $X$ , with  $D(S) \subset D(A)$ . Assume that for every  $u \in D(S)$  [there is  $f \in F(u)$  such that  $\text{Re}(Su, f) \geq 0$  and  $\text{Re}(Au, f) \geq 0$ . Assume further that

(i) for every  $u \in D(S)$  there is  $h \in F(Su)$  such that

$$(3.3) \quad \text{Re}(Au, h) \geq -c\|u\|^2 - a\|Su\|\|u\|,$$

where  $c$  and  $a$  are nonnegative constants;

(ii)  $D(S^*)$  is dense in  $X^*$ .

Then  $\text{ind}(\tilde{A} + \xi) \geq \text{ind}(S + \xi)$  for  $\xi > 0$ .

If in particular  $X$  is reflexive, then condition (ii) is redundant.

PROOF. First we note that  $A$  is closable and  $R(\tilde{A} + \xi)$  is closed for  $\xi > 0$ . Now let  $t > 0$ . Then it follows from (3.3) that

$$\text{Re}(Au, g(t)) \geq -tc\|u\|^2 - a\|tSu\|\|u\|,$$

where  $g(t) = th \in F(tSu)$ . Therefore, by Theorem 3.2,  $tS + A$  is closed and  $\text{ind}(tS + A + \xi) = \text{ind}(tS + \xi) = \text{ind}(S + \xi)$  for  $\xi > 0$ . Next, let  $v(t) \in R(tS + A + \xi)$  for  $t \in (0, 1]$ . Then there exists a unique family  $\{u(t)\}$  in  $D(S)$  such that

$$(3.4) \quad tSu(t) + Au(t) + \xi u(t) = v(t), \quad t \in (0, 1].$$

Taking a suitable  $h(t) \in F(Su(t))$ , we see from (3.3) that



$$\begin{aligned} t\|Su(t)\|^2 &\leq \operatorname{Re}((tS+A)u(t), h(t)) + a\|Su(t)\|\|u(t)\| + c\|u(t)\|^2 \\ &\leq (\|v(t)\| + (\xi + a)\|u(t)\|)\|Su(t)\| + c\|u(t)\|^2. \end{aligned}$$

Since  $\|u(t)\| \leq \xi^{-1}\|v(t)\|$ , it follows that

$$t\|Su(t)\|^2 \leq (2 + a\xi^{-1})\|Su(t)\|\|v(t)\| + c\xi^{-2}\|v(t)\|^2.$$

Solving this inequality, we have  $\|tSu(t)\| \leq [2 + (a + \sqrt{tc})\xi^{-1}]\|v(t)\|$ . Thus, we obtain (2.2) with  $M = 4 + (1 + a + \sqrt{c})\xi^{-1}$ . In view of (3.2), the conclusion follows from Proposition 2.2. The final assertion is well known (see e.g. [8], Theorem III-5.29). Q. E. D.

**COROLLARY 3.4.** *Let  $S$  be a densely defined and closed accretive operator in  $X$ . Let  $B$  be a linear accretive operator in  $X$ , with  $D(B) \supset D(S)$ . Assume that for every  $u \in D(S)$  there is  $f \in F(u)$  such that  $\operatorname{Re}(Su, f) \geq 0$  and  $\operatorname{Re}(Bu, f) \geq 0$ . Assume further that for every  $u \in D(S)$  there is  $h \in F(Su)$  such that*

$$(3.5) \quad \operatorname{Re}(Bu, h) \geq -c\|u\|^2 - a\|Su\|\|u\| - \|Su\|^2,$$

where  $c$  and  $a$  are nonnegative constants. Assume that  $D(S^*)$  is dense in  $X^*$  when  $X$  is non-reflexive. Then

$$\operatorname{ind}((S+B)^\sim + \xi) \geq \operatorname{ind}(S + \xi) \quad \text{for } \xi > 0$$

In fact, (3.5) implies (3.3) with  $A = S + B$ .

**REMARK 3.5.** (3.5) with  $c = 0$  is satisfied if

$$\|Bu\| \leq a\|u\| + \|Su\| \quad \text{for all } u \in D(S) \subset D(B).$$

In this case the same conclusion can be obtained under the assumption that  $D(B^*)$ , rather than  $D(S^*)$ , is dense in  $X^*$  (see Behncke-Focke [1], Theorem 2).

Combining Lemma 3.1 with Proposition 1.4, we obtain

**PROPOSITION 3.6.** *Let  $A$  be a densely defined and closed linear operator in  $X$ . Let  $B$  be a linear accretive operator in  $X$ , with  $D(B) \supset D(A)$ . Let  $A + tB$  be accretive ( $0 \leq t \leq 1$ ). Assume that for every  $u \in D(A)$  there is  $h \in F(Bu)$  such that (1.6) holds.*

*Then  $A + B$  is closed and  $\operatorname{ind}(A + B + \xi) = \operatorname{ind}(A + \xi)$  for  $\xi > 0$ .*

In fact, the  $A$ -boundedness of  $B$  was noted in the proof of Theorem 3.2.

Here, we present another application of Proposition 2.2.

**THEOREM 3.7.** *Let  $A$  be a linear accretive operator in  $X$ . Let  $S$  be a densely defined and closed linear accretive operator in  $X$ , with  $D(S) \subset D(A)$ . Assume that for every  $u \in D(S)$  there is  $f \in F(u)$  such that  $\operatorname{Re}(Su, f) \geq 0$  and  $\operatorname{Re}(Au, f) \geq 0$ . Assume further that*

(i) *for every  $u \in D(S)$  there is  $h \in F(Au)$  such that*

$$(3.6) \quad \operatorname{Re}(Su, h) \geq -c\|u\|^2 - a\|Au\|\|u\|,$$

where  $c$  and  $a$  are nonnegative constants;

(ii)  $D(S^*)$  is dense in  $X^*$ .

Then  $\text{ind}(\tilde{A} + \xi) \geq \text{ind}(S + \xi)$  for  $\xi > 0$ .

If in particular  $X$  is reflexive, then condition (ii) is redundant.

PROOF. As noted in the proof of Theorem 3.3, condition (i) of Proposition 2.2 is satisfied. Now let  $t > 0$ . Then it follows from (3.6) that

$$\text{Re}(tSu, h) \geq -tc\|u\|^2 - ta\|Au\|\|u\|.$$

Therefore, by Proposition 3.6,  $tS + A$  is closed and

$$\text{ind}(tS + A + \xi) = \text{ind}(tS + \xi) = \text{ind}(S + \xi), \quad \xi > 0.$$

It remains to show that  $\|Au(t)\|$  is bounded by  $\|v(t)\|$ , where  $u(t)$  is a unique solution of (3.4). By virtue of (3.6) there is  $g(t) \in F(Au(t))$  such that

$$\begin{aligned} \|Au(t)\|^2 &= (Au(t), g(t)) \\ &\leq \text{Re}((tS + A)u(t), g(t)) + tc\|u(t)\|^2 + ta\|Au(t)\|\|u(t)\| \\ &\leq [\|(tS + A)u(t)\| + ta\|u(t)\|]\|Au(t)\| + tc\|u(t)\|^2 \\ &\leq (2 + ta\xi^{-1})\|Au(t)\|\|v(t)\| + tc\xi^{-2}\|v(t)\|^2, \end{aligned}$$

where we have used  $\|u(t)\| \leq \xi^{-1}\|v(t)\|$ . Solving this inequality, we can obtain (2.2) with  $M = 2 + (1 + a + \sqrt{c})\xi^{-1}$ . Therefore, the conclusion follows from Proposition 2.2. Q. E. D.

REMARK 3.8. The  $m$ -accretive version of Theorem 3.7 improves Theorem 3.3 in [18] in which  $D(A^*)$  is also assumed to be dense in  $X^*$ .

REMARK 3.9. Let  $X$  be reflexive. Let  $A$  and  $B$  satisfy the assumption of Theorem 3.2. Then we have (3.5) with  $S$  replaced by  $A$ . Therefore, by Corollary 3.4,

$$\text{ind}(A + B + \xi) \geq \text{ind}(A + \xi), \quad \xi > 0;$$

note that  $A + B$  is closed (use Theorem 1.3). On the other hand, (3.5) (with  $S = A$ ) implies (3.6) with  $S$  replaced by  $A + B$ . So, by Theorem 3.7,

$$\text{ind}(A + \xi) \geq \text{ind}(A + B + \xi), \quad \xi > 0.$$

Thus, the conclusion of Theorem 3.2 holds. This means that Theorem 3.3 together with Theorem 3.7 generalizes Theorem 3.2.

For the later use, we want to state

THEOREM 3.10. Let  $X$  be reflexive. Let  $S$  and  $B$  be linear accretive operators in  $X$ , with  $D(S) \subset D(B)$ . Assume that for every  $u \in D(S)$  there is  $h \in F(Su)$  such that

$$\text{Re}(Bu, h) \geq -c\|u\|^2 - a\|Su\|\|u\| - b\|Su\|^2,$$

where  $a, b$  ( $b \leq 1$ ) and  $c$  are nonnegative constants.

In the case of  $b < 1$ ,  $S+B$  is  $m$ -accretive if and only if  $S$  is  $m$ -accretive. In the case of  $b=1$ ,  $(S+B)^\sim$  is  $m$ -accretive if  $S$  is  $m$ -accretive.

PROOF. First we note that an  $m$ -accretive operator in a reflexive space is necessarily densely defined (see Kato [7] or Yosida [24], VIII-§ 4). Suppose that  $S$  is  $m$ -accretive. Then we see from (3.1) that  $S+tB$  is accretive ( $0 \leq t \leq 1$ ). Therefore, the conclusion for the case of  $b < 1$  follows from Lemma 3.1 and Theorem 1.3. By virtue of (3.1) the case of  $b=1$  is contained in Corollary 3.4.

Q. E. D.

The following corollary is due to Nelson [14], Gustafson [6], Chernoff [2] and Okazawa [15].

COROLLARY 3.11. Let  $X$  be reflexive. Let  $S, B$  be linear accretive operators in  $X$ , with  $D(S) \subset D(B)$ . Assume that there are constants  $a, b \geq 0$  ( $b \leq 1$ ) such that for all  $u \in D(S)$ ,  $\|Bu\| \leq a\|u\| + b\|Su\|$ . Then the conclusion of Theorem 3.10 holds.

#### § 4. The sum of $m$ -accretive operators in a reflexive space.

Throughout this section  $X$  is assumed to be reflexive. Let  $A$  be a linear  $m$ -accretive operator in  $X$ . Then  $\{A_n\}$  denotes the Yosida approximation of  $A$ :

$$A_n = A \left(1 + \frac{1}{n}A\right)^{-1} = n \left[1 - \left(1 + \frac{1}{n}A\right)^{-1}\right] \in B(X), \quad n=1, 2, \dots,$$

where  $B(X)$  is the set of all bounded linear operators on  $X$  to  $X$ .  $A$  is approximated by  $\{A_n\}$  in the following sense:

$$\|Au - A_nu\| \longrightarrow 0 \quad (n \rightarrow \infty) \quad \text{for every } u \in D(A).$$

Let  $B$  be a linear  $m$ -accretive operator in  $X$  and consider the sequence  $\{A_n+B\}$ . Since  $A_n \in B(X)$ ,  $A_n+B$  is  $m$ -accretive by Corollary 3.11. Consequently, for every  $v \in X$  there is a unique  $u_n \in D(B)$  such that

$$(4.1) \quad A_nu_n + Bu_n + u_n = v, \quad n=1, 2, \dots.$$

LEMMA 4.1. Let  $A$  and  $B$  be linear  $m$ -accretive operators in  $X$ , and  $u_n$  be a unique solution of the equation (4.1). Assume that for every  $v \in X$ ,  $\{\|A_nu_n\|\}$  is bounded. Then  $A+B$  is also  $m$ -accretive.

A proof of this lemma will be found in [16].

Let  $A$  be an arbitrary closed linear operator in  $X$ . Then a linear manifold  $D$  contained in  $D(A)$  is called a *core* of  $A$  if the closure of the restriction of  $A$  to  $D$  is again  $A$ :  $(A|D)^\sim = A$ .

Let  $F$  be the duality map on  $X$  to  $X^*$ .

THEOREM 4.2. Let  $X$  be reflexive. Let  $A$  and  $B$  be linear  $m$ -accretive operators in  $X$ . Assume that for every  $u \in D(B)$  there is a sequence  $\{f_n\}$  such

that  $f_n \in F(A_n u)$  and

$$(4.2) \quad \operatorname{Re}(Bu, f_n) \geq -c\|u\|^2 - a\|A_n u\|\|u\| - b\|A_n u\|^2,$$

where  $a, b$  ( $b \leq 1$ ) and  $c$  are nonnegative constants.

If  $b < 1$  then  $A+B$  is  $m$ -accretive and  $D(A+B)$  is a core of  $A$ . If in particular  $b=0$  then  $D(A+B)$  is a core of both  $A$  and  $B$ . If  $b=1$  then  $(A+B)^\sim$  is  $m$ -accretive and

$$[(A+B)^\sim + \zeta]^{-1} = s\text{-}\lim_{n \rightarrow \infty} (A_n + B + \zeta)^{-1}, \quad \operatorname{Re} \zeta > 0.$$

PROOF. First let  $b < 1$ . Then it follows from Lemma 1.1 that for every  $u \in D(B)$ ,

$$\begin{aligned} \|A_n u\| &\leq (1-b)^{-1} \|(A_n + B)u\| + K\|u\| \\ &\leq (1-b)^{-1} \|A_n u + Bu + u\| + [K + (1-b)^{-1}]\|u\|. \end{aligned}$$

Now let  $u_n$  be a unique solution of (4.1). Then, since  $\|u_n\| \leq \|v\|$ , we obtain

$$\|A_n u_n\| \leq [K + 2(1-b)^{-1}]\|v\| \quad \text{for every } v \in X.$$

Therefore,  $A+B$  is  $m$ -accretive by Lemma 4.1.

Next, let  $b=1$ . Then (4.2) can be written as

$$(4.3) \quad \operatorname{Re}\left(\frac{1}{2}Bu, f_n\right) \geq -\frac{c}{2}\|u\|^2 - \frac{a}{2}\|A_n u\|\|u\| - \frac{1}{2}\|A_n u\|^2.$$

Thus,  $A + \frac{1}{2}B$  is  $m$ -accretive as shown above and hence  $D(A+B)$  is dense in  $X$ . Let  $u \in D(A+B)$ . Then, since  $\|f_n\| = \|A_n u\| \rightarrow \|Au\|$ ,  $\{f_n\}$  is bounded as  $n$  tends to infinity. Consequently, there exists  $f \in X^*$  such that  $f_{n_p} \rightarrow f$  ( $p \rightarrow \infty$ ) weakly and  $\|f\| \leq \liminf_{p \rightarrow \infty} \|f_{n_p}\| = \|Au\|$ . On the other hand,  $\|Au\|^2 = \lim_{p \rightarrow \infty} (A_{n_p} u, f_{n_p}) = (Au, f)$  and hence  $\|Au\| \leq \|f\|$ . Namely,  $f \in F(Au)$ . Going to the limit in (4.3) with  $n = n_p$ , we obtain for every  $u \in D(A+B)$ ,

$$(4.4) \quad \operatorname{Re}\left(\frac{1}{2}Bu, f\right) \geq -\frac{c}{2}\|u\|^2 - \frac{a}{2}\|Au\|\|u\| - \frac{1}{2}\|Au\|^2.$$

It then follows from Lemma 1.1 that for every  $g \in F\left(\left(A + \frac{1}{2}B\right)u\right)$ ,

$$\operatorname{Re}\left(\frac{1}{2}Bu, g\right) \geq -\left\|\left(A + \frac{1}{2}B\right)u\right\|^2 - (a + \sqrt{c})\left\|\left(A + \frac{1}{2}B\right)u\right\|\|u\|.$$

Applying Theorem 3.10, we see that the closure of  $A+B = \left(A + \frac{1}{2}B\right) + \frac{1}{2}B$  is  $m$ -accretive.

Here we mention the assertions on cores. It follows from (4.4) that

$$\operatorname{Re}((A+B)u, f) \geq -c\|u\|^2 - a\|Au\|\|u\|, \quad f \in F(Au).$$

Since  $A+B$  is  $m$ -accretive,  $D(A+B)$  is a core of  $A$  (see [18], Theorem 3.3). If  $b=0$  then we have

$$\operatorname{Re}(Bu, f) \geq -c\|u\|^2 - a\|Au\|\|u\|, \quad f \in F(Au).$$

So, we see from Lemma 1.1 that for every  $h \in F((A+B)u)$ ,

$$\operatorname{Re}(Bu, h) \geq -(a + \sqrt{c})\|(A+B)u\|\|u\|.$$

Therefore,  $D(A+B)$  is a core of  $B$  (see [18], Theorem 3.1).

The final assertion follows from the fact that  $(A_n+B)u \rightarrow (A+B)u$  ( $n \rightarrow \infty$ ) for every  $u \in D(A+B)$  and  $D(A+B)$  is a core of  $(A+B)^\sim$  (see Kato [8], Theorem VIII-1.5).  
Q. E. D.

REMARK 4.3. Let  $A, B$  be as in Theorem 4.2. Suppose that for every  $u \in D(B)$  there is  $f \in F(Bu)$  such that

$$\operatorname{Re}(A_n u, f) \geq -c\|u\|^2 - a\|Bu\|\|u\| - b\|Bu\|^2, \quad n=1, 2, \dots,$$

where  $a, b$  ( $b < 1$ ) and  $c$  are nonnegative constants.

Then  $A+B$  is  $m$ -accretive and  $D(A+B)$  is a core of  $B$ . If in particular  $b=0$  then  $D(A+B)$  is a core of both  $A$  and  $B$ .

In fact, by using Lemma 1.1 again, we can show that  $\{\|Bu_n\|\}$  (and hence  $\{\|A_n u_n\|\}$ ) in (4.1) is bounded for every  $v \in X$ . The proof for the statement on cores is similar to that in Theorem 4.2.

In the assumption of the next theorem, the Yosida approximation is not contained explicitly while the proof of it is based on Theorem 4.2.

THEOREM 4.4. Let  $A$  and  $B$  be linear  $m$ -accretive operators in  $X$ . Let  $D$  be a core of  $B$  such that  $(1+n^{-1}A)^{-1}D \subset D$  ( $n=1, 2, \dots$ ). Assume that for every  $u \in D_0 = (1+A)^{-1}D$  there is  $f \in F(Au)$  such that

$$\operatorname{Re}(Bu, f) \geq -c\|u\|^2 - a\|Au\|\|u\| - b\|Au\|^2,$$

where  $a, b$  ( $b \leq 1$ ) and  $c$  are nonnegative constants.

Then the conclusion of Theorem 4.2 holds.

PROOF. We shall show that the assumption of Theorem 4.2 is satisfied. Let  $v \in D$  and  $g_n \in F(A_n v)$ . Then

$$\frac{1}{n} g_n \in F\left(v - \left(1 + \frac{1}{n} A\right)^{-1} v\right)$$

and we have

$$\begin{aligned} (4.5) \quad & \operatorname{Re}(Bv, g_n) \\ &= n \operatorname{Re}\left(Bv - B\left(1 + \frac{1}{n} A\right)^{-1} v, \frac{1}{n} g_n\right) + \operatorname{Re}\left(B\left(1 + \frac{1}{n} A\right)^{-1} v, g_n\right) \end{aligned}$$

$$\geq \operatorname{Re}\left(B\left(1+\frac{1}{n}A\right)^{-1}v, g_n\right).$$

Noting that  $(1+n^{-1}A)^{-1}v \in D_0$ , we see by assumption that there is  $f_n \in F(A_nv)$  such that

$$\begin{aligned} & \operatorname{Re}\left(B\left(1+\frac{1}{n}A\right)^{-1}v, f_n\right) \\ & \geq -c\left\|\left(1+\frac{1}{n}A\right)^{-1}v\right\|^2 - a\|A_nv\|\left\|\left(1+\frac{1}{n}A\right)^{-1}v\right\| - b\|A_nv\|^2. \end{aligned}$$

Setting  $g_n = f_n$  in (4.5), we obtain (4.2) with  $u$  replaced by  $v \in D$ . Now let  $u \in D(B)$ . Then there exists a sequence  $\{u_p\}$  in  $D$  such that  $u_p \rightarrow u$  and  $Bu_p \rightarrow Bu$  as  $p \rightarrow \infty$ . Let  $f_n^{(p)} \in F(A_nu_p)$ . Then it follows that

$$(4.6) \quad \operatorname{Re}(Bu_p, f_n^{(p)}) \geq -c\|u_p\|^2 - a\|A_nu_p\|\|u_p\| - b\|A_nu_p\|^2.$$

In the same way as in the proof of Theorem 4.2, we can show that there is  $f_n \in F(A_nu)$  such that

$$f_n = \text{w-lim}_{k \rightarrow \infty} f_n^{(p_k)},$$

where  $\{f_n^{(p_k)}\}$  is a suitable subsequence of  $\{f_n^{(p)}\}$ . Going to the limit  $k \rightarrow \infty$  in (4.6) with  $p$  replaced by  $p_k$ , we obtain just (4.2). Q. E. D.

REMARK 4.5. As an example of  $D$  in the above theorem we have in mind  $D(A^\infty) = \bigcap_{n=1}^{\infty} D(A^n)$ . In this case  $D_0 = (1+A)^{-1}D = D(A^\infty)$  is also a core of  $A$ .

REMARK 4.6. When  $X$  is a Hilbert space, the difference of the above theorems from those in [17] is the fact that  $a \neq 0$ . This makes sense if  $b=1$ . We note further that Satz 2.1 in Sohr [20] corresponds to the case of  $b < 1$  and  $c=0$  in our Theorem 4.4.

If in particular  $X$  is a Hilbert space, then there is another corollary of Theorem 4.2.

Let  $H$  be a Hilbert space. Then we have

THEOREM 4.7. *Let  $A, B$  be linear  $m$ -accretive operators in  $H$ . Assume that there is a constant  $\alpha > 0$  such that  $\operatorname{Re}(Au, u) \geq \alpha\|u\|^2$  for all  $u \in D(A)$ . Assume further that there is a nonnegative constant  $b \leq 1$  such that for all  $u \in D(B^*)$ ,*

$$(4.7) \quad \operatorname{Re}(B^*u, A^{-1}u) \geq -b\|u\|^2,$$

where  $B^*$  is the adjoint of  $B$ .

Then we have (4.2) with  $c=a=0$  and  $f_n = A_nu$ .

PROOF. Since  $A^{-1}$  and  $B^*$  are also accretive, it follows from (4.7) that for  $m=1, 2, \dots$ ,

$$(4.8) \quad \operatorname{Re}\left(B^*u, A^{-1}\left(1+\frac{1}{m}B^*\right)u\right) \geq -b\|u\|^2$$

$$\geq -b \left\| \left( 1 + \frac{1}{m} B^* \right) u \right\|^2.$$

Let  $v \in H$ . Then  $\left( 1 + \frac{1}{m} B^* \right)^{-1} v \in D(B^*)$ ; note that  $B^*$  is  $m$ -accretive. Setting  $u = \left( 1 + \frac{1}{m} B^* \right)^{-1} v$  in (4.8), we have

$$(4.9) \quad \operatorname{Re} (B_m^* v, A^{-1} v) \geq -b \|v\|^2, \quad v \in H,$$

where  $B_m^*$  is the Yosida approximation of  $B^*$ .

Noting that  $A_n = A \left( 1 + \frac{1}{n} A \right)^{-1}$  and  $B_m = B \left( 1 + \frac{1}{m} B \right)^{-1} = m \left[ 1 - \left( 1 + \frac{1}{m} B \right)^{-1} \right]$ , we have  $A_n^{-1} = A^{-1} + \frac{1}{n}$  and  $(B_m)^* = B_m^*$ . Since  $B_m$  is also accretive, it follows that

$$\begin{aligned} \operatorname{Re} (v, B_m A_n^{-1} v) &= \operatorname{Re} \left( v, B_m \left( A^{-1} + \frac{1}{n} \right) v \right) \\ &\geq \operatorname{Re} (B_m^* v, A^{-1} v) \geq -b \|v\|^2, \quad v \in H, \end{aligned}$$

where we have used (4.9). Thus, for all  $u \in H$  we obtain

$$(4.10) \quad \operatorname{Re} (A_n u, B_m u) \geq -b \|A_n u\|^2, \quad m, n = 1, 2, \dots$$

Going to the limit  $m \rightarrow \infty$  in (4.10) with  $u \in D(B)$ , we obtain the desired inequality. Q. E. D.

REMARK 4.8. Going to the limit  $n \rightarrow \infty$  in (4.10) with  $u \in D(A)$ , we obtain an estimate of the form which is mentioned in Remark 4.3 (exchange  $A$  and  $B$ ).

REMARK 4.9. Theorem 4.7 generalizes a result of Sohr (see [21], Lemma 3.1) in which both  $A$  and  $B$  are assumed to be selfadjoint.

## § 5. Stability of selfadjointness.

Here we collect several stability theorems for (essential) selfadjointness. Let  $H$  be a Hilbert space. We begin with

THEOREM 5.1. *Let  $A$  be a symmetric operator in  $H$ . Let  $S$  be a nonnegative selfadjoint operator in  $H$ , with a core  $D$ . Assume that  $D \subset D(A)$  and*

(i) *there are constants  $a_0, b_0 \geq 0$  such that*

$$\|Au\| \leq a_0 \|u\| + b_0 \|Su\| \quad \text{for all } u \in D;$$

(ii) *there are constants  $c, a \geq 0$  such that for all  $u \in D$ ,*

$$(5.1) \quad |\operatorname{Im}(Au, Su)| \leq c \|u\| + a \|Su\| \|u\|.$$

*Then  $A$  is essentially selfadjoint on  $D$ .*

PROOF. By virtue of condition (i) we can obtain condition (ii) with  $A$  and  $D$  replaced by  $\tilde{A}$  and  $D(S)$ , respectively. Thus, (5.1) is equivalent to the in-

equality

$$\operatorname{Re}(\pm i\tilde{A}u, Su) \geq -a\|u\|^2 - b\|Su\|\|u\|, \quad u \in D(S).$$

Therefore,  $\tilde{A}$  is selfadjoint and  $D(S)$  is a core of  $\tilde{A}$  (see [18], Theorem 4.5). Namely,  $D$  is a core of  $\tilde{A}$  and  $A$  is essentially selfadjoint on  $D$ . Q. E. D.

**THEOREM 5.2.** *Let  $S$  and  $B$  be symmetric operators in  $H$ , with  $D(S) \subset D(B)$ . Assume that there are nonnegative constants  $a, b$  ( $b \leq 1$ ) and  $c$  such that for all  $u \in D(S)$ ,*

$$\operatorname{Re}(Bu, Su) \geq -c\|u\|^2 - a\|Su\|\|u\| - b\|Su\|^2.$$

*In the case of  $b < 1$ ,  $S+B$  is selfadjoint if and only if  $S$  is selfadjoint. In the case of  $b=1$ ,  $S+B$  is essentially selfadjoint on  $D(S)$  if  $S$  is selfadjoint.*

**PROOF.** Apply Theorem 3.10 to the pairs of  $\pm iS$  and  $\pm iB$ . A direct proof for the case of  $b=1$  will be found in Kuroda [12] (see also [18], Remark 4.3).

Q. E. D.

The following corollary is the well known Kato-Rellich theorem supplemented by Wüst [23].

**COROLLARY 5.3.** *Let  $S, B$  be symmetric operators in  $H$ , with  $D(S) \subset D(B)$ . Assume that there are nonnegative constants  $a$  and  $b$  ( $b \leq 1$ ) such that for all  $u \in D(S)$ ,  $\|Bu\| \leq a\|u\| + b\|Su\|$ . Then the conclusion of Theorem 5.2 holds.*

Here, we present a corollary of Theorem 4.2.

**THEOREM 5.4.** *Let  $A$  and  $B$  be nonnegative selfadjoint operators in  $H$ . Assume that there are nonnegative constants  $a, b$  ( $b \leq 1$ ) and  $c$  such that for all  $u \in D(A)$ ,*

$$(5.2) \quad \operatorname{Re}(Au, B_n u) \geq -c\|u\|^2 - a\|B_n u\|\|u\| - b\|B_n u\|^2,$$

where  $B_n = B\left(1 + \frac{1}{n}B\right)^{-1}$  is the Yosida approximation of  $B$ .

*If  $b < 1$  then  $A+B$  with  $D(A+B) = D(A) \cap D(B)$  is also selfadjoint. If in particular  $b=0$ , then  $D(A+B)$  is a core of both  $A$  and  $B$ . If  $b=1$  then  $A+B$  is essentially selfadjoint on  $D(A+B)$ .*

**COROLLARY 5.5.** *Let  $A, B$  be nonnegative selfadjoint operators in  $H$ . Assume that there are constants  $c, a \geq 0$  and  $k > -1$  such that for all  $u \in D(A)$ ,*

$$(5.3) \quad \operatorname{Re}(Au, B_n u) \geq k\|B_n u\|^2 - c\|u\|^2 - a\|B_n u\|\|u\|.$$

*Then  $A+tB$  is selfadjoint for  $-k < t \leq 1$  and  $A-kB$  is essentially selfadjoint on  $D(A+B)$ .*

**PROOF.** (5.3) implies (5.2) with  $b = -k < 1$ . Therefore,  $A+B$  is selfadjoint. Going to the limit  $n \rightarrow \infty$  in (5.3) with  $u \in D(A+B)$ , we have

$$(5.4) \quad \operatorname{Re}(Au, Bu) \geq k\|Bu\|^2 - c\|u\|^2 - a\|Bu\|\|u\|.$$

So, we see that for all  $u \in D(A+B)$ ,



$$\|Bu\| \leq (1+k)^{-1} \|(A+B)u\| + K\|u\|,$$

where  $K = a(1+k)^{-1} + [c(1+k)^{-1}]^{1/2}$  (cf. Lemma 1.1). Let  $t < 1$ . Then we have

$$\|(t-1)Bu\| \leq \frac{1-t}{1+k} \|(A+B)u\| + (1-t)K\|u\|.$$

Since  $(1-t)(1+k)^{-1} \leq 1$  for  $-k \leq t < 1$ , the conclusion follows from Corollary 5.3:  $A+tB = (A+B) + (t-1)B$  is selfadjoint for  $-k < t < 1$  and essentially selfadjoint for  $t = -k$ . Q. E. D.

REMARK 5.6. If in particular  $k > 0$  in (5.3), then  $B$  must be  $A$ -bounded. In fact, it follows from (5.4) that

$$\begin{aligned} k\|Bu\|^2 &\leq \operatorname{Re}(Au, Bu) + a\|Bu\|\|u\| + c\|u\|^2 \\ &\leq (\|Au\| + a\|u\|)\|Bu\| + c\|u\|^2. \end{aligned}$$

Solving this inequality, we have for all  $u \in D(A+B)$ ,

$$(5.5) \quad \|Bu\| \leq k^{-1}\|Au\| + [ak^{-1} + (ck^{-1})^{1/2}]\|u\|.$$

But, since  $D(A+B)$  is a core of  $A$ , we see from (5.5) that  $D(A) \subset D(B)$  and hence (5.5) holds for all  $u \in D(A)$ .

The following theorem is concerned with the essential selfadjointness of the difference of two nonnegative symmetric operators.

THEOREM 5.7. Let  $S$  and  $C$  be nonnegative symmetric operators in  $H$ . Let  $D$  be a linear manifold on which  $S+C$  is essentially selfadjoint:  $D$  is a core of  $[(S+C)|D]^\sim$ . Assume that for all  $u \in D$ ,

$$(5.6) \quad \|Su\| + \|Cu\| \leq a_0\|u\| + b_0\|(S+C)u\|,$$

$$(5.7) \quad |\operatorname{Im}(Cu, Su)| \leq c\|u\|^2 + a\|(S+C)u\|\|u\|,$$

where  $a_0, b_0, c$  and  $a$  are nonnegative constants.

Then  $S-C$  is also essentially selfadjoint on  $D$ .

PROOF. It follows from (5.6) that  $S-C$  is  $(S+C)$ -bounded. Also, we have

$$\operatorname{Im}((S-C)u, (S+C)u) = 2 \operatorname{Im}(Su, Cu)$$

and by (5.7)

$$|\operatorname{Im}((S-C)u, (S+C)u)| \leq 2c\|u\|^2 + 2a\|(S+C)u\|\|u\|, \quad u \in D.$$

Therefore, the conclusion follows from Theorem 5.1. Q. E. D.

## § 6. Applications to Schrödinger operators.

In this section we consider the (essential) selfadjointness of some simple Schrödinger operators in  $L^2 = L^2(\mathbb{R}^N)$ .

Let  $V(x) > 0$  be locally in  $L^2(\mathbf{R}^N \setminus \{0\})$  and set

$$V_n(x) = V(x) \left[ 1 + \frac{1}{n} V(x) \right]^{-1}, \quad n = 1, 2, \dots$$

Let  $B$  be the maximal multiplication operator by  $V(x)$ :

$$Bu(x) = V(x)u(x) \quad \text{for } u \in D(B) = \{u, V(x)u \in L^2\}.$$

Then  $B$  is a nonnegative selfadjoint operator in  $L^2$ . Let  $A$  be the minus Laplacian:

$$Au(x) = -\Delta u(x) \quad \text{for } u \in D(A) = H^2(\mathbf{R}^N).$$

Then  $A$  is also selfadjoint and nonnegative in  $L^2$ .

First we consider the (essential) selfadjointness of  $A+B = -\Delta + V(x)$  with  $D(A+B) = H^2(\mathbf{R}^N) \cap D(B)$ .

**THEOREM 6.1.** *Let  $A$  and  $B$  be as above. Assume that  $V_n(x)$  is a function of class  $C^1(\mathbf{R}^N)$  and*

$$(6.1) \quad |\text{grad } V_n(x)|^2 \leq cV_n(x) + a[V_n(x)]^2 + b[V_n(x)]^3, \quad x \in \mathbf{R}^N, \quad n \geq 1,$$

where  $a, b$  ( $b \leq 4$ ) and  $c$  are nonnegative constants.

If  $b < 4$  then  $A+B = -\Delta + V(x)$  is also selfadjoint in  $L^2$ . If  $b = 4$  then  $A+B = -\Delta + V(x)$  is essentially selfadjoint on  $D(A+B)$ .

**PROOF.** We shall show that for all  $u \in D(A)$ ,

$$(6.2) \quad \text{Re}(Au, B_n u) \geq -\frac{c}{4} \|u\|^2 - \frac{a}{4} (u, B_n u) - \frac{b}{4} \|B_n u\|^2.$$

So, we can apply Theorem 5.4. Let  $u(x) \in C_0^\infty(\mathbf{R}^N)$ . Then, since  $B_n u(x) = V_n(x)u(x)$ , we have

$$(Au, B_n u) = \int_{\mathbf{R}^N} V_n(x) |\text{grad } u(x)|^2 dx + \int_{\mathbf{R}^N} \overline{u(x)} \sum_{j=1}^N \frac{\partial V_n}{\partial x_j} \frac{\partial u}{\partial x_j} dx$$

and hence

$$\begin{aligned} \text{Re}(Au, B_n u) &= \int_{\mathbf{R}^N} V_n(x) |\text{grad } u(x)|^2 dx \\ &\geq - \int_{\mathbf{R}^N} |u(x)| \sum_{j=1}^N \left| \frac{\partial V_n}{\partial x_j} \right| \left| \frac{\partial u}{\partial x_j} \right| dx \\ &\geq - \int_{\mathbf{R}^N} V_n(x) |\text{grad } u(x)|^2 dx - \frac{1}{4} \int_{\mathbf{R}^N} \frac{|u(x)|^2}{V_n(x)} |\text{grad } V_n(x)|^2 dx. \end{aligned}$$

Therefore, (6.1) implies (6.2).

Q. E. D.

Let  $W(x) > 0$  be locally in  $L^2(\mathbf{R}^N \setminus \{0\})$ . Then we have

**COROLLARY 6.2.** *Let  $A$  and  $B$  be as in Theorem 6.1. Let  $C$  be the maximal multiplication operator by  $W(x)$ . Assume that  $V_n(x)$  and  $W_n(x)$  are functions of*

class  $C^1(\mathbf{R}^N)$  satisfying (6.1) with  $b < 4$ .

Then  $A+B+C=-\Delta+V(x)+W(x)$  is selfadjoint in  $L^2$ .

In fact, we have (6.2) with  $A$  and  $B$  replaced by  $A+B$  and  $C$ , respectively.

Next, we present a sufficient condition which guarantees the estimate (6.1).

**COROLLARY 6.3.** *Let  $A$  and  $B$  be as in Theorem 6.1. Assume instead of (6.1) that  $V(x) \geq 0$  is of class  $C^1(\mathbf{R}^N)$  and*

$$(6.3) \quad |\text{grad } V(x)|^2 \leq c + b[V(x)]^3, \quad x \in \mathbf{R}^N,$$

where  $b$  ( $b \leq 4$ ) and  $c$  are nonnegative constants.

Then the conclusion of Theorem 6.1 holds. If in particular  $b < 4$ , then  $C_0^\infty(\mathbf{R}^N)$  is a core of  $A+B$ .

**PROOF.** It suffices to show that  $A+B+1$  is (essentially) selfadjoint. So, we may assume that  $V(x) \geq 1$ . In fact,  $V(x)$  in (6.3) can be replaced by  $V(x)+1$ . Noting that  $V_n(x) = n - n \left[ 1 + \frac{1}{n} V(x) \right]^{-1}$ , we see from (6.3) that (6.1) with  $a=0$  is satisfied:

$$\begin{aligned} \frac{|\text{grad } V_n(x)|^2}{V_n(x)} &= \frac{|\text{grad } V(x)|^2}{V(x)} \left[ 1 + \frac{1}{n} V(x) \right]^{-3} \\ &\leq \frac{c + b[V(x)]^3}{V(x)} \left[ 1 + \frac{1}{n} V(x) \right]^{-2} \leq c + b[V_n(x)]^2. \end{aligned}$$

Since  $V(x) \geq 0$  is locally in  $L^2(\mathbf{R}^N)$ , the latter assertion follows from the famous result of Kato [10] (cf. also Reed-Simon [19], Theorem X. 28). Q. E. D.

It should be noted that the case of  $b < 4$  in Corollary 6.3 is nothing but the main result in Everitt-Giertz [3] (the case of  $G = \mathbf{R}^N$ ). In [3], however, the result is not formulated as an application of the perturbation theory for selfadjoint operators.

**REMARK 6.4.** The proof of Corollary 6.3 (the case of  $b < 4$ ) can be easily completed also by applying Lemma 3.1 in Sohr [21] (cf. Remark 4.9 above); this observation is due to the referee.

**EXAMPLE 6.5.** Let  $V(x) = |x|^2 = \sum_{j=1}^N x_j^2$ . Then  $|\text{grad } V(x)|^2 = 4V(x)$ . So, we have  $\text{Re}(Au, B_n u) \geq -\|u\|^2$  for  $u \in H^2(\mathbf{R}^N)$ . Therefore,  $-\Delta + |x|^2$  is selfadjoint in  $L^2$ .

**EXAMPLE 6.6.** Let  $V(x) = \beta |x|^{-2}$ , where  $\beta \geq 1$  is a constant. Then  $V_n(x) = \beta \left( |x|^2 + \frac{\beta}{n} \right)^{-1}$  and

$$(6.4) \quad |\text{grad } V_n(x)|^2 = 4\beta^2 |x|^2 \left( |x|^2 + \frac{\beta}{n} \right)^{-4} \leq \frac{4}{\beta} [V_n(x)]^3.$$

So, we have  $\text{Re}(Au, B_n u) \geq -\frac{1}{\beta} \|B_n u\|^2$  for  $u \in H^2(\mathbf{R}^N)$ . Therefore,  $A+B=$

$-\Delta + \beta|x|^{-2}$  ( $\beta > 1$ ) is selfadjoint in  $L^2$  and  $-\Delta + |x|^{-2}$  is essentially selfadjoint on  $D(A+B)$ . We note that in the case of  $\beta > N$  the selfadjointness of  $-\Delta + \beta|x|^{-2}$  was proved by Sohr (see [21], Folgerung 2.4).

To improve this result we prepare

LEMMA 6.7. *Let  $A$  and  $B$  be as in Theorem 6.1. If  $V_n(x) > 0$  is a function of class  $C^1(\mathbf{R}^N)$ , then we have for all  $u \in C_0^\infty(\mathbf{R}^N)$ ,*

$$(6.5) \quad \begin{aligned} \operatorname{Re}(Au, B_n u) = & \int_{\mathbf{R}^N} |\operatorname{grad}[u(x)\sqrt{V_n(x)}]|^2 dx \\ & - \frac{1}{4} \int_{\mathbf{R}^N} \frac{|u(x)|^2}{V_n(x)} |\operatorname{grad} V_n(x)|^2 dx. \end{aligned}$$

Furthermore, if  $N \geq 2$  then we have for all  $u \in C_0^\infty(\mathbf{R}^N)$ ,

$$(6.6) \quad \begin{aligned} \operatorname{Re}(Au, B_n u) \geq & \frac{(N-2)^2}{4} \int_{\mathbf{R}^N} \left(|x|^2 + \frac{1}{n}\right)^{-1} V_n(x) |u(x)|^2 dx \\ & - \frac{1}{4} \int_{\mathbf{R}^N} \frac{|u(x)|^2}{V_n(x)} |\operatorname{grad} V_n(x)|^2 dx. \end{aligned}$$

If in particular  $N=1$  then we have for all  $u \in C_0^\infty(\mathbf{R})$ ,

$$(6.7) \quad \begin{aligned} \operatorname{Re}(Au, B_n u) \geq & \frac{1}{4} \int_{-\infty}^{\infty} \left(x^2 + \frac{1}{n}\right)^{-1} V_n(x) |u(x)|^2 dx \\ & - \frac{3}{4n} \int_{-\infty}^{\infty} \left(x^2 + \frac{1}{n}\right)^{-2} V_n(x) |u(x)|^2 dx \\ & - \frac{1}{4} \int_{-\infty}^{\infty} \frac{|u(x)|^2}{V_n(x)} |V_n'(x)|^2 dx. \end{aligned}$$

PROOF. Let  $u \in C_0^\infty(\mathbf{R}^N)$ . Then we have

$$\begin{aligned} (Au, B_n u) = & - \int_{\mathbf{R}^N} \overline{u(x)} \sqrt{V_n(x)} \sqrt{V_n(x)} \Delta u(x) dx \\ = & \int_{\mathbf{R}^N} \sum_{j=1}^N \sqrt{V_n(x)} \frac{\partial u}{\partial x_j} \frac{\partial}{\partial x_j} [\overline{u(x)} \sqrt{V_n(x)}] dx \\ & + \frac{1}{2} \int_{\mathbf{R}^N} \overline{u(x)} \sum_{j=1}^N \frac{\partial V_n}{\partial x_j} \frac{\partial u}{\partial x_j} dx. \end{aligned}$$

So, we see that

$$\begin{aligned} (Au, B_n u) = & i \operatorname{Im} \int_{\mathbf{R}^N} \overline{u(x)} \sum_{j=1}^N \frac{\partial V_n}{\partial x_j} \frac{\partial u}{\partial x_j} dx + \int_{\mathbf{R}^N} |\operatorname{grad}[u(x)\sqrt{V_n(x)}]|^2 dx \\ & - \frac{1}{4} \int_{\mathbf{R}^N} \frac{|u(x)|^2}{V_n(x)} |\operatorname{grad} V_n(x)|^2 dx. \end{aligned}$$

Therefore, we obtain (6.5). Since (6.6) is trivial for  $N=2$ , it remains to show

that (6.6) with  $N \geq 3$  and (6.7) hold. Let  $N \geq 3$ . Then (6.6) is a consequence of the well known inequality for  $u \in H^1(\mathbf{R}^N)$ :

$$(6.8) \quad \int_{\mathbf{R}^N} |\text{grad } u(x)|^2 dx \geq \frac{(N-2)^2}{4} \int_{\mathbf{R}^N} \frac{|u(x)|^2}{|x|^2} dx.$$

We note that (6.8) follows from the identity:

$$\begin{aligned} & \sum_{j=1}^N \int_{\mathbf{R}^N} \left| \frac{\partial u}{\partial x_j} + \frac{N-2}{2} x_j |x|^{-2} u(x) \right|^2 dx \\ &= \int_{\mathbf{R}^N} |\text{grad } u(x)|^2 dx - \frac{(N-2)^2}{4} \int_{\mathbf{R}^N} \frac{|u(x)|^2}{|x|^2} dx, \quad u \in C_0^\infty(\mathbf{R}^N). \end{aligned}$$

Now, let  $N=1$ . Then we have for all  $u \in C_0^\infty(\mathbf{R})$ ,

$$\begin{aligned} & \int_{-\infty}^{\infty} \left| u'(x) - \frac{1}{2} x \left( x^2 + \frac{1}{n} \right)^{-1} u(x) \right|^2 dx = \int_{-\infty}^{\infty} |u'(x)|^2 dx \\ & - \frac{1}{4} \int_{-\infty}^{\infty} \left( x^2 + \frac{1}{n} \right)^{-1} |u(x)|^2 dx + \frac{3}{4n} \int_{-\infty}^{\infty} \left( x^2 + \frac{1}{n} \right)^{-2} |u(x)|^2 dx. \end{aligned}$$

So, we obtain (6.7).

Q. E. D.

**THEOREM 6.8.** Let  $V(x) = |x|^{-2}$ . Then  $A + tB = -\Delta + t|x|^{-2}$  is selfadjoint for  $t > -(N-4)N/4$  and essentially selfadjoint on  $D(A+B)$  for  $t = -(N-4)N/4$ . Furthermore, if  $N \geq 5$  then  $B$  is  $A$ -bounded:

$$\|Bu\| \leq \frac{4}{(N-4)N} \|Au\| \quad \text{for all } u \in H^2(\mathbf{R}^N).$$

**PROOF.** First we note that if  $t > 1$  then  $A + tB$  is selfadjoint (see Example 6.6). Since  $V_n(x) = \left( |x|^2 + \frac{1}{n} \right)^{-1}$  is of class  $C^\infty(\mathbf{R}^N)$ , we can apply Lemma 6.7. Let  $N \geq 3$ . Then by (6.4) we have  $|\text{grad } V_n(x)|^2 \leq 4[V_n(x)]^3$ . Therefore, it follows from (6.6) that for all  $u \in H^2(\mathbf{R}^N)$ ,

$$\text{Re}(Au, B_n u) \geq \frac{1}{4} [(N-2)^2 - 4] \|B_n u\|^2.$$

Next, let  $N=1$ . Then  $|V'_n(x)|^2 = 4x^2 \left( x^2 + \frac{1}{n} \right)^{-4}$  and we have

$$\frac{3}{4n} \left( x^2 + \frac{1}{n} \right)^{-2} V_n(x) + \frac{1}{4} \frac{|V'_n(x)|^2}{V_n(x)} \leq [V_n(x)]^3.$$

Therefore, it follows from (6.7) that for all  $u \in H^2(\mathbf{R})$ ,

$$\text{Re}(Au, B_n u) \geq -\frac{3}{4} \|B_n u\|^2.$$

So, in the case of  $N \neq 2$ , we obtain (5.3) with

$$k = \frac{N}{4}(N-4) > -1.$$

Therefore, the (essential) selfadjointness of  $A+tB$  follows from Corollary 5.5; for all  $u \in D(A+B)$ ,

$$\|Bu\| \leq \frac{4}{(N-2)^2} \|(A+B)u\|, \quad N \neq 2.$$

The case of  $N=2$  has already been treated in Example 6.6. Noting that  $(N-4)N/4 > 0$  if  $N \geq 5$ , we see from Remark 5.6 that  $B$  is  $A$ -bounded. Q. E. D.

In connection with the above theorem, it should be noted that if  $t \geq -(N-4)N/4$  then  $-\Delta + t|x|^{-2}$  is essentially selfadjoint on  $C_0^\infty(\mathbf{R}^N \setminus \{0\})$  (see e. g. [19], Theorem X.11). Consequently,  $C_0^\infty(\mathbf{R}^N \setminus \{0\})$  is a core of  $A+tB = -\Delta + t|x|^{-2}$  for  $t > -(N-4)N/4$ .

In the rest of this section we shall present an application of Theorem 5.7.

Let  $U(x) \leq 0$  be a function in  $L^p(\mathbf{R}^N)$ , where we assume that  $p \geq 2$  if  $N \leq 3$ ,  $p > 2$  if  $N=4$  and  $p \geq N/2$  if  $N \geq 5$ . Let  $B$  be the maximal multiplication operator by  $U(x)$ . Then  $B$  is selfadjoint in  $L^2$  and, moreover,  $B$  is  $A$ -bounded with  $A$ -bound zero, where  $A$  is the minus Laplacian (see e. g. Reed-Simon [19]). Hence it follows that for any  $\varepsilon > 0$  ( $0 \leq \varepsilon < 1$ ) there is a constant  $c(\varepsilon) > 0$  such that for all  $u \in H^1(\mathbf{R}^N)$ ,

$$(6.9) \quad |(u, Bu)| \leq (1-\varepsilon)(u, Au) + c(\varepsilon)\|u\|^2;$$

see Kato [8], VI-§ 1.7. Therefore,  $A+B+c(0)$  is nonnegative.

Next, let  $W(x) \geq 0$  be a function of class  $C^1(\mathbf{R}^N)$  and  $C$  be the maximal multiplication operator by  $W(x)$ . Then  $C$  is selfadjoint and nonnegative in  $L^2$ .

LEMMA 6.9. *Let  $A, B$  and  $C$  be as above. Assume that there are constants  $c_1, c_2 \geq 0$  such that*

$$(6.10) \quad |\text{grad } W(x)|^2 \leq c_1 + c_2 W(x), \quad x \in \mathbf{R}^N.$$

*Then for all  $u \in C_0^\infty(\mathbf{R}^N)$ ,*

$$(6.11) \quad 2 \operatorname{Re} ((C+1)u, [A+B+c(0)]u) \geq -c\|u\|^2$$

*where  $c(0)$  is a constant in (6.9) and  $c = (c_1 + c_2)/2$ .*

PROOF. Let  $u \in C_0^\infty(\mathbf{R}^N)$ . Then we see from (6.5) with  $B_n$  replaced by  $C+1$  that

$$\begin{aligned} \operatorname{Re} ((C+1)u, Au) &= \int_{\mathbf{R}^N} |\text{grad } [u(x)\sqrt{W(x)+1}]|^2 dx \\ &\quad - \frac{1}{4} \int_{\mathbf{R}^N} \frac{|u(x)|^2}{W(x)+1} |\text{grad } W(x)|^2 dx. \end{aligned}$$

Since we can write  $((C+1)u, Bu) = \int_{\mathbf{R}^N} U(x) |u(x) \sqrt{W(x)+1}|^2 dx$ , it follows from (6.9) with  $\varepsilon=0$  that

$$\begin{aligned} \operatorname{Re} ((C+1)u, (A+B)u) &\geq -c(0) \int_{\mathbf{R}^N} |u(x) \sqrt{W(x)+1}|^2 dx \\ &\quad - \frac{1}{4}(c_1+c_2) \int_{\mathbf{R}^N} |u(x)|^2 dx. \end{aligned}$$

So, we obtain (6.11).

Q. E. D.

Now let  $V(x) \geq 0$  be locally in  $L^2(\mathbf{R}^N)$  and replace  $U(x)$  by  $U(x)+V(x)$  in the definition of the operator  $B$ . Then it is obvious that Lemma 6.9 holds with this change.

**THEOREM 6.10.** *Let  $A$  be the minus Laplacian. Let  $B$  and  $C$  be the maximal multiplication operator by  $U(x)+V(x)$  and  $W(x)$ , respectively. Assume that (6.10) holds.*

*Then  $A+B-C = -\Delta + U(x)+V(x)-W(x)$  is essentially selfadjoint on  $C_0^\infty(\mathbf{R}^N)$ .*

**PROOF.** Since  $V(x)+W(x) \geq 0$  is locally in  $L^2(\mathbf{R}^N)$ ,  $A+B+C = -\Delta + U(x) + [V(x)+W(x)]$  is essentially selfadjoint on  $C_0^\infty(\mathbf{R}^N)$  (see e. g. Reed-Simon [19], Theorem X.29). Also, we see from (6.11) that for all  $u \in C_0^\infty(\mathbf{R}^N)$ ,

$$\|[A+B+c(0)]u\|^2 + \|(C+1)u\|^2 \leq \|u\|^2 + \|[A+B+C+1+c(0)]u\|^2.$$

So, we obtain (5.6) with  $S$  and  $D$  replaced by  $A+B+c(0)$  and  $C_0^\infty(\mathbf{R}^N)$ , respectively. On the other hand, we have

$$\operatorname{Im}(Cu, Au) = \operatorname{Im} \int_{\mathbf{R}^N} u(x) \sum_{j=1}^N \frac{\partial W}{\partial x_j} \frac{\partial u}{\partial x_j} dx, \quad u \in C_0^\infty(\mathbf{R}^N),$$

and hence

$$|\operatorname{Im}(Cu, Au)| \leq \frac{1}{2} \int_{\mathbf{R}^N} |\operatorname{grad} W(x)|^2 |u(x)|^2 dx + \frac{1}{2} \int_{\mathbf{R}^N} |\operatorname{grad} u(x)|^2 dx.$$

It then follows from (6.10) that

$$|\operatorname{Im}(Cu, [A+B+c(0)]u)| \leq \frac{c_1}{2} \|u\|^2 + \frac{c_2+1}{2} (u, (A+C)u).$$

Thus, we see from (6.9) that for all  $u \in C_0^\infty(\mathbf{R}^N)$ ,

$$|\operatorname{Im}(Cu, [A+B+c(0)]u)| \leq \left[ \frac{c_1}{2} + \frac{c(\varepsilon)}{\varepsilon} \right] \|u\|^2 + \frac{c_2+1}{2\varepsilon} (u, [A+B+C+c(0)]u).$$

Therefore, the conclusion follows from Theorem 5.7.

Q. E. D.

As a typical example of  $W(x)$  in Theorem 6.10 we have in mind  $W(x) = a+b|x-c|^2$ ,  $c \in \mathbf{R}^N$  (cf. Example 6.5).

**COROLLARY 6.11.** *Let  $V(x)$  be locally in  $L^2(\mathbf{R}^N)$  and assume that  $V(x) \geq -a-b|x-c|^2$  for some  $c \in \mathbf{R}^N$ , where  $a, b \geq 0$  are constants. Then  $-\Delta + U(x) + V(x)$  is essentially selfadjoint on  $C_0^\infty(\mathbf{R}^N)$ .*

In fact,  $V(x)$  can be written as

$$V(x)=[V(x)+a+b|x-c|^2]-(a+b|x-c|^2),$$

where the first term is nonnegative and locally in  $L^2(\mathbf{R}^N)$ .

REMARK 6.12. The case of  $c=0$  in Corollary 6.11 is treated by Faris-Lavine [4] (see also Reed-Simon [19], the first corollary of Theorem X.38).

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