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Fourier transform of L^p on real rank 1 semisimple Lie groups

By Takeshi KAWAZOE

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1. Introduction.

Let G be a real rank one, connected semisimple Lie group with finite center and $G=KAN$ an Iwasawa decomposition for G. Let τ be a unitary double representation of K on a finite dimensional Hilbert space V and $\mathcal{C}(G, \tau)$ the τ-spherical Schwartz space on G defined by Harish-Chandra [**1**]. Then $\mathcal{C}(G, \tau)$ can be written as the direct sum of ${}^{\circ}\mathcal{C}(G, \tau)$ and $\mathcal{C}_A(G, \tau)$, which consist of τ-spherical cusp forms on G and wave packets respectively (cf. [**3**, Theorem 27.2]). Here using the matrix coefficients of the discrete and principal series for G, we define the Fourier transform of $\mathcal{C}(G, \tau)$ as in the previous papers [**5, 6**]. Let $\mathcal{F}$ denote the dual space of the Lie algebra of A and $\mathcal{C}(\mathcal{F})$ the usual Schwartz space on $\mathcal{F}$. Then from Theorem 1 in [**6**], roughly speaking, the Fourier transform sets up a homeomorphism between $\mathcal{C}(G, \tau)$ and the direct sum of $\boldsymbol{C}^{n'}$, $n'=\dim {}^{\circ}\mathcal{C}(G, \tau)$, and the subspace $\mathcal{C}(\mathcal{F})_*^n$ of $\mathcal{C}(\mathcal{F})^n$ which consists of all elements satisfying the functional equations for the Weyl group of (G, A), where $n=\dim V^M$ (cf. § 3). Moreover from Theorem 2 in [**6**] the Fourier transform sets up a bijection between $C_c^\infty(G, \tau)$, the space of all τ-spherical C^∞-functions with compact support on G, and the subspace $\mathcal{H}(\mathcal{F})_*^n$ of $\mathcal{C}(\mathcal{F})_*^n$ which consists of all elements α in $\mathcal{C}(\mathcal{F})_*^n$ such that (i) each component of α extends to an entire holomorphic function on $\mathcal{F}_C$, the complexification of $\mathcal{F}$, which is an exponential type, (ii) α satisfies the functional equations for Eisenstein integrals on $\mathcal{F}_C$ (cf. § 5).

In this paper we shall characterize the Fourier transforms of $\mathcal{C}^p(G, \tau)$ which consists of all functions in $\mathcal{C}(G, \tau)$ with finite L^p-norm $(0<p\leqq 2)$. Obviously, for $0<p_1\leqq p_2\leqq 2$ $\mathcal{C}^{p_1}(G, \tau)\subset\mathcal{C}^{p_2}(G, \tau)\subset\mathcal{C}(G, \tau)$ and $C_c^\infty(G, \tau)\subset\bigcap_{0<p\leqq 2}\mathcal{C}^p(G, \tau)$. Here we put $\varepsilon=\dfrac{2}{p}-1$ and $\mathcal{F}(\varepsilon)=\{\nu\in\mathcal{F}_C;\, |\mathrm{Im}\,\nu|\leqq\varepsilon\rho\}$. Let ${}'\mathcal{H}_p^\tau$ denote the subspace of $\mathcal{C}(\mathcal{F})_*^n$ which consists of all elements α such that (i) each component of α extends to a holomorphic function on the interior of $\mathcal{F}(\varepsilon)$ which is rapidly decreasing on $\mathcal{F}(\varepsilon)$, (ii) α satisfies the functional equations for Eisenstein integrals on $\mathcal{F}(\varepsilon)$ (cf. § 5), where when $p=2$, these conditions are omitted, i.e., ${}'\mathcal{H}_2^\tau=\mathcal{C}(\mathcal{F})_*^n$. Then our main results can be stated as follows. Except a finite number of p the

Fourier transform sets up a bijection between $\mathcal{C}^p(G, \tau)$ and the direct sum of $\boldsymbol{C}^{i'_p}$, $i'_p=\dim {}^\circ\mathcal{C}(G, \tau)\cap\mathcal{C}^p(G, \tau)$, and $'\mathcal{H}^\tau_p$. To obtain this result we shall use the same method in the proof of an analogue of the Paley-Wiener theorem, that is, the characterization of the Fourier transforms of $C_c^\infty(G, \tau)$ (Theorem 5.1).

In the rest of the paper we shall study some topics in harmonic analysis of L^p-functions on G. For example, we show that the Fourier transforms of L^p $(1\leqq p<2)$ functions vanish at $\nu=\infty$ and moreover have polynomial growth on $\mathcal{F}(\varepsilon')$ $\left(0\leqq\varepsilon'\leqq\varepsilon=\dfrac{2}{p}-1\right)$ (Theorem 8.4). Next we obtain a formula for the Fourier transform of the convolution of two functions and, applying these results to a special case, we show the Kunze-Stein phenomenon for K-finite functions on G (Theorem 10.5).

2. Notations.

Let G be a connected semisimple Lie group with finite center and be of real rank one. Let $G=KAN$ be an Iwasawa decomposition for G and M (resp. M') denote the centralizer (resp. the normalizer) of A in K. Then $P=MAN$ is a minimal parabolic subgroup of G and $W=M'/M$ is the Weyl group for (G, A). For any subgroup of G we denote its Lie algebra by small German letter. As usual for any real vector space V, V_C (resp. V^*) denotes the complexification (resp. the dual space) of V. Let Δ denote the set of all roots of $(\mathfrak{g}_C, \mathfrak{a}_C)$, Δ^+ the set of positive roots in Δ such that $\mathfrak{n}_C=\sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the root space of α. Let $\mathfrak{a}^+$ denote the positive Weyl chamber in $\mathfrak{a}$ determined by Δ^+ and put $A^+=\exp\mathfrak{a}^+$. Since $\dim A=1$, there exists a unique positive reduced root α and $H_0\in\mathfrak{a}^+$ such that $\alpha(H_0)=1$. For simplicity we put $\mathcal{F}=\mathfrak{a}^*$ and $\mathcal{F}^+=\{\lambda\in\mathcal{F}\,;\,\lambda(H_0)>0\}$. For any real number ε, $\delta>0$ we define the subsets $\mathcal{F}(\varepsilon)$, $\mathcal{F}_\delta$, $\mathcal{F}_\delta^+(\varepsilon)$ of $\mathcal{F}_C$ as follows.

$$\mathcal{F}(\varepsilon)=\{\lambda\in\mathcal{F}_C\,;\,|\mathrm{Im}\,\lambda(H_0)|\leqq\varepsilon\rho(H_0)\},$$

$$\mathcal{F}_\delta=\{\lambda\in\mathcal{F}\,;\,|\lambda(H_0)|\geqq\delta\}\cup\{\lambda\in\mathcal{F}_C\,;\,|\lambda(H_0)|=\delta,\ \mathrm{Im}\,\lambda(H_0)\leqq 0\},$$

$$\mathcal{F}_\delta^+(\varepsilon)=\{\lambda\in\mathcal{F}_C\,;\,0\leqq\mathrm{Im}\,\lambda(H_0)\leqq\varepsilon\rho(H_0)\}\cup D_\delta,$$

where $\lambda=\mathrm{Re}\,\lambda+\sqrt{-1}\,\mathrm{Im}\,\lambda$ $(\mathrm{Re}\,\lambda,\ \mathrm{Im}\,\lambda\in\mathcal{F})$, $\rho=\dfrac{1}{2}\sum_{\beta\in\Delta^+}\beta$ and $D_\delta=\{\lambda\in\mathcal{F}_C\,;\,|\lambda(H_0)|\leqq\delta\}$. For any set S in a topological space, $\mathring{S}$ (resp. $\mathrm{CL}(S)$) denotes the interior (resp. the closure) of S. Then $\mathcal{F}+\sqrt{-1}\,\mathrm{CL}(\mathcal{F}^+)$ is the upper half plane of $\mathcal{F}_C$ and put $\mathcal{F}_\delta^+(\infty)=\mathcal{F}+\sqrt{-1}\,\mathrm{CL}(\mathcal{F}^+)\cup D_\delta$.

3. Fourier transform on the Schwartz space.

Let $\tau=(\tau_1, \tau_2)$ be a unitary double representation of K on a finite dimensional Hilbert space V. Here we assume that V satisfies the conditions in [**3**, §8]. Let $\mathcal{C}(G, \tau)$ denote the τ-spherical Schwartz space on G, $L_G={}^\circ\mathcal{C}(G, \tau)$ the space of τ-spherical cusp forms on G and $\mathcal{C}_A(G, \tau)$ the space of all C^∞, τ-spherical functions f on G such that $f^P \sim 0$ (see [**1**] for the definitions of these spaces). Then $\dim {}^\circ\mathcal{C}(G, \tau)<\infty$ and $\mathcal{C}(G, \tau)={}^\circ\mathcal{C}(G, \tau)\oplus\mathcal{C}_A(G, \tau)$ (direct sum). Since $M\subset K$ is compact, $L_M={}^\circ\mathcal{C}(M, \tau_M)=C^\infty(M, \tau_M)$, where τ_M is the restriction of τ to M, and the mapping $\phi\mapsto\phi(1)$ sets up a bijection between L_M and the subspace V^M of V consisting of all vectors v such that $\tau_1(m)v=v\tau_2(m)$ for all $m\in M$. Let ϕ_i^j $(1\leqq i\leqq n_j,\ 1\leqq j\leqq m)$ (resp. e_k $(1\leqq k\leqq n')$) denote the orthonormal basis for L_M (resp. L_G) chosen in the previous papers [**5**, **6**]. For simplicity we assume that $s\omega_j=\omega_j$ $(s\in W)$ for all j in this paper, that is, $L_M=\bigoplus_{j=1}^m L_M(\omega_j)$, where $L_M(\omega)$ is the set of all τ_M-spherical, V-valued extensions of the matrix coefficients of the discrete series ω of M. Then for f in $\mathcal{C}(G, \tau)$ the Fourier transform $F(f)$ of f is defined by

$$F(f)=((e_k, f))_{k=1}^{n'}\oplus\bigoplus_{j=1}^m(\hat{f}(\phi_i^j, \nu))_{i=1}^{n_j} \qquad (\nu\in\mathfrak{F}), \tag{1}$$

where $\hat{f}(\phi_i^j, \nu)=(c^2r)^{-1}(E(P:\phi_i^j:\nu:\cdot), f)$ for $\nu\in\mathfrak{F}$ (see [**1**] and [**3**, §11 and §2] for the definitions of the Eisenstein integral $E(P:\phi_i^j:\nu:x)$ and the constants $c=c(P)$, $r=r(P)$ respectively). Let $\mathcal{C}(\mathfrak{F})$ denote the Schwartz space on $\mathfrak{F}$ and $\mathcal{C}(\mathfrak{F})_*^n=\bigoplus_{j=1}^m\mathcal{C}(\mathfrak{F})_*^{n_j}$ $\left(n=\sum_{j=1}^m n_j\right)$ the closed subspace of $\mathcal{C}(\mathfrak{F})^n$ consisting of all elements $\alpha=\bigoplus_{j=1}^m\alpha_j$, $\alpha_j=(\alpha_1^j, \alpha_2^j, \cdots, \alpha_{n_j}^j)\in\mathcal{C}(\mathfrak{F})^{n_j}$, such that

$$\alpha_j(s^{-1}\nu)^t=\overline{{}^\circ C_{P|P}(s\,;s^{-1}\nu)}\alpha_j(\nu)^t \qquad \text{for all } s\in W \text{ and } \nu\in\mathfrak{F}, \tag{2}$$

where each α_j^t is the transposed vector of α_j and we regard the unitary operator ${}^\circ C_{P|P}(s\,;\nu)$ on $L_M(\omega_j)$, which is defined in [**3**, §17], as a matrix operator with respect to the basis ϕ_i^j $(1\leqq i\leqq n_j)$ (cf. [**5**, (1.5)]). The bar denotes the complex conjugate. Then we obtain the following theorem.

THEOREM 3.1 ([**6**]). *The Fourier transform sets up a homeomorphism between $\mathcal{C}(G, \tau)$ and $\mathbf{C}^{n'}\oplus\mathcal{C}(\mathfrak{F})_*^n$. Moreover for $f\in\mathcal{C}(G, \tau)$*

$$f(x)=\sum_{k=1}^{n'}(e_k, f)e_k(x)+\frac{1}{|W|}\sum_{j=1}^m\sum_{i=1}^{n_j}\int_{\mathfrak{F}}\mu(\omega_j, \nu)E(P:\phi_i^j:\nu:x)\hat{f}(\phi_i^j, \nu)d\nu, \tag{3}$$

*where each $\mu(\omega_j, \nu)$ is the μ-function corresponding to ω_j (see [**3**, §11]) and $d\nu$ is the usual Lebesgue measure on $\mathfrak{F}$.*

In what follows we shall characterize the subset of $C^{n'}\oplus\mathcal{C}(\mathfrak{F})_*^n$ which consists of the Fourier transforms of all functions in $\mathcal{C}(G, \tau)$ with finite L^p-norm $(0<p\leqq 2)$.

4. Singularities of expansions for Eisenstein integrals.

In this section we consider the singularities of meromorphic functions which appear in the Harish-Chandra expansions of $\mu(\omega_j, \nu)\ E(P: \phi_i^j: \nu: a)$ $(a\in A^+)$, that is, $I(j, i: \nu: a)=\Phi_0(\nu: a)C_{P|P}(1; \nu)^{*-1}\phi_i^j(1)$ (see (13) in § 6 for the definition). Then for a sufficiently small $\delta>0$ we know that the poles of $I(j, i: \nu: a)$ on $\mathfrak{F}_\delta^+(\infty)$ do not depend on $a\in A^+$, δ and moreover they are finite and pure imaginary. We fix such a $\delta>0$. Let $\xi_i^j(t)$ $(1\leqq t\leqq T_i^j)$ denote the poles of $I(j, i: \nu: a)$ on $\mathfrak{F}_\delta^+(\infty)$ and m_i^j the order of pole at $\xi_i^j(t)$. We may assume that $|\xi_i^j(t_1)|<|\xi_i^j(t_2)|$ for $1\leqq t_1<t_2\leqq T_i^j$. Now for $\varepsilon>0$ we put

$$T_i^j(\varepsilon)=\max\{t\,;\,\xi_i^j(t)\in\mathfrak{F}_\delta^+(\varepsilon)\}.$$

Obviously, $T_i^j(\varepsilon_1)\leqq T_i^j(\varepsilon_2)$ for $\varepsilon_1<\varepsilon_2$ and $T_i^j(\infty)=T_i^j$. Then we define the set S_ε as a collection of all functions on G such that

$$D^m(\xi_i^j(t))E(P: \phi_i^j: \nu: x)\quad \text{for } 0\leqq m\leqq m_i^j(t)-1,\ 1\leqq t\leqq T_i^j(\varepsilon),\ 1\leqq i\leqq n_j,\ 1\leqq j\leqq m,$$

where $D^m(\xi)=\left.\dfrac{d^m}{d\nu^m}\right|_{\nu=\xi}$. Here we note that these functions are real analytic on G. Let S_ε° denote a maximal linearly independent subset of S_ε, elements of which we denote by

$$E_p(x)=D^{m[p]}(\xi_{i[p]}^{j[p]}(t[p]))E(P: \phi_{i[p]}^{j[p]}: \nu: x)\,,\qquad 1\leqq p\leqq \gamma_\varepsilon.$$

For simplicity we put $D_p=D^{m[p]}(\xi_{i[p]}^{j[p]}(t[p]))$, $\phi[p]=\phi_{i[p]}^{j[p]}$ $(1\leqq p\leqq\gamma_\infty)$ and may assume that $S_{\varepsilon_1}^\circ\subset S_{\varepsilon_2}^\circ$ for $\varepsilon_1<\varepsilon_2$. Since E_p $(1\leqq p\leqq\gamma_\infty)$ are linearly independent and real analytic on G, there exist $h_p\in C_c^\infty(G, \tau)$ $(1\leqq p\leqq\gamma_\infty)$ such that

$$(E_q, h_p)=\delta_{pq}\qquad \text{for all } 1\leqq p,\ q\leqq\gamma_\infty.$$

Then in [6, Lemma 2] we obtained the following Lemma.

LEMMA 4.1. *$A_{p,k}=(e_k, h_p)$ $(1\leqq p\leqq\gamma_\infty, 1\leqq k\leqq n')$ do not depend on any choice of h_p $(1\leqq p\leqq\gamma_\infty)$.*

Now we note that e_k $(1\leqq k\leqq n')$ are V-valued extensions of the matrix coefficients of the discrete series for G. Thus, using the results in [7, 8], we can check the growth order of e_k. From this fact, for $0<p\leqq 2$ we may assume that e_k $(1\leqq k\leqq i_p)$ do not belong to $L^p(G, \tau)$ and the rest belongs to $L^p(G, \tau)$, where $L^p(G, \tau)$ is the space of all τ-spherical measurable functions f on G such that for any continuous seminorm s on V, ${}_s\|f\|_p=\left(\int_G |f(x)|_s^p dx\right)^{1/p}<\infty$. Obviously, e_k $(i_p+1\leqq k\leqq n')$ are contained in $\mathcal{C}^p(G, \tau)$ (see §5 for the definition) and $i_{p_1}\geqq i_{p_2}$ for $p_1<p_2$. Moreover $i_2=0$ and for a sufficiently small $p>0$, $i_p=n'$.

5. Statement of the main theorem.

Let $0<p\leqq 2$ and $\varepsilon=\dfrac{2}{p}-1$ $(\geqq 0)$. Let Ξ and σ be the spherical functions on G given in [1, §10], $\boldsymbol{Z}^+$ non-negative integers and $\mathcal{S}(V)$ the set of all continuous seminorms on V. As usual we regard an element in the universal enveloping algebra $U(\mathfrak{g}_C)$ of $\mathfrak{g}_C$ as a differential operator on G (cf. [1, §15]). Then let $\mathcal{C}^p(G, \tau)$ denote the space of all τ-spherical C^∞ functions on G satisfying the following conditions; for any $m\in\boldsymbol{Z}^+$, g_1, $g_2\in U(\mathfrak{g}_C)$ and $s\in\mathcal{S}(V)$,

$$\mu^p_{m, g_1, g_2, s}(f)=\sup_{x\in G}|f(g_1\,;\,x\,;\,g_2)|_s\Xi(x)^{-2/p}(1+\sigma(x))^m<\infty\,. \tag{5}$$

The seminorms $\mu^p_{m, g_1, g_2, s}$ convert $\mathcal{C}^p(G, \tau)$ into a Frechet space. Obviously, $\mathcal{C}^2(G, \tau)=\mathcal{C}(G, \tau)$ and for $0<p_1<p_2\leqq 2$, $C_c^\infty(G, \tau)\subset\mathcal{C}^{p_1}(G, \tau)\subset\mathcal{C}^{p_2}(G, \tau)$. Let $S(\mathfrak{F})$ denote the symmetric algebra over $\mathfrak{F}_C$. As usual we regard an element in $S(\mathfrak{F})$ as a differential operator on $\mathfrak{F}$. Then let $\mathcal{H}^\tau_p$ denote the space of all elements $(a_k)_{k=1}^{n'}\oplus\bigoplus_{j=1}^{m}(\alpha_i^j(\nu))_{i=1}^{n_j}$ in $\boldsymbol{C}^{n'}\oplus\mathcal{C}(\mathfrak{F})^n_*$ satisfying the following conditions; (i) each $\alpha_i^j(\nu)$ extends to a holomorphic function on $\mathring{\mathfrak{F}}(\varepsilon)$, (ii) for any $l\in\boldsymbol{Z}^+$ and $u\in S(\mathfrak{F})$

$$\zeta^p_{l,u}(\alpha_i^j)=\sup_{\nu\in\mathring{\mathfrak{F}}(\varepsilon)}|\alpha_i^j(\nu\,;\,u)|(1+|\nu|)^l<\infty\,, \tag{6}$$

where $|\nu|=|\nu(H_0)|$, (iii) if there exists a functional equation for Eisenstein integrals such that

$$\sum_{j=1}^{m}\sum_{i=1}^{n_j}\sum_{t=1}^{T_i^j(\varepsilon)}\sum_{r=0}^{m_i^j(t)-1}A(j, i, t, r)D^r(\xi_i^j(t))E(P: \phi_i^j: \nu: x)=0 \tag{7}$$

$(x\in G,\ A(j, i, t, r)\in\boldsymbol{C})$, then $\alpha_i^j(\nu)$ $(1\leqq i\leqq n_j,\ 1\leqq j\leqq m)$ satisfy the same equation, that is,

$$\sum_{j=1}^{m}\sum_{i=1}^{n_j}\sum_{t=1}^{T_i^j(\varepsilon)}\sum_{r=0}^{m_i^j(t)-1}A(j, i, t, r)D^r(\xi_i^j(t))\alpha_i^j(\nu)=0\,, \tag{8}$$

(iv)
$$a_k=\sum_{q=1}^{\gamma_\varepsilon}A_{q,k}D_q\alpha[q]\qquad(1\leqq k\leqq i_p), \tag{9}$$

where $\alpha[q]=\alpha^{j[q]}_{i[q]}$ for $1\leqq q\leqq\gamma_\varepsilon$. When $p=2$ $(\varepsilon=0)$, these conditions are omitted, i.e., $\mathcal{H}^\tau_2=\boldsymbol{C}^{n'}\oplus\mathcal{C}(\mathfrak{F})^n_*$.

The following theorem, which will be proved in §7, is our main result.

THEOREM 5.1. *Let $0<p\leqq 2$ and suppose that $\varepsilon\rho$ $\left(\varepsilon=\dfrac{2}{p}-1\right)$ is not equal to*

$\xi_i^j(T_i^j(\varepsilon))$ *for all* i *and* j. *Then the Fourier transform sets up a bijection between* $\mathcal{C}^p(G, \tau)$ *and* $\mathcal{H}_p^\tau$.

REMARK 1. When $p=0$, let $\bar{\mathcal{H}}_0^\tau$ denote the space of all $(a_k)_{k=1}^{n'} \oplus \bigoplus_{j=1}^{m} (\alpha_i^j(\nu))_{i=1}^{n_j}$ in $\boldsymbol{C}^{n'} \oplus \mathcal{C}(\mathfrak{F})_*^n$ satisfying the conditions (i), (iii), (iv) for $p=0$ ($\varepsilon=\infty$) and moreover the following condition (ii)$'$ instead of (ii); (ii)$'$ there exists an $R>0$ such that for any $N \in \boldsymbol{Z}^+$ there exists a constant c_N for which

$$|\alpha_i^j(\nu)| < c_N(1+|\nu|)^{-N} e^{R|\mathrm{Im}\,\nu|} \qquad (\nu \in \mathfrak{F}_C).$$

Then we obtained the following theorem in [**6**].

THEOREM 5.2 (an analogue of the Paley-Wiener theorem). *The Fourier transform sets up a bijection between* $C_c^\infty(G, \tau)$ *and* $\bar{\mathcal{H}}_0^\tau$.

REMARK 2. It follows from Theorem 5.1 that $\mathcal{H}_{p_1}^\tau \subset \mathcal{H}_{p_2}^\tau$ for $0<p_1<p_2 \leqq 2$. Therefore when e_k is not in $L^{p'}(G, \tau)$ $(0<p<p' \leqq 2)$, γ_ε and i_p in (9) can be replaced by $\gamma_{\varepsilon'}$ and $i_{p'}$ $\left(\varepsilon' = \frac{2}{p'} - 1\right)$ respectively.

6. Some results.

In this section we summarize some results which will be used in the proof of the main theorem.

First we recall the following properties of the spherical functions Ξ and σ (see [**1**, § 10]). There exist numbers $c_1>0$ and $r_1>0$ such that

$$e^{-\rho(\log(a))} \leqq \Xi(a) \leqq c_1(1+\sigma(a))^{r_1} e^{-\rho(\log(a))}$$

for all $a \in A^+$ and there exists $r_0>0$ such that

$$\int_G \Xi(x)^2(1+\sigma(x))^{-r_0} dx < \infty . \tag{10}$$

Moreover, $\sigma(xy) \leqq \sigma(x)+\sigma(y)$ $(x, y \in G)$.

Let θ denote the Cartan involution of $\mathfrak{g}$ induced by K and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ the corresponding Cartan decomposition of $\mathfrak{g}$. Let $\mathfrak{h}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{h} \cap \mathfrak{p} = \mathfrak{a}$ and $\mathfrak{h} \cap \mathfrak{k} \subset \mathfrak{m}$, where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Put $\mathfrak{h}_I = \mathfrak{h} \cap \mathfrak{k}$ and $\mathfrak{Z}$ denotes the center of $U(\mathfrak{g}_C)$. Then using the same arguments in [**9**, Lemma 3.5.3], we can prove that Eisenstein integrals satisfy the following facts. For any $u \in S(\mathfrak{F})$ and $\phi \in L_M(\omega)$ $(\omega \in \mathcal{E}(M))$, see the notation for [**2**, § 18], let $d(u)$ denote the degree of u and λ_ω an element of $\sqrt{-1}\,\mathfrak{h}_I^*$ which corresponds to the infinitesimal character of ω. Then if $z \in \mathfrak{Z}$,

$$E(P: \phi : \nu ; u : x ; z - \mu_{\mathfrak{g}/\mathfrak{h}}(z ; \lambda_\omega + \sqrt{-1}\,\nu)^{d(u)+1}) \equiv 0 \tag{11}$$

for all $\nu \in \mathfrak{F}$, where $\mu_{\mathfrak{g}/\mathfrak{h}}$ denotes the usual isomorphism of $\mathfrak{Z}$ into $\mathfrak{h}$ (see [**1**, § 11]).

Furthermore for any $g_1, g_2 \in U(\mathfrak{g}_C)$, $u \in S(\mathfrak{F})$ and $s \in \mathcal{S}(V)$ there exist constants $c_2 > 0$ and $r_2 \geqq 0$ such that

$$|E(P: \phi: \nu; u: g_1; x; g_2)|_s \leqq c_2 \|\phi\|_2 |(\nu, x)|^{r_2} \Xi(x)^{-\varepsilon+1} \tag{12}$$

for all $\nu \in \mathfrak{F}(\varepsilon)$ and $x \in G$, where $\|\cdot\|_2$ denotes the L^2-norm on M and $|(\nu, x)| = (1+|\nu|)(1+\sigma(x))$.

Next we recall that the Eisenstein integral satisfies the Harish-Chandra's expansion (see [10, Theorem 9.1.5.1]), that is, there exist uniquely determined $\mathrm{End}(V^M)$-valued meromorphic functions $C_{P|P}(s; \nu)$ $(s \in W)$ and rational functions $\Gamma_{n\alpha}$ $(n \in \boldsymbol{Z}^+)$ on $\mathfrak{F}_C$ such that

$$e^{\rho(\log(a))} E(P: \phi: \nu: a) = \sum_{s \in W} \Phi(s\nu: a) C_{P|P}(s; \nu) \phi(1) \qquad (a \in A^+),$$

where

$$\Phi(\nu: a) = e^{\sqrt{-1}\nu(\log(a))} \sum_{n \in \boldsymbol{Z}^+} \Gamma_{n\alpha}(\sqrt{-1}\,\nu - \rho) e^{-n\alpha(\log(a))} \tag{13}$$

and ν varies in a certain open dense subset $'\mathfrak{F}_C$ on $\mathfrak{F}_C$ (see [10, p. 288 and Theorem 9.1.4.1]). Put $\Phi_0(\nu: a) = e^{-\sqrt{-1}\nu(\log(a))} \Phi(\nu: a)$. Then we see that Φ_0 and $C_{P|P}{}^{*-1}$ satisfy the following estimates. Put $A_0^+ = \{a \in A^+; \log(a) - H_0 \in \mathfrak{a}^+\}$. Let D (resp. D') denote a domain in $\mathfrak{F}_C$ on which Φ_0 (resp. $C_{P|P}{}^{*-1}$) is holomorphic and whose imaginary part is bounded. Then for any $u \in S(\mathfrak{F})$ and $b \in U(\mathfrak{a}_C)$, the subalgebra in $U(\mathfrak{g}_C)$ generated by 1 and $\mathfrak{a}_C$, there exist constants c_3, $r_3 > 0$ such that

$$\|\Phi_0(\nu; u: a; b)\| \leqq c_3(1+|\nu|)^{r_3} \qquad (\nu \in D,\ a \in A_0^+), \tag{14}$$

where $\|\cdot\|$ denotes the operator norm in $\mathrm{End}(V^M)$ (cf. [4, Lemma 2.3]) and there exist constants c_4, $r_4 > 0$ such that

$$\|C_{P|P}(s; \nu)^{*-1}\| \leqq c_4(1+|\nu|)^{r_4} \qquad (\nu \in D') \tag{15}$$

(see [4, §3]). Last we recall that for each j $(1 \leqq j \leqq m)$ the Plancherel measure $\mu(\omega_j, \nu)$ $(\nu \in \mathfrak{F})$ extends to a meromorphic function on $\mathfrak{F}_C$ and satisfies the following relation (see [3, Lemma 17.1]);

$$\mu(\omega_j, \nu) C_{P|P}(s; \nu)^* C_{P|P}(s; \nu) = c(P)^2 \qquad (s \in W).$$

Furthermore there exists a sufficiently small $\delta > 0$ such that (i) $\mu(\omega_j, \nu)$ $(1 \leqq j \leqq m)$ are holomorphic on $\mathfrak{F}(\delta)$, (ii) there exist numbers c, $r > 0$ such that

$$|\mu(\omega_j, \nu)| \leqq c(1+|\mathrm{Re}\,\nu|)^r \tag{16}$$

for all $1 \leqq j \leqq m$ and $\nu \in \mathfrak{F}(\delta)$ (see [3, Theorem 25.1]).

7. The proof of Theorem 5.1.

We keep to the notations in the preceding sections. First we prove that for $f\in \mathcal{C}^p(G,\tau)$ $F(f)$ is contained in $\mathcal{H}_p^\tau$, that is, $\hat{f}(\phi_i^j,\nu)$ $(1\leqq i\leqq n_j,\ 1\leqq j\leqq m)$ and (e_k,f) $(1\leqq k\leqq n')$ satisfy the four conditions of the space $\mathcal{H}_p^\tau$.

C1. It follows from (10) and (12) that

$$\begin{aligned}|\hat{f}(\phi,\nu)| &\leqq (c^2r)^{-1}\int_G |f(x)|_s|E(P:\phi:\nu:x)|_s dx \qquad (s\in\mathcal{S}(V))\\ &\leqq (c^2r)^{-1}\mu^p_{r_2+r_0,1,1,s}(f)\int_G \Xi(x)^{2/p}(1+\sigma(x))^{-r_0-r_2}\\ &\qquad\qquad \times c_2\|\phi\|_2|(\nu,x)|^{r_2}\Xi(x)^{-\varepsilon+1}dx \qquad \left(\varepsilon=\frac{2}{p}-1\right)\\ &=(c^2r)^{-1}\mu^p_{r_2+r_0,1,1,s}(f)c_2\|\phi\|_2(1+|\nu|)^{r_2}\int_G \Xi(x)^2(1+\sigma(x))^{-r_0}dx\\ &<\infty\end{aligned}$$

for all $\phi\in L_M$ and $\nu\in\mathfrak{F}(\varepsilon)$. Therefore $\hat{f}(\phi,\nu)$ is well-defined on $\mathfrak{F}(\varepsilon)$ and obviously, holomorphic on $\mathring{\mathfrak{F}}(\varepsilon)$.

C2. First we note that the same argument as above and (12) show that for any $u\in S(\mathfrak{F})$ there exist integer $l_u\geqq 0$ and a continuous seminorm μ_u on $\mathcal{C}^p(G,\tau)$ such that

$$|\hat{f}(\phi,\nu;u)|\leqq(1+|\nu|)^{l_u}\mu_u(f)$$

for all $\nu\in\mathfrak{F}(\varepsilon)$ and $f\in\mathcal{C}^p(G,\tau)$. Thus using the same arguments in [**9**, Theorem 3.5.5] and (11), we can obtain the following result. For any $u\in S(\mathfrak{F})$ there exists an integer $l_u\geqq 0$ satisfying the following condition; for each integer $r\geqq 0$ there exists a continuous seminorm $\mu_{u,r}$ on $\mathcal{C}^p(G,\tau)$ such that

$$(1+|\nu|)^r|\hat{f}(\phi,\nu;u)|\leqq(1+|\nu|)^{l_u}\mu_{r,u}(f)$$

for all $f\in\mathcal{C}^p(G,\tau)$ and $\nu\in\mathring{\mathfrak{F}}(\varepsilon)$. Then since l_u does not depend on r, the desired relation (6) is obvious.

C3. We note that for $m\in \boldsymbol{Z}^+$ and $\xi\in\mathring{\mathfrak{F}}(\varepsilon)$

$$D^m(\xi)\hat{f}(\phi,\nu)=(c^2r)^{-1}(D^m(\xi)E(P:\phi:\nu:\cdot),f)$$

for $\nu\in\mathring{\mathfrak{F}}(\varepsilon)$ by C1. Therefore it is clear that if there exists a relation (7), then $\hat{f}(\phi_i^j,\nu)$ $(1\leqq i\leqq n_j,\ 1\leqq j\leqq m)$ satisfy the corresponding relation (8).

C4. In order to obtain the relation (9) we shall apply the method in the proof of the Paley-Wiener theorem (cf. [**5, 6**]). First we put

$$F(x)=f(x)-\sum_{q=1}^{r_\varepsilon} c_q h_q(x) \qquad (x\in G),$$

where $c_q=D_q\hat{f}(\phi[q], \nu)$ $(1\leqq q\leqq \gamma_\varepsilon)$. Here we note that c_q $(1\leqq q\leqq \gamma_\varepsilon)$ are well-defined, because for each i, j, $\varepsilon\rho$ is not equal to $\xi_i^j(T_i^j(\varepsilon))$ and $\hat{f}(\phi_i^j, \nu)$ is holomorphic on $\mathring{\mathcal{F}}(\varepsilon)$. Then $F\in\mathcal{C}^p(G, \tau)$ and satisfies the following Lemma.

LEMMA 7.1. *$D^m(\xi_i^j(t))\hat{F}(\phi_i^j, \nu)=0$ for all $0\leqq m\leqq m_i^j(t)-1$, $1\leqq t\leqq T_i^j(\varepsilon)$, $1\leqq i\leqq n_j$ and $1\leqq j\leqq m$.*

PROOF. Fix m, t, i and j. Since $S_\varepsilon^\circ=\{E_q;\, 1\leqq q\leqq \gamma_\varepsilon\}$ is a maximal linearly independent subset of S_ε, there exist constants a_q $(1\leqq q\leqq \gamma_\varepsilon)$ such that

$$\begin{aligned} D^m(\xi_i^j(t))E(P:\phi_i^j:\nu:x) &= \sum_{q=1}^{\gamma_\varepsilon} a_q E_q(x) \\ &= \sum_{q=1}^{\gamma_\varepsilon} a_q D_q E(P:\phi[q]:\nu:x)\,. \end{aligned}$$

Then from the condition (iii) of $\mathcal{H}_p^\tau$ which was obtained in C3 we have

$$D^m(\xi_i^j(t))\hat{F}(\phi_i^j, \nu)=\sum_{q=1}^{\gamma_\varepsilon} \overline{a_q} D_q\hat{F}(\phi[q], \nu)\,.$$

Here we recall that $(c^2r)D_q\hat{h}_s(\phi[q], \nu)=(D_qE(P:\phi[q]:\nu:\cdot), h_s)=(E_q, h_s)=\delta_{sq}$ for all $1\leqq s$, $q\leqq \gamma_\varepsilon$. Then we have

$$\begin{aligned} D_q\hat{F}(\phi[q], \nu) &= D_q\hat{f}(\phi[q], \nu)-\sum_{s=1}^{\gamma_\varepsilon} c_s D_q\hat{h}_s(\phi[q], \nu) \\ &= c_q-\sum_{s=1}^{\gamma_\varepsilon} c_s\delta_{sq} \\ &= 0\,. \end{aligned}$$

Therefore $D^m(\xi_i^j(t))\hat{F}(\phi_i^j, \nu)=0$. This completes the proof of Lemma. Q. E. D.

Put $F=F_0+F_1$, where $F_0\in{}^\circ\mathcal{C}(G, \tau)$ and $F_1\in\mathcal{C}_A(G, \tau)$. Let $\delta>0$ be a sufficiently small number satisfying the condition in § 4 and (16). Then using Theorem 3.1, (2), the results in § 6 and Cauchy's Theorem, we see that for $a\in A^+$,

$$\begin{aligned} |W|e^{\rho(\log(a))}F_1(a) &= \sum_{j=1}^{m}\sum_{i=1}^{n_j}\int_{\mathcal{F}} \mu(\omega_j, \nu)E(P:\phi_i^j:\nu:a)\hat{F}(\phi_i^j, \nu)d\nu e^{\rho(\log(a))} \\ &= \sum_{j=1}^{m}\sum_{i=1}^{n_j}\int_{\mathcal{F}_\delta} \mu(\omega_j, \nu)e^{\rho(\log(a))}E(P:\phi_i^j:\nu:a)\hat{F}(\phi_i^j, \nu)d\nu \\ &= \sum_{j=1}^{m}\sum_{i=1}^{n_j}\int_{\mathcal{F}_\delta}\sum_{s\in W}\Phi(s\nu:a)C_{P|P}(s;\nu)^{*-1}\phi_i^j(1)\hat{F}(\phi_i^j, \nu)d\nu \\ &= \sum_{j=1}^{m}\sum_{i=1}^{n_j}\sum_{s\in W}\int_{s(\mathcal{F}_\delta)}\Phi(\nu:a)C_{P|P}(1;\nu)^{*-1}\phi_i^j(1)\hat{F}(\phi_i^j, \nu)d\nu\,. \end{aligned}$$

Here we put $I(j, i:\nu:a)=\Phi_0(\nu:a)C_{P|P}(1;\nu)^{*-1}\phi_i^j(1)\hat{F}(\phi_i^j, \nu)$. Then

$$F_1(a)=\frac{1}{|W|}\sum_{j=1}^{m}\sum_{i=1}^{n_j}\sum_{s\in W}e^{-\rho(\log(a))}\int_{s(\mathcal{F}_\delta)}e^{\sqrt{-1}\nu(\log(a))}I(j, i:\nu:a)d\nu\,.$$

Now we note that for each i, j $\hat{F}(\phi_i^j, \nu)$ is holomorphic on $\mathring{\mathcal{F}}_\delta^+(\varepsilon)$ by C1 and has zero points $\xi_i^j(t)$ of order $m_i^j(t)$ $(1 \leqq t \leqq T_i^j(\varepsilon))$ by Lemma 7.1. Therefore from the definition of $\xi_i^j(t)$ we see that $I(j, i : \nu : a)$ is holomorphic on $\mathring{\mathcal{F}}_\delta^+(\varepsilon)$. Moreover since $\hat{F}(\phi_i^j, \nu)$ satisfies the condition (ii) of $\mathcal{H}_p^\tau$ by C2 and Φ_0, $C_{P|P}{}^{*-1}$ satisfy (14), (15) respectively, it follows that for any $u \in S(\mathcal{F})$, $s \in \mathcal{S}(V)$, $r \in \boldsymbol{Z}^+$ and $v \in U(\mathfrak{a}_C)$ there exists a constant $c_{u,v,s,r} > 0$ such that

$$|I(j, i : \nu ; u : a ; v)|_s \leqq c_{u,v,s,r}(1+|\nu|)^{-r}$$

for $\nu \in \mathcal{F}_\delta^+(\varepsilon)$ and $a \in A_0^+$. For any $m \in \boldsymbol{Z}^+$ we can choose a $u_m \in S(\mathcal{F})$ such that

$$(1+\sigma(a))^m \leqq u_m(\sqrt{-1}\log(a)) \qquad (a \in A^+).$$

Here we note that for any $g_1, g_2 \in U(\mathfrak{g}_C)$ and $s \in \mathcal{S}(V)$ there exist a constant $c>0$ and elements $a_1, a_2, \cdots, a_t \in U(\mathfrak{g}_C)$ such that

$$|F_1(g_1 ; x ; g_2)|_s \leqq c \sum_{k=1}^{t} |F_1(x ; a_k)|_s \qquad (x \in G)$$

(see [11, p. 344, Lemma 3]). Moreover we can easily prove that for each a_k $(1 \leqq k \leqq t)$ there exist elements $b_{k,l}$, $c_{k,l} \in U(\mathfrak{a}_C)$ and $f_{k,l} \in C^\infty(A^+)$ $(1 \leqq l \leqq m_k)$ satisfying $|f_{k,l}(a)| < Ce^{-\rho(\log(a))}$ for some $C>0$ such that for $a \in A^+$

$$|F_1(a ; a_k)|_s \leqq \sum_{k=1}^{t} \sum_{l=1}^{m_k} |F_1(a ; b_{k,l})|_s + |f_{k,l}(a)|\,|F_1(a ; c_{k,l})|_s .$$

Put $A_1^+ = \{a \in A_0^+ ; |f_{k,l}| \leqq 1 \text{ for all } 1 \leqq k \leqq t,\ 1 \leqq l \leqq m_k\}$. Then using these facts, we obtain that for any $g_1, g_2 \in U(\mathfrak{g}_C)$, $s \in \mathcal{S}(V)$ and $m \in \boldsymbol{Z}^+$

$$\begin{aligned}&\sup_{x \in KA_1^+K} |F_1(g_1 ; x ; g_2)|_s \Xi(x)^{-2/p}(1+\sigma(x))^m \\ &\leqq c_1 \sup_{a \in A_1^+} \sum_{h=1}^{l} |F_1(a ; b_h)|_s e^{(2/p)\rho(\log(a))}(1+\sigma(a))^m\end{aligned}$$

for certain $b_1, b_2, \cdots, b_l \in U(\mathfrak{a}_C)$. Hence for each h $(1 \leqq h \leqq l)$ there exist polynomials P_h on $\mathcal{F}_C$ and elements $v_h \in U(\mathfrak{a}_C)$ such that

$$\begin{aligned}&\leqq c_1 \sup_{a \in A_1^+} \sum_{j=1}^{m} \sum_{i=1}^{n_j} \sum_{h=1}^{l} \left| \int_{\mathcal{F}} u_m(e^{\sqrt{-1}\nu(\log(a))}) P_h(\sqrt{-1}\nu) \right. \\ &\qquad \left. \times I(j, i : \nu : a ; v_h)\, d\nu \right|_s e^{(2/p-1)\rho(\log(a))}\end{aligned}$$

and obviously, there exist elements $u_{1,q}$, $u_{2,q} \in S(\mathcal{F})$ $(1 \leqq q \leqq d)$ such that

$$\begin{aligned}&= c_1 \sup_{a \in A_1^+} \sum_{j=1}^{m} \sum_{i=1}^{n_j} \sum_{h=1}^{l} \sum_{q=1}^{d} \left| \int_{\mathcal{F}} e^{\sqrt{-1}\nu(\log(a))} P_h(\sqrt{-1}\nu ; u_{1,q}) \right. \\ &\qquad \left. \times I(j, i : \nu ; u_{2,q} : a ; v_h)\, d\nu \right|_s e^{\varepsilon\rho(\log(a))} \qquad \left(\varepsilon = \frac{2}{p} - 1\right).\end{aligned}$$

Then by Cauchy's Theorem

$$=c_1 \sup_{a\in A_1^+} \sum_{j=1}^{m} \sum_{i=1}^{n_j} \sum_{h=1}^{l} \sum_{q=1}^{d} \left| \int_{\mathcal{F}+\sqrt{-1}\varepsilon\rho} e^{\sqrt{-1}\nu(\log(a))} P_h(\sqrt{-1}\nu\,;\, u_{1,q}) \right.$$

$$\left. \times I(j,\, i:\, \nu\,;\, u_{2,q}:\, a\,;\, v_h) d\nu \right|_s e^{\varepsilon\rho(\log(a))}$$

$$\leqq c_1\Big(\sum_{j=1}^{m} \sum_{i=1}^{n_j} \sum_{h=1}^{l} \sum_{q=1}^{d} c_{u_2,q,v_h,s,d_{h,q}}\Big) \int_{\mathcal{F}+\sqrt{-1}\varepsilon\rho} (1+|\nu|)^{-2} d\nu < \infty\,,$$

where $d_{h,q}=2+\deg(P_h(\cdot\,;\, u_{1,q}))$ for all h, q.
On the other hand, since $\mathrm{CL}(A^+-A_1^+)$ is compact, we deduce that

$$\sup_{x\in G} |F_1(g_1\,;\, x\,;\, g_2)|_s \Xi(x)^{-2/p}(1+\sigma(x))^m < \infty\,.$$

Therefore this implies that F_1, and thus F_0 are contained in $\mathcal{C}^p(G, \tau)$. Here we note that

$$F_0 = \sum_{k=1}^{i_p} (e_k,\, F)e_k + \sum_{k=i_p+1}^{n'} (e_k, F)e_k$$

and e_k $(1\leqq k\leqq i_p)$ are not in $\mathcal{C}^p(G, \tau)$. Then using the linear independence of e_k and the results in [7, 8], we obtain that $(e_k,\, F)=0$ for $1\leqq k\leqq i_p$, that is, by the definition of F,

$$(e_k,\, f) = \sum_{q=1}^{r_\varepsilon} c_q(e_k,\, h_q)$$

$$= \sum_{q=1}^{r_\varepsilon} D_q\hat{f}(\phi[q],\, \nu) A_{q,k} \qquad (1\leqq k\leqq i_p).$$

This is the desired relation (9).

All the conditions of $\mathcal{H}_p^\tau$ have now been established and thus for any $f\in\mathcal{C}^p(G, \tau)$, $F(f)$ is contained in $\mathcal{H}_p^\tau$.

The injectivity of the mapping $f\mapsto F(f)$ of $\mathcal{C}^p(G, \tau)$ into $\mathcal{H}_p^\tau$ is clear by Theorem 3.1, because $\mathcal{C}^p(G, \tau)$ is contained in $\mathcal{C}(G, \tau)$. Thus it remains to prove the surjectivity.

Let $\alpha=(a_k)_{k=1}^{n'} \oplus \bigoplus_{j=1}^{m} (\alpha_i^j(\nu))_{i=1}^{n_j}$ be an arbitrary element in $\mathcal{H}_p^\tau$ and put

$$f(x) = \sum_{k=1}^{n'} a_k e_k(x) + \frac{1}{|W|} \sum_{j=1}^{m} \sum_{i=1}^{n_j} \int_{\mathcal{F}} \mu(\omega_j,\, \nu) E(P:\, \phi_i^j:\, \nu:\, x) \alpha_i^j(\nu) d\nu$$

$$= f_0(x) + f_1(x) \qquad (x\in G).$$

Then by Theorem 3.1 $f\in\mathcal{C}(G, \tau)$ and $F(f)=\alpha$, that is, $(e_k,\, f)=a_k$ $(1\leqq k\leqq n')$ and $\hat{f}(\phi_i^j,\, \nu)=\alpha_i^j(\nu)$ $(\nu\in\mathcal{F},\, 1\leqq i\leqq n_j,\, 1\leqq j\leqq m)$. Hence to prove the surjectivity it is enough to show that f belongs to $\mathcal{C}^p(G, \tau)$. Now we put

$$F(x)=f(x)-\sum_{q=1}^{\gamma_\varepsilon} c_q h_q(x)\,,$$

where $c_q=D_q\alpha[q](\nu)$ $(1\leqq q\leqq\gamma_\varepsilon)$, which are well-defined as before (cf. C4). Put $\mathcal{F}(F)=\beta$. Then, since each h_q has compact support and thus $h_q\in\mathcal{C}^p(G,\tau)$, $\mathcal{F}(h_q)$ satisfies the conditions of $\mathcal{H}_p^\tau$ by the above considerations and β also satisfies these conditions. Therefore, using the same arguments as above, we see that F_1 belongs to $\mathcal{C}^p(G,\tau)$. On the other hand, since α satisfies the relation (9),

$$\begin{aligned}(e_k, F)&=(e_k, f)-\sum_{q=1}^{\gamma_\varepsilon} c_q(e_k, h_p)\\&=a_k-\sum_{q=1}^{\gamma_\varepsilon} D_q\alpha[q](\nu)A_{q,k}\\&=0\end{aligned}$$

for $1\leqq k\leqq i_p$. In particular, $F_0=\sum_{k=i_p+1}^{n'}(e_k, F)e_k\in\mathcal{C}^p(G,\tau)$. Hence $F=F_0+F_1$ and thus f are contained in $\mathcal{C}^p(G,\tau)$. This is the desired assertion. Theorem 5.1 is thereby established. Q. E. D.

8. An analogue of the Riemann-Lebesgue's Lemma.

Let $L^1(\boldsymbol{R})$ denote the set of all complex valued measurable functions on $\boldsymbol{R}$ with finite L^1-norm and for $f\in L^1(\boldsymbol{R})$ $\hat{f}(x)$ $(x\in\boldsymbol{R})$ the usual Fourier transform of f on $\boldsymbol{R}$. Then the Riemann-Lebesgue's Lemma implies that

$$\lim_{|x|\to\infty}\hat{f}(x)=0\,.$$

In this section we shall obtain an analogue of this Lemma. The similar results were obtained by M. Eguchi and K. Kumahara in [**15**].

For simplicity we fix a continuous seminorm $s\in\mathcal{S}(V)$ and denote the L^p-norm ${}_s\|\cdot\|_p$ by $\|\cdot\|_p$.

PROPOSITION 8.1. *Let f be in $L^1(G,\tau)$. Then for all $\phi\in L_M$*

$$\lim_{|\nu|\to\infty}\hat{f}(\phi,\nu)=0 \qquad (\nu\in\mathcal{F}).$$

PROOF. First we note that there exists a constant $M_1\geqq 0$ such that

$$|E(P:\phi:\nu:x)|_s\leqq M_1\Xi(x) \qquad (\nu\in\mathcal{F},\ x\in G). \tag{17}$$

Obviously, $|E(P:\phi:\nu:x)|_s\leqq M_1$ for all $x\in G$ and $\nu\in\mathcal{F}$. Since $C_c^\infty(G,\tau)$ is dense in $L^1(G,\tau)$, for any $\delta>0$ there exists a function $g\in C_c^\infty(G,\tau)$ such that $\|f-g\|_1\leqq\delta/2M_1$. Then we have

$$|\hat{f}(\phi,\nu)-\hat{g}(\phi,\nu)|<\delta/2\,.$$

Moreover by Theorem 3.1 $\hat{g}(\phi, \nu)$ is contained in $C(\mathfrak{F})$. Hence there exists a constant $N>0$ such that $|\hat{g}(\phi, \nu)|<\delta/2$ for $|\nu|>N$. Thus $|\hat{f}(\phi, \nu)|<\delta$ for $|\nu|>N$. This proves Proposition. Q. E. D.

LEMMA 8.2. *Let* $2<q<\infty$. *Then* $\sup_{\nu\in\mathfrak{F}}\|E(P:\phi:\nu:x)\|_q=M_q<\infty$ $(\phi\in L_M, \nu\in\mathfrak{F})$.

PROOF. We note that for $\alpha>0$, $\beta\geqq 0$ there exists a constant $C_{\alpha,\beta}>0$ such that

$$\Xi(x)^\alpha(1+\sigma(x))^\beta\leqq C_{\alpha,\beta}(1+\sigma(x))^{-r_0} \qquad (\text{see } \S 6).$$

Therefore for $\nu\in\mathfrak{F}$

$$\begin{aligned} M_q^q &= \sup_{\nu\in\mathfrak{F}}\int_G |E(P:\phi:\nu:x)|_s^q dx \\ &\leqq M_1{}^q\int_G \Xi(x)^q dx \qquad \text{by (17)} \\ &\leqq M_1{}^q C_{q-2,0}\int_G \Xi(x)^2(1+\sigma(x))^{-r_0}dx<\infty \qquad \text{by (10).} \end{aligned}$$

Q. E. D.

PROPOSITION 8.3. *Let f be in $L^p(G, \tau)$ $(1<p<2)$. Then for all $\phi\in L_M$*

$$\lim_{|\nu|\to\infty}\hat{f}(\phi, \nu)=0 \qquad (\nu\in\mathfrak{F}).$$

PROOF. Put $q=\dfrac{p}{p-1}$. Then $\dfrac{1}{p}+\dfrac{1}{q}=1$ and $2<q<\infty$. Since $C_c^\infty(G, \tau)$ is dense in $L^p(G, \tau)$, for any $\delta>0$ there exists a function $g\in C_c^\infty(G, \tau)$ such that $\|f-g\|_p\leqq\delta/2M_q$. Then by Hölder's inequality we have

$$\begin{aligned} |\hat{f}(\phi, \nu)-\hat{g}(\phi, \nu)| &\leqq\int_G|f-g|_s|E(P:\phi:\nu:\cdot)|_s dx \\ &\leqq\|f-g\|_p\|E(P:\phi:\nu:\cdot)\|_q \\ &\leqq\delta/2. \end{aligned}$$

The rest of the proof is the same as before. Q. E. D.

THEOREM 8.4. *Let f be in $L^p(G, \tau)$ and put $\varepsilon=\dfrac{2}{p}-1$ $(1\leqq p<2)$. Then for any $0\leqq\varepsilon_0<\varepsilon$ there exists a constant $l_{\varepsilon_0}\geqq 0$, which does not depend on f, such that for all $\phi\in L_M$*

$$\lim_{|\mathrm{Re}\,\nu|\to\infty}\frac{\hat{f}(\phi, \nu)}{(1+|\nu|)^{l_{\varepsilon_0}}}=0 \qquad (\nu\in\mathfrak{F}(\varepsilon_0)),$$

where $l_0=0$ and $\mathfrak{F}(0)=\mathfrak{F}$, when $\varepsilon_0=0$.

PROOF. Obviously, since Proposition 8.1 and 8.3 imply the case of $\varepsilon_0=0$, we may assume that $\varepsilon_0>0$. Here we recall that there exist constants c, $l=l_{\varepsilon_0}$

and $r>0$ such that for all $\nu\in\mathfrak{F}(\varepsilon_0)$

$$|E(P:\phi:\nu:x)|_s\leqq c(1+|\nu|)^l(1+\sigma(x))^l\Xi(x)^{-\varepsilon_0+1} \qquad \text{(see (12))}.$$

Therefore for $q=\dfrac{p}{p-1}$ and $\nu\in\mathfrak{F}(\varepsilon_0)$

$$\begin{aligned} M_{q,\varepsilon_0}^q &= \int_G |E(P:\phi:\nu:x)|_s^q dx \\ &\leqq c^q(1+|\nu|)^{ql}\int_G \Xi(x)^{-q\varepsilon_0+q}(1+\sigma(x))^{ql}dx \\ &\leqq c^q C_{-q\varepsilon_0+q-2,\,ql}(1+|\nu|)^{ql}\int_G \Xi(x)^2(1+\sigma(x))^{-r_0}dx \\ &\leqq N^q(1+|\nu|)^{ql}. \end{aligned}$$

Here we note that $-q\varepsilon_0+q-2>-q\varepsilon+q-2=0$ and N does not depend on ν. Then for any $\delta>0$ there exists a function $g\in C_c^\infty(G,\tau)$ such that

$$\|f-g\|_p<\delta/2N.$$

Thus by Hölder's inequality we obtain that

$$\frac{|\hat{f}(\phi,\nu)-\hat{g}(\phi,\nu)|}{(1+|\nu|)^l}<\delta/2.$$

Therefore Theorem is obvious by the same arguments as before. Q.E.D.

9. Convolution.

Let f and g be in $\mathcal{C}^p(G,\tau)$ $(0\leqq p\leqq 2)$. Then it follows from the results in [7] that $f*g(x)=\int_G f(y)g(y^{-1}x)dy$ is contained in $\mathcal{C}^p(G,\tau)$. In this section we shall obtain the Fourier transform of $f*g$. Put $f=f_0+f_1$ and $g=g_0+g_1$, where $f_0, g_0\in{}^\circ\mathcal{C}(G,\tau)$ and $f_1, g_1\in\mathcal{C}_A(G,\tau)$ respectively. Then we have

LEMMA 9.1. $f*g=f_0*g_0+f_1*g_1$.

PROOF. We note that since f_0 is a cusp form,

$$f_0*E(P:\phi:\nu:\cdot)=E(P:(f_0)_\nu^{(P)}*\phi:\nu:\cdot)=0 \qquad (\phi\in L_M,\ \nu\in\mathfrak{F})$$

(see [3, Lemma 8.2]). Therefore, since g_1 can be written as the sum of wave packets (cf. (3)), f_0*g_1 must be equal to zero. By the same way, we obtain that $f_1*g_0=0$. Then the desired relation is clear. Q.E.D.

The space of cusp forms is an algebra under convolution and thus, there exist constants $C_{kk's}$ $(1\leqq k,k',s\leqq n')$ and $C_{ii'v}^{jj'u}$ $(1\leqq i,i',v\leqq n_j,\ 1\leqq j,j',u\leqq m)$ such that

$$
\begin{aligned}
e_k * e_{k'} &= \sum_{s=1}^{n'} C_{kk's} e_s \\
\phi_i^j * \phi_{i'}^{j'} &= \sum_{u=1}^{m} \sum_{v=1}^{n_j} C_{ii'v}^{jj'u} \phi_v^u .
\end{aligned}
\tag{18}
$$

PROPOSITION 9.2. *Let notations be as above. Then for $1 \leqq v \leqq n_u$, $1 \leqq u \leqq m$ and $1 \leqq s \leqq n'$ we have*

$$\widehat{f*g}(\phi_v^u, \nu) = (c^2 r)^{-1} \sum_{j,j'=1}^{m} \sum_{i,i'=1}^{n_j} C_{ii'v}^{jj'u} \hat{f}(\phi_i^j, \nu) \hat{g}(\phi_{i'}^{j'}, \nu) \qquad (\nu \in \mathcal{F}),$$

$$(e_s, f*g) = \sum_{k,k'=1}^{n'} C_{kk's} (e_k, f)(e_{k'}, g) .$$

PROOF. We note that for any $f \in \mathcal{C}(G, \tau)$

$$(f)_\nu^{(P)} = \sum_{j=1}^{m} \sum_{i=1}^{n_j} \hat{f}(\phi_i^j, \nu) \phi_i^j \qquad (\nu \in \mathcal{F})$$

$$f_0 = \sum_{k=1}^{n'} (e_k, f) e_k$$

(cf. [3, Theorem 20.1]). Then the second relation of Proposition is obvious by (18) and the fact that $(f*g)_0 = f_0 * g_0$. The first is easily obtained by the following relation.

$$
\begin{aligned}
\widehat{f*g}(\phi, \nu) &= (c^2 r)^{-1} (E(P : \phi : \nu : \cdot), f*g) \\
&= (c^2 r)^{-1} (\phi, (f*g)_\nu^{(P)}) \\
&= (c^2 r)^{-1} (\phi, (f)_\nu^{(P)} * (g)_\nu^{(P)}) \qquad (\nu \in \mathcal{F})
\end{aligned}
$$

(see [3, Lemma 8.1]). Q.E.D.

COROLLARY 9.3. *$\mathcal{C}^p(G, \tau)$ is commutative under convolution if and only if V^M is abelian.*

PROOF. Using Proposition 9.2, we can easily see that $C_c^\infty(G, \tau)$ is commutative if and only if L_G and L_M are commutative. On the other hand, since a compactly supported function is determined by its principal part, that is, wave packets (cf. Theorem 5.2 and (9)), L_G is commutative when L_M is commutative. Therefore the desired assertion is clear from the facts that $C_c^\infty(G, \tau)$ is dense in $\mathcal{C}^p(G, \tau)$ and the mapping $\phi \mapsto \phi(1)$ sets up a bijection between L_M and V^M. Q.E.D.

10. Special case.

Put $W = C^\infty(K \times K)$ and define a representation $\mu = (\mu_1, \mu_2)$ of K on W as follows;

$$\left.\begin{array}{l}\mu_1(k)v(k_1, k_2)=v(k_1k, k_2)\\ v(k_1, k_2)\mu_2(k)=v(k_1, kk_2)\end{array}\right. \quad (k_1, k_2, k\in K, v\in W).$$

Then it is clear that μ is differentiable and unitary with respect to the norm;

$$|v|^2=\int_{K\times K}|v(k_1, k_2)|^2dk_1dk_2 .$$

For any finite subset $F\subset\mathcal{E}(K)$ we denote by W_F the subspace of all $v\in W$ such that

$$v=\int_K \alpha_F(k)\mu_1(k)vdk=\int_K \alpha_F(k)v\mu_2(k)dk ,$$

where $\alpha_F=\sum_{\delta\in F}d(\delta)\bar{\chi}_\delta$, χ_δ is the character of δ and $d(\delta)=\chi_\delta(1)$. Then it is easily to verify that W_F is stable under μ and its dimension is finite. Let μ_F denote the restriction of μ on W_F. Moreover we define $\mathrm{tr}(v)$, v^* and the product $v\cdot w$ $(v, w\in W)$ as in [3, §9] and write (V, τ) for (W_F, μ_F). Let $\mathcal{C}(G)_F$ denote the subspace of the Schwartz space $\mathcal{C}(G)$ of G which consists of all $f\in\mathcal{C}(G)$ such that

$$f*\alpha_F(k)=\alpha_F*f(k)=f(k) \qquad (k\in K).$$

Then the mapping $f(x)\mapsto\tilde{f}(x)(k_1, k_2)=f(k_1xk_2)$ $(x\in G,\ k_1, k_2\in K)$ sets up a homeomorphism between $\mathcal{C}(G)_F$ and $\mathcal{C}(G, \tau)$.

Let $\mathfrak{H}_\omega$ $(\omega\in\mathcal{E}(M))$ denote the representation space of

$$\pi^P_{\omega,\nu}=\mathrm{Ind}^G_{MAN}(\omega\otimes e^{\sqrt{-1}\nu}\otimes 1) \qquad (\nu\in\mathfrak{F}_c)$$

and put $\mathfrak{H}^F_\omega=E_F(\mathfrak{H}_\omega)$, where $E_F=\int_K\alpha_F(k)\pi^P_{\omega,\nu}(k)dk$. Then the following results were obtained in [3].

LEMMA 10.1 (see [3, §7]). *For each T in* $\mathrm{End}(\mathfrak{H}^F_\omega)$ *we can associate a Ψ_T in $L_M(\omega)$ such that the mapping $T\mapsto d(\omega)^{1/2}\Psi_T$ sets up a linear isometry between* $\mathrm{End}(\mathfrak{H}^F_\omega)$ *with the Hilbert-Schmidt norm and $L_M(\omega)$ with the L^2-norm, where $d(\omega)$ is the formal degree of the class ω.*

LEMMA 10.2 (see [3, Lemma 9.1]). *Let $S, T\in\mathrm{End}(\mathfrak{H}^F_\omega)$. Then*

$$\Psi_S*\Psi_T=d(\omega)^{-1}\Psi_{TS} ,$$

$$(\Psi_S, \Psi_T)=d(\omega)^{-1}\mathrm{tr}(S*T) .$$

LEMMA 10.3 (see [3, Theorem 7.1]). *Let $T\in\mathrm{End}(\mathfrak{H}^F_\omega)$. Then*

$$E(P:\Psi_T:\nu:x)(k_1, k_2)=\mathrm{tr}(T\pi^P_{\omega,\nu}(k_1xk_2))$$

for $k_1, k_2\in K$ and $x\in G$.

Now we shall apply the arguments in the preceding sections to the pair $(V, \tau)=(W_F, \mu_F)$ and use the same notations as before.

For each ω_j $(1\leqq j\leqq m)$ let h_i^j $(1\leqq i\leqq m_j=\dim \mathfrak{H}_{\omega_j}^F)$ denote an orthonormal base for $\mathfrak{H}_{\omega_j}^F$ and $T_{k,l}^j$ $(1\leqq k,\ l\leqq m_j)$ elements in $\mathrm{End}(\mathfrak{H}_{\omega_j}^F)$ such that $T_{k,l}^j(h_i^j)=\delta_{ki}h_l^j$. Then using Lemmas 1 and 2, we see that

$$\{\phi_{k,l}^j=d(\omega_j)^{1/2}\Psi_{T_{k,l}^j};\ 1\leqq k,\ l\leqq m_j,\ 1\leqq j\leqq m\}$$

is an orthonormal base for $L_M(\omega_j)$. For simplicity we write ϕ_k^j for $\phi_{k,k}^j$. Here we note that

$$E(P: \phi_{i,l}^j : \nu : 1)=d(\omega_j)^{1/2}\mathrm{tr}(T_{k,l}^j)=\delta_{kl}d(\omega_j)^{1/2}$$

and

$$\phi_{k,l}^j*\phi_{k',l'}^{j'}=\delta_{jj'}\delta_{l'k}d(\omega_j)^{-1/2}\phi_{k',l}^j\,. \tag{19}$$

Then using these relations, we can easily deduce the following formulas.

PROPOSITION 10.4. *Let $f=f_0+f_1\in \mathcal{C}(G,\tau)$, where $f_0\in{}^\circ\mathcal{C}(G,\tau)$ and $f_1\in\mathcal{C}_A(G,\tau)$. Then*

$$f_1(1)=\frac{1}{|W|}\sum_{j=1}^m d(\omega_j)^{1/2}\sum_{k=1}^{m_j}\int_{\mathcal{F}}\mu(\omega_j,\ \nu)\hat{f}(\phi_k^j,\ \nu)d\nu\,,$$

$$\|f_1\|_2^2=\frac{1}{|W|}\sum_{j=1}^m\sum_{k,l=1}^{m_j}\int_{\mathcal{F}}\mu(\omega_j,\ \nu)|\hat{f}(\phi_{k,l}^j,\ \nu)|^2d\nu \tag{20}$$

and

$$\|f_0\|_2^2=\sum_{i=1}^{n'}|(f_0,\ e_i)|^2. \tag{21}$$

The following results was obtained for $G=SL(2, \boldsymbol{R})$ in [**12**], for $G=SL(n, \boldsymbol{C})$ in [**13**] and for the general case in [**14**]. Here we shall give a more direct proof for the K-finite case.

THEOREM 10.5 (the Kunze-Stein phenomenon). *There exists a constant A_p for each $1\leqq p<2$ such that the inequality*

$$\|f*g\|_2\leqq A_p\|f\|_p\|g\|_2$$

is valid for all f in $L^p(G, \tau)$ and g in $L^2(G, \tau)$.

PROOF. Using the standard limiting arguments, we may assume that f and g belong to $C_c^\infty(G, \tau)$. Thus we can apply the previous results to this case. First we note that $\|f*g\|_2^2=\|f_0*g_0\|_2^2+\|f_1*g_1\|_2^2$ (cf. Lemma 9.1). Then we see that

$$\begin{aligned}\|f_1*g_1\|_2^2&=\|(f*g)\|_2^2\\&=\frac{1}{|W|}\sum_{j=1}^m\sum_{k,l=1}^{m_j}\int_{\mathcal{F}}\mu(\omega_j,\ \nu)|\widehat{f*g}(\phi_{k,l}^j,\nu)|^2d\nu\end{aligned}$$

$$=\frac{1}{|W|}\sum_{j=1}^{m}\sum_{k,l=1}^{m_j}\int_{\mathcal{F}}\mu(\omega_j,\nu)|\sum_{s=1}^{m_j}\hat{f}(\phi_{s,l}^j,\nu)\hat{g}(\phi_{k,l}^j,\nu)|^2d\nu$$

(see Proposition 9.2 and (19))

$$\leqq\max_j\{m_j\max_{s,l}(\sup_{\nu\in\mathcal{F}}|\hat{f}(\phi_{s,l}^j,\nu)|^2)\}$$

$$\times\frac{1}{|W|}\sum_{j=1}^{m}\sum_{k,s=1}^{m_j}\int_{\mathcal{F}}\mu(\omega_j,\nu)|\hat{g}(\phi_{k,s}^j,\nu)|^2d\nu$$

$$\leqq A_{p,1}\|f\|_p^2\|g\|_2^2$$

by Hölder's inequality, where $A_{p,1}=\max_j\{m_j\max_{s,l}(\sup_{\nu\in\mathcal{F}}\|E(P:\phi:\nu:\cdot)\|_q^2)\}$ $\left(q=\frac{p}{p-1}\right)$ (cf. Lemma 8.2). Moreover

$$\|f_0*g_0\|_2^2=\|(f*g)_0\|_2^2$$

$$=\sum_{i=1}^{n'}|(f*g,e_i)|^2$$

$$\leqq\sum_{i,s,t=1}^{n'}|C_{ist}|^2|(f_0,e_s)|^2|(g_0,e_t)|^2$$

(see Proposition 9.2)

$$\leqq n'\max_{i,s,t}(|C_{ist}|^2)\sum_{s=1}^{n'}|(f_0,e_s)|^2\sum_{t=1}^{n'}|(g_0,e_t)|^2$$

$$\leqq A_{p,2}\|f\|_p^2\|g\|_2^2$$

by Hölder's inequality, where $A_{p,2}=n'\max_{i,s,t}(|C_{ist}|^2)\sum_{s=1}^{n'}\|e_s\|_q^2$. Here we put $A_p=A_{p,1}+A_{p,2}$. Then the desired relation is obvious. Q.E.D.

REMARK. The assumption that the real rank of G equals one is not essential for the arguments in Sections 8, 9 and 10. Therefore we can easily extend the results in these sections to the case of arbitrary rank.

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Takeshi KAWAZOE
Department of Mathematics
Faculty of Science and Technology
Keio University
Hiyoshi, Yokohama 223
Japan

Added in Proof.

After this paper was written, the author has found that P.C. Trombi has obtained Theorem 5.1 in J. Functional Analysis, **40** (1981), 84-125.