J. Math. Soc. Japan
Vol. 34, No. 3, 1982

Note on γ-dimension and products of real projective spaces

By Teiichi KOBAYASHI

(Received Oct. 11, 1980)
(Revised Jan. 19, 1981)

1. Introduction.

Let α be the stable class of a vector bundle over a complex X. The γ-dimension, $\dim_\gamma \alpha$, of α is defined as follows (cf. [**6**]):

$$\dim_\gamma \alpha = \sup\{i \in N \mid \gamma^i(\alpha) \neq 0\},$$

where N is the set of positive integers and γ^i is the i-th Grothendieck γ-operation (cf. [**2**]). Let $\tau_0(M)$ denote the stable class of the tangent bundle $\tau(M)$ of a differentiable manifold M. H. Suzuki [**5**] investigated $\dim_\gamma \tau_0(P^m \times P^n)$ and $\dim_\gamma(-\tau_0(P^m \times P^n))$, where P^n is the n-dimensional real projective space, and applied them to the problem of vector fields on $P^m \times P^n$ and to the problem of immersions and embeddings of $P^m \times P^n$ in Euclidean spaces. The purpose of this note is to improve Suzuki's results.

Let $\varphi(n)$ be the number of integers s such that $0 < s \leqq n$ and $s \equiv 0, 1, 2$ or $4 \bmod 8$, $[a]$ be the integral part of a real number a, and $\binom{k}{i}$ be a binomial coefficient $k!/(k-i)!\,i!$. Define integers $\delta(n)$ and $\delta(m, n)$ as follows:

$$\delta(n) = \max\left\{i > 0 \;\middle|\; 2^{i-1}\binom{n+1}{i} \not\equiv 0 \bmod 2^{\varphi(n)}\right\},$$

$$\delta(m, n) = \max\left\{i > 1 \;\middle|\; 2^{i-2}\left\{\binom{m+n+2}{i} - \binom{m+1}{i} - \binom{n+1}{i}\right\} \not\equiv 0 \bmod 2^{[l/2]}\right\},$$

where $l = \min\{m, n\}$. Then we prove

THEOREM 1. $\dim_\gamma \tau_0(P^m \times P^n) \geqq \delta(m, n)$.

If $m = n = 2^r - 2$ $(r \geqq 3)$, then $\delta(m, n) = 2^{r-1} = \delta(n) + 1 > \delta(n)$. Therefore Theorem 1 is a partial improvement of [**5**, (4.2)]. But if $m = 2^r - 2$ and $n \leqq 2^{r-1} - 2$ $(r \geqq 3)$ then $\delta(m, n) \leqq 2^{r-2} < 2^{r-1} - 1 = \delta(m)$ and hence in this case [**5**, (4.2)] is better than the theorem. Combining [**5**, (4.2)] and Theorem 1, we obtain

---

This research was partially supported by Grant-in-Aid for Scientific Research (No. C-55408), Ministry of Education.

COROLLARY 2. $\dim_\gamma \tau_0(P^m \times P^n) \geqq \max\{\delta(m), \delta(n), \delta(m, n)\}$.

Define other integers $\sigma(n)$ and $\sigma(m, n)$ as follows:

$$\sigma(n)=\max\left\{i>0 \,\middle|\, 2^{i-1}\binom{n+i}{i} \not\equiv 0 \bmod 2^{\varphi(n)}\right\},$$

$$\sigma(m, n)=\max\left\{i>1 \,\middle|\, 2^{i-2}\left\{\binom{m+n+1+i}{i}-\binom{m+i}{i}-\binom{n+i}{i}\right\} \not\equiv 0 \bmod 2^{[l/2]}\right\}.$$

Then we have

THEOREM 3. $\dim_\gamma(-\tau_0(P^m \times P^n)) \geqq \sigma(m, n)$.

If $m=n=2^r+6$ $(r \geqq 4)$, then $\sigma(m, n)=2^{r-1}+2=\sigma(n)+1>\sigma(n)$. Thus Theorem 3 is also a partial improvement of [**5**, (6.2)]. But if $m=2^r$ and $n \leqq 2^{r-1}$ $(r \geqq 3)$ then $\sigma(m, n) \leqq 2^{r-2}+1<2^{r-1}=\sigma(m)$ and hence in this case [**5**, (6.2)] is better than the theorem. Combining [**5**, (6.2)] and Theorem 3, we obtain

COROLLARY 4. $\dim_\gamma(-\tau_0(P^m \times P^n)) \geqq \max\{\sigma(m), \sigma(n), \sigma(m, n)\}$.

2. Preliminaries.

First, we recall the basic facts about the γ-operations γ^i in K_R-rings of the product space $P^m \times P^n$ according to [**5**]. Let ξ and η be the Hopf bundles over P^m and P^n respectively, let p_i be the projection of $P^m \times P^n$ on the i-th factor $(i=1, 2)$, and put $x=\xi-1$ $(\in \tilde{K}_R(P^m))$ and $y=\eta-1$ $(\in \tilde{K}_R(P^n))$. $\tilde{K}_R(P^m)$ and $\tilde{K}_R(P^n)$ are regarded as the direct summands of

$$\tilde{K}_R(P^m \times P^n) \cong \tilde{K}_R(P^m)+\tilde{K}_R(P^n)+\tilde{K}_R(P^m \wedge P^n) \tag{2.1}$$

by the ring homomorphisms p_1^* and p_2^* respectively, and so we denote p_1^*x and p_2^*y simply by x and y respectively. Put $\tau_0=\tau_0(P^m \times P^n)$. In [**5**, (4.1) and (6.1)], the values of γ^i on $\pm\tau_0$ are calculated as follows:

THEOREM (2.2) (H. Suzuki).

$$\gamma^i(\tau_0)=(-2)^{i-1}\binom{m+1}{i}x+(-2)^{i-1}\binom{n+1}{i}y+(-2)^{i-2}\sum_{j=1}^{i-1}\binom{m+1}{j}\binom{n+1}{i-j}xy.$$

$$\gamma^i(-\tau_0)=-2^{i-1}\binom{m+i}{i}x-2^{i-1}\binom{n+i}{i}y+2^{i-2}\sum_{j=1}^{i-1}\binom{m+j}{j}\binom{n+i-j}{i-j}xy.$$

3. Proofs of theorems.

Let $c: K_R(X) \to K_C(X)$ be the complexification.

LEMMA (3.1). *The order of the element* $c(xy)=c(x)c(y) \in \tilde{K}_C(P^m \wedge P^n)$ *is equal to* $2^{[l/2]}$, *where* $l=\min\{m, n\}$.

PROOF. From the Künneth formula (e.g. [**4**, Chapter IV, 3.27]) we have a short exact sequence:

$$0\to\tilde{K}_C(P^m)\otimes\tilde{K}_C(P^n)+\tilde{K}_C^1(P^m)\otimes\tilde{K}_C^1(P^n)\to\tilde{K}_C(P^m\wedge P^n)$$
$$\to\mathrm{Tor}(\tilde{K}_C^1(P^m),\ \tilde{K}_C(P^n))+\mathrm{Tor}(\tilde{K}_C(P^m),\ \tilde{K}_C^1(P^n))\to 0\,.$$

Since $\tilde{K}_C^1(P^m)\cong Z$ or 0 according as m is odd or even, the homomorphism $\kappa:\tilde{K}_C(P^m)\otimes\tilde{K}_C(P^n)\to\tilde{K}_C(P^m\wedge P^n)$ defined by $\kappa(x\otimes y)=xy$ gives an isomorphism of $\tilde{K}_C(P^m)\otimes\tilde{K}_C(P^n)$ onto $\tilde{K}_C(P^m\wedge P^n)$ if m or n is even, and of $\tilde{K}_C(P^m)\otimes\tilde{K}_C(P^n)$ onto the torsion subgroup of $\tilde{K}_C(P^m\wedge P^n)$ if both m and n are odd. Therefore the order of $c(x)c(y)$ $(\in\tilde{K}_C(P^m\wedge P^n))$ is equal to the order of $c(x)\otimes c(y)$ $(\in\tilde{K}_C(P^m)\otimes\tilde{K}_C(P^n))$, which is equal to $2^{[l/2]}$ (cf. [**1**, 7.3]).

PROOF OF THEOREM 1. If $\gamma^i(\tau_0)=0$, then

$$2^{i-2}\sum_{j=1}^{i-1}\binom{m+1}{j}\binom{n+1}{i-j}xy=0\,,$$

by the first formula of (2.2) and the direct sum decomposition (2.1). Applying the complexification $c:\tilde{K}_R(P^m\wedge P^n)\to\tilde{K}_C(P^m\wedge P^n)$ to the equality and using the identity

$$\sum_{j=0}^{i}\binom{m+1}{j}\binom{n+1}{i-j}=\binom{m+n+2}{i},$$

we have, by Lemma (3.1),

$$2^{i-2}\left\{\binom{m+n+2}{i}-\binom{m+1}{i}-\binom{n+1}{i}\right\}\equiv 0 \mod 2^{[l/2]}\,.$$

Thus $\dim_\gamma\tau_0\geqq\delta(m,\ n)$.

PROOF OF THEOREM 3. If $\gamma^i(-\tau_0)=0$, then

$$2^{i-2}\sum_{j=1}^{i-1}\binom{m+j}{j}\binom{n+i-j}{i-j}xy=0\,,$$

by the second formula of (2.2) and the direct sum decomposition (2.1). In the way similar to the proof of Theorem 1, we have

$$2^{i-2}\left\{\binom{m+n+1+i}{i}-\binom{m+i}{i}-\binom{n+i}{i}\right\}\equiv 0 \mod 2^{[l/2]}\,,$$

using this time the identity

$$\sum_{j=0}^{i}\binom{m+j}{j}\binom{n+i-j}{i-j}=\binom{m+n+1+i}{i}.$$

Therefore $\dim_\gamma(-\tau_0)\geqq\sigma(m,\ n)$.

4. Remarks.

Corollaries 2 and 4 can be easily extended to the case of a product space $P=\prod_{i=1}^{r}P^{n_i}$ of a finite number of real projective spaces P^{n_i}, $i=1, 2, \cdots, r$. Define

$$\delta=\max\{\delta(n_i),\ \delta(n_j,\ n_k)\,|\,1\leqq i\leqq r,\ 1\leqq j\leqq r,\ 1\leqq k\leqq r\},$$

$$\sigma=\max\{\sigma(n_i),\ \sigma(n_j,\ n_k)\,|\,1\leqq i\leqq r,\ 1\leqq j\leqq r,\ 1\leqq k\leqq r\}.$$

Then we obtain

THEOREM (4.1). $\dim_{\gamma}\tau_0(P)\geqq\delta$.

THEOREM (4.2). $\dim_{\gamma}(-\tau_0(P))\geqq\sigma$.

The proofs of Theorems (4.1) and (4.2) are similar to those of Corollaries 2 and 4 respectively. So we omit the details.

5. Applications.

As applications we have some informations about the number, Span$P$, of linearly independent vector fields on $P=\prod_{i=1}^{r}P^{n_i}$, and immersions and embeddings of P in Euclidean space R^k, by using Atiyah's method (cf. [**2**] and [**5**]). Recall the following useful properties of γ^i (cf. [**2**, (2.3), (3.3) and (4.3)]).

THEOREM (5.1) (M. F. Atiyah).

(i) *If* $\alpha\in\tilde{K}_R(X)$, *then* $\gamma^i(\alpha)=0$ *for* $i>$g.dim α.

(ii) *Let M be a compact differentiable manifold of dimension m. If M is immersible in R^{m+k}, then $\gamma^i(-\tau_0(M))=0$ for $i>k$. If M is embeddable in R^{m+k}, then $\gamma^i(-\tau_0(M))=0$ for $i\geqq k$.*

Let δ and σ be the numbers defined in § 4 and put $\sum_{i=1}^{r}n_i=p$. Then we have

THEOREM (5.2). Span$P\leqq p-\delta$.

PROOF. Suppose that Span$P\geqq p-\delta+1$. Then there is a $(\delta-1)$-dimensional vector bundle ζ such that $\tau(P)\cong(p-\delta+1)\oplus\zeta$. Thus g.dim $\tau_0(P)\leqq\delta-1$. Hence, by (5.1), (i), $\gamma^i(\tau_0(P))=0$ for $i\geqq\delta$, namely $\dim_{\gamma}\tau_0(P)\leqq\delta-1$. This contradicts (4.1).

THEOREM (5.3). *P cannot be immersed in $R^{p+\sigma-1}$ and cannot be embedded in $R^{p+\sigma}$.*

PROOF. Suppose that P is immersed in $R^{p+\sigma-1}$ or embedded in $R^{p+\sigma}$. Then $\gamma^i(-\tau_0(P))=0$ for $i\geqq\sigma$, by (5.1), (ii), that is, $\dim_{\gamma}(-\tau_0(P))\leqq\sigma-1$. This contradicts (4.2).

Y. Hayashi [**3**] and M. Yasuo [**6**] studied the non-immersibility and the non-embeddability of products of lens spaces by using Suzuki's technique.

References

[1] J.F. Adams, Vector fields on spheres, Ann. of Math., **75** (1962), 603-632.
[2] M.F. Atiyah, Immersions and embeddings of manifolds, Topology, **1** (1962), 125-132.
[3] Y. Hayashi, Non-immersions of the products of lens spaces, Research Rep. Kushiro Tech. College, **8** (1974), 181-183.
[4] M. Karoubi, *K*-Theory, Grundlehren der mathematischen Wissenschaften, **226**, Springer-Verlag, Berlin-Heidelberg-New York, 1978.
[5] H. Suzuki, Operations in *KO*-theory and products of real projective spaces, Mem. Fac. Sci. Kyushu Univ. Ser. A Math., **18** (1964), 140-153.
[6] M. Yasuo, γ-dimension and products of lens spaces, Mem. Fac. Sci. Kyushu Univ. Ser. A Math., **31** (1977), 113-126.

Teiichi KOBAYASHI
Department of Mathematics
Faculty of Science
Kochi University
Akebono, Kochi 780
Japan