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# Exceptional values for meromorphic solutions of some difference equations

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## 1. Introduction.

In this note, we consider the non-linear difference equation

$$y(x+1)=R(y(x)), \tag{1.1}$$

where $R(x)$ is a rational function with the degree $p$, $p\geqq 2$.

Julia [**1**, p. 158] proved that

either there is a number $\lambda$ such that

$$\lambda=R(\lambda), \qquad R'(\lambda)=1, \tag{1.2}$$

or there is a number $\lambda$ such that

$$\lambda=R(\lambda), \qquad |R'(\lambda)|>1. \tag{1.3}$$

In either case, the equation (1.1) has a meromorphic solution determined as follows.

Let $\lambda$ be a number for which (1.2) holds. Putting

$$y(x)=\lambda+1/w(x), \tag{1.2-1}$$

we obtain

$$\begin{aligned} w(x+1)&=w(x)\left[1-\frac{R^{(m+1)}(\lambda)}{(m+1)!}w(x)^{-m}+\cdots\right] \qquad (m\geqq 1) \\ &=R_1(w(x)), \quad \text{with a rational function } R_1(x). \end{aligned} \tag{1.2-2}$$

Further, if we put

$$\omega(x)=w(x)^m/A^m, \qquad A=\left[\frac{-m}{(m+1)!}R^{(m+1)}(\lambda)\right]^{1/m}, \tag{1.2-3}$$

then we get

$$\omega(x+1)=F(\omega(x)), \tag{1.2-4}$$

where

$$F(x)=x+1+\sum_{j\geq m+1} b_j x^{1-j/m}. \tag{1.2-5}$$

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The equation (1.2-4) was studied by Kimura [2], [3]. He obtained a local solution $\phi(x)$ such that

(1.2-6)
$$\begin{cases} \phi(x) \text{ is holomorphic in the domain} \\ \qquad D_l(B,\varepsilon)=\left\{|x|>B,\ |\arg x-\pi|<\dfrac{\pi}{2}-\varepsilon\right\} \\ \qquad\qquad \cup\{\operatorname{Im}[xe^{-i\varepsilon}]>B\}\cup\{\operatorname{Im}[xe^{i\varepsilon}]<-B\} \\ \text{and has an asymptotic expansion} \\ \qquad \phi(x)\sim x\left[1+\sum_{j+k\geq 1}\alpha_{jk}x^{-j/m}\left(\dfrac{\log x}{x}\right)^k\right] \quad \text{in} \quad D_l(B,\varepsilon), \\ \text{where } \alpha_{m0}=c \text{ is arbitrarily prescribed, } \varepsilon \text{ is an arbitrary positive} \\ \text{number, and } B \text{ is a sufficiently large number depending on } c \text{ and } \varepsilon. \end{cases}$$

Then we obtain a meromorphic solution $\tilde{\phi}(x)$ of (1.2-2) such that

(1.2-7)
$$\begin{cases} \tilde{\phi}(x) \text{ is holomorphic in the domain } D_l(B,\varepsilon) \text{ and} \\ \qquad \tilde{\phi}(x)\sim Ax^{1/m}\left[1+\sum_{j+k\geq 1}\alpha_{jk}x^{-j/m}\left(\dfrac{\log x}{x}\right)^k\right]^{1/m} \\ \qquad\quad =Ax^{1/m}\left[1+\sum_{j+k\geq 1}\tilde{\alpha}_{jk}x^{-j/m}\left(\dfrac{\log x}{x}\right)^k\right] \quad \text{in} \quad D_l(B,\varepsilon) \\ \text{and is continued analytically to } |x|<\infty, \text{ using (1.2-2).} \end{cases}$$

Further, a meromorphic solution $\psi(x)$ of (1.1) is obtained by (1.2-1):

(1.2-8)
$$\psi(x)=\lambda+1/\tilde{\phi}(x)\,.$$

Let $\lambda$ be a number for which (1.3) holds. Then there is a solution $\sigma_\lambda(x)$ of (1.1) such that $\sigma_\lambda(x)$ is holomorphic in $D(\rho)=\{x\,;\,|e^{ax}|<\rho\}$

(1.3-1)
$$\sigma_\lambda(x)=\lambda+\sum_{j=1}^{\infty}\kappa_j e^{jax}=f_\lambda(e^{ax}) \qquad \text{in} \quad D(\rho)$$

for sufficiently small $\rho$, where $e^a=R'(\lambda)$, and

(1.3-2)
$$f_\lambda(t)=\lambda+\sum_{j=1}^{\infty}\kappa_j t^j \qquad \text{converges in } |t|<\rho\,.$$

$\sigma_\lambda(x)$ is continued analytically to $|x|<\infty$, using (1.1). Thus $f_\lambda(t)$ is also meromorphic in $|t|<\infty$.

We say that a value $\mu$ is a *maximally fixed value* (mf-value) for $R(x)$ if $x=\mu$ is the only $p$-fold root of the equation $R(x)=\mu$; also we say that a pair $(\mu_1, \mu_2)$ $(\mu_1\neq\mu_2)$ is a *maximally fixed pair* (mf-pair) for $R(x)$ if $x=\mu_2$ and $x=\mu_1$ are the $p$-fold roots of the equations $R(x)=\mu_1$ and $R(x)=\mu_2$, respectively. If we suppose that there is a $\lambda$ for which (1.2) holds, then it is easy to see that $R(x)$ has no mf-pair, and may have at most one mf-value. See Lemma 2.1 below.

Shimomura [6] showed that, if $R(x)$ has an mf-value $\mu$, then any meromorphic solution $y(x)$ of (1.1) does not take $\mu$. See also [7]. For the convenience of readers, we will prove this in Lemma 2.2.

In this respect, it would be natural to conjecture that: if there is a value $c$ such that $x=c$ is the $k$-fold root of $R(x)=c$ $(1\leqq k\leqq p-1)$, then there would be some values which are taken relatively sparsely by a meromorphic solution $y(x)$ of (1.1).

To consider this problem, we use some tools from the value distribution theory of Nevanlinna [4], [5]. Let $T(r)=T(r\,;f)$, $N(r, a)=N(r, a\,;f)$, and $m(r, a)=m(r, a\,;f)$ be the Nevanlinna characteristic, the counting function for the value $a$, and the proximity function for $a$ $(|a|\leqq\infty)$, respectively, of a meromorphic function $f(x)$ [4, pp. 165-167]. Then

$$T(r)=N(r, a)+m(r, a)+O(1), \qquad [4, \text{ p. } 166]. \tag{1.5}$$

We define (see [4, p. 266], [5, p. 147])

$$\delta(a, f)=\delta(a)=\liminf_{r\to\infty}\frac{m(r, a)}{T(r)}=1-\limsup_{r\to\infty}\frac{N(r, a)}{T(r)}. \tag{1.6}$$

$\delta(a)$ is called the (Nevanlinna) *deficiency* of the value $a$, for $f(x)$. If $\delta(a)>0$, then $a$ is said to be *deficient* or to be *Nevanlinna exceptional value*. $\delta(a)$ is a measure of the frequency in which the value $a$ is taken by $f(x)$. Thus, it would be natural to inquire whether there may be some deficient values or not. Our answer to this problem is:

THEOREM 1. *Suppose $\lambda$ is a value for which* (1.2) *holds. Then, the solution $\phi(x)$ in* (1.2-8) *has no Nevanlinna exceptional value other than a (possible) mf-value.*

THEOREM 2. *Suppose $\lambda$ is a value for which* (1.3) *holds. Then, the function $f_\lambda(x)$ in* (1.3-1) *and* (1.3-2) *has no Nevanlinna exceptional value other than (possible) mf-values or mf-pair.*

For the proof of Theorems 1 and 2, we need the following theorems which are of independent interest.

THEOREM 3. *We have*

$$\lim_{r\to\infty}\frac{T(r+1)}{T(r)}=p, \qquad \textit{where} \quad T(r)=T(r\,;\phi). \tag{1.7}$$

THEOREM 4. *We have*

$$\lim_{r\to\infty}\frac{T(cr)}{T(r)}=p, \qquad \textit{where} \quad T(r)=T(r\,;f_\lambda), \tag{1.8}$$

*where* $c=|e^a|=|R'(\lambda)|>1$. $(p=\deg[R(x)]$.)

## 2. Preliminaries.

LEMMA 2.1. *Suppose there is a $\lambda$ for which* (1.2) *holds. Then $R(x)$ has no mf-pair, and may have at most one mf-value.*

PROOF. Suppose $R(x)$ has an mf-pair $(\mu_1, \mu_2)$. Put

$$v(x)=[y(x)-\mu_1]/[y(x)-\mu_2], \tag{2.1}$$

then the equation (1.1) is transformed to

$$v(x+1)=R_2(v(x)), \tag{2.1'}$$

in which $R_2(x)$ has an mf-pair $(0, \infty)$. Then, $R_2(x)$ must be of the form $R_2(x)=K/x^p$, $p\geqq 2$ ($K$=const.). Put $\lambda_2=(\lambda-\mu_1)/(\lambda-\mu_2)$. Obviously $\lambda_2\neq 0, \infty$. Then $\lambda_2=R_2(\lambda_2)=K/\lambda_2^p$, i.e., $\lambda_2=K^{1/(p+1)}$. Then $R'(\lambda)=R_2'(\lambda_2)=-pK/\lambda_2^{p+1}=-p\neq 1$, which contradicts (1.2).

Suppose $\mu_1$ and $\mu_2$ are two mf-values. Then, putting as in (2.1), we get the equation (2.1′), where $R_2(x)$ is of the form $R_2(x)=Kx^p$, $p\geqq 2$, from which we again obtain a contradiction as above. Q.E.D.

LEMMA 2.2. *Let $\mu$ (or $(\mu_1, \mu_2)$) be an mf-value (or mf-pair for $R(x)$). Then, any meromorphic solution $y(x)$ of* (1.1) *does not take $\mu$ (any of $\mu_j$).*

PROOF. Put $u(x)=y(x)-\mu$. Then (1.1) becomes

$$u(x+1)=R_3(u(x)), \tag{2.1''}$$

in which $R_3(x)$ has the mf-value 0, hence of the form $R_3(x)=x^p/Q(x)$, $Q(0)\neq 0$. Suppose $u(x)$ has a zero of order $k$ at $x=x_0$. Then, by (2.1″), we see that $x_0-1$ must be also a zero point of order $k/p$. In general, $x_0-n$ must be a zero point of order $k/p^n$, which leads to a contradiction since $0<k/p^n<1$ if $n$ is sufficiently large.

The proof for mf-pair is similar, using the equation (2.1′). Q.E.D.

Further, we need the following

LEMMA 2.3 (Kimura). *Let $F(x)$ be the function in* (1.2-5). *Put*

$$F^n(x)=x+n+\chi_n(x)+\zeta_n(x),$$

*where $F^n(x)$ is the n-th iterate of $F(x)$, and*

$$\begin{cases} \chi_n(x)=\chi_n^{(1)}(x)+\cdots+\chi_n^{(m)}(x), \\ \chi_n^{(j)}(x)=\sum_{\nu=0}^{n-1} b_{j+m}(x+\nu)^{-j/m}, \qquad j=1, \cdots, m. \end{cases} \tag{2.2}$$

*Then, if $x\in\tilde{D}^B$ for a sufficiently large $B$, where*

$$\tilde{D}^B=\{|x|>B, \operatorname{Re} x>0\}, \tag{2.3}$$

*then we have*

(2.4) $$|\zeta_n(x)| \leqq K \sum_{\nu=0}^{n-1} \frac{|\chi_{\nu+1}^{(1)}(x)| + \cdots + |\chi_{\nu+1}^{(m)}(x)| + 1}{|x+\nu|^{1+1/m}},$$

*where $K$ is a constant.*

For the proof, see [**2**, p. 222, Theorem 9.1].

LEMMA 2.4. *Let $\delta$ be an arbitrary positive number. If $B$ is sufficiently large in* (2.3), *then we have*

(2.5) $$|\phi(x)/x - 1| < \delta \qquad \textit{for} \quad x \in D^B,$$

*where $\phi(x)$ is the meromorphic function defined by $\phi(x) = [\tilde{\phi}(x)/A]^m$ with the meromorphic solution $\tilde{\phi}(x)$ in* (1.2-7), *and $D^B$ is defined by*

(2.5′) $$D^B = \{|x| > B, \text{ Re } x < 0\} \cup \{|\text{Im } x| > B\}.$$

PROOF. By the equation (1.2-4), we have from Lemma 2.3

$$\phi(x+n) = \phi(x) + n + \chi_n(\phi(x)) + \zeta_n(\phi(x)).$$

By easy estimations of (2.2) and (2.4), we have

(2.6) $$\phi(x+n)/(x+n) \longrightarrow 1 \qquad \text{as} \quad n \longrightarrow \infty$$

if $x \in D_l(B, \varepsilon) \cap \{\text{Re } x > 0\}$. By asymptotic expansion (1.2-6) and (2.6), we obtain (2.5). Q. E. D.

## 3. Proof of Theorem 3.

Obviously, it suffices to prove the theorem for the function $\tilde{\phi}(x)$ in (1.2-7). In this section, $T(r)$, $N(r, a)$, and $m(r, a)$ denote the corresponding functions for $\tilde{\phi}(x)$.

We consider the equation

(1.2-2) $$w(x+1) = R(w(x)),$$

in which $R_1(x)$ is written as $R(x)$ for simplicity. $\tilde{\phi}(x)$ is a meromorphic solution of (1.2-2), admitting asymptotic expansion (1.2-7).

By Lemma 2.4, we have

LEMMA 3.1. *For any $\delta > 0$, we have*

(3.1) $$|\tilde{\phi}(x)/Ax^{1/m} - 1| < \delta \qquad \text{for} \quad x \in D^B,$$

*provided $B$ is sufficiently large.*

By [**4**, p. 276], we have

(3.2) $$T(r) \sim N(r, a) \qquad \text{for} \quad a \notin E,$$

where $E$ is a set of inner capacity 0. Since

$$E'=\{a\,;\ a=R(b),\ b\in E\}$$

is also of inner capacity 0, we can choose a value $a$ such that

if $a=R(a'_j)$, $j=1,\cdots,p$,

then $a, a'_1,\cdots,a'_p$ are mutually distinct, and

$$a\notin E\,,\quad a'_j\notin E\,,\quad j=1,\cdots,p\,.$$

By Lemma 3.1, $a$ as well as $a'_j$ are not taken by $\tilde{\phi}(x)$ in $D^B$, if $\delta>0$ and $B$ are suitably chosen. Thus, if $r$ is sufficiently large, then

$$\sum_{j=1}^{p}N(r,\ a'_j)\leqq N(r+1,\ a)+O(\log r)\leqq\sum_{j=1}^{p}N(r+\varepsilon(r),\ a'_j)\,, \tag{3.3}$$

where

$$\varepsilon(r)=\sqrt{[\sqrt{(r+1)^2-B^2}-1]^2+B^2}-r=O(1/r^2)\,. \tag{3.3'}$$

By (3.2) and (3.3), with the supposition on $a$, $a'_j$, we obtain

$$p(1+o(1))T(r)\leqq T(r+1)+O(\log r)\leqq p(1+o(1))T(r+\varepsilon(r))\,. \tag{3.4}$$

Especially, we get

$$T(r+1/2)\leqq p(1+o(1))T(r)\,. \tag{3.4'}$$

Since $T(r)$ is a convex function of $\log r$, we obtain for $0<\varepsilon<1/2$,

$$T(r+\varepsilon)\leqq\frac{[\log(r+1/2)-\log(r+\varepsilon)]T(r)+[\log(r+\varepsilon)-\log r]T(r+1/2)}{\log(r+1/2)-\log r} \tag{3.5}$$

$$\leqq T(r)+(p-1)\frac{\log(1+\varepsilon/r)}{\log(1+1/2r)}T(r)+o(1)\frac{\log(1+\varepsilon/r)}{\log(1+1/2r)}T(r)\,.$$

Since $\log(1+\varepsilon(r)/r)/\log(1+1/2r)\to 0$ as $r\to\infty$, we obtain by (3.5)

$$T(r+\varepsilon(r))/T(r)\longrightarrow 1\qquad\text{as}\quad r\longrightarrow\infty\,, \tag{3.5'}$$

hence by (3.4), we obtain (1.7) for $\tilde{\phi}(x)$ instead of $\phi(x)$. Q.E.D.

## 4. Proof of Theorem 4.

In this section, $T(r)$, $N(r,\ a)$, and $m(r,\ a)$ denote the corresponding functions for $f_\lambda(t)$ in (1.3-2). $f_\lambda(t)$ satisfies the equation

$$f_\lambda(e^\alpha t)=R(f_\lambda(t))\,,\qquad e^\alpha=R'(\lambda)\,. \tag{4.1}$$

As in § 3, we take a set $E$ of inner capacity 0 and a value $a$, as in (3.2) and (3.2'), respectively. Write $c=|e^\alpha|$. By (4.1), we have

$$\sum_{j=1}^{p} N(r, a_j') \sim N(cr, a) \tag{4.2}$$

as easily seen, from which we have (1.8) as in § 3. Q. E. D.

## 5. Some preliminary lemmas.

LEMMA 5.1. *Let $g(x)$ be a meromorphic function such that its characteristic $T(r)=T(r; g)$ satisfies* (1.7) *or* (1.8). *Then the inequality in the second fundamental theorem of Nevanlinna* [**4**, p. 246] *holds without any exception of values $r$.*

PROOF. Let $g(x)=c_\nu x^\nu + c_{\nu+1}x^{\nu+1}+\cdots$ $(c_\nu \neq 0)$. Then, in [**4**, p. 244, Lemma 1], we have

$$m\left(r, \frac{g'}{g}\right) < 11 + 3\log|1/c_\nu| + 2\log^+(1/r) + 4\log^+\rho + 3\log^+\frac{1}{\rho - r} + 4\log^+ T(\rho, g) \tag{5.2}$$

for all values of $r$ and $\rho$ $(0<r<\rho<\infty)$. If (1.7) or (1.8) holds, take $\rho = r+1$ or $\rho = cr$ in (5.2), respectively. Then the Theorem on the logarithmic derivative [**4**, p. 245] is valid without any exceptions, and our Lemma follows. Q. E. D.

For an integer $m \geqq 1$, let $R^m(x)$ be the $m$-th iterate of $R(x)$. For a value $a$, we denote by $A_m(a)$ the set of roots ($p^m$ in number) of the equation $a = R^m(x)$, counting multiple roots according its multiplicities. Write

$$A_m(a) = \{a_j^{(m)},\ j=1, \cdots, p^m\}, \tag{5.3}$$

and

$$A(a) = \bigcup_{m=1}^{\infty} A_m(a) \cup \{a\}. \tag{5.3'}$$

Then, obviously

$$\sum_{j=1}^{p^m} N(r, a_j^{(m)}) \leqq N(r+m, a) + O(\log r). \tag{5.4}$$

Let $r_n \uparrow \infty$. We have the following dichotomy: either

(5.5) there is an increasing sequence $\{k_h\}$ of positive integers with the property: for each $h$, there is a subsequence $\{r_n^{(h)}\}$ of $\{r_n\}$ such that $\{r_n^{(h+1)}\}$ is a subsequence of $\{r_n^{(h)}\}$ and

$$m(r_n^{(h)} + k_m, a) \geqq m(r_n^{(h)}, a), \quad n=1, 2, \cdots \quad \text{for} \quad m=1, \cdots, h, \tag{5.5'}$$

where $\{k_h\} = \{k_h(a)\}$ depending on $a$, or

(5.6) there is a subsequence $\{r_n^*\}$ of $\{r_n\}$ for which we can find an integer $k_0 = k_0(a)$ such that, for each $k \geqq k_0$,

$$m(r_n^* + k, a) < m(r_n^*, a) \tag{5.6'}$$

if $n \geqq n_k$, where $n_k$ is a sufficiently large number depending on $k$.

PROPOSITION 5.2. *Let $a$ be a value such that $A(a)$ in* (5.3′) *consists of mutually distinct values. Let $r_n \uparrow \infty$.*

(i) *Suppose* (5.5) *holds for $\{r_n+k\}$, $0 \leqq k < k^{\#}$ ($k^{\#} \leqq \infty$) and $a$. Then*

$$\lim_{n\to\infty} \frac{N(r_n+k,\ a)}{T(r_n+k)} = 1, \quad 0 \leqq k < k^{\#}. \quad (k^{\#} \text{ is a positive integer or } \infty.) \tag{5.7}$$

(ii) *Suppose* (5.6) *holds for $\{r_n\}$ and $a$. If*

$$\lim_{n\to\infty} \frac{N(r_n^*,\ a)}{T(r_n^*)} = 1-\delta, \qquad \delta \geqq 0, \tag{5.7′}$$

*then*

$$\liminf_{n\to\infty} \frac{N(r_n^*+k,\ a)}{T(r_n^*+k)} \geqq 1-\delta/p^k \qquad \textit{when} \quad k \geqq k_0. \tag{5.7″}$$

PROOF. (i) Obviously, it suffices to prove for the case $k=0$. Assume, taking a subsequence if necessary,

$$\lim_{n\to\infty} (N(r_n,\ a)/T(r_n)) = 1-\delta, \qquad \delta > 0. \tag{5.8}$$

Take $h$ so large that

$$h\delta > 2. \tag{5.9}$$

Write $\{r_n\}$ for $\{r_n^{(h)}\}$ for simplicity. By

$$T(r_n+k_m) - N(r_n+k_m,\ a) \geqq T(r_n) - N(r_n,\ a) + O(1),$$

$$1 - \frac{N(r_n+k_m,\ a)}{T(r_n+k_m)} \geqq \frac{T(r_n)}{T(r_n+k_m)}\left[1 - \frac{N(r_n,\ a)}{T(r_n)}\right].$$

Letting $n \to \infty$, we get

$$\limsup_{n\to\infty} (N(r_n+k_m,\ a)/T(r_n+k_m)) \leqq 1-\delta/p^{k_m}.$$

Then, by (5.4)

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{T(r_n)} \sum_{j=1}^{p^{k_m}} N(r_n,\ a_j^{(k_m)}) &\leqq \limsup_{n\to\infty} \frac{T(r_n+k_m)}{T(r_n)} \frac{N(r_n+k_m,\ a)}{T(r_n+k_m)} \\ &\leqq p^{k_m} - \delta.\end{aligned} \tag{5.10}$$

Let $q = p^{k_1} + \cdots + p^{k_h}$. By the second fundamental theorem [**4**, p, 246],

$$(q-2)T(r) < \sum_{m=1}^{h} \sum_{j=1}^{p^{k_m}} N(r,\ a_j^{(k_m)}) - N_1(r) + S(r),$$

where $N_1(r) \geqq 0$ and $S(r) = O(\log [rT(r)])$.

Let $r = r_n$ and $n \to \infty$, then we obtain by (5.9)

$$q-2 \leqq \sum_{m=1}^{h} (p^{k_m} - \delta) = q - h\delta,$$

which contradicts (5.9).

(ii) By (5.6′), we have

$$\frac{N(r_n^*+k,\ a)}{T(r_n^*+k)}=1-\frac{m(r_n^*+k,\ a)}{T(r_n^*+k)}+\frac{O(1)}{T(r_n^*+k)}$$

$$\geqq 1-\frac{m(r_n^*,\ a)}{T(r_n^*)}\frac{T(r_n^*)}{T(r_n^*+k)}+\frac{O(1)}{T(r_n^*+k)}.$$

Letting $n\to\infty$, we have (5.7″). Q.E.D.

In particular, we have

COROLLARY 5.3. *Let $a$ be a value such that the set $A(a)$ in* (5.3′) *consists of mutually distinct values. Then, $a$ is not a deficient value.*

## 6. Proof of Theorem 1—the first case.

Suppose there is a deficient value $a$. In this section, we suppose that

$$a\neq R^m(a) \qquad \text{for any} \quad m\geqq 1. \tag{6.1}$$

LEMMA 6.1. *Under the hypothesis* (6.1), *we have*

$$A_m(a)\cap A_{m'}(a)=\text{void} \qquad \text{if} \quad m\neq m'.$$

PROOF. Suppose $a_j^{(m)}=a_{j'}^{(m')}$ with $m>m'$. Then

$$a=R^m(a_j^{(m)})=R^{m-m'}(R^{m'}(a_{j'}^{(m')}))=R^{m-m'}(a),$$

which contradicts the hypothesis (6.1). Q.E.D.

LEMMA 6.2. *Suppose* (6.1) *holds. If $m$ is sufficiently large and $b\in A_m(a)$, then $A(b)$* (*see* (5.3′)) *consists of mutually distinct values.*

Proof is obvious from the fact that the equation $c=R(x)$ has multiple roots only for finitely many $c$. Q.E.D.

PROPOSITION 6.3. *Suppose* (6.1) *holds. Let $r_n\uparrow\infty$. There is a number $\delta'$, $0\leqq\delta'\leqq 1$, such that*

$$\liminf_{n\to\infty}(N(r_n+m+k,\ a)/T(r_n+m+k))\geqq 1-\delta'/p^k \tag{6.2}$$

*for any $k\geqq 1$, provided $m$ is sufficiently large.*

PROOF. Let $m$ be so large that Lemma 6.2 holds, and let $b_1,\cdots,b_h$ be all distinct values in $A_m(a)$, which appear in the multiplicities $\mu_1,\cdots,\mu_h$, respectively. Then $\mu_1+\cdots+\mu_h=p^m$ and

$$\sum_{j=1}^{h}\mu_j N(r_n,\ b_j)\leqq N(r_n+m,\ a)+O(\log r).$$

We can suppose, taking a subsequence if necessary,

$$\lim_{n\to\infty}(N(r_n,\ b_j)/T(r_n))=1-\delta_j, \qquad \delta_j\geqq 0, \quad j=1,\cdots,h.$$

Then, by Proposition 5.2 (i) and (ii), we have

$$\sum_{j=1}^{h}\mu_j N(r_n+k,\ b_j)\leqq N(r_n+m+k,\ a)+O(\log r)$$

and

$$\sum_{j=1}^{h}\mu_j(1-\delta_j/p^k)p^{-m}\leqq\liminf_{n\to\infty}(N(r_n+m+k,\ a)/T(r_n+m+k)).$$

Thus

$$1-p^{-k}\sum_{j=1}^{h}(\mu_j\delta_j/p^m)\leqq\liminf_{n\to\infty}(N(r_n+m+k,\ a)/T(r_n+m+k)). \quad \text{Q. E. D.}$$

In particular, we have

COROLLARY 6.4. *Let a be a value for which* (6.1) *holds. Then, a is not a deficient value.*

## 7. Proof of Theorem 1——the final case.

Suppose there is a deficient value $a$. We suppose that $a$ satisfies $a=R^m(a)$ for some $m\geqq 1$. Considering $R^m(x)$ instead of $R(x)$, we can suppose

$$a=R(a). \tag{7.1}$$

Let $b_0$ $(=a)$, $b_1, \cdots, b_h$ be all distinct values in $A_1(a)$, in the multiplicities $\mu_0$, $\mu_1, \cdots, \mu_h$, respectively. Then, each $b_j$, $j\geqq 1$, satisfies obviously the hypothesis (6.1). We note that $\mu_0+\mu_1+\cdots+\mu_h=p$.

Then

$$\sum_{j=0}^{h}\mu_j N(r,\ b_j)\leqq N(r+1,\ a)+O(\log r).$$

For each $b_j$, $j=1, \cdots, h$, let $m_j$ be the integer for which Proposition 6.3 holds with $b_j$ instead of $a$. Put $m=\max(m_1, \cdots, m_h)$. Take $r_n\uparrow\infty$. Suppose

$$\liminf_{n\to\infty}(N(r_n+m+k,\ a)/T(r_n+m+k))\geqq 1-\delta, \qquad \delta>0,$$

for a $k$. Then, by Proposition 6.3,

$$\begin{aligned} p^{-1}\Big[\mu_0(1-\delta)+\sum_{j=1}^{h}\mu_j(1-\delta_j'/p^k)\Big]&=1-\frac{\mu_0\delta}{p}-p^{-k-1}\sum_{j=1}^{h}\mu_j\delta_j' \\ &\leqq\liminf_{n\to\infty}(N(r_n+m+k+1,\ a)/T(r_n+m+k+1)). \end{aligned} \tag{7.2}$$

Thus, if we write

$$\delta^{(1)}=p^{-1}\Big[\mu_0\delta+p^{-k}\sum_{j=1}^{h}\mu_j\delta_j'\Big],$$

$$\delta^{(l)}=p^{-1}\Big[\mu_0\delta^{(l-1)}+p^{-k-l+1}\sum_{j=1}^{h}\mu_j\delta_j'\Big], \qquad l\geqq 2,$$

we obtain, using the arguments in (7.2) repeatedly,

$$\liminf_{n\to\infty}(N(r_n+m+k+l,\ a)/T(r_n+m+k+l))\geqq 1-\delta^{(l)}\,.$$

Therefore we obtain

$$\limsup_{n\to\infty}(N(\rho_n,\ a)/T(\rho_n))=1$$

for some sequence $\{\rho_n\}$, which completes the proof of Theorem 1. Q. E. D.

## 8. Proof of Theorem 2.

Proof is almost the same as in §§ 5$\sim$7, in which $(r+1)$ is replaced by $cr$ $(c>1)$, using (1.8) instead of (1.7).

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