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A condition for holomorphic maps of C^2 into C^2 to be algebraic

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1. In this paper we shall give a condition for holomorphic maps of C^2 into C^2 to be algebraic.

DEFINITION. A polynomial $P(X, Y)$ is said to be *of type* (g, n), if the level curve $P_c:=\{(X, Y)\in C^2 \mid P(X, Y)=c\}$ is of genus g and has n boundary points in the two dimensional projective space P^2 for almost every $c\in C$. In particular $P(X, Y)$ is said to be of general type, if $g\geqq 1$ or $n\geqq 3$.

THEOREM. *Let*

$$\Phi: X=f(x, y), \qquad Y=g(x, y)$$

be a holomorphic map of C^2 into C^2, where $f(x, y)$ and $g(x, y)$ are entire functions. If there exists a polynomial $P(X, Y)$ of general type such that the composite function

$$q(x, y):=P[f(x, y), g(x, y)]$$

is a polynomial, then $f(x, y)$ and $g(x, y)$ are polynomials.

REMARK. For a polynomial $P(X, Y)$ of type (0, 1) or (0, 2) the theorem is incorrect. In fact, we have the following counterexamples:

1) $P(X, Y)=X$ when $f(x, y)$ is a polynomial and $g(x, y)$ is a transcendental entire function.

2) $P(X, Y)=XY$ when $f(x, y)=e^x$ and $g(x, y)=e^{-x}$.

REMARK. T. Kizuka ([1]) proved the above theorem under the condition that Φ is an automorphism of C^2.

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2. Key lemma.

We introduce a lemma which is essential in this paper.

LEMMA (Nishino [2], see also [1]). *Let (V, π, Γ) be an analytic family of compact analytic curves of genus g on the disc Γ: $|z|<1$. Suppose every fibre on $z\neq 0$ is irreducible, non-singular and of genus g. If an unramified finitely many-valued analytic section η on the punctured disc Γ': $0<|z|<1$ satisfies one* *of the following conditions* (1), (2) *and* (3), *then* η *can be extended to* Γ:

1) $g \geqq 2$.

2) $g=1$. *There exists on* Γ *a finitely many-valued analytic section* ξ *which is unramified on* Γ', *and each branch of* ξ *doesn't intersect any branch of* η *on* Γ'.

3) $P=0$. *There exists on* Γ *a finitely many-valued analytic section* ξ *which satisfies the same condition of* 2), *and the number of sheets is greater than or equal to* 3.

3. Construction of an analytic family.

Let $P(X, Y)$ be a polynomial and suppose $P_c := \{(X, Y) \in C^2 \mid P(X, Y)=c\}$ is irreducible for almost every $c \in C$. Let M be the hypersurface $\{(X, Y, Z) \in C^3 \mid Z=P(X, Y)\}$. We regard M as the union of P_c's.

LEMMA. *There exist a compactification* $\tilde{M}$ *of* M, (ι *denotes the embedding* $M \hookrightarrow \tilde{M}$) *and a holomorphic map* $\pi: \tilde{M} \to C^1 \cup \{\infty\}$ *such that*

1) $\pi^{-1}(c)$ *is a non-singular compact curve for all but finite* c.

2) ι, *restricted on* P_c, *induces an embedding* $P_c \hookrightarrow \pi^{-1}(c)$, $c \neq \infty$, *which gives the compactification of* P_c.

PROOF. Let $(U:V:W)$ be a homogeneous coordinate of P^2. We understand (X, Y) an inhomogeneous coordinate $(U/W, V/W)$ of the affine part $\{(U:V:W) \in P^2 \mid W \neq 0\}$ of P^2. Let m be a degree of $P(X, Y)$. We can obtain the rational surface $\tilde{P}^2$ such that the closure of P_c in $\tilde{P}^2$ is non-singular for all but finite c, by making use of blowing-ups at indeterminations of the rational function $W^m P(U/W, V/W)$ on P^2. Let $\tilde{M}$ denote the closure of M in $P^1 \times \tilde{P}^2$ and π the restriction on $\tilde{M}$ of the first projection $P^1 \times \tilde{P}^2 \to P^1$. Then we can easily check that $\tilde{M}$ and π satisfy the conditions of the lemma.

NOTE. When $P(X, Y)$ is of type (g, n), $\pi^{-1}(c)-P_c$ consists of n points for all but finite $c \neq \infty$, and

$$\bigcup_{c \in C} (\pi^{-1}(c)-P_c)=\tilde{M}-\pi^{-1}(\infty)-M .$$

Hence the closure $\overline{\bigcup_{c \in C}(\pi^{-1}(c)-P_c)}$ is an algebraic curve in $\tilde{M}$ and defines n-valued analytic section ξ on P^1.

4. Proof of the theorem.

It is known that for every polynomial $Q(X, Y)$ there exist a polynomial $P(X, Y)$ and a polynomial $g(z)$ of one variable such that $Q(X, Y)=q[P(X, Y)]$ and that, $P_c := \{(X, Y) \in C^2 \mid P(X, Y)=c\}$ is irreducible for almost every $c \in C$ (cf. Stein [3]). Hence we may suppose that the level curve P_c is irreducible for almost every c.

Suppose $f(x, y)$ is a transcendental entire function. Then there exist a line

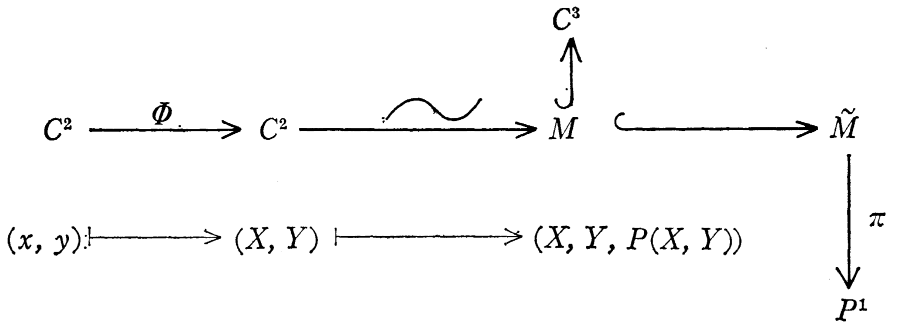
$$l_{a_0} := \{(x, y) \in C^2 \mid y = a_0 x\}$$

and an analytic curve

$$\nu_{\alpha_0} := \{(x, y) \in C^2 \mid f(x, y) = \alpha_0\}$$

such that $l_{a_0} \not\subset \nu_{\alpha_0}$ and l_{a_0} intersects ν_{α_0} at infinitely many points.

We identify C^2 with M and obtain the following diagram:



Put $q_c := \{(x, y) \in C^2 \mid q(x, y) = P[f(x, y), g(x, y)] = c\}$. We conclude that $\Phi(l_{a_0}) \cap P_c = \Phi(l_{a_0} \cap q_c)$. In fact, if $\Phi(l_{a_0}) \cap P_c \supsetneqq \Phi(l_{a_0} \cap q_c)$, then there exists a point t_1 in $\Phi^{-1}(\Phi(l_{a_0})) \cap q_c - l_{a_0} \cap q_c$ such that $\Phi(t_1) \notin \Phi(l_{a_0} \cap q_c)$. Hence $\Phi^{-1}(\Phi(t_1)) \cap l_{a_0} - q_c$ is not empty, because $\Phi(t_1) \in \Phi(l_{a_0})$. This contradicts the fact that $\Phi^{-1}(P_c) = q_c$. Since $q(x, y)$ is a polynomial, q_c and l_{a_0} intersect at finitely many points. Hence the curves P_c and $\Phi(l_{a_0})$ intersect at finitely many points in M, and $\Phi(l_{a_0})$ defines a finitely many-valued analytic section η of M on $P^1 - \{\infty\}$.

Put $V = \tilde{M}$ and $\Gamma = \{c \in C \cup \{\infty\} \mid |c| \gg 1\}$ and recall that $P(X, Y)$ is of general type. Then we see that η, in the neighborhood of ∞, satisfies one of the three conditions in the lemma in § 2. Therefore the closure $\bar{\eta}$ of η in M is a compact analytic curve. Note that the closure of $\Phi(\nu_{\alpha_0}) = \{(X, Y) \in C^2 \mid X = \alpha_0\}$ in $\tilde{M}$ is also a compact analytic curve. This contradicts the fact that $\Phi(l_{a_0})$ and $\Phi(\nu_{\alpha_0})$ intersect at infinitely many points. This completes the proof.

References

[1] T. Kizuka, Analytic automorphisms and algebraic automorphisms of C^2, Tohoku Math. J., **31** (1979), 553-565.

[2] T. Nishino, Nouvelles recherches sur les fonctions entieres de plusieurs variables complexes (V), J. Math. Kyoto Univ., **15** (1975), 527-553.

[3] K. Stein, Analytische Projektion Komplexer Mannigfaltigkeiten, Colloque Bruxell, 1953.

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