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On the boundary behavior of superharmonic functions in a half space

By Yoshihiro MIZUTA

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1. Introduction.

A non-negative superharmonic function u in the half space $D=\{x=(x_1, \cdots, x_n) \in R^n ; x_n>0\}$, $n \geqq 2$, is represented as

$$u(x)=a x_n+\int_D G(x, y) d\mu(y)+\int_{\partial D} P(x, y) d\nu(y), \qquad x \in D,$$

where a is a non-negative number, μ (resp. ν) is a non-negative measure on D (resp. ∂D), G is the Green function for D and P is the Poisson kernel for D. It is known in [4] that

$$\lim_{x \to O,\, x \in D-E} x_n^{-1} u(x)=a+b_n \int \frac{y_n}{|y|^n} d\mu(y)+c_n \int \frac{1}{|y|^n} d\nu(y),$$

$$\lim_{x \to O,\, x \in D-E} x_n^{-1}|x|^n\{u(x)-a x_n\}=c_n \nu(\{O\})$$

for a Borel set $E \subset D$ which is minimally thin at O, where

$$b_n=\begin{cases} 2(n-2) & \text{if } n \geqq 3, \\ 2 & \text{if } n=2, \end{cases} \qquad c_n=\pi^{-n/2} \Gamma(n/2).$$

Our aim in this note is to show that $x_n^{-\beta}|x|^{\beta+\gamma}\{u(x)-a x_n\}$, $0 \leqq \beta \leqq 1$, $-1 \leqq \gamma \leqq n-1$, has a limit as $x \to O$ with an exceptional set, for which we shall give a metrical estimate of Wiener type. To do this, we shall study the boundary behavior of the Green potential $G_\alpha(x, \mu)=\int_D G_\alpha(x, y) d\mu(y)$, where

$$G_\alpha(x, y)=\begin{cases} |x-y|^{\alpha-n}-|\bar{x}-y|^{\alpha-n} & \text{in case } 0<\alpha<n, \\ \log(|\bar{x}-y|/|x-y|) & \text{in case } \alpha=n=2, \end{cases}$$

$\bar{x}$ denoting the reflection of x with respect to the surface ∂D, i.e.,

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$$\bar{x}=(x_1, \cdots, x_{n-1}, -x_n) \quad \text{for } x=(x_1, \cdots, x_{n-1}, x_n).$$

As an application, we shall prove the existence of radial limits of $G_\alpha(x, \mu)$.

2. Main results.

We first note the following elementary lemma.

LEMMA 1. *There exist positive constants* c_1 *and* c_2 *such that*

$$c_1 \frac{x_n y_n}{|x-y|^{n-\alpha}|\bar{x}-y|^2} \leqq G_\alpha(x, y) \leqq c_2 \frac{x_n y_n}{|x-y|^{n-\alpha}|\bar{x}-y|^2} \quad \textit{in case } \alpha<n,$$

$$c_1 \frac{x_n y_n}{|\bar{x}-y|^2} \leqq G_2(x, y) \leqq c_2 \frac{x_n y_n}{|x-y|^2}$$

for $x=(x_1, \cdots, x_n)$ *and* $y=(y_1, \cdots, y_n)$ *in* D.

Set

$$k_{\alpha,\beta,\delta}(x, y)=x_n^{-\delta} y_n^{-\beta} G_\alpha(x, y) \quad \text{for } x, y \in D.$$

If $\beta=\delta=1$, then $k_\alpha=k_{\alpha,1,1}$ is extended to be continuous on $\bar{D}\times\bar{D}$ in the extended sense, where $\bar{D}=D\cup\partial D$.

Following Fuglede [2], we set

$$k(x, \mu)=\int_E k(x, y)d\mu(y) \quad \text{and} \quad k(\mu, y)=\int_E k(x, y)d\mu(x)$$

for a non-negative Borel measurable function k on $R^n\times R^n$ and a non-negative measure μ on a Borel set $E\subset R^n$. Define the capacity

$$C_k(E)=\sup \mu(R^n), \quad E\subset D,$$

where the supremum is taken over all non-negative measures μ on D such that S_μ (the support of μ) is contained in E and

$$k(x, \mu)\leqq 1 \quad \text{for every } x\in D.$$

The following lemma can be proved by using Fuglede [2; Théorème 7.8].

LEMMA 2. *For a Borel set* E *in* D, *we have*

$$C_{k_{\alpha,\beta,\delta}}(E)=\inf \lambda(D) \quad (\textit{resp.}\ \inf \lambda(\bar{D})) \quad \textit{if } \delta<1\ (\textit{resp.}\ \delta=1),$$

where the infimum is taken over all non-negative measures λ *on* D (*resp.* $\bar{D}$) *such that* $k_{\alpha,\beta,\delta}(\lambda, y)\geqq 1$ *for every* $y\in E$.

By Lemma 1, we obtain the following lemma.

LEMMA 3. *Let* μ *be a non-negative measure on* D. *Then* $G_\alpha(x, \mu)\not\equiv\infty$ *if and only if* $\int_D (1+|y|)^{\alpha-n-2} y_n d\mu(y)<\infty$.

Let μ be a non-negative measure on D such that $G_\alpha(x, \mu)\not\equiv\infty$, and define

$d\lambda(y)=y_n d\mu(y)$. Then λ is a measure on $\bar{D}$ by Lemma 3, and $G_\alpha(x, \mu)=x_n k_\alpha(x, \lambda)$.

For a non-negative measure λ on $\bar{D}$, we write $k_\alpha(x, \lambda)=k'_\alpha(x, \lambda)+k''_\alpha(x, \lambda)$, where

$$k'_\alpha(x, \lambda)=\int_{\{y\in\bar{D};|x-y|\geqq|x|/2\}} k_\alpha(x, y)d\lambda(y),$$

$$k''_\alpha(x, \lambda)=\int_{\{y\in\bar{D};|x-y|<|x|/2\}} k_\alpha(x, y)d\lambda(y).$$

LEMMA 4. *Let* $-1\leqq\gamma\leqq n-\alpha+1$ *and* $\int_{\bar{D}}|y|^{\alpha+\gamma-n-1}d\lambda(y)<\infty$. *Then*

$$\lim_{x\to O,\, x\in D} x_n^{1-\beta}|x|^{\beta+\gamma}k'_\alpha(x, \lambda)$$

$$=\begin{cases} k_\alpha(O, \lambda) & \text{if } \beta=1 \text{ and } \gamma=-1, \\ d_\alpha\lambda(\{O\}) & \text{if } \beta=1 \text{ and } \gamma=n-\alpha+1, \\ 0 & \text{if } 0\leqq\beta\leqq1 \text{ and } -1<\gamma<n-\alpha+1, \end{cases}$$

where $d_\alpha=2(n-\alpha)$ *if* $\alpha<n$ *and* $=2$ *if* $\alpha=n$.

PROOF. If $x, y\in D$ and $|x-y|\geqq|x|/2$, then

$$x_n^{1-\beta}|x|^{\beta+\gamma}k_\alpha(x, y)\leqq\text{const.}\,|y|^{\alpha+\gamma-n-1},$$

so that Lebesgue's dominated convergence theorem gives the lemma.

LEMMA 5. *Let* $-1\leqq\gamma\leqq n-\alpha+\delta$ *and* λ *be a non-negative measure on* D *(resp.* $\bar{D}$*) satisfying*

$$\int_D|y|^{\alpha+\gamma-\delta-n}y_n^{\delta-1}d\lambda(y)<\infty, \qquad \delta<1,$$

$$\left(\text{resp. } \int_{\bar{D}}|y|^{\alpha+\gamma-n-1}d\lambda(y)<\infty, \qquad \delta=1\right).$$

Then there exists a Borel set $E\subset D$ *with the properties:*

i) $\lim_{x\to O,\, x\in D-E} x_n^{1-\beta}|x|^{\beta+\gamma}k''_\alpha(x, \lambda)=0;$

ii) $\sum_{i=1}^{\infty}2^{i(n-\alpha+\beta+\delta)}C_{k_\alpha,\beta,\delta}(E^{(i)})<\infty,$

where $E^{(i)}=\{x\in E;\, 2^{-i}\leqq|x|<2^{-i+1}\}$.

PROOF. We shall prove only the case $\delta=1$, because the case $\delta<1$ can be proved similarly. Let $\{a_i\}$ be a sequence of positive numbers such that $\lim_{i\to\infty}a_i=\infty$ but $\sum_{i=1}^{\infty}a_ib_i<\infty$, where

$$b_i=\int_{\{y\in\bar{D};2^{-i-1}<|y|<2^{-i+2}\}}|y|^{\alpha+\gamma-n-1}d\lambda(y).$$

Consider the sets

$$E_i=\{x\in D\,;\,2^{-i}\leqq|x|<2^{-i+1},\ x_n^{1-\beta}k''_\alpha(x,\lambda)\geqq a_i^{-1}2^{i(\beta+\gamma)}\}$$

for $i=1, 2, \cdots$. If μ is a non-negative measure on D such that $S_\mu\subset E_i$ and $k_{\alpha,\beta,1}(x,\mu)\leqq 1$ for $x\in D$, or equivalently, for $x\in\bar{D}$ by the lower semicontinuity of $k_{\alpha,\beta,1}$, then

$$\begin{aligned}\int_D d\mu &\leqq a_i 2^{-i(\beta+\gamma)}\int x_n^{1-\beta}k''_\alpha(x,\lambda)d\mu(x)\\ &\leqq a_i 2^{-i(\beta+\gamma)}\int_{\{y\in\bar{D};2^{-i-1}<|y|<2^{-i+2}\}}k_{\alpha,\beta,1}(y,\mu)d\lambda(y)\\ &\leqq a_i 2^{-i(\beta+\gamma)}\int_{\{y\in\bar{D};2^{-i-1}<|y|<2^{-i+2}\}}d\lambda(y)\\ &\leqq 4^{n-\alpha-\gamma+1}2^{-i(n-\alpha+\beta+1)}a_i b_i\,,\end{aligned}$$

which yields

$$C_{k_{\alpha,\beta,1}}(E_i)\leqq 4^{n-\alpha-\gamma+1}2^{-i(n-\alpha+\beta+1)}a_i b_i\,.$$

Thus the set $E=\bigcup_{i=1}^{\infty}E_i$ has the required properties.

THEOREM 1. *Let $\gamma\geqq -1$, $\delta\leqq 1$ and $n-\alpha-\gamma+\delta\geqq 0$. Let μ be a non-negative measure on D satisfying*

$$\int_D|y|^{\alpha+\gamma-\delta-n}y_n^\delta d\mu(y)<\infty\,.$$

Then there exists a Borel set $E\subset D$ such that

$$\lim_{x\to 0,\,x\in D-E}x_n^{-\beta}|x|^{\beta+\gamma}G_\alpha(x,\mu)=\begin{cases}d_\alpha\int|y|^{\alpha-n-2}y_n d\mu(y) & \text{if } \beta=1 \text{ and } \gamma=-1\,,\\ 0 & \text{if } 0\leqq\beta\leqq 1 \text{ and } \gamma>-1\,;\text{ and}\end{cases}$$

$$\text{(A)}\qquad \sum_{i=1}^{\infty}2^{i(n-\alpha+\beta+\delta)}C_{k_{\alpha,\beta,\delta}}(E^{(i)})<\infty\,,$$

where $E^{(i)}=\{x\in E\,;\,2^{-i}\leqq|x|<2^{-i+1}\}$.

This theorem follows readily from Lemmas 4 and 5.

Let $R_\alpha(x,y)=|x-y|^{\alpha-n}$ if $\alpha<n$ and $=\log(|\bar{x}-y|/|x-y|)$ if $\alpha=n$. For simplicity, we write C_α for C_{R_α}. We denote by Λ_l the l-dimensional Hausdorff measure.

COROLLARY. *Let $\alpha+\gamma-1>1$, and μ be a non-negative measure on D such that $G_\alpha(x,\mu)\not\equiv\infty$. Then we can find a set $E\subset\partial D$ such that $C_{\alpha+\gamma-1}(E)=0$ and for each $\xi\in\partial D-E$, there exists a Borel set $E_\xi\subset D$ with the properties:*

i) $\lim_{x\to\xi,\, x\in D-E_\xi} x_n^{-\beta}|x-\xi|^{\beta+\gamma}G_\alpha(x, \mu)$

$$=\begin{cases} d_\alpha\int_D|\xi-y|^{\alpha-n-2}y_n d\mu(y) & \text{if } \beta=1 \text{ and } \gamma=-1, \\ 0 & \text{if } 0\leqq\beta\leqq 1 \text{ and } \gamma>-1; \end{cases}$$

ii) $\sum_{i=1}^{\infty} 2^{i(n-\alpha+\beta+1)}C_{k_{\alpha,\beta,1}}(E_\xi^{(i)})<\infty$,

where $E_\xi^{(i)}=\{x\in E; 2^{-i}\leqq|x-\xi|<2^{-i+1}\}$.

In fact, it is seen that

$$E=\left\{\xi\in\partial D; \int_{B_+(\xi,1)}|\xi-y|^{\alpha+\gamma-n-1}y_n d\mu(y)=\infty\right\}$$

has the required properties, where $B_+(\xi, r)=\{x\in D; |x-\xi|<r\}$, $r>0$. Since $C_{\alpha+\gamma-1}(E)=0$, $\Lambda_{n-1}(E)=0$.

The case $\alpha+\gamma-1=1$ is treated in the following.

PROPOSITION 1. *If* $-1<\gamma\leqq n-\alpha+1$ *and* $G_\alpha(x, \mu)\not\equiv\infty$, *then there exists a set* $E\subset\partial D$ *such that* $\Lambda_{n-\alpha-\gamma+1}(E)=0$ *and to each* $\xi\in\partial D-E$, *there corresponds a set* $E_\xi\subset D$ *with the properties:*

i)′ $\lim_{x\to\xi,\, x\in\Gamma(\xi,a)-E_\xi}|x-\xi|^{\gamma}G_\alpha(x, \mu)=0$;

ii)′ $E_\xi\cap\Gamma(\xi, a)$ *is* α-*thin at* ξ, *i.e.*,

$$\sum_{i=1}^{\infty} 2^{i(n-\alpha)}C_\alpha(E_\xi^{(i)}\cap\Gamma(\xi, a))<\infty,$$

where $\Gamma(\xi, a)=\{x=(x_1, \cdots, x_n); |x-\xi|<ax_n\}$, $a>1$.

To prove this, we need the following lemmas.

LEMMA 6. *Let* $-1<\gamma\leqq n-\alpha+1$ *and* μ *be a non-negative measure on* D *such that* $G_\alpha(x, \mu)\not\equiv\infty$. *Then the following are equivalent:*

a) $\lim_{x\to\xi,\, x\in\Gamma(\xi,a)} x_n^{\gamma}\int_{\{y\in D; |x-y|\geqq b^{-1}x_n\}}G_\alpha(x, y)d\mu(y)=0$ *for* a *and* $b>1$;

b) $\lim_{r\downarrow 0} r^{\gamma+1}\int_{B_+(\xi,1)}\frac{y_n d\mu(y)}{(r+|y-\xi|)^{n-\alpha+2}}=0$;

c) $\lim_{r\downarrow 0} r^{\alpha+\gamma-n-1}\int_{B_+(\xi,r)}y_n d\mu(y)=0$.

PROOF. By Lemma 1, a) is equivalent to b). Clearly, b) implies c). It is not difficult to see that c) implies b). Thus the lemma is obtained.

REMARK. Let $A=\left\{\xi\in\partial D; \limsup_{r\downarrow 0} r^{l-n}\int_{B_+(\xi,r)}y_n d\mu(y)>0\right\}$. If $G_\alpha(x, \mu)\not\equiv\infty$, then by Lemma 3 and [5; p. 165], $\Lambda_{n-l}(A)=0$.

LEMMA 7. *Let μ be a non-negative measure on D satisfying*

$$\int_{B_+(O,r)} y_n^{\delta} d\mu(y) < \infty \qquad \text{for any } r>0, \tag{1}$$

and set

$$A_{\delta'} = \left\{\xi \in \partial D ; \int_{B_+(\xi,1)} |\xi - y|^{\alpha+\gamma-\delta'-n} y_n^{\delta'} d\mu(y) = \infty\right\}, \qquad \delta' > \delta .$$

Then $\Lambda_{n-\alpha-\gamma+\delta}(A_{\delta'})=0$.

PROOF. If $n-\alpha-\gamma+\delta \leqq 0$, then

$$\int_{B_+(\xi,1)} |\xi - y|^{\alpha+\gamma-\delta'-n} y_n^{\delta'} d\mu(y) \leqq \int_{B_+(\xi,1)} y_n^{\delta} d\mu(y) < \infty ,$$

which implies that $A_{\delta'}$ is empty. Let $n-\alpha-\gamma+\delta>0$, and suppose $\Lambda_{n-\alpha-\gamma+\delta}(A_{\delta'})>0$. Then by [**1**; Theorems 1 and 3 in § II] we can find a positive measure ν with compact support in $A_{\delta'}$ such that $\nu(B(x, r)) \leqq r^{n-\alpha-\gamma+\delta}$ for any $x \in R^n$ and $r>0$, $B(x, r)$ denoting the open ball with center at x and radius r. Note that

$$\int |\xi - y|^{\alpha+\gamma-\delta'-n} d\nu(\xi) \leqq \text{const.}\, y_n^{\delta-\delta'} , \qquad y \in D .$$

Taking $N>0$ such that $S_\nu \subset B(O, N)$, we obtain

$$\begin{aligned} \infty &= \int \left\{ \int_{B_+(\xi,1)} |\xi - y|^{\alpha+\gamma-\delta'-n} y_n^{\delta'} d\mu(y) \right\} d\nu(\xi) \\ &\leqq \int_{B_+(O,N+1)} \left\{ \int |\xi - y|^{\alpha+\gamma-\delta'-n} d\nu(\xi) \right\} y_n^{\delta'} d\mu(y) \\ &\leqq \text{const.} \int_{B_+(O,N+1)} y_n^{\delta} d\mu(y) < \infty , \end{aligned}$$

which is a contradiction. Thus $\Lambda_{n-\alpha-\gamma+\delta}(A_{\delta'})=0$, and our lemma is proved.

PROOF OF PROPOSITION 1. Define A with $l=\alpha+\gamma-1$ and $A_{\delta'}$, $\delta'>1$, as above. Then $\Lambda_{n-l}(A \cup A_{\delta'})=0$ by Lemma 7 and the remark given after Lemma 6. Let $\xi \in \partial D - (A \cup A_{\delta'})$, and write

$$\begin{aligned} G_\alpha(x, \mu) &= \int_{\{y \in D ; |x-y| \geqq x_n/2\}} G_\alpha(x, y) d\mu(y) + \int_{\{y ; |x-y| < x_n/2\}} G_\alpha(x, y) d\mu(y) \\ &= G'(x) + G''(x) . \end{aligned}$$

Then Lemma 6 implies that $\lim_{x \to \xi,\, x \in \Gamma(\xi, a)} |x-\xi|^{\gamma} G'(x) = 0$.

For $a>1$, take $b>1$ such that $\{y ; |x-y| < x_n/2\} \subset \Gamma(\xi, b)$ whenever $x \in \Gamma(\xi, a)$. Note that if $x \in \Gamma(\xi, a)$, then

$$|x-\xi|^{\gamma} G''(x) \leqq \text{const.} \int_{\{y ; |x-y| < x_n/2\}} |x-y|^{\alpha+\gamma-n} d\mu(y) .$$

Since $\int_{B_+(\xi,1)\cap\Gamma(\xi,b)}|\xi-y|^{\alpha+\gamma-n}d\mu(y)<\infty$, in the same way as the proof of Lemma 5, we can find a set $E_{\xi,a}$ such that $E_{\xi,a}$ is α-thin at ξ and

$$\lim_{x\to\xi,\, x\in\Gamma(\xi,a)-E_{\xi,a}}|x-\xi|^{\gamma}G''(x)=0.$$

One easily finds a sequence $\{r_a\}$ of positive numbers such that $E_\xi=\bigcup_{a=1}^{\infty}(E_{\xi,a}\cap B(\xi, r_a))$ satisfies ii)$''$. Clearly, i)$'$ holds for this E_ξ, and hence the proof of Proposition 1 is complete.

PROPOSITION 2. *Let $\delta<1$ and $-1<\gamma\leqq n-\alpha+\delta$. Let μ be a non-negative measure on D such that $G_\alpha(x, \mu)\not\equiv\infty$ and μ satisfies* (1). *Define $A_{\delta'}$, $\delta<\delta'<1$, as in Lemma 7. Then for each $\xi\in\partial D-A_{\delta'}$, there exists a Borel set $E_{\xi,\delta'}\subset D$ with the properties:*

i) $\displaystyle\lim_{x\to\xi,\, x\in D-E_{\xi,\delta'}} x_n^{-\beta}|x-\xi|^{\beta+\gamma}G_\alpha(x, \mu)=0$;

ii) $\displaystyle\sum_{i=1}^{\infty}2^{i(n-\alpha+\beta+\delta')}C_{k_{\alpha,\beta,\delta'}}(E_{\xi,\delta'}^{(i)})<\infty$,

where $E_{\xi,\delta'}^{(i)}=\{x\in E_{\xi,\delta'}\,;\, 2^{-i}\leqq|x-\xi|<2^{-i+1}\}$.

This is an easy consequence of Theorem 1. We note here that $\Lambda_{n-\alpha-\gamma+\delta}(A_{\delta'})=0$ on account of Lemma 7.

Let u be a non-negative superharmonic function on D, and write

$$u(x)=ax_n+G_2(x, \mu)+P(x, \nu)=ax_n+x_nk_2(x, \lambda).$$

THEOREM 2. *If $-1\leqq\gamma\leqq n-1$ and*

$$\int_{\bar{D}}|y|^{1+\gamma-n}d\lambda(y)<\infty, \tag{2}$$

then there exists a Borel set $E\subset D$ with the properties:

i) $\displaystyle\lim_{x\to O,\, x\in D-E} x_n^{-\beta}|x|^{\beta+\gamma}[u(x)-ax_n]$

$$=\begin{cases} b_n\displaystyle\int\frac{y_n}{|y|^n}d\mu(y)+c_n\int\frac{1}{|y|^n}d\nu(y) & \textit{in case } \beta=1 \textit{ and } \gamma=-1, \\ c_n\nu(\{O\}) & \textit{in case } \beta=1 \textit{ and } \gamma=n-1, \\ 0 & \textit{in case } 0\leqq\beta\leqq1 \textit{ and } -1<\gamma<n-1; \end{cases}$$

ii) $\displaystyle\sum_{i=1}^{\infty}2^{i(n+\beta-1)}C_{k_{2,\beta,1}}(E^{(i)})<\infty$.

This theorem follows readily from Lemmas 4 and 5.

REMARK. In case $\beta=1$, property ii) is equivalent to the condition that E is minimally thin at O.

PROPOSITION 3. *If $0\leqq\gamma\leqq n-1$ and u is a non-negative superharmonic func-*

tion in D, then there exists a set $E\subset\partial D$ *with* $\Lambda_{n-\gamma-1}(E)=0$ *such that for each* $\xi\in\partial D-E$, *there correspond a number* c_ξ *and a set* $E_\xi\subset D$ *with the properties:*

i) $\lim_{x\to\xi,\, x\in\Gamma(\xi, a)-E_\xi} |x-\xi|^\gamma u(x)=c_\xi$;

ii) $E_\xi\cap\Gamma(\xi, a)$ *is 2-thin at* ξ,

for every $a>1$.

REMARK. In case $\gamma>0$, $c_\xi=0$.

PROOF. If $\gamma=0$, Proposition 3 follows from Proposition 1 and Fatou's theorem (cf. [3; Theorem 3.9]). The case $\gamma>0$ can be proved in a way similar to the proof of Proposition 1.

Theorem 2 is best possible as to the size of the exceptional set. In fact we have the next result.

THEOREM 3. *Let E be a Borel set in D which satisfies* ii) *in Theorem* 2. *Then there exists a non-negative measure* λ *on* $\bar{D}$ *satisfying* (2) *such that* $\lim_{x\to 0,\, x\in E} x_n^{-\beta}|x|^{\beta+\gamma}u(x)=\infty$, *where* $u(x)=x_n k_2(x, \lambda)$.

PROOF. On account of Lemma 2, one can find non-negative measures λ_i on $\bar{D}$ such that $\lambda_i(\bar{D})<C_{k_{2,\beta,1}}(E^{(i)})+2^{-i(n+\beta)}$ and $k_{2,\beta,1}(\lambda_i, z)\geqq 1$ on $E^{(i)}$. Denote by λ'_i the restriction of λ_i to the set $\{x\in\bar{D};\ 2^{-i-1}<|x|<2^{-i+2}\}$. If $z\in E^{(i)}$, then Lemma 1 gives

$$k_{2,\beta,1}(\lambda'_i, z)\geqq 1-\int_{\{|x|\leqq 2^{-i-1}\}\cup\{|x|\geqq 2^{-i+2}\}} k_{2,\beta,1}(x, z)d\lambda_i(x)$$
$$\geqq 1-c_2 2^{(i+1)(n+\beta-1)}\{C_{k_{2,\beta,1}}(E^{(i)})+2^{-i(n+\beta)}\}.$$

Let $\{a_i\}$ be a sequence of positive numbers such that $\lim_{i\to\infty} a_i=\infty$ and

$$\sum_{i=1}^{\infty} a_i 2^{i(n+\beta-1)}\{C_{k_{2,\beta,1}}(E^{(i)})+2^{-i(n+\beta)}\}<\infty .$$

Define

$$\lambda=\sum_{i=1}^{\infty} a_i 2^{i(\beta+\gamma)}\lambda'_i .$$

Let $u(z)=z_n k_2(z, \lambda)$. Then we have

$$z_n^{-\beta}|z|^{\beta+\gamma}u(z)\geqq a_i 2^{-|\beta+\gamma|}k_{2,\beta,1}(\lambda'_i, z)$$

for $z\in E^{(i)}$, and hence

$$\lim_{z\to 0,\, z\in E} z_n^{-\beta}|z|^{\beta+\gamma}u(z)=\infty .$$

On the other hand,

$$\int_{\bar{D}}|x|^{1+\gamma-n}d\lambda(x)=\sum_{i=1}^{\infty} a_i 2^{i(\beta+\gamma)}\int|x|^{1+\gamma-n}d\lambda'_i(x)$$

$$\leqq \sum_{i=1}^{\infty} 2^{n-\gamma-1} a_i 2^{i(n+\beta-1)} \{C_{k_{2,\beta,1}}(E^{(i)})+2^{-i(\beta+\gamma)}\} < \infty .$$

Thus our theorem is established.

REMARK. Let h be a positive non-decreasing function on $(0, \infty)$, and set

$$E_h = \{x=(x_1, \cdots, x_n);\ 0<x_n<h(|x|)\}.$$

Then E_h satisfies ii) in Theorem 2 if and only if

$$\int_0^1 \left(\frac{h(r)}{r}\right)^{\beta} \frac{dr}{r} < \infty .$$

To prove this fact, we have only to establish the next lemma.

LEMMA 8. *Let E be a non-empty bounded open set in ∂D. Then there exists $c>0$ such that*

$$c^{-1}t^{\beta} \leqq C_{k_{2,\beta,1}}(E\times(0, t)) \leqq ct^{\beta} \qquad \text{for } t>0 .$$

For a proof, it suffices to note that

$$C_1 t^{-\beta} \leqq \int_{E(t)} k_{2,\beta,1}(x, y)dS(y) \leqq C_2 t^{-\beta}$$

whenever $x\in E\times(0, t)$, where C_1 and C_2 are positive constants independent of t, and $E(t)=\{\xi+(0, \cdots, 0, t/2);\ \xi\in E\}$.

3. Radial limits.

By the definition of $C_{k_{\alpha,\beta,\delta}}$, we have the following lemma.

LEMMA 9. (1) *For $r>0$, let $T_rE=\{rx;\ x\in E\}$, $E\subset R^n$. Then*

$$C_{k_{\alpha,\beta,\delta}}(T_rE)=r^{n-\alpha+\beta+\delta}C_{k_{\alpha,\beta,\delta}}(E), \qquad E\subset D .$$

(2) *There exists $M>0$ such that*

$$M^{-1}C_{\alpha}(E) \leqq C_{k_{\alpha,\beta,\delta}}(E) \leqq MC_{\alpha}(E)$$

whenever $E\subset\Gamma(O, a)\cap B(O, 2)-B(O, 1)$.

For a set $E\subset D$, we define

$$E^{\sim}=\{\zeta\in S_+;\ r\zeta\in E \quad \text{for some } r>0\},$$

where $S_+=\{x\in D;\ |x|=1\}$.

COROLLARY. *If $E\subset D$ satisfies* (A) *in theorem* 1, *then*

$$C_{\alpha}\Big(\bigcap_{j=1}^{\infty}\Big(\bigcup_{i=j}^{\infty} E^{(i)}\Big)^{\sim}\Big)=0 . \tag{3}$$

PROOF. First note that if $E\subset D$ is α-thin at O, then E satisfies (3). If E satisfies (A) in Theorem 1, then $E\cap\Gamma(O, a)$ is α-thin at O for any $a>1$ on account of Lemma 9, so that E satisfies (3).

By using this corollary and Propositions 1, 2, we have the following radial limit theorem.

THEOREM 4. *Let* $\delta\leqq 1$ *and* $-1<\gamma\leqq n-\alpha+\delta$. *Let* μ *be a non-negative measure on* D *satisfying* (1) *such that* $G_\alpha(x, \mu)\not\equiv\infty$. *Then there exists a set* $E\subset\partial D$ *with* $\Lambda_{n-\alpha-\gamma+\delta}(E)=0$ *such that to each* $\xi\in\partial D-E$, *there corresponds a set* $E_\xi\subset S_+$ *with the properties:*

i) $\lim_{r\downarrow 0} r^\gamma G_\alpha(\xi+r\zeta, \mu)=0$ *for every* $\zeta\in S_+-E_\xi$;

ii) $C_\alpha(E_\xi)=0$.

In view of Proposition 3 and the corollary to Lemma 9, we can establish the following result.

PROPOSITION 4. *Let* u *be a non-negative superharmonic function on* D, *and* $0\leqq\gamma\leqq n-1$. *Then we can find a set* $E\subset\partial D$ *with* $\Lambda_{n-\gamma-1}(E)=0$ *such that for each* $\xi\in\partial D-E$, *there exist a number* c_ξ *and a set* $E_\xi\subset S_+$ *with the properties:*

i) $\lim_{r\downarrow 0} r^\gamma u(\xi+r\zeta)=c_\xi$ *for every* $\zeta\in S_+-E_\xi$;

ii) $C_2(E_\xi)=0$.

REMARK. Let $v(x)=G_2(x, \mu)+P(x, \nu)$. Then there exists a set $E\subset S_+$ such that $C_2(E)=0$ and

$$\lim_{r\downarrow 0} r^{-1}v(r\zeta)=\zeta_n\left(b_n\int\frac{y_n}{|y|^n}d\mu(y)+c_n\int\frac{1}{|y|^n}d\nu(y)\right)$$

for every $\zeta\in S_+-E$. Note here that the right hand side may be infinity; in this case it is trivial that $\lim_{r\downarrow 0} r^{-1}v(r\zeta)=\infty$ for every $\zeta\in S_+$.

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Yoshihiro MIZUTA

Department of Mathematics
Faculty of Integrated Arts and Sciences
Hiroshima University
Hiroshima 730
Japan