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Elimination of certain Thom-Boardman singularities of order two

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§0. Introduction.

In this paper we will study the problem of deforming a differentiable map of a differentiable manifold N into a differentiable manifold P in the homotopy class to a differentiable map which does not admit particularly complicated Thom-Boardman singularities of order two.

Let $\Sigma^I(N, P)$ denote the Thom-Boardman singularity with symbol I which is defined in the jet-space $J^r(N, P)$ where I denote either (i) for $r=1$, or (i, j) for $r=2$ ([2], [11] and [18]). Let $\bar{\Sigma}^I(N, P)$ denote the closure of $\Sigma^I(N, P)$ in $J^r(N, P)$, and ν_I the codimension of $\Sigma^I(N, P)$ in $J^r(N, P)$. Let N be a closed differentiable manifold, $n=\dim N$ and $p=\dim P$. The canonical fiber of $\Sigma^I(N, P)$ over $N\times P$ will be denoted by $\Sigma^I(n, p)$. We will define in §2 and §6 the dual class $[\bar{\Sigma}^I(N, P)]$ in $H^{\nu_I}(N\times P; G)$ of the Thom-Boardman singularity $\Sigma^I(N, P)$. The coefficient group G denotes either $\boldsymbol{Z}$ or $\boldsymbol{Z}_2$ depending on whether $\Sigma^I(n, p)$ is orientable or not. When G is $\boldsymbol{Z}$, we assume N and P to be orientable manifolds. For a differentiable map $f: N\to P$ we denote the class $(\mathrm{id}_N\times f)^*([\bar{\Sigma}^I(N, P)])$ in $H^{\nu_I}(N; G)$ by $c^I(TN, f^*(TP))$. We will give in §5 a formula to calculate the dual class $c^I(TN, f^*(TP))$ in a finite process in terms of the characteristic classes of N and P. The $\boldsymbol{Z}_2$-reduction of these dual classes coincides with those which have been defined in [15] under the sheaf homology (cohomology resp.) groups with closed supports. We will use the singular homology (cohomology resp.) groups in our definition. We will show the following two applications of the dual classes.

Let $\Omega^I(N, P)$ denote the union of all Thom-Boardman singularities with symbol smaller than or equal to I in the lexicographic order. Let $C^\infty_{\Omega^I}(N, P)$ denote the space of all differentiable maps, $f: N\to P$ such that the image of $j^r f: N\to J^r(N, P)$ is contained in $\Omega^I(N, P)$ with C^∞-topology. Let $\Gamma_{\Omega^I}(N)$ denote the space of all continuous sections of the fiber bundle of $J^r(N, P)$ over N such that the image of a section is contained in $\Omega^I(N, P)$ equipped with compact-open topology. Let $\Omega(N, P)$ denote $\Omega^I(N, P)\backslash\Sigma^I(N, P)$ and we consider similarly

$C_{\Omega}^{\infty}(N, P)$ and $\Gamma_{\Omega}(N)$. Then we have a canonical cotinuous map $j^{r}: C_{\Omega}^{\infty}(N, P)\rightarrow \Gamma_{\Omega}(N)$. We will say that $\Omega(N, P)$ is *π_0-integrable* when the induced map $(j^{r})^{*}: \pi_0(C_{\Omega}^{\infty}(N, P))\rightarrow\pi_0(\Gamma_{\Omega}(N))$ is surjective and that a symbol I is *good* when every fibre of $\Omega^{I}(N, P)$ and of $\Omega(N, P)$ over $N\times P$ are connected and simply connected. The π_0-integrability has been discussed in [**4**] and [**14**] extensively and also in [**1**]. Note that I is good if $\operatorname{codim}\Sigma^{K}>2$ for every K with $K\geqq I$, and hence I is good in most case. The following theorem will be proved in § 6.

THEOREM 1. *Let N, P and f be as above. Let I be any good symbol such that* $\operatorname{codim}\Sigma^{I}(N, P)=n$ *except for the case that $r=2$, $p-n+i=1$ and $i>j$. Let $\Omega(N, P)$ be π_0-integrable. Then there exists a differentiable map g of $C_{\Omega}^{\infty}(N, P)$ such that $j^{r}g$ is homotopic to $j^{r}f$ as maps of N into $\Omega^{I}(N, P)$ (hence g is homotopic to f) if and only if the dual class $c^{I}(TN, f^{*}(TP))$ vanishes.*

In § 7 we will treat continuous maps which are not homotopic to any C^{∞}-stable map. J. N. Mather has proved in [**10**] that C^{∞}-stable maps are dense in the space of all proper differentiable maps of $C^{\infty}(N, P)$ in the 'nice range'. We will obtain the following result outside of the nice range.

THEOREM 2. *Let f be a continuous map of a closed differentiable manifold N into a differentiable manifold P. We assume that there exists an integer i such that*

(1) $(p-n+i)\left\{i+\frac{1}{2}i(i+1)\right\}-i^{2}-(p-n+i)^{2}+1>n,$ *and*

(2) *either* (i) *the determinant of the following $(p-n+i)$-matrix whose (s, t) component is the Stiefel-Whitney class $W_{i+s-t}(\gamma)$ of $\gamma=TN-f^{*}(TP)$*

$$\begin{pmatrix} W_i(\gamma) & W_{i-1}(\gamma) & & \\ W_{i+1}(\gamma) & \ddots & \ddots & \\ & \ddots & \ddots & W_{i-1}(\gamma) \\ & & W_{i+1}(\gamma) & W_i(\gamma) \end{pmatrix}$$

is not zero, or

(ii) *both of $n-p$ and i are even, both of N and P are orientable and the determinant of the following $(p-n+i)/2$-matrix whose (s, t) component is the Pontrjagin class $P_{(i/2)+s-t}(\gamma)$*

$$\begin{pmatrix} P_{i/2}(\gamma) & P_{i/2-1}(\gamma) & & \\ P_{i/2+1}(\gamma) & \ddots & \ddots & \\ & \ddots & \ddots & P_{i/2-1}(\gamma) \\ & & P_{i/2+1}(\gamma) & P_{i/2}(\gamma) \end{pmatrix}$$

is not an element of 2-torsion.
Then f is not homotopic to any C^∞-stable map (especially when P is $\boldsymbol{R}^p$, there exists no C^∞-stable map in $C^\infty(N, \boldsymbol{R}^p)$).

This theorem follows from more general Theorem 7.4. In §7 we will give examples of manifolds N and P such that there exists no C^∞-stable map in $C^\infty(N, P)$.

All manifolds, fiber bundles and maps will be differentiable of class C^∞ unless otherwise stated. All manifolds will be paracompact and Hausdorff.

§1. Notations and preliminaries.

Let ξ and η denote respectively differentiable real vector bundles, $E \to X$ and $F \to X$ of dimensions n and p over a manifold X. Let E_x and F_x be respectively fibres of E and F over a point x in X. Let $\mathrm{Hom}(\xi, \eta)$ denote the union of all linear maps of E_x into F_x, $\bigcup_{x \in X} \mathrm{Hom}(E_x, F_x)$, which becomes naturally a real vector bundle over X.

We begin with recalling the definition of Thom-Boardman singularities (see [2], [11] and especially [15]). For convenience we put $J^1(\xi, \eta) = \mathrm{Hom}(\xi, \eta)$ and $J^2(\xi, \eta) = \mathrm{Hom}(\xi, \eta) \oplus \mathrm{Hom}(\xi \circ \xi, \eta)$ where $\xi \circ \xi$ denote the symmetric product of ξ. In the sequel we will denote an element of $J^2(\xi, \eta)$ over a point x in X by (α, β) where α (resp. β) is an element of $\mathrm{Hom}(E_x, F_x)$ (resp. $\mathrm{Hom}(E_x \circ E_x, F_x)$). An element (α, β) of $J^2(E_x, F_x)$ determines a linear map $\tilde{\beta}: \mathrm{Ker}(\alpha) \to \mathrm{Hom}(\mathrm{Ker}(\alpha), \mathrm{Cok}(\alpha))$ which is induced from the projection of F_x onto $\mathrm{Cok}(\alpha)$ and the isomorphism of $\mathrm{Hom}(E_x \otimes E_x, F_x)$ onto $\mathrm{Hom}(E_x, \mathrm{Hom}(E_x, F_x))$.

DEFINITION 1.1. Let $\Sigma^i(\xi, \eta)$ denote the space of all elements of $J^1(\xi, \eta)$ such that the dimension of $\mathrm{Ker}(\alpha)$ is i. Let $\Sigma^{i,j}(\xi, \eta)$ denote the space of all elements (α, β) of $J^2(\xi, \eta)$ such that $\alpha \in \Sigma^i(\xi, \eta)$ and that the dimension of $\mathrm{Ker}(\tilde{\beta})$ is j.

In the sequel I means either (i) or (i, j). $\Sigma^{i,j}(\xi, \eta)$ is nonempty if and only if (i) $n \geqq i \geqq j \geqq 0$ (ii) $i \geqq n-p$ and (iii) $i=j$ for $i=n-p$. We call $\Sigma^I(\xi, \eta)$ the *Thom-Boardman singularity with symbol I*. When ξ and η are trivial bundles $\boldsymbol{R}^n$ and $\boldsymbol{R}^p$ over a single point, we simply write $\Sigma^I(n, p)$ (resp. $J^r(n, p)$) in place of $\Sigma^I(\xi, \eta)$ (resp. $J^r(\xi, \eta)$). It is clear that $\Sigma^I(\xi, \eta) = \bigcup_{x \in X} \Sigma^I(E_x, F_x)$. It is shown in [2], [11] and [15] that $\Sigma^I(\xi, \eta)$ is a regular submanifold of $J^r(\xi, \eta)$. The codimension of $\Sigma^i(\xi, \eta)$ in $J^1(\xi, \eta)$ is $i(p-n+i)$ and the codimension of $\Sigma^{i,j}(\xi, \eta)$ in $J^2(\xi, \eta)$, $(i+i \circ j)(p-n+i)-j(i-j)$ where an integer $i \circ j$ denotes the dimension $j(i-j)+(i/2)j(j+1)$ of $\boldsymbol{R}^i \circ \boldsymbol{R}^j$.

If we provide ξ and η with metrics, then we can provide $J^r(\xi, \eta)$ with a metric. Let $S^r(\xi, \eta)$ denote the associated sphere bundle of $J^r(\xi, \eta)$. We put $\Sigma_0^I(\xi, \eta) = \Sigma^I(\xi, \eta) \cap S^r(\xi, \eta)$. Then $\Sigma_0^I(\xi, \eta)$ is empty for $I=(n)$ and $I=(n, n)$.

If t is a non-zero real number, then it is easy to see that $t\alpha\in\Sigma^{i}(\xi, \eta)$ (resp. $(t\alpha, t\beta)\in\Sigma^{i,j}(\xi, \eta)$) if and only if $\alpha\in\Sigma^{i}(\xi, \eta)$ (resp. $(\alpha, \beta)\in\Sigma^{i,j}(\xi, \eta)$). Hence $\Sigma^{I}(\xi, \eta)$ is diffeomorphic to $\Sigma_{0}^{I}(\xi, \eta)\times \boldsymbol{R}^{+}$ unless $I=(n)$ and (n, n). Here $\boldsymbol{R}^{+}$ is the set of all positive real numbers.

Let $\bar{\Sigma}^{I}(\xi, \eta)$ denote the topological closure of $\Sigma^{I}(\xi, \eta)$ in $J^{r}(\xi, \eta)$. Then $\bar{\Sigma}^{I}(\xi, \eta)\cap S^{r}(\xi, \eta)$ is equal to the closure of $\Sigma_{0}^{I}(\xi, \eta)$ in $S^{r}(\xi, \eta)$. Let $\Omega^{K}(\xi, \eta)$ denote the union of all Thom-Boardman singularities $\Sigma^{L}(\xi, \eta)$ with symbols L either smaller than or equal to K in the lexicographic order. We put $\Omega_{0}^{K}(\xi, \eta)=\Omega^{K}(\xi, \eta)\cap S^{r}(\xi, \eta)$ and $\Omega(\xi, \eta)=\Omega^{I}(\xi, \eta)\backslash\Sigma^{I}(\xi, \eta)$. When ξ and η are trivial bundles $\boldsymbol{R}^{n}$ and $\boldsymbol{R}^{p}$, we will write simply Ω^{I} and Ω for $\Omega^{I}(\xi, \eta)$ and $\Omega(\xi, \eta)$ respectively.

Next we will fix notations about grassmann bundles which will be often used. Let $G_{i,n-i}(V)$ be the grassmann manifold of all i-planes in an n-dimensional vector space V and $G_{i,j,n-i}(V)$, the iterated grassmann manifold of all pairs (a, b) where a is an i-dimensional space of V and b a j-dimensional space of a. We often write simply $G_{i,n-i}$ and $G_{i,j,n-i}$ when there is no confusion. Let ν be a vector bundle over a space M. Then $G_{i,n-i}(\nu)$ (resp. $G_{i,j,n-i}(\nu)$) denote the associated (resp. iterated) grassmann bundle over M whose total space is $\bigcup_{x\in M} G_{i,n-i}(\nu_{x})$ (resp. $\bigcup_{x\in M} G_{i,j,n-i}(\nu_{x})$) where ν_{x} is the fiber of ν over a point x of M. We will denote by θ^{m} any trivial bundle of dimension m over any base space.

Here we give an outline of the proof of Theorem 1. Let $\pi_{N}: N\times P\to N$ and $\pi_{P}: N\times P\to P$ be respectively the projection maps. Let $\xi=(\pi_{N})^{*}(TN)$ and $\eta=(\pi_{P})^{*}(TP)$ where TN and TP denote the tangent bundles of N and P. We can identify $J^{r}(N, P)$ with $J^{r}(\xi, \eta)$ ([**15**, § 3]). Let $f: N\to P$ be a given map and consider $J^{r}(TN, f^{*}(TP))$ which is induced from $J^{r}(\xi, \eta)$ by the map $\mathrm{id}_{N}\times f: N\to N\times P$. Then the primary obstruction class for $j^{r}f: N\to\Omega^{I}(TN, f^{*}(TP))$ to be homotopic to a section of $\Omega(TN, f^{*}(TP))$ exists in $H^{\nu_{I}}(N; \pi_{\nu_{I}}(\Omega^{I}, \Omega))$ where ν_{I} is the codimension of $\Sigma^{I}(n, p)$ in $J^{r}(n, p)$. Let d_{I} denote the dimension of $\Sigma_{0}^{I}(n, p)$. The essential part of the proof is to compute $\pi_{\nu_{I}}(\Omega^{I}, \Omega)$ and $H_{d_{I}}(\bar{\Sigma}_{0}^{I}(n, p); \boldsymbol{Z})$. In most cases these groups are isomorphic to either $\boldsymbol{Z}$ or $\boldsymbol{Z}_{2}$ depending on whether $\Sigma^{I}(n, p)$ is orientable or not. This enables us to define the fundamental class of $\bar{\Sigma}_{0}^{I}(TN, f^{*}(TP))$, and its dual class becomes the above primary obstruction class.

§ 2. The closure of $\Sigma^{I}(n, p)$.

In this section we will determine the homology and cohomology groups, $H_{d_{I}}(\bar{\Sigma}_{0}^{I}(n, p))$ and $H^{d_{I}}(\bar{\Sigma}_{0}^{I}(n, p))$ with the coefficient group $\boldsymbol{Z}$ or $\boldsymbol{Z}_{2}$ where $d_{I}=\dim\Sigma_{0}^{I}(n, p)$. We will simply write Σ^{I} in place of $\Sigma^{I}(n, p)$ if there is no confusion.

Let (α, β) be an element of $\Sigma^{l,m}(n, p)$. Let K be an i-dimensional subspace of $\mathrm{Ker}(\alpha)$ and L a j-dimensional subspace of K. Then (α, β) induces a homomorphism $\alpha' : \boldsymbol{R}^n/K \to \boldsymbol{R}^p$ and a restriction map $\beta' : K \circ L \subset \boldsymbol{R}^n \circ \boldsymbol{R}^n \to \boldsymbol{R}^p$.

LEMMA 2.1. *Let (α, β) be an element of $J^2(n, p)$ such that $\alpha \in \Sigma^l(n, p)$. Then (α, β) belongs to $\bar{\Sigma}^{i,j}(n, p)$ if and only if* (1) $l \geqq i$ *and* (2) *there exist an i-dimensional subspace K in* $\mathrm{Ker}(\alpha)$ *and a j-dimensional subspace L in K such that the dimension of the kernel of $(\alpha', \beta') : \boldsymbol{R}^n/K \oplus K \circ L \to \boldsymbol{R}^p$ is not less than $i \circ j$.*

PROOF. Let (α, β) belong to $\bar{\Sigma}^{i,j}$. Then there exists a sequence (α_t, β_t) in $\Sigma^{i,j}$ which converges to (α, β). Since the sequence $\{\alpha_t\}$ converges to α, it is clear that $l \geqq i$. Since $\dim \mathrm{Ker}(\alpha_t) = i$, $\dim \mathrm{Ker}(\tilde{\beta}_t) = j$ and $\mathrm{Ker}(\alpha_t) \supset \mathrm{Ker}(\beta_t)$, we obtain an element $(\mathrm{Ker}(\alpha_t), \mathrm{Ker}(\beta_t))$ of $G_{i,j,n-i}$ for each t. Then there exist subspaces K_0 and L_0 and a subsequence $\{t_s\}$ of $\{t\}$ such that the sequence $(\mathrm{Ker}(\alpha_{t_s}), \mathrm{Ker}(\tilde{\beta}_{t_s}))$ converges to (K_0, L_0) in $G_{i,j,n-i}$. Let ξ_i and ξ_j denote the canonical vector bundles of dimension i and j respectively over $G_{i,j,n-i}$. We consider the vector bundle $\mathrm{Hom}(\theta^n/\xi_i \oplus \xi_i \circ \xi_j, \theta^p)$ over $G_{i,j,n-i}$ in which we have a sequence $\{(\alpha'_{t_s}, \beta'_{t_s})\}$, where $(\alpha'_{t_s}, \beta'_{t_s}) : \boldsymbol{R}^n/\mathrm{Ker}(\alpha_{t_s}) \oplus \mathrm{Ker}(\alpha_{t_s}) \circ \mathrm{Ker}(\tilde{\beta}_{t_s}) \to \boldsymbol{R}^p$ is induced from $\{(\alpha_{t_s}, \beta_{t_s})\}$. Since K_0 is contained in $\mathrm{Ker}(\alpha)$, we can also define an element $(\alpha', \beta') : \boldsymbol{R}^n/K_0 \oplus K_0 \circ L_0 \to \boldsymbol{R}^p$ in $\mathrm{Hom}(\theta^n/\xi_i \oplus \xi_i \circ \xi_j, \theta^p)$. Then it follows from the construction that the sequence $\{(\alpha'_{t_s}, \beta'_{t_s})\}$ converges to (α', β'). Therefore the dimension of the kernel of (α', β') is not less than $i \circ j$ since the dimension of the kernel of $(\alpha'_{t_s}, \beta'_{t_s})$ is $i \circ j$ for each t_s.

Conversely we assume that the conditions (1) and (2) are satisfied by (α, β). Since $\dim(\mathrm{Ker}(\alpha', \beta')) \geqq i \circ j$, we know that the dimension of $\mathrm{Ker}(p \circ \beta')$ where p is the projection of $\boldsymbol{R}^p$ onto $\boldsymbol{R}^p/\mathrm{Im}(\alpha)$ is not less than $i \circ j - (l - i)$. We consider the usual metrics on $\boldsymbol{R}^n$ and $\boldsymbol{R}^p$. Let V_1 be the orthogonal space of K_0 in $\mathrm{Ker}(\alpha)$ and V_2 the orthogonal space of $\mathrm{Im}(\alpha)$ in $\boldsymbol{R}^p$. Then $\dim V_1 = l - i$, $\dim V_2 = p - n + l$ and $\dim V_2 \geqq \dim V_1$. Hence there exists a homomorphism $\gamma : \boldsymbol{R}^n \to \boldsymbol{R}^p$ such that γ maps V_1 injectively into V_2 and annihilates both of K_0 and the orthogonal complement of $\mathrm{Ker}(\alpha)$. We put $\alpha_t = \alpha + (1/t)\gamma$. Then it is clear that $\mathrm{Ker}(\alpha_t)$ is constantly K_0 and $\mathrm{Im}(\alpha_t)$ the direct sum of $\mathrm{Im}(\alpha)$ and $\mathrm{Im}(\gamma)$. Then we have that $\beta'(K_0 \circ L_0) \subset \mathrm{Im}(\alpha_t)$. Hence the dimension of the kernel of $\tilde{\beta} : K_0 \to \mathrm{Hom}(K_0, \boldsymbol{R}^p/(\mathrm{Im}(\alpha) \oplus \mathrm{Im}(\gamma)))$ induced from (α_t, β) is not less than j. Since we can construct easily a sequence $\{\beta_s\}$ which converges to β such that the dimension of the kernel of $\beta_s : K_0 \to \mathrm{Hom}(K_0, \boldsymbol{R}^p/(\mathrm{Im}(\alpha) \oplus \mathrm{Im}(\beta)))$ is equal to j, there exists a sequence $\{(\alpha_t, \beta_s)\}$ in $\Sigma^{i,j}$ which converges to (α, β).

Q. E. D.

Let V^I be the union of all Thom-Boardman singularities $\Sigma^K(n, p)$ with $K \geqq I$. Then as a trivial corollary of lemma 2.1, we have the following

COROLLARY 2.2. (1) $\bar{\Sigma}^{i,j}(n, p)$ *is contained in* $V^{i,j}$.

(2) *If* $h\geqq j$, *then* $\Sigma^{i,h}(n, p)$ *is contained in* $\bar{\Sigma}^{i,j}(n, p)$.

It is clear that $\bar{\Sigma}^{i}(n, p)$ is exactly the union V^{i} of all $\Sigma^{k}(n, p)$ with $k\geqq i$ and that it is an algebraic set of $J^{1}(n, p)$. For $r=2$, V^{I} is represented as a set of all elements z of $J^{2}(n, p)$ which satisfy the following conditions under the notations of [**11**, § 3 and § 4]

$$\operatorname{rk} \mathfrak{Z}(z)\leqq n-i \quad \text{and} \quad \operatorname{rk} \Delta^{i}\mathfrak{Z}(z)\leqq n-j .$$

Hence V^{I} is an algebraic set of $J^{2}(n, p)$. The closure of $\Sigma^{I}(n, p)$ is also an algebraic set of $J^{2}(n, p)$ ([**15**, Proposition 4.1]).

For a given symbol I, we define a symbol I' as follows. We put $I'=(i+1)$ for $r=1$. For $r=2$ we put $I'=(i+1, 0)$ when $i=j$ and $I'=(i, j+1)$ when $i>j$. Then it follows from Corollary 2.2 that $\bar{\Sigma}^{I}\backslash\Sigma^{I}$ is equal to $V^{I'}\cap\bar{\Sigma}^{I}$. Hence $\bar{\Sigma}^{I}\backslash\Sigma^{I}$ is an algebraic set. Next we will estimate the dimension of $\bar{\Sigma}_{0}^{I}\backslash\Sigma_{0}^{I}$ as an algebraic set.

PROPOSITION 2.3. *Let* I *be either* (i) *or* (i, j) *and* $p-n+i\geqq 1$. *Then the dimension of* $\bar{\Sigma}_{0}^{I}(n, p)\backslash\Sigma_{0}^{I}(n, p)$ *is smaller than* $d_{I}-1$ *except for the case of* $p-n+i=1$ *and* $j=0$. *When* $p-n+i=1$ *and* $j=0$, *it is equal to* $d_{I}-1$.

PROOF. It is easy to prove for $r=1$ by the formula $\operatorname{codim}\Sigma^{k}=k(p-n+k)$. For $r=2$ it is enough to estimate the dimension of $\bar{\Sigma}^{I}\cap(\Sigma^{l}\times\operatorname{Hom}(\boldsymbol{R}^{n}\circ\boldsymbol{R}^{n}, \boldsymbol{R}^{p}))$ for each $l\geqq i$. If $l=i$ and $h>j$, then we have

$$\begin{aligned}\operatorname{codim}\Sigma^{i,h}-\operatorname{codim}\Sigma^{i,j}&=i(p-n+i)+i\circ h(p-n+i)-h(i-h)\\&\quad -i(p-n+i)-i\circ j(p-n+i)+j(i-j)\\&\geqq(1/2)h(h+1)-(i/2)j(j+1)\,.\end{aligned}$$

Hence the difference is greater than 1 unless $p-n+i=1$, $h=1$ and $j=0$. In the last case the difference is equal to 1.

Let l be greater than i. Let ξ denote the differentiable vector bundle of dimension l over $\Sigma^{l}\times\operatorname{Hom}(\boldsymbol{R}^{n}\circ\boldsymbol{R}^{n}, \boldsymbol{R}^{p})$ whose total space is the set of all pairs $((\alpha, \beta), v)$ such that $\alpha\in\Sigma^{l}$ and v is a vector of $\operatorname{Ker}(\alpha)$. Then we can consider the iterated grassmann bundle $G_{i,j,l-i}(\xi)$ over $\Sigma^{l}\times\operatorname{Hom}(\boldsymbol{R}^{n}\circ\boldsymbol{R}^{n}, \boldsymbol{R}^{p})$ whose total space is the set of all triples $((\alpha, \beta), a, b)$ where a is an i-dimensional subspace of $\operatorname{Ker}(\alpha)$ and b a j-dimensional subspace of a. The projection p maps $((\alpha, \beta), a, b)$ onto (α, β). Clearly its fibre is $G_{i,j,l-i}$. Let ξ_{1} and ξ_{2} denote the canonical vector bundles of dimensions i and j respectively over $G_{i,j,l-i}(\xi)$. The total space of ξ_{1} (resp. ξ_{2}) is the set of all quadruples $((\alpha, \beta), a, b, v)$ such that v is a vector of a (resp. b) for an element $((\alpha, \beta), a, b)$ of $G_{i,j,l-i}(\xi)$. We have a vector bundle $\operatorname{Hom}(\theta^{n}/\xi_{1}\oplus\xi_{1}\circ\xi_{2}, \theta^{p})$ over $G_{i,j,l-i}(\xi)$. Then there exists its cross section s which is defined as follows

$$s(((\alpha, \beta), a, b))=(\alpha', \beta')$$

where $\alpha': \boldsymbol{R}^n/a \to \boldsymbol{R}^p$ and $\beta': a\circ b \to \boldsymbol{R}^p$ are the homomorphisms canonically induced from α and β respectively. It follows from Lemma 2.1 that $p(s^{-1}(\bar{\Sigma}^{i\circ j}(\theta^n/\xi_1\oplus\xi_1\circ\xi_2,\ \theta^p)))$ is equal to $\Sigma^{i,j}\cap(\Sigma^l\times\mathrm{Hom}(\boldsymbol{R}^n\circ\boldsymbol{R}^n,\ \boldsymbol{R}^p))$. In fact, let (α, β) belong to the last space. Then there exist subspaces a and b which satisfy the condition (2) of Lemma 2.1. Therefore (α, β) belongs to $p(s^{-1}(\bar{\Sigma}^{i\circ j}(\theta^n/\xi_1\oplus\xi_1\circ\xi_2,\ \theta^p)))$. Conversely if we have an element $((\alpha, \beta), a, b)$ such that α belongs to Σ^l and that the dimension of the kernel of (α', β') is not less than $i\circ j$, then (α, β) is clearly contained in $\bar{\Sigma}^{i,j}$.

Since the section s is transverse to each $\Sigma^h(\theta^n/\xi_1\oplus\xi_1\circ\xi_2,\ \theta^p)$, the codimension of $s^{-1}(\bar{\Sigma}^{i\circ j}(\theta^n/\xi_1\oplus\xi_1\circ\xi_2,\ \theta^p))$ in $G_{i,j,l-i}(\xi)$ is equal to $i\circ j(p-n+i)$. Now we can estimate the codimension of $\bar{\Sigma}^{i,j}\backslash\Sigma^{i,j}$ as follows.

$$\begin{aligned}
&\mathrm{codim}\,(\bar{\Sigma}^{i,j}\cap(\Sigma^l\times\mathrm{Hom}\,(\boldsymbol{R}^n\circ\boldsymbol{R}^n,\ \boldsymbol{R}^p)))\\
&\quad=\dim J^2(n,\ p)-\dim\,(\bar{\Sigma}^{i,j}\cap(\Sigma^l\times\mathrm{Hom}\,(\boldsymbol{R}^n\circ\boldsymbol{R}^n,\ \boldsymbol{R}^p)))\\
&\quad\geqq\dim J^2(n,\ p)-\dim s^{-1}(\bar{\Sigma}^{i\circ j}(\theta^n/\xi_1\oplus\xi_1\circ\xi_2,\ \theta^p))\\
&\quad=\dim J^2(n,\ p)-\dim G_{i,j,l-i}(\xi)+i\circ j(p-n+i)\\
&\quad=\dim J^2(n,\ p)-\dim\,(\Sigma^l\times\mathrm{Hom}\,(\boldsymbol{R}^n\circ\boldsymbol{R}^n,\ \boldsymbol{R}^p))-\dim G_{i,j,l-i}+i\circ j(p-n+i)\\
&\quad=\mathrm{codim}\,\Sigma^l-\dim G_{i,j,l-i}+i\circ j(p-n+i)\,.
\end{aligned}$$

Since $\dim G_{i,j,l-i}=i(l-i)+j(i-j)$, we have

$$\begin{aligned}
&\mathrm{codim}\,(\bar{\Sigma}^{i,j}\cap(\Sigma^l\times\mathrm{Hom}\,(\boldsymbol{R}^n\circ\boldsymbol{R}^n,\ \boldsymbol{R}^p)))-\mathrm{codim}\,\Sigma^{i,j}\\
&\quad\geqq\mathrm{codim}\,\Sigma^l-\dim G_{i,j,l-i}+i\circ j(p-n+i)-(p-n+i)(i+i\circ j)+j(i-j)\\
&\quad=(l-i)(p-n+l)\,.
\end{aligned}$$

Since $l>i$ and $p-n+i\geqq 1$, we have that $(l-i)(p-n+l)$ is greater than 1.

Q. E. D.

LEMMA 2.4. *Let P be a finite simplicial complex with* $\dim P=p$ *and let Q be a subcomplex of P. Let $P\backslash Q$ be a connected differentiable manifold without boundary with* $\dim P-\dim Q\geqq 2$. *Then the following holds.*

(1) *If $P\backslash Q$ is orientable, then we have*

$$H_p(P;\ \boldsymbol{Z})\cong\boldsymbol{Z} \quad\text{and}\quad H^p(P;\ \boldsymbol{Z})\cong\boldsymbol{Z}\,.$$

(2) *If $P\backslash Q$ is nonorientable, then we have*

$$H_p(P;\ \boldsymbol{Z})\cong\{0\} \quad\text{and}\quad H^p(P;\ \boldsymbol{Z})\cong\boldsymbol{Z}_2\,.$$

PROOF. By the universal coefficient theorem it is enough to show that if $P\backslash Q$ is orientable, then $H_p(P;\ \boldsymbol{Z})\cong\boldsymbol{Z}$ and the torsion of $H_{p-1}(P;\ \boldsymbol{Z})\cong\{0\}$ and that if $P\backslash Q$ is nonorientable, then $H_p(P;\ \boldsymbol{Z})\cong\{0\}$ and the torsion part of

$H_{p-1}(P;\boldsymbol{Z})\cong\boldsymbol{Z}_2$. A proof for $H_p(P;\boldsymbol{Z})$ is similar as the proof for the case of manifolds. We give a proof for the torsion part of $H_{p-1}(P;\boldsymbol{Z})$ which is also similar to the manifold case. So we give an elementary proof. Consider the chain complex $0\to C_p(P)\to C_{p-1}(P)\to\cdots$. We assume that there exist a $(p-1)$-cycle a and a p-chain b such that $\partial(b)=la$ for a positive integer l. Let Λ_1 (resp. Λ_2) be the set of all $(p-1)$-simplexes (resp. p-simplexes). Then we may write as

$$b=\sum_{d_t\in\Lambda_2}\mu_t d_t \qquad (d_s\neq d_t \text{ if } s\neq t).$$

If we put $b'=\sum\gamma_t d_t$ and $a'=a-\partial(\sum\mu_t' d_t)$ where $\mu_t=l\mu_t'+\gamma_t$, $0\leqq\gamma_t<l$, then we have $\partial(b')=la'$. Let a' be written as

$$a'=\sum_{c_i\in\Lambda_1}\lambda_i c_i \qquad (c_i\neq c_j \text{ if } i\neq j).$$

Let c_i be a $(p-1)$-simplex. Then there exist exactly two p-simplexes, say d_s and d_t, such that c_i is a common face of d_s and d_t. It follows from the equality $\partial(b')=la'$ that

$$\gamma_s[d_s:c_i]+\gamma_t[d_t:c_i]=l\lambda_i$$

where the bracket means the incidence number. The following (A) follows from the fact that $|[d_s:c_i]|=|[d_t:c_i]|=1$ and $l>\gamma_t$, $\gamma_s\geqq 0$.

(A) λ_i is equal to $+1$, -1 or 0. If $|\lambda_i|=1$, then $\gamma_s+\gamma_t=l$ and $[d_s:c_i]$, $[d_t:c_i]$ and λ_i have the same sign. If $\lambda_i=0$, then $\gamma_s=\gamma_t$ and $[d_s:c_i]+[d_t:c_i]=0$.

Now we take a p-simplex d_t and put $\gamma_t=\gamma$. Let d_u be any other p-simplex. Then there exists a sequence of p-simplexes $d_{t_0},\cdots,d_{t_r}$ such that $d_{t_0}=d_t$ and $d_{t_r}=d_u$ and that d_{t_i} and $d_{t_{i+1}}$ have a common $(p-1)$-face. Then we have either $\gamma_{t_i}+\gamma_{t_{i+1}}=l$ or $\gamma_{t_i}=\gamma_{t_{i+1}}$ by (A). Hence we can prove inductively that every γ_u is equal to either $l-\gamma$ or γ.

If $l-\gamma\neq\gamma$, then we put P_1 (resp. P_2)$=\{d_u\in\Lambda_2\,|\,d_u=\gamma$ (resp. $l-\gamma$)$\}$. Then clearly $\Lambda_2=P_1\cup P_2$. If we consider

$$\sum_{d_t\in P_1}\gamma_t d_t=\sum_{d_t\in P_1}\gamma d_t \quad\text{and}\quad \sum_{d_t\in P_2}\gamma_t d_t=\sum_{d_t\in P_2}(l-\gamma)d_t,$$

then it follows from (A) that $\partial(\sum_{d_t\in P_1}d_t)=(\sum_{d_t\in P_2}d_t)=a'$. Hence a' is a bounding cycle, that is, a is also a bounding cycle.

Next we consider the case of $l-\gamma=\gamma$ (this means that l must be even and γ is positive). If $P\backslash Q$ is orientable, then we can choose the incidence number such that the identity $[d_s:c_i]+[d_t:c_i]=0$ holds for all pairs of p-simplexes which have a common $(p-1)$-face. Hence it follows from (A) that a' must be zero, therefore a is a bounding cycle. Thus the torsion of $H_{p-1}(P;\boldsymbol{Z})$ is zero in the case where $P\backslash Q$ is orientable.

Now let $P\backslash Q$ be nonorientable. Then for any choice of orientations of simplexes there exists a $(p-1)$-simplex c_i such that $[d_s: c_i]=[d_t: c_i]$. If we put $d_0=\sum_{d_t\in\Lambda_2} d_t$, then it follows from (A) that $\partial(d_0)=2(\sum\lambda_i c_i)=2a'$ since $l=2\gamma$. Since $(2a')=\partial^2(d_0)$, we know that a' determines a unique homology class. This shows that the torsion of $H_{p-1}(P; \boldsymbol{Z})$ is $\boldsymbol{Z}_2$. Q.E.D.

Since $\bar{\Sigma}_0^I(n, p)$ and $\bar{\Sigma}_0^I(n, p)\backslash\Sigma_0^I(n, p)$ are respectively an algebraic set and its subalgebraic set, we can triangulate $\bar{\Sigma}_0^I(n, p)$ by [7, Theorem 1] such that $\bar{\Sigma}_0^I(n, p)\backslash\Sigma_0^I(n, p)$ becomes a subcomplex of $\bar{\Sigma}_0^I(n, p)$. In §4 we will show that either every connected component of $\Sigma^I(n, p)$ is orientable or every connected component is nonorientable. Let c_I denote the number of connected components of $\Sigma_0^I(n, p)$. Then the following theorem follows from Proposition 2.3 and Lemma 2.4 by the Mayer-Vietoris sequence.

THEOREM 2.5. *Let I be as in Proposition* 2.3 *except for the case of $p-n+i=1$ and $j=0$. Let G denote either $\boldsymbol{Z}$ or $\boldsymbol{Z}_2$ depending on whether $\Sigma^I(n, p)$ is orientable or not. Then*

$$H_{d_I}(\bar{\Sigma}_0^I(n, p);\ G)\cong c_I G \quad \text{and} \quad H^{d_I}(\bar{\Sigma}_0^I(n, p);\ \boldsymbol{Z})\cong c_I G$$

where $c_I G$ is the direct sum of c_I copies of G.

REMARK 2.6. In Proposition 4.1 of §4 we will describe the number c_I and when $\Sigma^I(n, p)$ is orientable or not. If $\Sigma_0^I(n, p)$ is nonorientable, then $H_{d_I}(\bar{\Sigma}_0^I(n, p);\ \boldsymbol{Z})\cong\{0\}$. When $p-n+i=0$, it is easy to see that $\bar{\Sigma}_0^I(n, p)=S^r(n, p)$. When $p-n+i=1$ and $j=0$, we know that $\bar{\Sigma}^I(n, p)=\Sigma^{n-p+1}(n, p)\times \mathrm{Hom}(\boldsymbol{R}^n\circ\boldsymbol{R}^n, \boldsymbol{R}^p)$.

We will say that I is *a good symbol* if both of the spaces $\Omega^I(n, p)$ and $\Omega(n, p)$ are connected and simply connected. If $\operatorname{codim}\Sigma^K$ is greater than 2 for any K such that $K\geqq I$, then the symbol I is clearly good.

COROLLARY 2.7. *Let I be as in Theorem* 2.5 *and we assume that I is a good symbol. Then*

$$\pi_{\nu_I}(\Omega^I, \Omega)\cong c_I G\,.$$

PROOF. The complement of Ω^I (resp. Ω) is $V^{I'}$ (resp. V^I). Then we may triangulate the triple of algebraic sets, $(S^r(n, p), V_0^I, V_0^{I'})$ by [7, Theorem 1] so that V_0^I (resp. $V_0^{I'}$) becomes a subcomplex of $S^r(n, p)$ (resp. V_0^I). We also triangulate the pair of algebraic sets $(\bar{\Sigma}_0^I, \bar{\Sigma}_0^I\backslash\Sigma_0^I)$ so that $\bar{\Sigma}_0^I\backslash\Sigma_0^I$ becomes a subcomplex of $\bar{\Sigma}_0^I$. Since $\pi_i(\Omega^I, \Omega)$ is $\{0\}$ for $i<\nu_I$ and I is good, we have that $\pi_{\nu_I}(\Omega^I, \Omega)$ is isomorphic to $H_{\nu_I}(\Omega^I, \Omega;\ \boldsymbol{Z})$. If I is neither (n) nor (n, n), then (Ω^I, Ω) is diffeomorphic to $(\Omega_0^I, \Omega_0)\times\boldsymbol{R}^+$. Now using the Alexander duality theorem for $\Omega_0^I=S^r(n, p)\backslash V_0^{I'}$ and $\Omega_0=S^r(n, p)\backslash V_0^I$, we have

$$\begin{aligned} H_{\nu_I}(\Omega^I, \Omega;\ \boldsymbol{Z})&\cong H_{\nu_I}(\Omega_0^I, \Omega_0;\ \boldsymbol{Z}) \\ &=H_{\nu_I}(S^r(n, p)\backslash V_0^{I'}, S^r(n, p)\backslash V_0^I;\ \boldsymbol{Z}) \end{aligned}$$

$$\cong H^{d_I}(V_0^I, V_0^{I'}; \boldsymbol{Z})$$

$$\cong H^{d_I}(V_0^I/V_0^{I'}, *; \boldsymbol{Z})$$

$$= H^{d_I}(\bar{\Sigma}_0^I/(\bar{\Sigma}_0^I \backslash \Sigma_0^I), *; \boldsymbol{Z})$$

$$\cong H^{d_I}(\bar{\Sigma}_0^I, \bar{\Sigma}_0^I \backslash \Sigma_0^I; \boldsymbol{Z}). \qquad \text{Q.E.D.}$$

We now define the fundamental class of $\bar{\Sigma}_0^I(\xi, \eta)$. In the sequel of the paper we assume that a manifold X is compact without boundary and that vector bundles ξ, η and a manifold X are orientable when $\Sigma^I(n, p)$ is orientable. Using a spectral sequence technique, we see that there exists the *fundamental class* of $\bar{\Sigma}_0^I(\xi, \eta)$ in $H_{d_I+\dim X}(\bar{\Sigma}_0^I(\xi, \eta); G)$ for every symbol I by Theorem 2.5 and Remark 2.6 which we denote by $[\bar{\Sigma}_0^I(\xi, \eta)]$. We choose orientations of ξ, η and X together with an orientation of $\Sigma^I(n, p)$. Then the fundamental class is the sum of generators associated with the orientations of groups G since $H_{d_I+\dim X}(\bar{\Sigma}_0^I(\xi, \eta); G)$ is the direct sum of a certain number of copies of G. We will show in § 4 that each connected components of $\Sigma_0^I(n, p)$ is invariant under coordinate transformations of $\boldsymbol{R}^n$ and $\boldsymbol{R}^p$. Let $\Sigma_0^I(n, p)_t$ $(1 \leqq t \leqq c_I)$ denote a connected component of $\Sigma_0^I(n, p)$ and we put $\Sigma_0^I(\xi, \eta)_t = \bigcup_{x \in X} \Sigma_0^I(\xi_X, \eta_X)_t$. If I is as in Proposition 2.3, then we can similarly define the fundamental class of $\bar{\Sigma}_0^I(\xi, \eta)_t$ which is denoted by $[\bar{\Sigma}_0^I(\xi, \eta)_t]$. Then $[\bar{\Sigma}_0^I(\xi, \eta)]$ is the sum $\sum_{t=1}^{c_I} [\bar{\Sigma}_0^I(\xi, \eta)_t]$.

Next we consider the dual classes. Let $i: \bar{\Sigma}_0^I(\xi, \eta) \to S^r(\xi, \eta)$ be the inclusion. Since $S^r(\xi, \eta)$ is a closed manifold, we may use the Poincaré duality theorem. We will denote the dual classes of $(i)_*([\bar{\Sigma}_0^I(\xi, \eta)])$ and $(i)_*([\bar{\Sigma}_0^I(\xi, \eta)_t])$ in $H^{\nu_I}(S^r(\xi, \eta); G)$ by $[\bar{\Sigma}_0^I(\xi, \eta)]^c$ and $[\bar{\Sigma}_0^I(\xi, \eta)_t]^c$ respectively. We denote the projection of $S^r(\xi, \eta)$ onto X by π. Then we know by the Gysin sequence that π induces an isomorphism of $H^{\nu_I}(X; G)$ to $H^{\nu_I}(S^r(\xi, \eta); G)$. We put

$$c^I(\xi, \eta) = (\pi^*)^{-1}([\bar{\Sigma}_0^I(\xi, \eta)]^c) \qquad (I \neq (n), (n, n))$$

and for a symbol I as in Proposition 2.3

$$c^I(\xi, \eta)_t = (\pi^*)^{-1}([\bar{\Sigma}_0^I(\xi, \eta)_t]^c).$$

When $I = (n)$, or (n, n), we define $c^I(\xi, \eta)$ to be the Euler class of the vector bundle $J^r(\xi, \eta)$. We will call $c^I(\xi, \eta)$ the *dual class of Thom-Boardman singularity with symbol I*. We should note that $c^I(\xi, \eta)$ depends on a choice of orientations.

REMARK. 2.8. The existence of fundamental classes and dual classes of $\Sigma^I(\xi, \eta)$ under the coefficient group $\boldsymbol{Z}_2$ has been shown in [**15**]. The $\boldsymbol{Z}_2$-reduction of our dual classes of Thom-Boardman singularities coincides with those in [**15**].

§ 3. The primary obstruction class.

Let s be a continuous section of the fibre bundle $\Omega_0^I(\xi, \eta)$ over X. In this section we will consider the primary obstruction class to deform s homotopically to a continuous section of the fibre bundle $\Omega_0(\xi, \eta)$ over X. We will denote this primary obstruction class of s by $c(s)$ ([**17**]). The following proposition will be used in § 6 to prove Theorem 1.

PROPOSITION 3.1. *Let I be as in Theorem 2.5 and a good symbol. Then the primary obstruction class $c(s)$ of a section of $\Omega^I(\xi, \eta)$ over X is equal to $(c^I(\xi, \eta)_1, \cdots, c^I(\xi, \eta)_{c_I})$ of $H^{\nu_I}(X, \pi_{\nu_I}(\Omega^I, \Omega))$ under the identification $\pi_{\nu_I}(\Omega^I, \Omega) \cong c_I G$.*

PROOF. If I is either (n) or (n, n), then the proposition is well known ([**13**]). Let $\Omega_t = \Omega^I(n, p) \backslash \Sigma^I(n, p)_t$ and $\Omega_0(\xi, \eta)_t = \Omega_0^I(\xi, \eta) \backslash \Sigma_0^I(\xi, \eta)_t$. Let $c_t(s)$ denote the primary obstruction class for s to be homotopic to a section of $\Omega(\xi, \eta)_t$ over X. Consider the natural homomorphism of $H^{\nu_I}(X, \pi_{\nu_I}(\Omega^I, \Omega))$ into $H^{\nu_I}(X, \pi_{\nu_I}(\Omega^I, \Omega_t))$ induced from the inclusion of Ω into Ω_t. Then $c(s)$ is mapped onto $c_t(s)$ by the definition of the primary obstruction class ([**17**]). Therefore we know that $c(s)$ is the sum of all $c_t(s)$ under the identification $\pi_{\nu_I}(\Omega^I, \Omega) \cong c_I G$.

Next we show that $c_t(s)$ is equal to the dual class $c^I(\xi, \eta)_t$. We may consider $\Omega_0^I(\xi, \eta)$ in place of $\Omega^I(\xi, \eta)$ and regard s as a section of $\Omega_0^I(\xi, \eta)$ over X. Note that $\Omega_0^I(\xi, \eta) \cap \bar{\Sigma}_0^I(\xi, \eta)_t = \Sigma_0^I(\xi, \eta)_t$. We may consider that the section s is differentiable and transverse to the manifold $\Sigma_0^I(\xi, \eta)_t$. We take a sufficiently small normal disk bundle ν_Σ of $\Sigma_0^I(\xi, \eta)_t$ in $\Omega_0^I(\xi, \eta)$ and denote its associated sphere bundle by $\partial\nu_\Sigma$. We put $N = s^{-1}(\Sigma_0^I(\xi, \eta))$, $\nu_N = s^{-1}(\nu_\Sigma)$ and $\partial\nu_N = s^{-1}(\partial\nu_\Sigma)$. Then we have sections, $s|_{(\nu_N, \partial\nu_N)} : (\nu_N, \partial\nu_N) \rightarrow (\nu_\Sigma, \nu_\Sigma \backslash \Sigma_0^I(\xi, \eta)_t)$ and $s|_{(X, X \backslash \mathring{\nu}_N)} : (X, X \backslash \mathring{\nu}_N) \rightarrow (\Omega_0^I(\xi, \eta), \Omega_0^I(\xi, \eta) \backslash \mathring{\nu}_\Sigma)$. Let $c(s|_{(\nu_N, \partial\nu_N)})$ (resp. $c(s|_{(X, X \backslash \mathring{\nu}_N)})$) be the primary obstruction class to be homotopic to a section of $\nu_\Sigma \backslash \Sigma_0^I(\xi, \eta)_t$ over ν_N (resp. $\Omega_0^I(\xi, \eta) \backslash \Sigma_0^I(\xi, \eta)_t$ over X) relative to $\partial\nu_N$ (resp. $X \backslash \mathring{\nu}_N$). Let $i_N : (\nu_N, \partial\nu_N) \rightarrow (X, X \backslash \mathring{\nu}_\Sigma)$ be the inclusion. It follows from the naturality of the primary obstruction class that $(i_N)^*(c(s|_{(X, X \backslash \mathring{\nu}_N)}) = c(s|_{(\nu_N, \partial\nu_N)})$, where $i_N^* : H^{\nu_I}(X, X \backslash \mathring{\nu}_N) \rightarrow H^{\nu_I}(\nu_N, \partial\nu_N)$ is the excision isomorphism and that $(j_X^*)(s|_{(X, X \backslash \mathring{\nu}_N)}) = c_t(s)$, where $j_X^* : H^{\nu_I}(X, X \backslash \mathring{\nu}_N) \rightarrow H^{\nu_I}(X)$. It is easy to see that $c(s|_{(\nu_N, \partial\nu_N)})$ is equal to the Thom class of the vector bundle ν_N over N (cf. [**13**] and [**17**]). If we take a sufficiently small disk bundle ν_Σ, we can extend ν_Σ to a tubular neighbourhood T of $\bar{\Sigma}_0^I(\xi, \eta)_t$ in $S^r(\xi, \eta)$ with collapsing map p so that $\partial T \supset \partial\nu_\Sigma$ ([**6**, § 5 of Ch. II]). We put $T' = p^{-1}(\bar{\Sigma}_0^I(\xi, \eta)_t \backslash \Sigma_0^I(\xi, \eta)_t)$. In the following commutative diagram we simply write Ω_0^I and S^r in place of $\Omega_0^I(\xi, \eta)$ and $S^r(\xi, \eta)$ and use the given notations such as the inclusions, i_T, i_Ω, i_ν, i_N and i_Σ. The coefficient group is G under the identification $\pi_{\nu_I}(\Omega^I, \Omega) \cong G$.

$$(*)\qquad \begin{array}{ccccc}
H^{\nu I}(S^r) & \xleftarrow{\;j_S^*\;} & H^{\nu I}(S^r, S^r\backslash \mathring{T}) & \xrightarrow[\cong]{\;i_T^*\;} & H^{\nu I}(T, \partial T) \\
\downarrow{\scriptstyle i_\Omega^*} & & \downarrow & & \downarrow{\scriptstyle i_\nu^*} \\
H^{\nu I}(\Omega_0^I) & \longleftarrow & H^{\nu I}(\Omega_0^I, \Omega_0^I\backslash \mathring{\nu}_\Sigma) & \xrightarrow[\cong]{} & H^{\nu I}(\nu_\Sigma, \partial\nu_\Sigma) \\
\downarrow{\scriptstyle s^*} & & \downarrow & & \downarrow \\
H^{\nu I}(X) & \xleftarrow{\;j_X^*\;} & H^{\nu I}(X, X\backslash \mathring{\nu}_N) & \xrightarrow[\cong]{\;i_N^*\;} & H^{\nu I}(\nu_N, \partial\nu_N)
\end{array}$$

$$\begin{array}{ccccc}
H^{\nu I}(T, \partial T) & \xleftarrow{\;P\;} & H_{d_I}(T) & \xleftarrow{\;(i_\Sigma)_*\;} & H_{d_I}(\bar{\Sigma}_0^I(\xi, \eta)_t) \\
\downarrow{\scriptstyle (i_\nu)^*} & & \downarrow{\scriptstyle (j_T)_*} & & \cong\downarrow{\scriptstyle (j_\Sigma)_*} \\
H^{\nu I}(\nu_\Sigma, \partial\nu_\Sigma)=H^{\nu I}(T\backslash T', \partial T\backslash T') & \xleftarrow{\;P'\;} & H_{d_I}(T, T') & \xleftarrow{\;(i_\Sigma)_*\;} & H_{d_I}(\bar{\Sigma}_0^I(\xi, \eta)_t, \bar{\Sigma}_0^I(\xi, \eta)_t\backslash\Sigma_0^I(\xi, \eta)_t)
\end{array}$$

In $(*)$ P and P' denote the isomorphisms of the Poincaré duality. Then it follows from the definition of the Poincaré duality and $(*)$ that $[\bar{\Sigma}_0^I(\xi, \eta)_t]$ is mapped onto $[\bar{\Sigma}_0^I(\xi, \eta)_t]^c$ by $j_S^*\circ(i_T^*)^{-1}\circ P\circ(i_\Sigma)_*$ and onto the Thom class of the normal bundle ν_Σ over $\Sigma_0^I(\xi, \eta)_t$ by $i_\nu^*\circ P\circ(i_\Sigma)_*$. We know by the commutativity of $(*)$ that $c_t(s)$ is equal to $(s^*\circ i_\Omega^*)([\bar{\Sigma}_0^I(\xi, \eta)_t]^c)$ which is equal to $c^I(\xi, \eta)_t$ since $\pi_r\circ i_\Omega\circ s=\mathrm{id}_X$. Q.E.D.

§4. Orientability and connected components of $\Sigma^I(n, p)$.

In this section we will describe the number of connected components of $\Sigma^I(n, p)$ and whether $\Sigma^I(n, p)$ is orientable or not. We will omit to refer to the case of $p-n+i=0$ in the following proposition because it is easy and is not important for our purpose.

PROPOSITION 4.1. *Let* $p-n+i\geqq 1$. (1) $\Sigma^i(n, p)$ *is orientable if and only if* (i) $n+p\equiv 0(2)$ *or* (ii) $i=0$ *or* n.

(2) $\Sigma^i(n, p)$ *is always connected.*

(3) *We put* $w(i)=(p+n+j)+j(p-n+i)+i\circ j$ *and* $w(j)=i+(p-n+i)(i+j+1)$. *Then* $\Sigma^{i,j}(n, p)$ *is orientable if and only if one of the following conditions is satisfied.*

(*Case* 1: $p-n+i>1$ *and* $i>j$)

(i) $n>i>j>0$, $w(i)\equiv 0(2)$ *and* $w(j)\equiv 0(2)$,

(ii) $n=i>j>0$ *and* $w(j)\equiv 0(2)$,

(iii) $n>i>j=0$ *and* $w(i)\equiv 0(2)$,

(iv) $n=i>j=0$,

(*Case* 2: $p-n+i=1$ *and* $i>j$)

(i) $i-j\equiv 1(2)$, $j>0$ *and* $w(j)\equiv 0(2)$,
(ii) $i-j\equiv 1(2)$ *and* $j=0$,
(iii) $i-j\equiv 0(2)$, $n>i>j>0$, $w(i)\equiv 0(2)$ *and* $w(j)\equiv 0(2)$,
(iv) $i-j\equiv 0(2)$, $n=i>j>0$ *and* $w(j)\equiv 0(2)$,
(v) $i-j\equiv 0(2)$ *and* $n=i>j=0$,

(*Case* 3: $p-n+i\geqq 1$ *and* $i=j$)

(i) $n>i=j>0$ *and* $w(i)+w(j)\equiv 0(2)$,
(ii) $n=i=j>0$,
(iii) $n>i=j=0$.

(4) $\Sigma^{i,j}(n, p)$ *is connected except for the case of* $p-n+i=1$ *and* $i>j$. *If* $p-n+i=1$, $i>j$ *and* $i-j\equiv 0(2)$, *then the number of connected components of* $\Sigma^{i,j}(n, p)$ *is* $(i-j)$. *If* $p-n+i=1$ *and* $i-j=1(2)$, *then it is* $(1/2)(i-j+1)$.

PROOF. (1) Let ξ denote the canonical vector bundle of dimension i over $G_{i,n-i}$. We define a differentiable fibre bundle $p: \Sigma^i \to G_{i,n-i}$ by $p(\alpha)=\mathrm{Ker}(\alpha)$, where $\alpha\in\Sigma^i$. We define a map $\varphi: \Sigma^i \to \mathrm{Hom}(\theta^n/\xi, \theta^p)$ by $\varphi(\alpha)=\alpha'$. Then we have the following commutative diagram,

$$\begin{array}{ccc} \Sigma^i & \xrightarrow{\ \varphi\ } & \mathrm{Hom}(\theta^n/\xi, \theta^p) \\ \downarrow & & \downarrow \\ G_{i,n-i} & = & G_{i,n-i}\,. \end{array}$$

It is clear that Σ^i is mapped bijectively onto $\Sigma^0(\theta^n/\xi, \theta^p)$ by φ. Hence we consider the last space in place of Σ^i. Let $p': \Sigma^0(\theta^n/\xi, \theta^p)\to G_{i,n-i}$ be the restriction map of the projection. Let $T(*)$ denote the tangent bundle and $T_f(*)$ the tangent bundle along the fibre of a differentiable fibre bundle. Then we have that $T(\Sigma^0(\theta^n/\xi, \theta^p))$ is isomorphic to the Whitney sum $(p')^*T(G_{i,n-i})\oplus T_f(p')$. Since $\mathrm{Hom}(\theta^n/\xi, \theta^p)$ is a vector bundle, the vector bundle $T_f(p')$ is isomorphic to $(p')^*(\mathrm{Hom}(\theta^n/\xi, \theta^p))$. Let $W_1(*)$ denote the first Stiefel-Whitney class. Since $T(G_{i,n-i})$ is isomorphic to $\mathrm{Hom}(\xi, \theta^n/\xi)$, we have

$$\begin{aligned} W_1(G_{i,n-i}) &= W_1(\mathrm{Hom}(\xi, \theta^n/\xi)) \\ &= (n-i)W_1(\xi)+iW_1(\theta^n/\xi) \\ &= nW_1(\xi)\,. \end{aligned}$$

Hence we have

$$W_1(\Sigma^0(\theta^n/\xi, \theta^p))=(p')^*(W_1(G_{i,n-i})+W_1(\mathrm{Hom}(\theta^n/\xi, \theta^p)))$$

$$=(p')^*(nW_1(\xi)+pW_1(\xi))$$
$$=(n+p)(p')^*W_1(\xi)\,.$$

If $p-n+i\geqq 1$, then $\Sigma^0(n-i,\, p)$ is connected since the codimension of $\bar{\Sigma}^1(n-i,\, p)$ is $p-n+i+1$. Hence the homomorphism $(p')^*\colon H^1(G_{i,n-i};\, \boldsymbol{Z}_2)\to H^1(\Sigma^0(\theta^n/\xi,\, \theta^p);\, \boldsymbol{Z}_2)$ is injective. Hence $\Sigma^0(\theta^n/\xi,\, \theta^p)$ is orientable if and only if either (i) $p+n\equiv 0(2)$ or (ii) ξ is trivial, that is, $i=0$ or n. This proves (1).

(2) is clear.

(3) Let $\bar{\xi}$ denote the induced bundle $p^*(\xi)$ over Σ^i and let $G_{j,i-j}(\bar{\xi})$ denote the associated grassmann bundle with projection g which is identified with the space

$$\{(\alpha,\, b)\,|\,\alpha\in\Sigma^i \text{ and } b \text{ is a } j\text{-dimensional subspace of } \operatorname{Ker}(\alpha)\}\,.$$

Then there exists a map $q\colon \Sigma^{i,j}\to G_{j,i-j}(\bar{\xi})$ defined by $q((\alpha,\, \beta))=(\alpha,\, \operatorname{Ker}(\tilde{\beta}))$. Let ξ_1 and ξ_2 be respectively the canonical vector bundles of dimensions i and j over $G_{j,i-j}(\bar{\xi})$. If we define a map $\bar{g}$ of $G_{j,i-j}(\bar{\xi})$ onto $G_{i,j,n-i}$ by $\bar{g}((\alpha,\, b))=(\operatorname{Ker}(\alpha),\, b)$, then $\xi_1=(\bar{g})^*\xi_i$ and $\xi_2=(\bar{g})^*\xi_j$ (ξ_i and ξ_j are as in the proof of Lemma 2.1). Let $h\colon G_{j,i-j}(\bar{\xi})\times\boldsymbol{R}^n\to G_{j,i-j}(\bar{\xi})\times\boldsymbol{R}^p$ be a bundle homomorphism defined by $h((\alpha,\, b),\, v))=((\alpha,\, b),\, \alpha(v))$ for $v\in\boldsymbol{R}^n$. Since h is of constant rank $(n-i)$, the image of h (denoted by η) is a vector bundle. Now we consider the bundle $\operatorname{Hom}(\theta^n\circ\theta^n,\, \theta^p)$ over $G_{j,i-j}(\bar{\xi})$. Then there exists an injective map $\phi\colon \Sigma^{i,j}\to\operatorname{Hom}(\theta^n\circ\theta^n,\, \theta^p)$ defined by $\phi((\alpha,\, \beta))=((\alpha,\, \operatorname{Ker}(\tilde{\beta})),\, \beta)$. Then we have the following commutative diagram

$$\begin{array}{ccc} \Sigma^{i,j} & \xrightarrow{\quad\phi\quad} & \operatorname{Hom}(\theta^n\circ\theta^n,\, \theta^p) \\ \downarrow{\scriptstyle q} & & \downarrow \\ G_{j,i-j}(\bar{\xi}) & = & G_{j,i-j}(\bar{\xi})\,. \end{array}$$

Now we determine the image of ϕ. Consider the two surjective bundle homomorphisms, $h_1\colon \operatorname{Hom}(\theta^n\circ\theta^n,\, \theta^p)\to\operatorname{Hom}(\xi_1\circ\xi_1,\, \theta^p/\eta)$ and $h_2\colon \operatorname{Hom}(\xi_1\circ\xi_1,\, \theta^p/\eta)\to\operatorname{Hom}(\xi_1\circ\xi_2,\, \theta^p/\eta)$ which are induced from the restrictions and the projection of θ^p onto θ^p/η. Then the bundle $\operatorname{Ker}(h_2)$ is canonically identified with $\operatorname{Hom}((\xi_1/\xi_2)\circ(\xi_1/\xi_2),\, \theta^p/\eta)$. Let H^0 be the subspace of all elements of $\operatorname{Hom}((\xi_1/\xi_2)\circ(\xi_1/\xi_2),\, \theta^p/\eta)$ whose corresponding elements of $\operatorname{Hom}(\xi_1/\xi_2,\, \operatorname{Hom}(\xi_1/\xi_2,\, \theta^p/\eta))$ are of rank $(i-j)$. Let K^0 be the subspace of $\operatorname{Ker}(h_2)$ which corresponds to H^0 by the above identification. Then it follows from the definition of $\Sigma^{i,j}$ that $\Sigma^{i,j}$ is mapped diffeomorphically onto $(h_1)^{-1}(K^0)$ by ϕ. Hence we will consider $(h_1)^{-1}(K^0)$ in place of $\Sigma^{i,j}$. We denote the projection of $(h_1)^{-1}(K^0)$ onto $G_{j,i-j}(\bar{\xi})$ by q'. The tangent bundle of $(h_1)^{-1}(K^0)$ is isomorphic to the Whitney sum

$(q')^*T(G_{j,i-j}(\bar{\xi}))\oplus T_f(q')$ and $T_f(q')$ is isomorphic to $(q')^*(\mathrm{Ker}\,(h_2\circ h_1))$. Since $T(G_{j,i-j}(\bar{\xi}))$ is isomorphic to $g^*(T(\Sigma^i)\oplus\mathrm{Hom}\,(\xi_2,\ \xi_1/\xi_2))$, we have

$$\begin{aligned}W_1(G_{j,i-j}(\bar{\xi}))&=g^*(W_1(\Sigma^i)+W_1(\mathrm{Hom}\,(\xi_2,\ \xi_1/\xi_2)))\\&=g^*(W_1(\Sigma^i)+(i-j)W_1(\xi_2)+j(W_1(\xi_1)-W_1(\xi_2)))\\&=g^*(p^*((n+p+j)W_1(\xi))+iW_1(\xi_2))\\&=(\bar{g})^*((n+p+j)W_1(\xi_1)+iW_1(\xi_2))\,.\end{aligned}$$

Since η is isomorphic to θ^n/ξ_1, we have

$$\begin{aligned}W_1(\mathrm{Ker}\,(h_2\circ h_1))&=W_1(\mathrm{Hom}\,(\xi_1\circ\xi_2,\ \theta^p/\eta))\\&=(p-n+i)W_1(\xi_1\circ\xi_2)+(i\circ j)W_1(\eta)\\&=(p-n+i)W_1(\xi_2\circ\xi_2\oplus(\xi_1/\xi_2)\otimes\xi_2)+i\circ jW_1(\xi_1)\\&=(p-n+i)((i+1)W_1(\xi_2)+j(W_1(\xi_1)-W_1(\xi_2))+(i-j)W_1(\xi_2))\\&\qquad+i\circ jW_1(\xi_1)\\&=(p-n+i)(i+j+1)W_1(\xi_2)+(j(p-n+i)+i\circ j)W_1(\xi_1)\\&=(\bar{g})^*((p-n+i)(i+j+1)W_1(\xi_j)+(j(p-n+i)+i\circ j)W_1(\xi_i))\,.\end{aligned}$$

Hence we have

$$\text{(A)}\quad \begin{aligned}W_1(h_1^{-1}(K^0))=(\bar{g}\circ q')^*\{&(i+(p-n+i)(i+j+1))W_1(\xi_j)\\&+((p+n+j)+j(p-n+i)+i\circ j)W_1(\xi_i)\}\,.\end{aligned}$$

Since H^0 is identified with $\Sigma^{i-j,0}(\xi_1/\xi_2,\ \theta^p/\eta)$, the codimension of the complement of H^0 is equal to $(i-j)(p-n+i-1)+1$. Hence if $i>j$ and $p-n+i>1$, then the fibre of $\bar{g}\circ q'$ is connected and the homomorphism $(\bar{g}\circ q')^*\colon H^1(G_{i,j,n-i}\,;\ \boldsymbol{Z}_2)\to H^1(h_1^{-1}(K^0)\,;\ \boldsymbol{Z}_2)$ is injective. Therefore the result of Case 1 follows from (A).

(Case 2) The codimension of $h_1^{-1}(K^0)$ is 1. We consider a map *det*: $\mathrm{Hom}\,((\xi_1/\xi_2)\circ(\xi_1/\xi_2),\ \theta^p/\eta)\to\boldsymbol{R}$ which is defined by determinants of quadratic forms. However the map *det* is not well defined when $i-j$ is odd. For, let x be an element of $h_1^{-1}(K^0)$ and $h_1(x)=l$ with $q'(x)=(\alpha,\ \mathrm{Ker}\,(\tilde{\beta}))$. Then the sign of $det(l)$ changes depending on a choice of orientation of $\mathrm{Cok}\,(\alpha)$. Therefore x determines the orientation (which is denoted by $o(\alpha)$) of $\mathrm{Ker}\,(\alpha)$ such that $det(l)$ is positive. This enables us to define a map $\tilde{q}'\colon h_1^{-1}(K^0)\to\tilde{G}_{j,i-j}(\bar{\xi})$ where the last space denote the set of all triples $(\alpha,\ o,\ b)$ such that o is an orientation of $\mathrm{Ker}\,(\alpha)$. We put $q'(x)=(\alpha,\ o(\alpha),\ \mathrm{Ker}\,(\tilde{\beta}))$. Then we have the following commutative diagram

$$\begin{array}{ccc} h_1^{-1}(K^0) & = & h_1^{-1}(K^0) \\ \downarrow \tilde{q}' & & \downarrow q' \\ \tilde{G}_{j,i-j}(\bar{\xi}) & \longrightarrow & G_{j,i-j}(\bar{\xi}) . \end{array}$$

Since the choice of a basis of ξ_1/ξ_2 does not change the signature of quadratic forms, it is easy to see that the number of connected components of $h_1^{-1}(K^0)$ is equal to the number of connected components of the fibre of q'. This is equal to $(1/2)(i-j+1)$ which is the order of the set of signatures of nonsingular quadratic forms with rank $(i-j)$ and positive determinants. Clearly we have that $(\bar{g}\circ q')^*W_1(\xi_i)=0$ and that $(\bar{g}\circ q')^*W_1(\xi_j)=0$ if and only if $j=0$. If $i-j$ is positive and even, the map *det* is well defined. Hence we obtain similarly that the number of connected components of $h_1^{-1}(K^0)$ is equal to $i-j$ which is the order of the set of signatures of nonsingular quadratic forms. It is clear that $(\bar{g}\circ q')^*W_1(\xi_i)=0$ if and only if $i=0$ or n and that $(\bar{g}\circ q')^*W_1(\xi_j)=0$ if and only if $j=0$.

(Case 3) In the case the result follows from the fact that the fibre of $\bar{g}\circ q'$ is connected. Q. E. D.

§ 5. Calculation of the dual classes.

In this section we will calculate the dual classes of Thom-Boardman singularities. When the coefficient group is $\boldsymbol{Z}_2$, the dual classes coincide with those which are defined and calculated in [**15**]. We should also mention [**16**]. Therefore we only consider the case that $\Sigma^I(n, p)$ is orientable. We will follow the method of desingularization in [**15**] under the singular (co)homology groups in place of sheaf (co)homology groups with closed supports in [3]. We also need to consider orientations in our desingularization.

We denote the projection of $J^r(\xi, \eta)$ onto X by the common letter π for $r=1, 2$ and we put $\xi'=\pi^*(\xi)$ and $\eta'=\pi^*(\eta)$. We consider the (iterated) grassmann bundles associated with ξ and ξ' as in § 1. We denote the canonical vector of dimension i over $G_{i,n-i}(\xi)$ and $G_{i,j,n-i}(\xi)$ by the letter ξ_i. Let ξ_j be the canonical vector bundle of dimension j over $G_{i,j,n-i}(\xi)$. Let p be the projection $G_{i,j,n-i}(\xi)\rightarrow X$. Then we put $\xi_{i,j}=(p^*\xi/\xi_i)\oplus\xi_i\circ\xi_j$ and consider $G_{i\circ j,n-i}(\xi_{i,j})$ whose projection onto $G_{i,n-i}(\xi)$ will be denoted by g. The projection of $G_{i,n-i}(\xi)$ onto X will be denoted by p_i. We put $p_{i,j}=g\circ p_i$. We similarly define maps, $p_i'\colon G_{i,n-i}(\xi')\rightarrow J^r(\xi, \eta)$ and $g'\colon G_{i\circ j,n-i}(\xi_{i,j}')\rightarrow G_{i,n-i}(\xi')$ and put $p_{i,j}'=p_i'\circ g'$. Then we have the following diagram,

(5.A)

$$
\begin{array}{ccccc}
 & & G_{i\circ j,n-i}(\xi) & \longleftarrow & G_{j\circ j,n-i}(\xi') \\
 & & \downarrow & & \downarrow \\
 & g \searrow & G_{i,j,n-i}(\xi) & \longleftarrow & G_{i,j,n-i}(\xi') \quad \swarrow g' \\
 & & \downarrow & & \downarrow \\
G_{i,n-i}(\xi') & \longrightarrow & G_{i,n-i}(\xi) & \longleftarrow & G_{i,n-i}(\xi') \\
\downarrow p_i' & & \downarrow p_i & & \downarrow p_i' \\
J^1(\xi,\eta) & \xrightarrow{\pi} & X & \xleftarrow{\pi} & J^2(\xi,\eta)\,.
\end{array}
$$

(1) The desingularization of $\bar{\Sigma}^i(\xi, \eta)$ is defined in $G_{i,n-i}(\xi')$ by

$$\tilde{\Sigma}^i(\xi, \eta)=\{(\alpha, a)\,|\,\alpha\in\bar{\Sigma}^i(\xi, \eta) \text{ and } a \text{ is an } i\text{-dimensional subspace of } \operatorname{Ker}(\alpha)\}.$$

Let (α, a) be an element of $G_{i,n-i}(\xi')$ with $\pi(\alpha)=x$. Then it induces a homomorphism $\alpha'\colon \xi_x/a\rightarrow\eta_x$. We define a section s_1 of a vector bundle $\operatorname{Hom}(\xi_i', (p_i')^*(\eta'))$ over $G_{i,n-i}(\xi')$ by $s_1((\alpha, a))=\alpha'$. Then s_1 is transverse to the zero-section and we have $\tilde{\Sigma}^i(\xi, \eta)=(s_1)^{-1}$(zero-section).

(2) We denote the following subset of $G_{i,n-i}(\xi')$ over $J^2(\xi, \eta)$ by the same notation.

$$\tilde{\Sigma}^i(\xi, \eta)=\{((\alpha, \beta), a)\,|\,\alpha\in\bar{\Sigma}^i(\xi, \eta) \text{ and } a \text{ is an } i\text{-dimensional subspace of } \operatorname{Ker}(\alpha)\}.$$

Let $((\alpha, \beta), a, b)$ be an element of $G_{i,j,n-i}(\xi')$ such that $((\alpha, \beta), a)\in\tilde{\Sigma}^i(\xi, \eta)$ with $\pi((\alpha, \beta))=x$. Then it induces a homomorphism $(\alpha', \beta')\colon \xi_x/a\oplus a\circ b\rightarrow\eta_x$ where α' is defined as in (1) and β' is a restriction map. In [**15**] the desingularization $\tilde{\Sigma}^{i,j}(\xi, \eta)$ of $\bar{\Sigma}^{i,j}(\xi, \eta)$ is defined in $G_{i,j,n-i}(\xi')$ as follows.

$$\tilde{\Sigma}^{i,j}(\xi, \eta)=\{((\alpha, \beta), a, b)\,|\,((\alpha, \beta), a)\in\tilde{\Sigma}^i(\xi, \eta) \text{ and } \dim(\operatorname{Ker}((\alpha', \beta')))\geqq i\circ j\}.$$

However $\tilde{\Sigma}^{i,j}(\xi, \eta)$ is not generally a nonsingular manifold (nevertheless, we can still use the calculation in [**15**]). For our purpose we need nonsingular desingularization because we must consider orientations. So we have considered $G_{i\circ j,n-i}(\xi_{i,j}')$. We define a subset $\tilde{\tilde{\Sigma}}^{i,j}(\xi, \eta)$ as follows,

$$\{((\alpha, \beta), a, b, c)\,|\,((\alpha, \beta), a, b)\in\tilde{\Sigma}^{i,j}(\xi, \eta) \text{ and } c \text{ is an } i\circ j\text{-dimensional subspace of } \operatorname{Ker}((\alpha', \beta'))\}\,.$$

This is defined analogously as $\tilde{\Sigma}^i(\xi, \eta)$ in (1). Let $(\xi_{i,j}')_{i\circ j}$ be the canonical vector bundle of dimension $i\circ j$ over $G_{i\circ j,n-i}(\xi_{i,j}')$. Then there exists a section

s_2 of a vector bundle Hom$((\xi'_{i,j})_{i\circ j}, (p'_{i,j})^*(\eta'))$ restricted on $(g')^{-1}\tilde{\Sigma}^i(\xi, \eta)$. s_2 is defined by

$$s_2(((\alpha, \beta), a, b, c))=(\alpha', \beta')|_c\,.$$

Then the following facts will be shown similarly as in (1). s_2 is transverse to the zero-section and $\tilde{\tilde{\Sigma}}^{j,j}(\xi, \eta)$ is equal to $(s_2)^{-1}$(zero-section) (hence $\tilde{\tilde{\Sigma}}^{i,j}(\xi, \eta)$ is nonsingular). Since $\tilde{\tilde{\Sigma}}^{i,j}(\xi, \eta)$ is mapped onto $\bar{\Sigma}^{i,j}(\xi, \eta)$ by the forgetting map of c, we know that the restriction of $p'_{i,j}$ to $\tilde{\tilde{\Sigma}}^{i,j}(\xi, \eta)$ define the desingularization of $\bar{\Sigma}^{i,j}(\xi, \eta)$.

We may restrict our construction of the desingularization onto $S^r(\xi, \eta)$. Let

$$\tilde{\Sigma}^i_0(\xi, \eta)=\tilde{\Sigma}^i(\xi, \eta)\cap(p'_i)^{-1}(S^1(\xi, \eta))\,,$$

$$\tilde{\tilde{\Sigma}}^{i,j}_0(\xi, \eta)=\tilde{\tilde{\Sigma}}^{i,j}(\xi, \eta)\cap(p'_{i,j})^{-1}(S^2(\xi, \eta))\,.$$

The (iterated) grassmann manifolds are not always orientable. So we next need to consider the oriented (iterated) grassmann manifolds. We define $\tilde{G}_{i,n-i}(\nu)$ and $\tilde{G}_{i,j,n-i}(\nu)$ similarly as in §1 by considering oriented subspaces of dimensions i and j. We have a double covering space of $\tilde{G}_{i,n-i}(\nu)$ onto $G_{i,n-i}(\nu)$ and a quadruple covering space of $\tilde{G}_{i,j,n-i}(\nu)$ onto $G_{i,j,n-i}(\nu)$. Then we may consider the similar diagram as in (5.A) with the given notations and restrict it from $J^r(\xi, \eta)$ to $S^r(\xi, \eta)$.

(5.B)
$$\begin{array}{ccccc}
 & & \tilde{G}_{i\circ j,n-i}(\xi_{i,j}) & \longleftarrow & \tilde{G}_{i\circ j,n-i}(\xi'_{i,j}) \\
 & & \downarrow \tilde{g} & & \downarrow \tilde{g}' \\
\tilde{G}_{i,n-i}(\xi') & \longrightarrow & \tilde{G}_{i,n-i}(\xi) & \longleftarrow & \tilde{G}_{i,n-i}(\xi') \\
\downarrow \tilde{p}'_i & & \downarrow \tilde{p}_i & & \downarrow \tilde{p}'_i \\
S^1(\xi, \eta) & \xrightarrow{\ \pi\ } & X & \xleftarrow{\ \pi\ } & S^2(\xi, \eta)
\end{array}$$

We put $\xi'=\pi^*(\xi)$, $\eta'=\pi^*(\eta)$ and $\tilde{p}_{i,j}=\tilde{g}\circ\tilde{p}_i$, $\tilde{p}'_{i,j}=\tilde{g}'\circ\tilde{p}'_i$.

Let $c_1: \tilde{G}_{i,n-i}(\xi)\rightarrow G_{i,n-i}(\xi)$ and $c_2: \tilde{G}_{i\circ j,n-i}(\xi_{i,j})\rightarrow G_{i\circ j,n-i}(\xi)$ be respectively the covering maps of degree 2 and 8. Let

$$(\tilde{p}_i)!: H^*(\tilde{G}_{i,n-i}(\xi);\, \boldsymbol{Z})\longrightarrow H^*(X;\, \boldsymbol{Z})\,,$$

$$(\tilde{p}_{i,j})!: H^*(\tilde{G}_{i\circ j,n-i}(\xi);\, \boldsymbol{Z})\longrightarrow H^*(X;\, \boldsymbol{Z})$$

denote the Gysin homomorphisms which are the dual maps of $(\tilde{p}_i)_*$ and $(\tilde{p}_{i,j})_*$.

THEOREM 5.1. *Let I denote either (i) or (i, j). Let $\Sigma^I(n, p)$ be orientable. Then we have*

(1) $c^i(\xi, \eta)=\pm\dfrac{1}{2}(\tilde{p}_i)!\,(\chi(\mathrm{Hom}\,(c_1^*(\xi_i),\, \tilde{P}_i^*(\eta))))\,,$

(2) $c^{i,j}(\xi, \eta)=\pm\frac{1}{8}(\tilde{p}_{i,j})_!\{\chi(\mathrm{Hom}(c_2^*((\xi_{i,j})_{i\circ j}), \tilde{p}_{i,j}^*(\eta)))$

$$\cdot\tilde{g}(\chi(\mathrm{Hom}(c_1^*(\xi_i), \tilde{P}_i^*(\eta)))))\}.$$

PROOF. We again denote the covering maps, $\tilde{G}_{i,n-i}(\xi')\rightarrow G_{i,n-i}(\xi')$ and $\tilde{G}_{i\circ j,n-i}(\xi'_{i,j})\rightarrow G_{i\circ j,n-i}(\xi'_{i,j})$ by the same letters c_1 and c_2. We put

$$s^i(\xi, \eta)=c_1^{-1}(\tilde{\Sigma}_0^i(\xi, \eta)), \qquad s^{i,j}(\xi, \eta)=c_2^{-1}(\tilde{\Sigma}_0^{i,j}(\xi, \eta)).$$

Since $s^I(\xi, \eta)$ are always orientable, we provide those with orientations which come from $\Sigma_0^I(\xi, \eta)$. We have sections

$$\tilde{s}_1:\ \tilde{G}_{i,n-i}(\xi')\longrightarrow \mathrm{Hom}\,(c_1^*(\xi'_i), (\tilde{p}'_i)(\eta'))$$

$$\tilde{s}_2:\ (\tilde{g}')^{-1}(\tilde{s}^i(\xi, \eta))\longrightarrow \mathrm{Hom}\,(c_2^*((\xi'_{i,j})_{i\circ j}), (\tilde{p}'_{i,j})(\eta'))$$

which are induced from s_1 and s_2. Then $s^I(\xi, \eta)$ is equal to $\tilde{s}_r^{-1}$(zero-section). Then the fundamental class of $s^i(\xi, \eta)$ (resp. $s^{i,j}(\xi, \eta)$) is mapped onto 2 (resp. 8) multiple of $[\tilde{\Sigma}_0^i(\xi, \eta)]$ (resp. $[\tilde{\Sigma}_0^{i,j}(\xi, \eta)]$).

(1) It follows from the construction of $s^i(\xi, \eta)$ that the dual class of $s^i(\xi, \eta)$ is $\chi(\mathrm{Hom}\,(c_1^*(\xi'_i), (\tilde{p}'_i)^*(\eta')))$. By the definition of the Gysin homomorphism we have the dual class of $(\tilde{p}'_i)^*([s^i(\xi, \eta)])$ is $(\tilde{p}'_i)_!\,\chi(\mathrm{Hom}\,(c_1^*(\xi'_i), (\tilde{p}'_i)^*(\eta')))$. This means that

$$2[\tilde{\Sigma}_0^i(\xi, \eta)]^c=(\tilde{p}'_i)_!\,(\chi(\mathrm{Hom}(c_1^*(\xi'_i), (\tilde{p}'_i)^*(\eta')))).$$

Thus the first formula follows from the naturality of the Gysin homomorphism and $\pi^*(c^i(\xi, \eta))=[\tilde{\Sigma}_0^i(\xi, \eta)]^c$ by considering the diagram (5.B).

(2) Let $i_1:\ (\tilde{g}')^{-1}(s^i(\xi, \eta))\rightarrow \tilde{G}_{i\circ j,n-i}(\xi'_{i,j})$ and $i_2:\ s^{i,j}(\xi, \eta)\rightarrow(\tilde{g}')^{-1}(s^i(\xi, \eta))$ be the inclusions. Then the dual class of $s^{i,j}(\xi, \eta)$ is the cup product of the dual class of $(i_1)_*([(\tilde{g}')^{-1}(s^i(\xi, \eta))])$ and $\chi\,(\mathrm{Hom}\,(c_2^*((\xi'_{i,j})_{i\circ j}), (\tilde{p}'_{i,j})^*(\eta')))$ since the last class is mapped onto $[(i_2)_*([s^{i,j}(\xi, \eta)])]^c$ by $(i_1)^*$. The first class is equal to $(\tilde{g}')^*\,(\chi\,(\mathrm{Hom}\,(c_1^*(\xi'_i), (\tilde{p}'_i)^*(\eta'))))$ as in (1). Therefore

$$8[\tilde{\Sigma}_0^{i,j}(\xi, \eta)]^c=(\tilde{p}'_{i,j})_!\,\{\chi(\mathrm{Hom}\,(c_2^*((\xi'_{i,j})_{i\circ j}), (\tilde{p}'_{i,j})^*(\eta')))$$

$$\cdot(\tilde{g}')^*(\chi\,(\mathrm{Hom}\,(c_1^*(\xi'_i), (\tilde{p}'_i)^*(\eta'))))\}.$$

The second formula again follows from the naturality of the Gysin homomorphism and $\pi^*([\tilde{\Sigma}_0^{i,j}(\xi, \eta)]^c)=c^{i,j}(\xi, \eta)$ by (5.B). Q.E.D.

Let $\zeta\rightarrow X$ be an orientable vector bundle of dimension p', and consider $\Sigma^I(\xi\oplus\zeta, \eta\oplus\zeta)$. Here we should note by Proposition 4.1 that $\Sigma^I(n, p)$ and $\Sigma^I(n+p', p+p')$ become orientable (resp. non-orientable) at the same time. Let ζ be a vector bundle such that $\eta\oplus\zeta$ is trivial and $f:\ X\rightarrow\tilde{G}_{n+p',N}$ a classifying map of $\xi\oplus\zeta$. Let γ be a universal bundle over $\tilde{G}_{n+p',N}$. Then if we consider $c^I(\gamma, \theta^{p+p'})$ for vector bundles γ and $\theta^{p+p'}$ over $\tilde{G}_{n+p',N}$, then we can prove the following proposition similarly as in [**15**, Theorem 4.4].

PROPOSITION 5.2. *Let I be as in Theorem* 5.1. *Then we have*

$$c^{i}(\xi, \eta)=f^{*}(c^{i}(\gamma, \theta^{p+p'})) \quad and \quad c^{i,j}(\xi, \eta)=f^{*}(c^{i,j}(\gamma, \theta^{p+p'})).$$

COROLLARY 5.3. *Let I be as in Theorem* 5.1.

(1) *If i is odd, then $c^{i}(\xi, \eta)$ is an element of* 2-*torsion.*

(2) *If either i or $i\circ j$ is odd, then $c^{i,j}(\xi, \eta)$ is an element of* 2-*torsion.*

PROOF. It is enough to prove the corollary for $c^{I}(\gamma, \theta^{p+p'})$ by Proposition 5.2. We note that

$$\chi(\mathrm{Hom}(c_1^{*}(\gamma_i), \theta^{p+p'}))=(\chi(c_1^{*}(\gamma_i)))^{p+p'},$$

$$\chi(\mathrm{Hom}(c_2^{*}((\gamma_{i,j})_{i\circ j}), \theta^{p+p'}))=(\chi(c_2^{*}(\gamma_{i,j})_{i\circ j}))^{p+p'}.$$

Then the corollary follows from Theorem 5.1 and the fact that the Euler class of an orientable vector bundle of odd dimension is of order 2 ([**13**]). Q. E. D.

When $p-n$ and i are even, we can represent $c^{i}(\xi, \eta)$ by Pontrjagin classes of ξ and η. This has been already mentioned in [**16**] in a slightly different form.

PROPOSITION 5.4 ([**16**]). *If $p-n$ and i are even, then $c^{i}(\xi, \eta)$ is equal modulo* 2-*torsion to the determinant of the following $(p-n+i)/2$-matrix, whose (s, t) component is $P_{(i/2)+(s-t)}(\xi-\eta)$.*

$$\begin{pmatrix} P_{i/2}(\xi-\eta) & P_{i/2-1}(\xi-\eta) & & \\ P_{i/2+1}(\xi-\eta) & \ddots & \ddots & \\ & \ddots & \ddots & P_{i/2-1}(\xi-\eta) \\ & & P_{i/2+1}(\xi-\eta) & P_{i/2}(\xi-\eta) \end{pmatrix}$$

REMARK 5.5. The Z_2-reduction of $c^{i}(\xi, \eta)$ has been also represented in [**15**] by Stiefel-Whitney classes as the determinant of the following $(p-n+i)$-matrix whose (s, t) component is $W_{i+s-t}(\xi-\eta)$.

$$\begin{pmatrix} W_{i}(\xi-\eta) & W_{i-1}(\xi-\eta) & & \\ W_{i+1}(\xi-\eta) & \ddots & \ddots & \\ & \ddots & \ddots & W_{i-1}(\xi-\eta) \\ & & W_{i+1}(\xi-\eta) & W_{i}(\xi-\eta) \end{pmatrix}$$

REMARK 5.6. Theorem 5.1 and Proposition 5.2 enable us to calculate $c^{i,j}(\xi, \eta)$ modulo 2-torsion in a finite process in terms of Pontrjagin classes of ξ and η. Although this method is quite similar to that given in [**15**, Ch. II], our adaptation require much preparation. Hence we will give only an exam ple for $I=(4, 4)$ and $p-n+2=0$, as follows.

$$c^I(\xi,\eta)=P_5d(0,0)+5P_4d(1,0)+4P_3d(2,0)+15P_3d(0,1)$$
$$+10P_2d(1,1)+16P_1d(2,1)-59P_1d(0,2)-16d(3,1)-107d(1,2)$$

where $P_i=P_i(\xi-\eta)$ and $d(j,a)=(-1)^j\begin{vmatrix}\bar{P}_{a+1+j} & \bar{P}_a \\ \bar{P}_{a+2+j} & \bar{P}_{a+1}\end{vmatrix}$ for $\bar{P}=P^{-1}$.

§ 6. Singularities of differentiable maps.

Let N and P be respectively manifolds of dimensions n and p. Let $\pi_P: N\times P\to P$ and $\pi_N: N\times P\to N$ be the canonical projections. Then the jet space $J^r(N,P)$ is identified with $J^r(\xi,\eta)$ over $N\times P$ for $\xi=\pi_N^*(TN)$ and $\eta=\pi_P^*(TP)$ and $\Sigma^I(N,P)$ corresponds to $\Sigma^I(\xi,\eta)$ under this identification ([**15**, § 3]). Consider the commutative diagram,

$$\begin{array}{ccc} J^r(TN, f^*(TP)) & \longrightarrow & J^r(\xi,\eta) \\ \downarrow & & \downarrow \\ N & \xrightarrow{\ \mathrm{id}_N\times f\ } & N\times P \end{array}$$

where f is a differentiable map of N into P. Then $j^rf: N\to J^r(N,P)$ $(=J^r(\xi,\eta))$ induces a map of N into $J^r(TN, f^*(TP))$ which we denote by d^rf. We consider two dual classes $c^I(TN, f^*(TP))$ and $c^I(\xi,\eta)$. Then we have the relation $c^I(TN, f^*(TP))=(\mathrm{id}_N\times f)^*(c_I(\xi,\eta))$. Therefore $c^I(TN, f^*(TP))$ depends only on the homotopy class of f.

PROPOSITION 6.1. *Let N be a closed manifold. When $\Sigma^I(n,p)$ is orientable, we assume that N and P are orientable. Let f be a differentiable map of $C^\infty_{\Omega^I}(N,P)$. If the dual class $c^I(TN, f^*(TP))$ does not vanish, then there exists no differentiable map g of $C^\infty_\Omega(N,P)$ such that d^rg is homotopic to d^rf in $\Gamma_{\Omega^I}(N)$.*

PROOF. We assume that such a map g exists and that j^rg is transverse to $\Sigma^I(\xi,\eta)$ where ξ and η are as above. Then j^rf and j^rg are homotopic in $\Omega^I(\xi,\eta)$. Since f and g are homotopic, we have $c^I(TN, g^*(TP))=c^I(TN, f^*(TP))$. On the other hand $c^I(TN, g^*(TP))$ is equal to the dual class of $(j^rg)^{-1}(\Sigma^I(\xi,\eta))$ $(=(d^rg)^{-1}(\Sigma^I(TN, g^*(TP))))$ which is empty. Hence $c^I(TN, f^*(TP))$ must be zero. This is a contradiction. Q.E.D.

PROOF OF THEOREM 1. We may deform f so that $d^rf: N\to\Omega^I(TN, f^*(TP))$ is transverse to $\Sigma^I(TN, f^*(TP))$. It follows from Proposition 3.1 that the primary obstruction class is $c^I(TN, f^*(TP))$ for d^rf to be homotopic to a section of $\Omega(TN, f^*(TP))$ in $\Omega^I(TN, f^*(TP))$. So there exists such a section $s: N\to\Omega(TN, f^*(TP))$ if the dual class vanishes. Let s' be the corresponding section of $\Omega(\xi,\eta)$ $(=\Omega(N,P))$ over N where ξ and η are as above. Since $\Omega(N,P)$ is π_0-integrable for N and P, there exists a differentiable map g in

$C^\infty_\Omega(N, P)$ such that $j^r g$ is homotopic to s' in $\Gamma_\Omega(N)$.

Conversely let g be a differentiable map of $C^\infty_\Omega(N, P)$ such that $j^r g$ is homotopic in $\Omega^I(N, P)$ to $j^r f$. Then we get $c^I(TN, f^*(TP))=c^I(TN, g^*(TP))$ as in the proof of Proposition 6.1. Since $j^r g^{-1}(\Sigma^I(N, P))$ is empty, $c^I(TN, g^*(TP))$ must be zero. Q. E. D.

The condition for $\Omega(N, P)$ to be π_0-integrable have been discussed in [**1**], [**4**] and [**14**]. Especially if $k>n-p$, then $\Omega^k(N, P)$ is π_0-integrable for any N and P ([**4**]). For $r=2$, $\Omega^{k,l}(N, P)$ is π_0-integrable for any N and P if $l>n-p-d$ where d denotes either 1 for $k-l>1$ or 0 for $k-l\leqq 1$ ([**14**]). As an application of Theorem 1 to k-mersions ([**4**]) we have the following proposition.

PROPOSITION 6.2. *Let f be a continuous map of a closed manifold N into a manifold P. Let $i(p-n+i)=n$ and $i>n-p+1$. Then there exists an $(n-i+1)$-mersion which is homotopic to f if and only if*

(i) *$n+p$ is odd and $c^i(TN, f^*(TP))=0$,*

(ii) *$n+p$ is even and i is odd, or*

(iii) *$n+p$ and i are even and $c^i(TN, f^*(TP))=0$.*

PROOF. It is clear that f is homotopic to a $(n-i)$-mersion since $\operatorname{codim}\Sigma^{i+1}(n, p)=(i+1)(p-n+i+1)$. Hence the proposition follows from Theorem 1. Q. E. D.

§ 7. C^∞ unstable maps.

We will consider differentiable maps which are not homotopic to any C^∞ stable map. Here we will recall the results due to J. Mather and use the same notations given in [**9**]. Let $\theta(f)_x$ denote the set of C^∞ vector fields along a germ $f: (N, x)\to(P, f(x))$, i.e., C^∞ map germs $\zeta: (N, x)\to TP$ such that $\zeta(x')$ is in $TP_{f(x')}$ for any point x' of N. We put $\theta(N)_x=\theta(\mathrm{id}_N)_x$, $\theta(P)_y=\theta(\mathrm{id}_P)_y$ and let $tf: \theta(N)_x\to\theta(f)_x$ and $\omega f: \theta(P)_{f(x)}\to\theta(f)_x$ be defined by $tf(\xi)=Tf\circ\xi$ and $\omega f(\eta)=\eta\circ f$. The following theorem is only a part of Theorem 4.1 of [**9**].

THEOREM 7.1 ([**9**]). *If $f: N\to P$ is a proper C^∞ stable map, then we have*

$$(*)\quad \theta(f)_x=tf(\theta(N)_x)+\omega f(\theta(P)_{f(x)})$$

for every point x of N.

Let $n-i_1$ be the rank of f at x. We can choose local coordinates $x_1, \cdots, x_n$ for N, null at x and $y_1, \cdots, y_p$ for P, null at $f(x)$ such that f has the form

$$(**)\quad \begin{cases} y_i\circ f=x_i, & i\leqq n-i_1 \\ d(y_i\circ f)(x)=0, & n-i_1+1\leqq i\leqq p \end{cases}$$

where d denotes the differential. Let $\mathcal{E}$ (resp. $\mathcal{E}'$) denote the ring of germs at 0 of C^∞ functions in the variables $x_1, \cdots, x_n$ (resp. $x_{n-i_1+1}, \cdots, x_n$). Let $\mathcal{E}'^{p-n+i_1}$ be the $(p-n+i_1)$-fold product $\mathcal{E}'\times\cdots\times\mathcal{E}'$ of $\mathcal{E}'$. For any u of $\mathcal{E}$, let u' denote the element of $\mathcal{E}'$ defined by $u'(x_{n-i_1+1}, \cdots, x_n)=u(0, \cdots, 0, x_{n-i_1+1}, \cdots, x_n)$. There exists a canonical identification of $\theta(f)$ with the p-fold product $\mathcal{E}^p$ of $\mathcal{E}$. We can define a certain projection of $\theta(f)$ onto $\mathcal{E}'^{p-n+i_1}$ under this identification ([8, §1]). Then $tf(\theta(N)_x)+\omega f(\theta(P)_{f(x)})$ corresponds by the projection to the sum of modules $\Omega(f'_{p-n+i_1}, \cdots, f'_p)$ and $[\partial f]$ defined as follows. Let $\Omega(f'_{p-n+i_1}, \cdots, f'_p)$ be the $\mathcal{E}'$-submodule of $\mathcal{E}'^{p-n+i_1}$ generated by the i_1 vectors

$$[f'_{n-i_1+1}/\partial x_j, \cdots, f'_p/\partial x_j] \qquad (j=n-i_1+1, \cdots, n)$$

and the $(p-n+i_1)$-fold product $\mathfrak{J}(f')\times \cdots \times\mathfrak{J}(f')$ of $\mathfrak{J}(f')$ where $\mathfrak{J}(f')$ means the ideal generated by $f'_{p-n+i_1}, \cdots, f'_p$. Let $[\partial f]$ denote the $\boldsymbol{R}$-vector subspace of $\mathcal{E}'^{p-n+i_1}$ spanned by the $(n-i_1)$ vectors

$$[(f_{n-i_1+1}/\partial x_j)', \cdots, (f_p/\partial x_j)'] \qquad (j=1, \cdots, n-i_1)\,.$$

Let $\mathfrak{m}'$ be the maximal ideal of $\mathcal{E}'$. If V is any subset of $\mathcal{E}'^{p-n+i_1}$, we denote the image of V under the projection of $\mathcal{E}'^{p-n+i_1}$ onto $\mathcal{E}'^{p-n+i_1}/\mathfrak{m}'^k\cdot(\mathcal{E}'^{p-n+i_1})$ by $V^{(k-1)}$.

THEOREM 7.2 ([8]). *Let $k>p$. For a germ $f:(N, x)\rightarrow(P, f(x))$ the relation* (∗) *of Theorem* 7.1 *holds if and only if*

$$(*)\quad \Omega(f'_{p-n+i_1}, \cdots, f'_p)^{(k-1)}+[\partial f]^{(k-1)}=(\mathfrak{m}'\mathcal{E}'^{p-n+i_1})^{(k-1)}.$$

The above theorem says that whether the relation (∗) of Theorem 7.1 holds or not for a germ f depends only on the $p+1$ jet of a germ f. Let $k>p$. We will say that a germ f (or a k-jet of f) is *unstable* if and only if the relation (∗) does not hold for f. Let $\Sigma(n, p)$ denote the set of unstable k-jets in $J^k(n, p)$ which has been defined in [**9**, **10**]. Then $\Sigma(n, p)$ becomes an algebraic subset of $J^k(n, p)$ by Theorem 7.2. We consider $\Sigma_{x,y}(N, P)$ similarly in $J^k_{x,y}(N, P)$ and put $\Sigma(N, P)=\bigcup_{x\in N, y\in P}\Sigma_{x,y}(N, P)$ in $J^k(N, P)$. The following corollary is a direct consequence of Theorem 7.1.

COROLLARY 7.3. *If f is a proper C^∞ stable map, then the following holds for $k>p$.*

$$j^kf(N)\cap\Sigma(N, P)=\emptyset\,.$$

Our purpose is to obtain topological conditions for N, P and f that for any map g which is homotopic to f, the intersection $j^kg(N)\cap\Sigma(N, P)$ is not empty. Since the set of unstable jets $\Sigma(n, p)$ is unfortunately so difficult to observe, we will consider the Thom-Boardman singularity $\Sigma^I(n, p)$ with symbol $I=(i_1, i_2, \cdots, i_k)$ such that $\Sigma(n, p)\supset\Sigma^I(n, p)$ in place of $\Sigma(n, p)$ (see the definition of the Thom-Boardman singularity with general symbol I of [**11**]). It is clear that if $\Sigma(n, p)\supset\Sigma^I(n, p)$, then $\Sigma(n, p)\supset\bar{\Sigma}^I(n, p)$.

For a symbol $I=(i)$ (resp. (i, j)) we will simply write $\Sigma^{(I,0,\cdots,0)}(n, p)$ for

$\Sigma^{(i,0,\cdots,0)}(n, p)$ (resp. $\Sigma^{(i,j,0,\cdots,0)}(n, p)$). Let π_r be the canonical projection of $J^k(n, p)$ onto $J^r(n, p)$ $(k \geqq r)$. Then it follows from the definition of the Thom-Boardman singularities that $(\pi_r)^{-1}(\bar{\Sigma}^I(n, p)) = \bar{\Sigma}^{(I,0,\cdots,0)}(n, p)$. In the rest of the section we assume that N is a closed manifold.

THEOREM 7.4. *Let N be a closed manifold and $f: N \to P$ a differentiable map. We assume N and P to be orientable when $\Sigma^I(n, p)$ is an orientable manifold. If there exists a symbol I such that $\Sigma(n, p) \supset \Sigma^{(I,0,\cdots,0)}(n, p)$ and that the dual class $c^I(TN, f^*(TP))$ does not vanish, then f is not homotopic to any C^∞ stable map (especially when P is $\boldsymbol{R}^p$, there exists no C^∞ stable map in $C^\infty(N, \boldsymbol{R}^p)$).*

PROOF. Consider a C^∞ stable map g and a stratification of $\bar{\Sigma}^I(N, P)$ which satisfies the Whitney's condition (b) ([**12**]). Then $j^r g: N \to J^r(N, P)$ is transverse to the stratification since g is C^∞ stable. Then we can show by the elementary property of the stratification the existence of the fundamental class of $\bar{\Sigma}^I(g)$ similarly to the case of $\bar{\Sigma}^I(N, P)$. The dual class of the fundamental class of $\bar{\Sigma}^I(g)$ is equal to $c^I(TN, g^*(TP))$ (cf. [**15**, §4]). Let f be homotopic to a C^∞ stable map g. Then $\bar{\Sigma}^I(g)$ is empty since $(j^k g)(N) \cap \Sigma(N, P) = \emptyset$ by Corollary 7.3 and $\Sigma(N, P)$ contains $\bar{\Sigma}^{(I,0,\cdots,0)}(N, P)$. Therefore $c^I(TN, g^*(TP))$ must be zero which is a contradiction since $c^I(TN, f^*(TP))$ is equal to $c^I(TN, g^*(TP))$.

Q. E. D.

We have showed in §5 that $c^i(TN, f^*(TP))$ is explicitly represented by characteristic classes and showed the formula to calculate $c^{i,j}(TN, f^*(TP))$. Another problem is to show when $\Sigma^I(n, p)$ is contained in $\Sigma(n, p)$.

Let I be $(i_1, \cdots, i_k)$. Consider an ideal

$$\mathfrak{J}' = (x_{n-i_1+1}, \cdots, x_{n-i_2})^2 + (x_{n-i_1+1}, \cdots, x_{n-i_3})^3 + \cdots + (x_{n-i_1+1}, \cdots, x_{n-i_k})^k$$

of $\mathcal{E}'/\mathfrak{m}'^{k+1}$ and its elements $g_{n-i_1+1}, \cdots, g_p$. We denote the vector $[\partial g_{n-i_1+1}/\partial x_j, \cdots, \partial g_p/\partial x_j]$ by $[\partial g/\partial x_j]$. Let $\mathfrak{J}(g)$ be the ideal generated by $g_{n-i_1+1}, \cdots, g_p$. We define a $\mathcal{E}'$-submodule,

$$\Omega(g) = \mathcal{E}'[\partial g/\partial x_{n-i_1+1}] + \cdots + \mathcal{E}'[\partial g/\partial x_p] + \mathfrak{J}(g)^{p-n+i_1}.$$

We define $d(I)$ to be the minimal number of

$$\dim_{\boldsymbol{R}}(\mathfrak{m}'\mathcal{E}'^{p-n+i_1})^{(k-1)}/\Omega(g)^{(k-1)}$$

where $g_{n-i_1+1}, \cdots, g_p$ vary in $\mathfrak{J}'$.

PROPOSITION 7.5. *Let $k > p$ and $I = (i_1, \cdots, i_k)$. Then $\Sigma(n, p) \supset \Sigma^I(n, p)$ if and only if $d(I) > n - i_1$.*

PROOF. Let $\Sigma(n, p)$ contain $\Sigma^I(n, p)$. This means that $\Sigma(n, p)$ contains $\bar{\Sigma}^I(n, p)$. Suppose that $d(I) \leqq n - i_1$. Then there exist elements $g_{n-i_1+1}, \cdots, g_p$ of $\mathfrak{J}'$ such that

$$\dim_R(\mathfrak{m}'\mathcal{E}'^{p-n+i_1})^{(k-1)}/\Omega(g)^{(k-1)})\leqq n-i_1 .$$

So we can take $(n-i_1)$ vectors $[h^s_{n-i_1+1}, \cdots, h^s_p]$ $(1\leqq s\leqq n-i_1)$ of $(\mathfrak{M}'\mathcal{E}'^{p-n+i_1})^{(k-1)}$ such that their images span $(\mathfrak{m}'\mathcal{E}'^{p-n+i_1})^{(k-1)}/\Omega(g)^{(k-1)}$ by the canonical projection. We define a germ $f: (N, x)\rightarrow(P, y)$ by

$$(***)\quad \begin{cases} y_i\circ f=x_i & i\leqq n-i_1 \\ y_i\circ f=g_i+\sum_{s=1}^{n-i_1} x_s h^s_i & n-i_1<i\leqq p . \end{cases}$$

It follows from the definition of the Thom-Boardman singularities that $j^k f\in \Sigma^I(n, p)$. Therefore f is not stable. On the other hand we have $[\partial f_{n-i_1+1}/\partial x_s, \cdots, \partial f_p/\partial x_s]=[h^s_{n-i_1+1}, \cdots, h^s_p]$. Hence,

$$\Omega(g_{n-i_1+1}, \cdots, g_p)^{(k-1)}+[\partial f]^{(k-1)}=(\mathfrak{M}'\mathcal{E}'^{p-n+i_1})^{(k-1)} .$$

Therefore f must be stable by Theorem 7.2 which is a contradiction.

Let $d(I)$ be greater than $n-i_1$. Let z be represented by a germ $f: (N, x)\rightarrow(P, y)$. Then we can choose coordinates $(\bar{x}_1, \cdots, \bar{x}_n)$ and $(\bar{y}_1, \cdots, \bar{y}_p)$ of N and P for $z\in\Sigma^I(n, p)$ such that the relation (**) holds for the coordinates. Since $\Sigma(n, p)$ and $\Sigma^I(n, p)$ is invariant under any coordinate transformation of N and P, we may consider only germs f represented as in (**). Then we have

$$\begin{aligned} &\dim(\Omega(f'_{p-n+i_1}, \cdots, f'_p)^{(k-1)}+[\partial f]^{(k-1)}) \\ &\quad\leqq\dim(\Omega(f'_{p-n+i_1}, \cdots, f'_p)^{(k-1)})+\dim[\partial f]^{(k-1)} \\ &\quad\leqq\dim(\mathfrak{m}'\mathcal{E}'^{p-n+i_1})^{(k-1)}-d(I)+n-i_1 \\ &\quad<\dim(\mathfrak{m}'\mathcal{E}'^{p-n+i_1})^{(k-1)}. \end{aligned}$$

That is, $\Omega(f'_{p-n+i_1}, \cdots, f'_p)^{(k-1)}+[\partial f]^{(k-1)}\subsetneqq(\mathfrak{m}'\mathcal{E}'^{p-n+i_1})^{(k-1)}$. Q. E. D.

COROLLARY 7.6. *Let* $k>p$, $k\geqq l+1$ *and* $I=(\underbrace{i, i, \cdots, i}_{l}, 0, \cdots, 0)$. *If*

$$(p-n+i)\dim(\mathfrak{m}'/\mathfrak{m}'^{l+2})-(i+i^2+(p-n+i)^2-1)>n-i,$$

then $\Sigma(n, p)$ *contains* $\Sigma^I(n, p)$.

PROOF. It is enough from Proposition 7.5 to show that $d(I)$ is greater than or equal to

$$(p-n+i)\dim(\mathfrak{m}'/\mathfrak{m}'^{l+2})-(i+i^2+(p-n+i)^2-1) .$$

In our case $\mathfrak{J}'$ is equal to $(x_{n-i+1}, \cdots, x_n)^{l+1}$ which is $\mathfrak{m}'^{l+1}/\mathfrak{m}'^{k+1}$. Let $d(I)^{(l)}$ be the minimal number of

$$\{\dim[(\mathfrak{m}'/\mathfrak{m}'^{l+2})^{p-n+i}/\Omega(g)^{(l+1)}]\}$$

where $g_{p-n+i}, \cdots, g_p$ vary in $\mathfrak{J}'$. Clearly $d(I)$ is not less than $d(I)^{(l)}$. Let g_j be any element of $\mathfrak{J}'$. Then

$$g_j \equiv \sum_{h=n-i+1}^{n} x_h(\partial g_j/\partial x_h) \qquad (\text{modulo } \mathfrak{m}'^{l+2}).$$

Therefore the vector $[g_{n-i+1}, \cdots, g_p]^{(l+1)}$ is contained in $(\mathcal{E}'[\partial g/\partial x_{n-i_1+1}]+\cdots+\mathcal{E}'[\partial g/\partial x_p])^{(l+1)}$. It is not difficult to show that

$$\dim \mathfrak{J}(g)^{(l+1)} \leqq p-n+i \qquad \text{and}$$

$$\dim (\mathcal{E}'[\partial g/\partial x_{n-i_1+1}]+\cdots+\mathcal{E}'[\partial g/\partial x_p])^{(l+1)} \leqq i+i^2.$$

So we have that $\dim \Omega(g)^{(l+1)} \leqq i+i^2+(p-n+i)^2-1$. Hence,

$$d(I)^{(l)} \geqq (p-n+1)\dim(\mathfrak{m}'/\mathfrak{m}'^{l+2})-(i+i^2+(p-n+i)^2-1).$$

Q. E. D.

We are now ready to prove Theorem 2.

PROOF OF THEOREM 2. We consider the case of $l=1$ in Corollary 7.6. Then we have that if $(p-n+i)\{i+(1/2)i(i+1)\}-i-i^2-(p-n+i)^2+1>n-i$, then $\Sigma(n, p) \supset \Sigma^{(i,0,\cdots,0)}(n, p)$. On the other hand the dual class $c^I(TN, f^*(TP))$ has been calculated as in Proposition 5.4 and Remark 5.5 for $\xi=TN$ and $\eta=f^*(TP)$. Therefore Theorem 2 follows from Theorem 7.4 and the above fact.

Q. E. D.

EXAMPLE 7.7. Consider maps of $\boldsymbol{R}P^n$ into $\boldsymbol{R}^n$ for $n=2^l\cdot a-1$ where a is odd, $2^l \leqq a \leqq 2^{l-1}(2^l-1)$ and $l \geqq 2$. Then there exists no C^∞ stable map in $C^\infty(\boldsymbol{R}P^n, \boldsymbol{R}^n)$. In fact, we know that

$$W(\boldsymbol{R}P^n)=(1+W_1)^{2^l\cdot a}=(1+W_1^{2^l})^a=1+aW_1^{2^l}+\cdots.$$

We take an integer 2^l as i of Theorem 2. Then we have that $W_j=0$ for $1 \leqq j<2^l$. Therefore

$$\begin{vmatrix} W_i & & & 0 \\ W_{i+1} & \ddots & & \\ & \ddots & \ddots & \\ & & W_{i+1} & W_i \end{vmatrix} = (W_i)^i = W_1^{2^l\cdot 2^l},$$

and $W_1^{2^l\cdot 2^l}$ is not zero by $2^l<a$ (c.f. [5, §6 of Ch. VI]).

We may construct such examples for $C^\infty(\boldsymbol{C}P^n, \boldsymbol{R}^p)$ by the formula $P(\boldsymbol{C}P^n)=(1+h_n^2)^{n+1}$ where h_n is a generator of $H^2(\boldsymbol{C}P^n; \boldsymbol{Z})$ ([13]). For example, there exists no C^∞ stable map in $C^\infty(\boldsymbol{C}P^4, \boldsymbol{R}^6)$.

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