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Ultrafilters in a product of spaces

By Tsugunori NOGURA

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1. Introduction.

Let N denote the natural numbers and let βN denote the Stone-Čech compactification of N. For each $M\subset N$, we denote $M^*=Cl_{\beta N}M-N$. Let F be a closed subset in N^*. We introduce a topology in $X=N\cup\{F\}$ as follows; each point of N is isolated and a neighborhood filter of $\{F\}$ in X is $\{(N\cap U)\cup\{F\}: U\in\mathfrak{U}_F\}$, where $\mathfrak{U}_F=\{U\}$ is the neighborhood filter of F in βN.

A countable space with one non-isolated point is denoted by $N\cup\{q\}$. Here q is the non-isolated point and its filter of neighborhoods restricted to N is denoted by $\mathfrak{F}_q=\{F_\alpha:\alpha\in A\}$. We denote $F_q=\cap\{Cl_{\beta N}F_\alpha:\alpha\in A\}$ and call F_q the representation of q in βN. Clearly $N\cup\{F_q\}$ is homeomorphic to $N\cup\{q\}$. Each countable space with one non-isolated point is denoted by the form $N\cup\{F\}$, where F is a closed subset in N^*. In this paper we sometimes use $N\cup\{F\}$ as a countable space with one non-isolated point.

Let p denote a free ultrafilter on N. Let $\mathfrak{X}$ denote a certain nice class of spaces such that each $X\in\mathfrak{X}$ cannot contain $N\cup\{p\}$ as a subspace. Then does finite (or countable) product of elements of $\mathfrak{X}$ contain $N\cup\{p\}$ as a subspace? We have much concern with this problem.

In the previous paper ([5]), we showed that, assuming the continuum hypothesis (CH), there exist Fréchet spaces (see Definition 2-2) X and Y such that $X\times Y$ contains $N\cup\{p\}$ as a subspace. In this paper, we shall show the following;

1 (CH). There exist strongly Fréchet spaces (see Definition 2-2) X and Y such that $X\times Y$ contains $N\cup\{p\}$ as a subspace.

2. Let X be a bi-sequential space (see Definition 2-2) and Y be any topological space. If $X\times Y$ contains $N\cup\{p\}$ as a subspace, then Y contains $N\cup\{p\}$ as a subspace.

3. There exists a non-metrizable Lašnev space T such that countable product of T does not contain $N\cup\{p\}$ as a subspace, where a *Lašnev space* is the closed continuous image of a metric space.

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In this paper all spaces are assumed to be topological spaces.

2. Properties of Lašnev spaces.

DEFINITION 2-1 ([1]). A space X is said to be *Fréchet* if, whenever $x\in Cl_X A$ for some $A\subset X$, there exists a sequence $\{x(n): n\in N\}\subset A$ such that $\lim_{n\to\infty} x(n)=x$.

DEFINITION 2-2 ([4]). A space X is said to be *bi-sequential* if, whenever $\mathfrak{F}$ is a filter in X with a cluster point x, then there exists a countable filter base $\mathfrak{H}$ in X which converges to x and all of whose elements intersect all elements of $\mathfrak{F}$. If the definition of a bi-sequential space is modified by restricting $\mathfrak{F}$ to be a countable filter base, the resulting concept is said to be *strongly Fréchet*.

LEMMA 2-1 ([3]). (1) *$N\cup\{F\}$ is a Fréchet space if and only if $F=Cl_{\beta N}(\mathrm{Int}_{N^*}F)$.*

(2) *$N\cup\{F\}$ is a strongly Fréchet space if and only if $F=\{x\in N^*:$ for each zero set Z in N^* such that $x\in Z$, $Z\cap\mathrm{Int}_{N^*}F\neq\emptyset\}$.*

(3) *$N\cup\{F\}$ is a bi-sequential space if and only if F is the union of zero sets in N^*.*

A family $\mathfrak{H}=\{H_\alpha:\alpha\in A\}$ of subsets of a space X is said to be *hereditarily closure preserving* if for each $B\subset A$ and $K_\alpha\subset H_\alpha$, $\cup\{Cl_X K_\alpha:\alpha\in B\}=Cl_X(\cup\{K_\alpha:\alpha\in B\})$. A family $\mathfrak{H}=\{H_\alpha:\alpha\in A\}$ of subsets of a space X is said to be a network at $x\in X$ if, for each open neighborhood U of x, there exists $H_\alpha\in\mathfrak{H}$ such that $x\in H_\alpha\subset U$. $\mathfrak{H}$ is said to be a *network of* X if it is a network at each point of X.

DEFINITION 2-3. Let X be a space. A sequence $\{\mathfrak{H}_n: n\in N\}$ of closed coverings of X is said to be a *Lašnev sequence* if the following three conditions are satisfied.

(1) $\mathfrak{H}_n$ is hereditarily closure preserving for each $n\in N$.

(2) If $x\in X$ and if for each $n\in N$, $H_n\in\mathfrak{H}_n$ and $x\in H_n$ then $\{H_n: n\in N\}$ is hereditarily closure preserving or a network at the point x.

(3) $\cup\{\mathfrak{H}_n: n\in N\}$ is a network of X.

LEMMA 2-2 ([2]). *A space X is Lašnev if and only if X is Fréchet and has a Lašnev sequence.*

LEMMA 2-3. *Let $\{U_i: i\in N\}$ be a family of clopen subsets in N^*. Then $\cap\{U_i: i\in N\}=Cl_{N^*}(\mathrm{Int}_{N^*}\cap\{U_i: i\in N\})$.*

PROOF. Since $\mathrm{Int}_{N^*}\cap\{U_i: i\in N\}\subset U_i$, $Cl_{N^*}(\mathrm{Int}_{N^*}\cap\{U_i: i\in N\})\subset\cap\{U_i: i\in N\}$. We shall show the converse implication. Choose

$$x\in\cap\{U_i: i\in N\}-Cl_{N^*}(\mathrm{Int}_{N^*}\cap\{U_i: i\in N\}).$$

Let V be a clopen subset of N^* such that

$$x \in V \quad \text{and} \quad V \cap Cl_{N^*}(\text{Int}_{N^*} \cap \{U_i : i \in N\}) = \emptyset .$$

Then $x \in V \cap \cap \{U_i : i \in N\}$ and

$$\text{Int}_{N^*}(V \cap \cap \{U_i : i \in N\}) = V \cap \text{Int}_{N^*} \cap \{U_i : i \in N\} = \emptyset .$$

This is impossible since each non-empty zero set in N^* has non-empty interior in N^*. The proof is completed.

LEMMA 2-4. *Let $X = N \cup \{F\}$ be a Lašnev space and let $\mathfrak{H}_n = \{H_\alpha : \alpha \in A_n\}$ $n \in N$ be a Lašnev sequence of X. Put $H^*_\alpha = Cl_{\beta N}(H_\alpha - \{F\}) - N$ and $\mathfrak{H}^*_n = \{H^*_\alpha : \alpha \in A_n,\ H^*_\alpha \cap F \neq \emptyset\}$. Then we have*

(1) *$\mathfrak{H}^*_n$ is a locally finite covering of* $\text{Int}_{N^*}F$ *for each $n \in N$.*

(2) *$\mathfrak{H}^*_n$ is countable.*

PROOF. We shall show that $\cup \mathfrak{H}^*_n$ is dense in $\text{Int}_{N^*}F$ for each $n \in N$, where $\cup \mathfrak{H}^*_n = \cup \{H_\alpha : H_\alpha \in \mathfrak{H}^*_n\}$. Since X is Fréchet and $H^*_\alpha \cap F \neq \emptyset$, $H^*_\alpha \cap \text{Int}_{N^*}F \neq \emptyset$ by Lemma 2-1. Assume that $\cup \mathfrak{H}^*_n$ is not dense in $\text{Int}_{N^*}F$. Then we can choose K $(\subset N)$ such that $\emptyset \neq K^* \cap \text{Int}_{N^*}F$ and $K^* \cap H^* = \emptyset$ for each $H \in \mathfrak{H}_n$. Then $K \cap H$ is finite for each $H \in \mathfrak{H}_n$. Put $H'_\alpha = K \cap H_\alpha$ for each $\alpha \in A_n$. Then H'_α is closed in X and $K = \cup \{H'_\alpha : \alpha \in A_n\}$. Since $\{F\} \in Cl_X K$ and $\{F\} \notin Cl_X H_\alpha$ for each $\alpha \in A_n$, $\mathfrak{H}_n$ is not hereditarily closure preserving. This is a contradiction.

Now we shall show that $\mathfrak{H}^*_n$ is locally finite in $\text{Int}_{N^*}F$. Assume that $\mathfrak{H}^*_n$ is not locally finite at $x \in \text{Int}_{N^*}F$. Choose $K_x \subset N$ such that $x \in K^*_x \subset \text{Int}_{N^*}F$. Then $\{H \in \mathfrak{H}_n : K^*_x \cap H^* \neq \emptyset\}$ is infinite. Choose $K' = \{k_1, k_2, \cdots\} \subset K_x$ such that $k_i \in H_{n(i)} \in \mathfrak{H}_n$ and $H_{n(i)} \neq H_{n(j)}$ if $i \neq j$. Then $K'^* \subset K^*_x \cap \text{Int}_{N^*}F$. $\{F\} \in Cl_X K'$ and $\{F\} \notin \{k_i : i = 1, 2, \cdots\}$. This is a contradiction since $\mathfrak{H}_n$ is hereditarily closure preserving.

Since $\cup \mathfrak{H}^*_n$ is dense in $\text{Int}_{N^*}F$ and $\mathfrak{H}^*_n$ is locally finite in $\text{Int}_{N^*}F$, $\mathfrak{H}^*_n$ is a covering of $\text{Int}_{N^*}F$.

Next we shall show that $\mathfrak{H}^*_n$ is countable for each $n \in N$. Assume that $\mathfrak{H}^*_n$ is uncountable for some $n \in N$. For each $H^*_\alpha \in \mathfrak{H}_n$, choose $K_\alpha \subset H_\alpha \cap N$ such that $K^*_\alpha \neq \emptyset$ and $K^*_\alpha \subset \text{Int}_{N^*}F$. Put $K = \cup \{K_\alpha : H^*_\alpha \in \mathfrak{H}^*_n\}$. For each $m \in K$, there exists $K_{\alpha(m)}$ such that $m \in K_{\alpha(m)}$. Fix such $K_{\alpha(m)}$ for each $m \in K$ and put

$$\mathfrak{B}_m = \{K_\alpha : K_\alpha \cap K_{\alpha(m)} \text{ is infinite}\} .$$

Since $\mathfrak{H}^*_n$ is locally finite in $\text{Int}_{N^*}F$ and $K_{\alpha(m)}{}^* \subset \text{Int}_{N^*}F$, $\mathfrak{B}_n$ is finite for each $m \in K$. Pick $K_\alpha \in \{K_\beta : H^*_\beta \in \mathfrak{H}^*_n\} - \cup \{\mathfrak{B}_m : m \in K\}$. Then $K_\alpha \cap K_{\alpha(m)}$ is finite for each $m \in K$ and $K_\alpha = \cup \{K_\alpha \cap K_{\alpha(m)} : m \in K\}$. Clearly $\{F\} \in Cl_X K_\alpha$ but $\{F\} \notin K_\alpha \cap K_{\alpha(m)}$ for each $m \in K$. This is a contradiction since $\mathfrak{H}_n$ is hereditarily closure preserving. The proof is completed.

THEOREM 2-1. *Let $X = N \cup \{F\}$ be a Lašnev space. Then, for each $p \in F$,* *there exists a zero set* Z_p *in* N^* *such that* $p \in Z_p \subset F$ *or otherwise* $p \in Z_p \subset N^* - \mathrm{Int}_{N^*} F$.

PROOF. Let $\mathfrak{H}_n = \{H_\alpha : \alpha \in A_n,\ n \in N\}$ be a Lašnev sequence of X. Assume that there exists $p \in F$ such that the condition of the theorem is not satisfied. Then we shall show that there exists $H_{\alpha(n)} \in \mathfrak{H}_n$ such that $p \in H^*_{\alpha(n)}$ for each $n \in N$. If there exists $n \in N$ such that $p \notin H^*$ for each $H \in \mathfrak{H}_n$, put $Z_p = \cap \{N^* - H^* : H^* \in \mathfrak{H}^*_n\}$. Then Z_p is a zero set in N^* since $\mathfrak{H}^*_n$ is countable by Lemma 2-4. Moreover, $Z_p \subset N^* - \mathrm{Int}_{N^*} F$ since $\mathfrak{H}^*_n$ is a covering of $\mathrm{Int}_{N^*} F$. This contradicts our assumption.

We shall show that $\{H_{\alpha(n)} : n \in N\}$ is neither a network at $\{F\}$ nor hereditarily closure preserving. This contradicts that $\mathfrak{H}_n$ is a Lašnev sequence.

(I) $\{H_{\alpha(n)} : n \in N\}$ is not a network at $\{F\}$.

By Lemma 2-3, $Cl_{N^*}(\mathrm{Int}_{N^*} \cap \{H^*_{\alpha(n)} : n \in N\}) = \cap \{H^*_{\alpha(n)} : n \in N\}$. By our assumption, $\cap \{H^*_{\alpha(n)} : n \in N\} \cap (N^* - F) \neq \emptyset$. Choose $K \subset N$ such that

$$\emptyset \neq K^* \subset (\mathrm{Int}_{N^*} \cap \{H^*_{\alpha(n)} :\ n \in N\}) \cap (N^* - F)\,.$$

Put $V = (N - K) \cup \{F\}$. Then V is a neighborhood of $\{F\}$ in X. Since $H_{\alpha(n)} - V$ is infinite for each $n \in N$, $H_{\alpha(n)} \not\subset V$ for each $n \in N$. This shows that $\{H_{\alpha(n)} : n \in N\}$ is not a network at $\{F\}$.

(II) $\{H_{\alpha(n)} : n \in N\}$ is not hereditarily closure preserving.

By Lemma 2-3 and by our assumption, we obtain $\mathrm{Int}_{N^*}(\cap \{H^*_{\alpha(n)} : n \in N\} \cap F) \neq \emptyset$. Choose $K = \{k_1, k_2, \cdots\} \subset N$ such that $k_n < k_{n+1}$, $k_n \in H_{\alpha(n)}$ for each $n \in N$ and $K^* \subset (\mathrm{Int}_{N^*} \cap \{H^*_{\alpha(n)} : n \in N\}) \cap F$. Then $Cl_X K = K \cup \{F\}$. Therefore $\{H_{\alpha(n)} : n \in N\}$ is not hereditarily closure preserving. The proof is completed.

In the following sections we shall sometimes use $M \cup \{p\}$ instead of $N \cup \{p\}$ to avoid the confusion. If $M \cup \{p\}$ can be embedded in a certain space, then we identify $M \cup \{p\}$ with the image of the embedding.

3. Bi-sequential and strongly Fréchet spaces.

LEMMA 3-1. *Let* $X_i = N \cup \{F_i\}$ *for each* $i = 1, 2, \cdots, n$. *If there exists* $M \subset N^n$ *such that the neighborhood filter of* $\prod_{i=1}^{n} \{F_i\}$ *restricted to* M *is an ultrafilter on* M *and moreover if* $p = (p_1, p_2, \cdots, p_n) \in (Cl_{(\beta N)^n} M) \cap \prod_{i=1}^{n} F_i$, *then* $M \cap \prod_{i=1}^{n} K_i$ *is an element of the ultrafilter for each* $K_i \subset N$ *and* $p_i \in K^*_i$.

PROOF. Let $\mathfrak{M}$ be the ultrafilter on M mentioned in the theorem. Let $\mathfrak{F}_i = \{F_\alpha : \alpha \in A_i\}$ be the filter on N such that $F_i = \cap \{Cl_{\beta N} F_\alpha : \alpha \in A_i\}$. We shall show $M \cap \prod_{i=1}^{n} F_{\alpha(i)} \cap \prod_{i=1}^{n} K_i \neq \emptyset$ for each $(\alpha(1), \alpha(2), \cdots, \alpha(n)) \in \prod_{i=1}^{n} A_i$. This shows $M \cap \prod_{i=1}^{n} K_i \in \mathfrak{M}$. Since $p_i \in F^*_{\alpha(i)} \cap K^*_i$, there exists $L_i \subset F_{\alpha(i)} \cap K_i$ such that $p_i \in L^*_i$

for each $i=1, 2, \cdots, n$. Since $p \in Cl_{(\beta N)^n} M$, $M \cap \prod_{i=1}^{n} L_i \neq \emptyset$. Therefore $\emptyset \neq M \cap \prod_{i=1}^{n} L_i \subset M \cap \prod_{i=1}^{n} F_{\alpha(i)} \cap \prod_{i=1}^{n} K_i$. The proof is completed.

LEMMA 3-2. *Let $X_i = N \cup \{F_i\}$ for each $i=1, 2, \cdots, n$. If there exists $M \subset N^n$ such that the neighborhood filter of $\prod_{i=1}^{n} \{F_i\}$ restricted to M is an ultrafilter on M, then $(Cl_{(\beta N)^n} M) \cap \prod_{i=1}^{n} F_i$ is a singleton.*

PROOF. Let $\mathfrak{M}$ be an ultrafilter on M mentioned in the lemma. Assume that $(Cl_{(\beta N)^n} M) \cap \prod_{i=1}^{n} F_i$ is not a singleton. Choose $p, q \in (Cl_{(\beta N)^n} M) \cap \prod_{i=1}^{n} F_i$, $p = (p_1, p_2, \cdots, p_n)$, $q = (q_1, q_2, \cdots, q_n)$ and $p \neq q$. Without loss of generality, we assume $p_1 \neq q_1$. Let K be a subset of N such that $p_1 \in K^*$ and $q_1 \notin K^*$. Since $(K \times \prod_{i=2}^{n} N_i) \cap L \neq \emptyset$ for each $L \in \mathfrak{M}$ by Lemma 3-1, $(K \times \prod_{i=2}^{n} N_i) \cap M \in \mathfrak{M}$, where N_i is a copy of N for each $i \in N$. Similarly, $((N-K) \cap \prod_{i=2}^{n} N_i) \cap M \in \mathfrak{M}$. This is a contradiction. The proof is completed.

LEMMA 3-3. *Let $\mathfrak{F}_n = \{F_\alpha : \alpha \in A_n\}$ be a filter on N for each $n \in N$ and let $\mathfrak{F}_n \subset \mathfrak{F}_{n+1}$. If $\mathfrak{F} = \cup \{\mathfrak{F}_n : n \in N\}$ is a free ultrafilter on N, then there exists $n(0) \in N$ such that $\mathfrak{F} = \mathfrak{F}_{n(0)}$.*

PROOF. Put $F_n = \cap \{Cl_{\beta N} F_\alpha : \alpha \in A_n\}$. Since $\mathfrak{F}$ is an ultrafilter, $\cap \{F_n : n \in N\}$ is a singleton. Assume $\mathfrak{F}_n$ is not the ultrafilter $\mathfrak{F}$ for each $n \in N$. Then we can choose $\{F_{n(k)} : k \in N\}$ such that $F_{n(k+1)} \subsetneqq F_{n(k)}$ for each $k \in N$. Choose $x(k) \in F_{n(k)} - F_{n(k+1)}$. Then $Cl_{\beta N} \{x(k) : k \in N\} - \{x(k) : k \in N\}$ is homeomorphic to N^*. On the other hand, $Cl_{\beta N} \{x(k) : k \in N\} - \{x(k) : k \in N\} \subset \cap \{F_n : n \in N\} =$ singleton. This is a contradiction. The proof is completed.

DEFINITION 3-1 ([**1**]). A subset U of a space X is said to be *sequentially open* if each sequence in X converging to a point in U is eventually in U. X is said to be a *sequential space* if each sequentially open subset of X is open. A space is said to be *subsequential* if it can be embedded in a sequential space.

LEMMA 3-4 ([**5**]). *$N \cup \{p\}$ is not subsequential for each free ultrafilter p on N.*

Let X be a space and $p \in X$. We denote by X_p, the space with the same underlying set as X, for which each point of $X - \{p\}$ is isolated and the neighborhoods of the point p in X_p is the same as p in X.

The following Lemma 3-5 is easy to prove, so we omit the proof.

LEMMA 3-5. (1) *Let X be a Lašnev space. Then X_p is Lašnev for each $p \in X$.*

(2) *Let X be a bi-sequential space. Then X_p is bi-sequential for each $p \in X$.*

LEMMA 3-6. *Let $p \in N^*$. Let K be a subset of N such that $p \in Cl_{\beta N} K$. Then $N \cup \{p\}$ is homeomorphic to $K \cup \{p\}$*

PROOF. Let L be an infinite subset of K such that $K-L$ is infinite and $p\in Cl_{\beta N}L$. Define ϕ as follows;

$$\phi(n)=n \quad \text{for each } n\in L\,,$$

$$\phi(p)=p\,,$$

$$\phi/N-L \text{ is a one to one and onto map from } N-L \text{ to } K-L\,.$$

Then clearly ϕ is a homeomorphism from $N\cup\{p\}$ to $K\cup\{p\}$. The proof is completed

THEOREM 3-1. *Let X be a bi-sequential space and Y be any space. If $M\cup\{p\}$ can be embedded in $X\times Y$, then $M\cup\{p\}$ can be embedded in Y, where p is a free ultrafilter on M.*

PROOF. Put $M_1=M\cap(\{p_1\}\times Y)$ and $M_2=M\cap(X\times\{p_2\})$, where $p=(p_1, p_2)$. If $p\in Cl_{X\times Y}(M_2-\{p\})$, then $M\cup\{p\}$ can be embedded in X by Lemma 3-6, which is impossible by Lemma 3-4. Thus, without loss of generality, we can assume $M_1\cup M_2=\emptyset$. Let π_X and π_Y be the projections from $X\times Y$ to X and Y, respectively. Put $\pi_X(M\cup\{p\})\cap X_{p_1}=N\cup\{p_1\}$ and $\pi_Y(M\cup\{p\})\cap X_{p_2}=N\cup\{p_2\}$. Let F_1 and F_2 be the representations of p_1 and p_2 in βN, respectively. By Lemma 3-2, $(Cl_{(\beta N)^2}M)\cap(F_1\times F_2)=q=(q_1, q_2)$. Since $N\cup\{F_1\}$ is bi-sequential by Lemma 3-5, then there exists a zero set Z in N^* such that $q_1\in Z\subset F_1$ by Lemma 2-1. Let $\{K_n : n\in N\}$ be a sequence of subsets of N such that $K_{n+1}\subset K_n$ and $Z=\cap\{Cl_{\beta N}K_n : n\in N\}$. Let $\mathfrak{G}_n$ be the filter generated by the filter base $\{M\cap(K\times F) : K_n\subset K,\ F\in\mathfrak{F}_2\}$. Then $\mathfrak{G}_n\subset\mathfrak{G}_{n+1}$ for each $n\in N$. We shall show that $\cup\{\mathfrak{G}_n : n\in N\}$ is an ultrafilter on M. Choose $F\in\mathfrak{F}_2$, then, since $Z\subset F_1$ and $F_1\subset F^*$ by the definition of F_1 (see Introduction), there exists K_n such that $K_n\subset F$. This shows $M\cap(F\times F_\beta)\in\mathfrak{G}_n$ for each $F_\beta\in\mathfrak{F}_2$. Thus $p\subset\cup\{\mathfrak{G}_n : n\in N\}$. Since p is an ultrafilter and $\cup\{\mathfrak{G}_n : n\in N\}$ is a filter, $p=\cup\{\mathfrak{G}_n : n\in N\}$.

By Lemma 3-3, there exists $n(0)$ such that $\mathfrak{G}_{n(0)}=p$. Put $L=\pi_Y(M\cap(K_{n(0)}\times Y))$. We shall show that $L\cup\{L\cap F_\beta : F_\beta\in\mathfrak{F}_2\}$ is homeomorphic to $M\cup\{p\}$. Assume that, for each $F_\beta\in\mathfrak{F}_2$, there exists $k_\beta\in F_\beta$ such that $|M\cap\pi_Y^{-1}(k_\beta)|\geqq 2$. It is easy to choose $n_\beta\in M\cap\pi_Y^{-1}(k_\beta)$ and $m_\gamma\in M\cap\pi_Y^{-1}(k_\gamma)$ such that $n_\beta\neq m_\gamma$. Put $A=\{n_\beta : F_\beta\in\mathfrak{F}_2\}$ and $B=\{m_\beta : F_\beta\in\mathfrak{F}_2\}$. Then $A\cup B\subset M$ and $A\cap B=\emptyset$. By the definition of A and B, $A\cap(K_{n_0}\times F_\beta)\neq\emptyset$ and $B\cap(K_{n_0}\times F_\beta)\neq\emptyset$ for each $F_\beta\in\mathfrak{F}_2$. These are impossible since $\mathfrak{G}_{n_0}$ is an ultrafilter and $A\cap B=\emptyset$. Hence, we can assume that there exists $F_\beta\in\mathfrak{F}_2$ such that $|M\cap\pi_Y^{-1}(n)|=1$ for each $n\in F_\beta$. Then, clearly, $L\cup\{L\cap F_\beta : F_\beta\in\mathfrak{F}_2\}$ is homeomorphic to $M\cup\{p\}$ by Lemma 3-6. The proof is completed.

THEOREM 3-2 (CH). *There exist strongly Fréchet spaces X, Y and $p\in N^*$ such that $N\cup\{p\}$ can be embedded in $X\times Y$, where p is a free ultrafilter on N.*

PROOF. V. I. Malyhin ([3]) used the continuum hypothesis to construct a strongly Fréchet space $X=N\cup\{F\}$ which has the following properties;

(1) $Bdy_{N^*}F=\{p\}$, where p is a P-point in N^*.

(2) $F-\{p\}$ is a clopen subset of $N^*-\{p\}$ and F is a closed subset of N^*. Put $Y=N\cup\{G\}$, where $G=(N^*-F)\cup\{p\}$. Then Y is strongly Fréchet by Lemma 2-1.

Put $p=\{P_\alpha:\alpha\in A\}$. Note that $F-P_\alpha^*$ and $G-P_\alpha^*$ are clopen in N^* for each $\alpha\in A$. Since $(F-P_\alpha^*)\cap(G-P_\alpha^*)=\emptyset$, there exist disjoint subsets F_α and G_α of N such that $F_\alpha^*=F-P_\alpha^*$ and $G_\alpha^*=G-P_\alpha^*$, respectively. Put $\mathfrak{F}=\{F_\alpha\cup P_\alpha\cup\{F\}:\alpha\in A\}$ and $\mathfrak{G}=\{G_\alpha\cup P_\alpha\cup\{G\}:\alpha\in A\}$. Then clearly $\mathfrak{F}$ and $\mathfrak{G}$ are neighborhood filters of $\{F\}$ in X and $\{G\}$ in Y, respectively. Define $\phi: N\cup\{p\}\to X\times Y$ as follows;

$$\phi(n)=(n, n) \quad \text{and} \quad \phi(p)=\{F\}\times\{G\}.$$

We shall show that ϕ is an embedding. The implication

$$\phi^{-1}(((F_\alpha\cup P_\alpha)\times(G_\beta\cup P_\beta))\cap\Delta)\supset P_\alpha\cap P_\beta\,,$$

implies ϕ is continuous, where $\Delta=\{(n, n): n\in N\}$. We shall show ϕ is an open map. Since $F_\alpha\cap G_\alpha=F_\alpha\cap P_\alpha=G_\alpha\cap P_\alpha=\emptyset$, $\phi(P_\alpha)=((F_\alpha\cup P_\alpha)\times(G_\alpha\cup P_\alpha))\cap\Delta$. The above equality implies that ϕ is an open map. Clearly ϕ is one to one, hence ϕ is an embedding. The proof is completed.

Theorem 3 of [5] is strengthened as follows.

COROLLARY 3-1 (CH). *There exist strongly Fréchet spaces X and Y such that $X\times Y$ is not subsequential.*

PROOF. By Lemma 3-4, $N\cup\{p\}$ is not subsequential. Hence this corollary is a direct consequence of Theorem 3-2.

The author does not know Theorem 3-2 and Corollary 3-1 are still true without the continuum hypothesis.

4. Lašnev space T.

Let $R=\{0\}\cup\{1/n: n\in N\}$ be a convergent sequence and let $S=\bigoplus\{R(n): n\in N\}$, where $\bigoplus$ denotes the disjoint union and $R(n)$ denotes a copy of R for each $n\in N$. Let $A=\{0(n)\in R(n): 0(n)=0,\ n\in N\}$ and let $T=S/A$, the quotient space obtained from S by identifying A to a point $\{A\}$. It is easy to show that the quotient map $\nu: S\to T$ is closed and hence T is a Lašnev space.

THEOREM 4-1. *T^n is sequential for each $n\in N$.*

PROOF. Clearly T^1 is sequential. Assume T^k is sequential for each $k\leqq n-1$ $(n\geqq 2)$. We shall show that each sequentially open subset of T^n is open in T^n. Let U be a sequentially open subset in T^n and $(x_1, x_2, \cdots, x_n)\in U$.

Case I. $x_i\neq\{A\}_i$ for each $i\leqq n$.

In this case $\pi_i^{-1}(x_i)$ is an open subspace of T^n and is homeomorphic to T^{n-1}, where π_i is the projection from T^n to T_i, $T_i=T$. Hence there exists an open neighborhood W of $(x_1, x_2, \cdots, x_n)$ such that $W \subset U$ by the inductive assumption

Case II. $x_i = \{A\}_i$ for each $i \leqq n$.

Since $U \cap \prod_{i=1}^{n} \nu(R(k_i))$ is a sequentially open subset of $\prod_{i=1}^{n} \nu(R(k_i))$ and $\prod_{i=1}^{n} \nu(R(k_i))$ is a metrizable subspace of T^n, $U \cap \prod_{i=1}^{n} \nu(R(k_i))$ is open in $\prod_{i=1}^{n} \nu(R(k_i))$. We can choose inductively a sequence $\{t_m : m \in N\}$ of increasing natural numbers satisfying the following condition;

$$\prod_{i=1}^{n} \nu([t_{k_i}]) \subset U \cap \prod_{i=1}^{n} \nu(R(k_i))$$

for each $k_i \leqq m$, where $[t_{k_i}] = \{1/s : s \geqq t_{k_i}\} \cup \{0\}$.

Put $U(k_1, k_2, \cdots, k_n) = \prod_{i=1}^{n} \nu([t_{k_i}])$ and put $W = \cup \{U(k_1, k_2, \cdots, k_n) : k_i \in N, i \leqq n\}$. Then $W \subset U$ and W is a neighborhood of $\prod_{i=1}^{n} \{A\}_i$ in T^n since $U_k = \nu(\bigoplus_{i=1}^{\infty} [t_i])$ is a neighborhood of $\{A\}_k$ in T_k, $W = \prod_{k=1}^{n} \nu(U_k)$ and $\nu^{-1}(\nu(U_k)) = U_k$. By I and II, U is open in T^n. The proof is completed.

THEOREM 4-2. *Let $\{X_n : n \in N\}$ be a family of spaces. If $N \cup \{p\}$ can be embedded in $\prod_{i=1}^{\infty} X_n$, then there exists $n(0) \in N$ such that $N \cup \{p\}$ can be embedded in $\prod_{n=1}^{n(0)} X_n$, where p is a free ultrafilter on N.*

PROOF. Put $p = (p_1, p_2, \cdots)$. Let $\mathfrak{U}_n = \{U_\beta : \beta \in B_n\}$ be the neighborhood filter of p_n in X_n for each $n \in N$. Put

$$\mathfrak{F}_n = \{N \cap (U_{\beta(1)} \times U_{\beta(2)} \times \cdots \times U_{\beta(n)} \times \prod_{k=n+1}^{\infty} X_k) : (\beta(1), \beta(2), \cdots, \beta(n)) \in \prod_{i=1}^{n} B_i\}.$$

Then $\mathfrak{F}_n \subset \mathfrak{F}_{n+1}$ and $\mathfrak{F}_n$ is a filter on N for each $n \in N$. Clearly $\cup \{\mathfrak{F}_n : n \in N\}$ is the ultrafilter p. Therefore, by Lemma 3-3, there exists $n(0) \in N$ such that $\mathfrak{F}_{n(0)}$ is the ultrafilter p. Then $N \cup \{p\}$ can be embedded in $\prod_{n=1}^{n(0)} X_n$. The proof is completed.

COROLLARY 4-1. *Let p be a free ultrafilter on N. Then $N \cup \{p\}$ cannot be embedded in T^ω.*

PROOF. Since T^n is sequential for each $n \in N$ by Theorem 4-1, $N \cup \{p\}$ cannot be embedded in T^n for each $n \in N$ by Lemma 3-4. Hence this corollary is a direct consequence of Theorem 4-2. The proof is completed.

REMARK 4-1. According to Y. Tanaka ([**6**], Theorem 1-3), T^ω is not sequential. The author does not know whether T^ω is subsequential or not.

PROBLEM 4-1. Can $N \cup \{p\}$ not be embedded in a countable product of Lašnev spaces?

Perhaps Theorem 2-1 is useful to solve the above problem. The author

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Tsugunori NOGURA
Department of Mathematics
Ehime University
Bunkyo-cho, Matsuyama 790
Japan