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# Interpolation, trivial and non-trivial homomorphisms in $H^{\infty}$

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## 1. Introduction.

Let $D$ be the open unit disc in the complex plane. We assume that the reader is somewhat familiar with the theory of $H^{\infty}=H^{\infty}(D)$ as a function algebra (see [4]) including Hoffman's paper [6] on the parts of $H^{\infty}$.

We recall Carleson's fundamental characterization of interpolating sequences for $H^{\infty}$ [1]. In terms of the *pseudo hyperbolic* metric, $\chi(z, w)=|z-w|/|1-\bar{z}w|$, a sequence $\{z_n\}$ in $D$ (possibly finite) is interpolating if it is *uniformly separated*, that is

$$\inf_{n} \prod_{k\neq n} \chi(z_k, z_n)>0 .$$

Garnett [3] found a characterization of interpolating sequences, which is more geometric in nature. It is stated in terms of the concept of a *Carleson measure,* namely, a finite measure $\mu$ on $D$ for which there exists a constant $K$ such that $\mu(S_{\theta,h})\leqq Kh$ for every set of the form $S_{\theta,h}=\{re^{i\varphi}: 0<1-r<h, |\varphi-\theta| \leqq h/2\}$ (equivalently, for all sufficiently small $h$).

THEOREM (Garnett). *Let $\{z_n\}$ be a separated sequence in D, that is, one for which $\inf_{n\neq k} \chi(z_n, z_k)>0$. Let $\mu$ be the measure on D assigning point mass $1-|z_n|$ to $z_n$ for each n and* 0 *elsewhere. Then, $\{z_n\}$ is interpolating if and only if $\mu$ is a Carleson measure.*

Note that the condition that $\{z_n\}$ be separated is a natural (and simple) one since every uniformly separated sequence is necessarily separated.

It was Hoffman [5], [6] who characterized the Gleason parts of the maximal ideal space $\mathcal{M}$ of $H^{\infty}$ as being either singleton points or analytic discs and who saw the connection with interpolating sequences. A homomorphism in $\mathcal{M}$ is *non-trivial* (its part is an analytic disc) or *trivial* (a one point part) corresponding to whether it lies in the closure in $\mathcal{M}$ of some interpolating sequence or not.

1) Some of the results of this paper form a portion of this author's doctoral thesis (1978) submitted to the University of California, Santa Barbara.

In this paper we are interested in the class $\mathcal{C}$ of all subsets $C$ of $D$ for which a sequence in $C$ is interpolating if (and only if) it is separated; and, the class $\mathcal{C}'$ of all subsets $C$ of $D$ whose closures in $\mathcal{M}$ contain only non-trivial homomorphisms. In particular in §3 we observe that they are the same class, $\mathcal{C}=\mathcal{C}'$, and obtain other useful characterizations of the class $\mathcal{C}$. For example, let $\rho(z, w)=(1/2)\, ln(1+\chi(z, w))/(1-\chi(z, w))$ be the *hyperbolic metric* on $D$ and let $d\rho$ give hyperbolic length on $[0, 1)$. For $S\subset D$ and $0<\varepsilon<1$ let $N(S, \varepsilon)$ denote the pseudo hyperbolic $\varepsilon$-neighborhood of $S$. Then, $S\in\mathcal{C}$ if and only if the product measure $d\gamma=d\lambda d\rho$ restricted to $N(S, \varepsilon)$ is a Carleson measure, where $d\lambda$ gives Lebesgue measure on the unit circle.

In §4 we investigate curves in $D$ ending on the boundary. We use Garnett's theorem to show that such a curve is in the class $\mathcal{C}$ if and only if it is pseudo hyperbolically close to one for which arc length is a Carleson measure. Another application allows us to characterize the nontrivial points in the relative interior of the fiber of $\mathcal{M}$ lying above 1.

## 2. Preliminaries.

Let $z\in D$, $0<\delta<1$, $S\subset D$. Throughout the paper we will use the notation

$$D(z, \delta)=\{w\in D : \chi(z, w)<\delta\}, \qquad N(S, \delta)=\cup\{D(z, \delta) : z\in S\}$$

for the pseudo hyperbolic $\delta$-neighborhoods of a point and of a set.

We need a series of concepts and elementary lemmas concerning the pseudo hyperbolic geometry of $D$.

DEFINITION 2.1. Let $0<\delta<1$. A set $S\subset D$ is said to be *$\delta$-separated* if $\chi(z, w)\geqq\delta$ for every pair $z$, $w$ of distinct points of $S$. Given a set $V\subset D$ we say that *$S$ $\delta$-covers $V$* if $N(S, \delta)\supset V$.

The next lemma follows from an easy maximality argument and the succeeding lemma from a standard compactness argument.

LEMMA 2.2. *Given a set $V\subset D$ and $\delta\in(0, 1)$ there is a $\delta$-separated subset $S$ of $V$ such that $S$ $\delta$-covers $V$.*

LEMMA 2.3. *Let $0<\delta<1$, $0<\varepsilon<1$, and $z\in D$. Then there exists a finite constant $M=M(\delta, \varepsilon)$ such that the number of points of a $\delta$-separated subset of $D(z, \varepsilon)$ does not exceed $M$.*

The next lemma isolates some of the technical calculations we encounter with the use of the pseudo hyperbolic metric.

LEMMA 2.4. *Let $0<\varepsilon<1$. Then, there exist positive constants $k_\varepsilon$ and $K_\varepsilon$ such that for each $re^{i\theta}$ in $D$, the following hold:*

(a) *For each $\rho e^{i\varphi}$ in $D(re^{i\theta}, \varepsilon)$*

$$k_\varepsilon(1-r)\leqq(1-\rho)\leqq K_\varepsilon(1-r).$$

(b) *If* $\chi(\rho e^{i\varphi}, re^{i\theta})<\varepsilon$, *then*

$$k_{\varepsilon}(1-r)\leqq|\rho e^{i\varphi}-re^{i\theta}|\leqq K_{\varepsilon}(1-r).$$

(c) *If* $A$ *is the area of* $D(re^{i\theta}, \varepsilon)$, *then*

$$k_{\varepsilon}(1-r)^2\leqq A\leqq K_{\varepsilon}(1-r)^2.$$

(d) *For* $0<h<1-\varepsilon$ *and for every angle* $\varphi$, *if* $K_{\varepsilon}h<1$ *and* $re^{i\theta}\in S_{\varphi,h}$, *then*

$$D(re^{i\theta}, \varepsilon)\subset S_{\varphi,K_{\varepsilon}h}.$$

PROOF. The reader should have no difficulties calculating a), b), c) using the fact that $D(re^{i\theta}, \varepsilon)$ is the Euclidean disc with radius $\varepsilon(1-r^2)/(1-\varepsilon^2r^2)$ and center $r(1-\varepsilon^2)e^{i\theta}/(1-\varepsilon^2r^2)$.

Part (d) is slightly more technical and we include its proof. Suppose $0<h<1-\varepsilon$, $K_{\varepsilon}h<1$ and $re^{i\theta}\in S_{\varphi,h}$ for some $\varphi$. Since $1-h>\varepsilon$, any pseudo hyperbolic disc $D(re^{i\theta}, \varepsilon)$ whose center has modulus $\geqq 1-h$ will fail to contain 0. Since $K_{\varepsilon}h<1$, the same is true of $S_{\varphi,K_{\varepsilon}h}$. Let $A$ denote the $\varepsilon$-neighborhood of $S_{\varphi,h}$. We show that $A\subset S_{\varphi,K_{\varepsilon}h}$ which implies the result. The angle centered at $\varphi$ subtended by $A$ equals $h$ increased by the angle subtended by $D(1-h, \varepsilon)$, namely, $h+2\sin^{-1}[\varepsilon h(2-h)/(1-h)(1-\varepsilon^2)]$. This, in turn, does not exceed $h+\pi\varepsilon(2-h)h/(1-h)(1-\varepsilon^2)$ which is less than or equal to $(1+2\pi(1-\varepsilon^2)^{-1})h\leqq K_{\varepsilon}h$. Furthermore, the minimum modulus of points of $A$ is $(1-h-\varepsilon)/(1-\varepsilon+\varepsilon h)$ which is easily seen to be no smaller than $1-(1+2\pi(1-\varepsilon^2)^{-1})h\geqq 1-K_{\varepsilon}h$. Thus $A\subset S_{\varphi,K_{\varepsilon}h}$ as required.

Later on we will require the following alternative description of the measure $\gamma$ mentioned in the introduction.

LEMMA 2.5. *Let* $\gamma$ *be the product measure on* $D$ *given by* $d\gamma=d\lambda d\rho$ *where* $d\lambda$ *gives Lebesgue measure on the unit circle and* $d\rho$ *gives hyperbolic length on* $[0, 1)$. *Then, if* $dA$ *represents two dimensional Lebesgue measure on* $D$ *and* $S$ *is a measurable subset of* $D$,

$$\gamma(S)=\int_S\frac{dA}{|z|(1-|z|^2)}.$$

PROOF. It is enough to verify the equation for sets of the form $S=\{re^{i\theta}: r_1\leqq r\leqq r_2, \theta_1\leqq\theta\leqq\theta_2\}$ in $D$. Then, changing to polar coordinates,

$$\int_S\frac{dA}{|z|(1-|z|^2)}=\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2}\frac{1}{1-r^2}drd\theta=\int_{\theta_1}^{\theta_2}\left[\frac{1}{2}\log\frac{1+r_2}{1-r_2}-\frac{1}{2}\log\frac{1+r_1}{1-r_1}\right]d\theta$$

$$=\int_{\theta_1}^{\theta_2}\rho(r_1, r_2)d\theta=\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2}d\rho d\theta=\gamma(S).$$

## 3. Characterizations of the class $\mathcal{C}$.

Before proceeding to the main theorem of this section we require some lemmas. The first of these is well known.

LEMMA 3.1. *Let $0<\delta<1$ and let $S$ be a $\delta$-separated sequence in $D$ which is a finite disjoint union of interpolating sequences. Then, $S$ is interpolating.*

LEMMA 3.2. *Let $\{z_n\}$ be an interpolating sequence in $D$ and let $0<\delta<1$, $0<\varepsilon<1$. If $S$ is a $\delta$-separated subsequence of $N(\{z_n\}, \varepsilon)$, then $S$ is an interpolating sequence.*

PROOF. By Lemma 2.3 there are at most $M_{\delta,\varepsilon}$ points of $S$ in each disc $D(z_n, \varepsilon)$. Therefore, $S$ can be decomposed into the union of at most $M_{\delta,\varepsilon}$ sets, $S_i$, each having at most one point in each disc $D(z_n, \varepsilon)$. By Lemma 3.1 it is enough to show each $S_i$ is interpolating.

Let $K_\varepsilon$ be the constant of Lemma 2.4, fix $h$ so that $0<h<1-\varepsilon$, $K_\varepsilon h<1$ and choose $\theta$. If $w\in S_i\cap S_{\theta,h}$, then there is an $m$ such that $\chi(w, z_m)<\varepsilon$. By part (d) of Lemma 2.4 we see that $z_m\in S_{\theta,K_\varepsilon h}$ and by part (a) of the same lemma, $1-|w|\leqq K_\varepsilon(1-|z_m|)$. Hence,

$$\sum_{w\in S_{\theta,h}\cap S_i}(1-|w|)\leqq \sum_{z_m\in S_{\theta,K_\varepsilon h}} K_\varepsilon(1-|z_m|).$$

and the result is clear from Garnett's theorem.

Hoffman [6] showed that the closure of an interpolating sequence in $\mathcal{M}$ contains only non-trivial homomorphisms. The next result shows that interpolating sequences are the only separated sequences with that property.

LEMMA 3.3. *Let $S$ be a separated sequence in $D$ which is not interpolating. Then, the closure of $S$ in $\mathcal{M}$ contains a trivial homomorphism.*

PROOF. Let $S$ be as above. Since by Lemma 3.1 $S$ is not the finite union of interpolating sequences there is a net $\{w_\alpha\}$ in $S$ which is eventually out of every interpolating sequence and which converges to some homomorphism $h$.

Suppose $h$ is nontrivial. Then $h$ is in the closure of some interpolating sequence $\{z_n\}$. Choose some $0<\varepsilon<1$. By Lemma 3.2 the sequence $S\cap N(\{z_n\}, \varepsilon)$ is interpolating so the net $\{w_\alpha\}$ is eventually out of this sequence, that is, $h$ is in the closure of $S-N(\{z_n\}, \varepsilon)$. One of Hoffman's characterizations of a nontrivial homomorphism in [6] implies that if $h$ is in the closures of two subsets $S_1$ and $S_2$ of $D$, then $\chi(S_1, S_2)=0$. But $\chi(S-N(\{z_n\}, \varepsilon), \{z_n\})\geqq\varepsilon$, which brings a contradiction. Thus $h$ must be a trivial homomorphism.

The main result of this section gives several necessary and sufficient conditions for a set to belong to the class $\mathcal{C}$ of the introduction.

THEOREM 3.4. *If $S$ is a subset of $D$, then the following are equivalent:*

(a) *The closure of $S$ in $\mathcal{M}$ contains only nontrivial homomorphisms.*

(b) *For each $\delta\in(0, 1)$, each $\delta$-separated subset of $S$ is an interpolating* *sequence.*

(c) *For some interpolating sequence* $\{z_n\}$ *and some* $\varepsilon\in(0, 1)$, $S\subset N(\{z_n\}, \varepsilon)$.

(d) *For each (or for some)* $\varepsilon\in(0, 1)$ *the measure* $\gamma$ *restricted to* $N(S, \varepsilon)$ *is a Carleson measure.*

PROOF. (a) $\Rightarrow$ (b): This is the content of Lemma 3.3.

(b) $\Rightarrow$ (c): By Lemma 2.2 there is a $\delta$-separated set $T$ which $\delta$-covers $S$. Then, $S\subset N(T, \delta)$ and by (b), $T$ must be an interpolating sequence.

(c) $\Rightarrow$ (a): Let $\{w_\alpha\}$ be a net in $S$ converging to a homomorphism $h$. Since $S\subset N(\{z_n\}, \varepsilon)$ there is a net $\{z_{n(\alpha)}\}$ in $\{z_n\}$ such that for each $\alpha$, $\chi(w_\alpha, z_{n(\alpha)})<\varepsilon$. Since $\chi$ is lower semicontinuous on $\mathcal{M}\times\mathcal{M}$ [6],

$$\chi(h, \lim_\alpha z_{n(\alpha)})\leqq\lim_\alpha \chi(w_\alpha, z_{n(\alpha)})\leqq\varepsilon<1\,.$$

Since the interpolating sequence $\{z_n\}$ contains only nontrivial homomorphisms in its closure, $h$ is in a nontrivial part, and this is true for all $h$ in the closure of $S$.

(b) $\Rightarrow$ (d): Choose $\varepsilon\in(0, 1)$ and $0<\delta<1-\varepsilon$. Let $S_0$ be a $\delta$-separated sequence in $S$ which $\delta$-covers $S$. By (b) $S_0$ is an interpolating sequence $\{z_n\}$ and by the triangle inequality if $\sigma=\varepsilon+\delta$, then $\{z_n\}$ $\sigma$-covers $N(S, \varepsilon)$.

Let $K_\sigma$ be the constant of Lemma 2.4, fix $h$ with $0<h<1-\sigma$, $K_\sigma h<1$, and fix $\theta$. For each $w\in N(S, \varepsilon)\cap S_{\theta, h}$ there is a $z_m$ such that $\chi(w, z_m)<\sigma$. By Lemma 2.4(d) and the choice of $h$, since $w\in S_{\theta, h}$ we have $z_m\in S_{\theta, K_\sigma h}$. Thus,

$$N(S, \varepsilon)\cap S_{\theta, h}\subset\bigcup_{z_m\in S_{\theta, K_\sigma h}} D(z_m, \sigma)\,.$$

Once and for all choose $\tau$ such that $K_\sigma h\leqq\tau<1$. Then, the above inclusion implies, using Lemma 2.5 and the fact that $|z|\geqq1-K_\sigma h\geqq1-\tau$ for $z\in S_{\theta, h}$, that

$$\gamma(N(S, \varepsilon)\cap S_{\theta, h})\leqq\sum_{z_m\in S_{\theta, K_\sigma h}}(1-\tau)^{-1}\int_{D(z_m, \sigma)}(1-|z|)^{-1}dA\,.$$

Now, by Lemma 2.4 (a), (b) we have the facts that if $z\in D(z_m, \sigma)$, then $1-|z|\geqq k_\sigma(1-|z_m|)$, and the area of $D(z_m, \sigma)$ is bounded above by $K_\sigma(1-|z_m|)^2$. Thus, we may continue the above estimate

$$\sum_{z_m\in S_{\theta, K_\sigma h}}\int_{D(z_m, \sigma)}(1-|z|)^{-1}dA\leqq k_\sigma^{-1}K_\sigma\sum_{z_m\in S_{\theta, K_\sigma h}}(1-|z_m|)\,.$$

Since $\{z_n\}$ is interpolating, the result follows by Garnett's theorem.

(d) $\Rightarrow$ (b): Clearly if (b) holds for some $\delta\in(0, 1)$ it holds for all such $\delta$. Given $\varepsilon\in(0, 1)$ from (d), let $0<\delta_0<\varepsilon$ and $\delta=\delta_0/2$. Let $\{z_n\}$ be a $\delta_0$-separated sequence in $S$. Fix $0<h'<1-\delta$ and $\theta$, let $K_\delta$ be the constant of Lemma 2.4 and let $h=K_\delta^{-1}h'$.

We now notice that $D(z_n, \delta)\subset N(S, \varepsilon)$ since $z_n\in S$. By Lemma 2.4 (d) if

$z\in S_{\theta,h}$ then $D(z, \delta)\subset S_{\theta,h'}$. Thus,

$$V=\bigcup_{z_n\in S_{\theta,h}} D(z_n, \delta)\subset S_{\theta,h'}\cap N(S, \varepsilon).$$

Since the $z_n$'s are $2\delta$-separated the triangle inequality implies that the union $V$ is disjoint. From Lemma 2.4 (a), (c) we have $1-|z|\leqq K_\delta(1-|z_n|)$ for $z\in D(z_n, \delta)$ and the area of $D(z_n, \delta)$ is bounded below by $k_\delta(1-|z_n|)^2$. Using this and keeping Lemma 2.5 in mind we estimate

$$\begin{aligned}\gamma(V)&=\sum\gamma(D(z_n, \delta))\\ &\geqq\sum\frac{1}{2}K_\sigma^{-1}(1-|z_n|)^{-1}\int_{D(z_n,\delta)}dA\\ &\geqq\frac{1}{2}K_\sigma^{-1}k_\delta\sum(1-|z_n|)\end{aligned}$$

where the sum ranges over those $z_n\in S_{\theta,h}$. Thus, with the same convention on the sum, for some $K'$,

$$\begin{aligned}\sum(1-|z_n|)&\leqq 2K_\delta k_\delta^{-1}\gamma(V)\\ &\leqq 2K_\delta k_\delta^{-1}\gamma(N(S, \delta)\cap S_{\theta,h'})\\ &\leqq 2K_\delta k_\delta^{-1}K'h'=(2K_\delta^2 k_\delta^{-1}K')h\end{aligned}$$

where the last step follows from the hypothesis that $\gamma$ restricted to $N(S, \delta)$ is a Carleson measure. Hence $\{z_n\}$ is an interpolating sequence by Garnett's theorem.

## 4. Curves belonging to $\mathcal{C}$.

The main result of this section is Theorem 4.1 below which characterizes those curves in $D$ which belong to the family $\mathcal{C}$. We will first give the simpler direction of the proof. The other direction will be accomplished by a series of lemmas.

We note that the curve described in Theorem 4.1 must tend to a single point of the unit circle, although this fact is not used in the proof. If it accumulates at more than one point, it must accumulate on an arc. It is easily shown that such a curve would contain trivial homomorphisms from the Šilov boundary of $\mathcal{M}$ in its closure.

THEOREM 4.1. *Let $\Gamma$ be a curve in $D$ ending at the unit circle. Then, $\Gamma$ belongs to $\mathcal{C}$ if and only if there is a (necessarily rectifiable) curve $\Gamma'$ in $D$ such that*

(a) $\Gamma\subset N(\Gamma', \varepsilon)$ *for some* $0<\varepsilon<1$.

(b) *The measure $\nu(E)$ measuring the total arclength of $\Gamma'$ contained in $E$ is a Carleson measure.*

PROOF. Suppose the existence of a curve $\Gamma'$ as described in the statement of the theorem. Since $\Gamma \subset N(\Gamma', \varepsilon)$, we see that for any net $\{z_\alpha\}$ in $\Gamma$ there is a corresponding net $\{w_\alpha\}$ in $\Gamma'$ such that eventually $\chi(z_\alpha, w_\alpha) < \varepsilon$. If $\{z_\alpha\}$ converges to a homomorphism $h$ in $\mathcal{M}$ then, by the lower semicontinuity of $\chi$, $\{w_\alpha\}$ must converge to a homomorphism $h'$ in the same part as $h$. Hence, using the equivalence of (a) and (b) of Theorem 3.4 it is enough to show that $\Gamma'$ belongs to $\mathcal{C}$.

Let $0<\delta_0<1$, set $\delta=\delta_0/2$ and let $S$ be a $\delta_0$-separated set in $\Gamma'$. By the triangle inequality $D(z, \delta)\cap D(w, \delta)=\emptyset$ for each pair $z$, $w$ of distinct points of $S$. Also, by Lemma 2.4 (d), for small enough $h$ and any $\theta$, for each $z\in S_{\theta,h}$ we have $D(z, \delta)\subset S_{\theta, K_\delta h}$. Thus,

$$V=\bigcup\{D(z, \delta):\ z\in S_{\theta,h}\cap S\}\subset S_{\theta, K_\delta h}\,.$$

Since $\nu$ is a Carleson measure, there is a $K$ such that

$$K(K_\delta h)\geqq \nu(S_{\theta, K_\delta h})\geqq \nu(V)=\sum \nu(D(z, \delta))$$

the last sum being over all $z\in S_{\theta,h}\cap S$. Since $\Gamma'$ ends on the unit circle, for any $z\in S$, $\Gamma'$ eventually leaves $D(z, \delta)$. Thus, the total arclength of $\Gamma'$ in $D(z, \delta)$ exceeds $k_\delta(1-|z|)$ by Lemma 2.4 (b). Therefore, with the same convention on the sum as above

$$K(K_\delta h)\geqq \sum k_\delta(1-|z|)\,.$$

In terms of the measure of Garnett's theorem $\mu(S_{\theta,h})\leqq(KK_\delta k_\delta^{-1})h$ so by that theorem $S$ is an interpolating sequence. The same holds for each separated set in $\Gamma'$ so $\Gamma'$ belongs to $\mathcal{C}$ as required.

Before proving the necessity of the condition in Theorem 4.1 we will give several lemmas to be used in the construction of $\Gamma'$. The first is a well-known fact from general topology; the second follows from an easy compactness argument.

LEMMA 4.2. *Let $\mathcal{V}$ be a cellection of open, connected subsets of a topological space such that $\bigcup\mathcal{V}$ is connected. If $V_1, V_2\in\mathcal{V}$, then there is a finite subfamily $\mathcal{V}_0$ of $\mathcal{V}$ such that $V_1, V_2\in\mathcal{V}_0$ and $\bigcup\mathcal{V}_0$ is connected.*

LEMMA 4.3. *Let $\Gamma$ be a curve in $D$ ending on the unit circle, and for some $\delta\in(0, 1)$ let $\{z_n\}$ be a $\delta$-separated sequence in $\Gamma$ which $\delta$-covers $\Gamma$. If $I_0$ is a finite set of indices and $W=\bigcup\{D(z_k, \delta): k\notin I_0\}$, then there is one component of $W$ containing all but a finite subset of $\{z_n\}$.*

LEMMA 4.4. *Let $\Gamma$ be a curve in $D$ ending on the unit circle and let $\{z_n\}$ be a $\delta$-separated sequence in $\Gamma$ which $\delta$-covers $\Gamma$, $0<\delta<1$. Then, there is a partition of the indices into finite subsets $I_n$, $n=1, 2, \cdots$ such that for each $n$*

(a) $\bigcup\{D(z_k, \delta): k\in I_n\}$ *is connected.*

(b) $\bigcup\{D(z_k, \delta): k\in I_n\}\cap\bigcup\{D(z_k, \delta): k\in I_{n+1}\}\neq\emptyset$.

PROOF. We construct the family $\{I_n\}$ inductively to have the following properties in addition to (a) of the statement: (c) $m<n$ implies $I_m\cap I_n=\emptyset$, (d) $\{1, \cdots, n\}\subset I_1\cup\cdots\cup I_n$, (e) if $J_m=I-(I_1\cup\cdots\cup I_m)$, then $\bigcup\{D(z_k, \delta): k\in I_m\}\cap\bigcup\{D(z_k, \delta): k\in J_m\}\neq\emptyset$, (f) $\bigcup\{D(z_k, \delta): k\in J_m\}$ is connected. In particular we give the method for constructing $I_1$ and for $I_p$, $p>1$, assuming $I_1, \cdots, I_{p-1}$ have already been constructed.

If $p>1$ we know $\bigcup\{D(z_k, \delta): k\in I_{p-1}\}\cap\bigcup\{D(z_k, \delta): k\in I_{p-1}\}\neq\emptyset$ and we choose an index $k_1\in J_{p-1}$ such that $\bigcup\{D(z_k, \delta): k\in I_{p-1}\}\cap D(z_{k_1}, \delta)\neq\emptyset$. If $p=1$ choose $k_1$ to be any index. For $p>1$ let $k_2=\inf\{k: k\in J_{p-1}\}$ and if $p=1$ choose $k_2=1$.

Now, $\bigcup\{D(z_k, \delta): k\in J_{p-1}\}$ is connected so, by Lemma 4.2, there is a finite set $L$ containing $\{k_1, k_2\}$ such that $V=\bigcup\{D(z_k, \delta): k\in L\}$ is connected. If we let $L_0=L\cup I_1\cup\cdots\cup I_{p-1}$ (if $p=1$, $L_0=L$), then $L_0$ is a finite set and, by Lemma 4.3, $W=\bigcup\{D(z_k, \delta): k\notin L_0\}$ has all but a finite subset $\{z_k: k\in L_1\}$ of $\{z_n\}$ in one component, $V_1$. We note here that $L_1$ is contained in $J_{p-1}$.

Every component $V_1, \cdots, V_n$ of $W$ must meet $V$, because if one of them, $V_{i_0}$, did not meet $V$, then $V_{i_0}$ and $V\cup\bigcup\{V_k: k\neq i_0\}$ would be disjoint nonempty open sets whose union is the connected set $\bigcup\{D(z_k, \delta): k\in J_{p-1}\}$. Thus, $\bigcup\{D(z_k, \delta): k\in L\cup L_1\}=V\cup(V_2\cup\cdots\cup V_n)$ is a connected set. We set $I_p=L\cup L_1$. The reader can easily check that $I_p$ has the desired properties and that $\{I_n\}$ is the required partition of the indices.

LEMMA 4.5. *Let* $\{z_1, \cdots, z_n\}\subset D$, $0<\delta_0<1$, $\delta=\delta_0/2$. *Suppose* $\bigcup\{D(z_k, \delta): k=1, \cdots, n\}$ *is connected. Then, there is a segmental path beginning at* $z_1$, *ending at* $z_n$, *joining all the points* $\{z_1, \cdots, z_n\}$ *such that if* $z_i$ *and* $z_j$ *are adjacent points on the path,*

(a) $\chi(z_i, z_j)<\delta_0$,

(b) *the segment between* $z_i$ *and* $z_j$ *is traversed no more than twice.*

PROOF. Let $L$ be the set of indices $k$ such that there exists a segmental path beginning at $z_1$, ending at $z_k$, joining some of the points $\{z_1, \cdots, z_n\}$ and satisfying (a) and (b) above. Suppose $L_1=\{1, \cdots, n\}-L\neq\emptyset$. Then, $\bigcup\{D(z_k, \delta): k\in L\}$ and $\bigcup\{D(z_k, \delta): k\in L_1\}$ are nonempty open sets whose union is the connected set $\bigcup\{D(z_k, \delta): k=1, \cdots, n\}$ so there would exist $i_0\in L_1$, $j_0\in L$ such that $D(z_{i_0}, \delta)\cap D(z_{j_0}, \delta)\neq\emptyset$. Thus, a segmental path joining $z_1$ to $z_{j_0}$ can be extended in the obvious way to $z_{i_0}$. Since by the triangle inequality, $\chi(z_{i_0}, z_{j_0})<\delta_0$, this extended path would satisfy (a) and (b) above. Thus $i_0\in L$ and this contradiction proves $L=\{1, \cdots, n\}$.

Next, consider the family $\mathcal{F}$ of all subsets $\{1, i_1, i_2, \cdots, i_m, n\}$ of $\{1, 2, \cdots, n\}$ such that $\{z_1, z_{i_1}, z_{i_2}, \cdots, z_{i_m}, z_n\}$ can be joined by a segmental path beginning at $z_1$, ending at $z_n$ and satisfying (a) and (b).

We have already shown $\mathcal{F}$ to be nonempty. Partially order $\mathcal{F}$ by inclusion and let $F$ be a maximal set in the finite family $\mathcal{F}$. Suppose $F \neq \{1, \cdots, n\}$. Then, as above there is an $i_0 \in F$ and $j_0 \in \{1, \cdots, n\} - F$ such that $D(z_{i_0}, \delta) \cap D(z_{j_0}, \delta) \neq \emptyset$. Let $\gamma$ be the path which joins the points $\{z_k : k \in F\}$ as described above. Let $\gamma'$ be a path which is the same as $\gamma$ from $z_1$ to $z_{i_0}$, traverses the segment from $z_{i_0}$ to $z_{j_0}$ and back and then is the same as $\gamma$ from $z_{i_0}$ to $z_n$. By the triangle inequality $\chi(z_{i_0}, z_{j_0}) < \delta_0$ and also the segment between $z_{i_0}$ and $z_{j_0}$ is traversed only twice. Thus, $\gamma'$ is the desired type of path and the maximality of $F$ would be contradicted. Hence $F = \{1, \cdots, n\}$ and the lemma is proved.

We are now in a position to complete the proof of Theorem 4.1.

Proof of necessity in Theorem 4.1. Let $\Gamma \in \mathcal{C}$, let $0 < \delta_0 < 1$, $\delta = \delta_0/2$ and let $\{z_n\}$ be a $\delta$-separated sequence in $\Gamma$ which $\delta$-covers $\Gamma$. Applying Lemma 4.4 we get a partition $\{I_n\}$ of the positive integers for which the conditions of that lemma hold. From Lemma 4.4 (b) for each $i$ we may choose $k_i \in I_i$ and $j_{i+1} \in I_{i+1}$ such that

$$D(z_{k_i}, \delta) \cap D(z_{j_{i+1}}, \delta) \neq \emptyset .$$

Since condition (a) of Lemma 4.4 holds, we can apply Lemma 4.5 to find a segmental path $\Gamma_i$ joining the points $\{z_k : k \in I_i\}$. This path starts at $z_{j_i}$ (for $i=1$, $j_i$ is any index in $I_1$) and ends at $z_{k_i}$ such that conditions (a) and (b) of the lemma hold. If we extend $\Gamma_i$ segmentally from $z_{k_i}$ to $z_{j_{i+1}}$, then it still satisfies the conditions (a) and (b) since $\chi(z_{k_i}, z_{j_{i+1}}) < \delta_0$. This extended path $\Gamma_i$ ends at $z_{j_{i+1}}$ and the (similarly extended) path $\Gamma_{i+1}$ begins at $z_{j_{i+1}}$ so all the paths $\Gamma_i$ can be joined together to give a path $\Gamma'$ containing all the points of $\{z_n\}$ and satisfying conditions (a) and (b) of Lemma 4.5. It is this path $\Gamma'$ which satisfies the present theorem as we now show.

Since $\{z_n\}$ $\delta$-covers $\Gamma$ and $\{z_n\} \subset \Gamma'$ we have $\Gamma \subset N(\{z_n\}, \delta) \subset N(\Gamma', \delta)$ and $\Gamma'$ satisfies (a) of the present theorem.

Next, let $\Gamma'_m$ be the union of all segments of $\Gamma'$ which have $z_m$ as an endpoint and let $L_m$ be the total length of all these segments. Since each point $z_{n_0}$ adjacent to $z_m$ on the path $\Gamma'$ satisfies $\chi(z_{n_0}, z_m) < \delta_0$ we see from Lemma 2.4 (b) that

$$|z_{n_0} - z_m| < K_{\delta_0}(1 - |z_m|) .$$

Since $\{z_n\}$ is $\delta$-separated, there are at most $M_{\delta, \delta_0}$ points of $\{z_n\}$ which are adjacent to $z_m$ on $\Gamma'$. Also, by construction, the segment between $z_m$ and each adjacent point is traversed no more than twice. Thus, $L_m < K_1(1 - |z_m|)$ with $K_1 = 2M_{\delta, \delta_0} K_{\delta_0}$.

Fix $h$ sufficiently small and fix $\theta$. If $w \in \Gamma'$, then for some $m_0$ $w$ is on a segment with $z_{m_0}$ as endpoint. Thus, $\chi(w, z_{m_0})<\delta_0$. If, in addition, $w\in S_{\theta,h}$, then, by Lemma 2.4 (d), $z_{m_0}\in S_{\theta,K_{\delta_0}h}$. Hence,

$$\Gamma'\cap S_{\theta,h}\subset\bigcup\{\Gamma'_m : z_m\in S_{\theta,K_{\delta_0}h}\}$$

and, therefore,

$$\nu(S_{\theta,h})\leqq\sum L_m\leqq K_1\sum(1-|z_m|)$$

the sums extending over $z_m\in S_{\theta,K_{\delta_0}h}$. Because $\{z_n\}$ is separated and $\Gamma\in\mathcal{C}$, $\{z_n\}$ is an interpolating sequence. Thus, applying Garnett's theorem we are finished.

Although one should verify it analytically (and we do so below in a general situation) it is intuitively clear that arclength is a Carleson measure for curves ending, say, at 1 which are convex (as in [**10**]) or, more generally, monotone in almost any reasonable sense. Thus, such curves have only nontrivial homomorphisms in their closures and each separated subsequence on such curves is an interpolating sequence.

THEOREM 4.6. *Let $\Gamma$ be a curve in $D$ ending at* 1. *Suppose for $z=re^{i\theta}=x+iy$ on $\Gamma$ that $\theta\geqq0$, $\theta$ is a nonincreasing function and $x$ is a nondecreasing function of the parameter of the curve as $z\to1$. Then, $\Gamma\in\mathcal{C}$. In addition, suppose $\{z_n=re^{i\theta_n}=x_n+iy_n\}$ is a sequence in $D$ tending to* 1 *such that $\theta_n\geqq\theta_{n+1}\geqq0$, $x_n\leqq x_{n+1}$ and $\chi(z_n, z_{n+1})\geqq\varepsilon>0$ for some $\varepsilon\in(0, 1)$, $n=1, 2, 3, \cdots$. Then, $\{z_n\}$ is an interpolating sequence.*

PROOF. Given a sequence $\{z_n\}$ as described in the statement of the theorem, the segmental curve $\Gamma$ joining the points of the sequence in order satisfies the description of $\Gamma$ in the statement of the theorem. If we first show that $\{z_n\}$ is separated, then by Theorems 4.1 and 3.4 the proof will be completed as soon as we show that the arclength of $\Gamma$ is a Carleson measure.

If we show that the conditions on the points $\{z_n\}$ are enough to prove that eventually $\chi(z_n, z_{n+2})\geqq\varepsilon$, then a simple induction shows that $\{z_n\}$ is separated. To begin with it is not hard to verify that for all large $n$ the ray of smallest angle tangent to $D(z_n, \varepsilon)$ intersects it at a point whose real part is less than $x_n$. From this it is clear geometrically that for the cases where $y_{n+1}\leqq y_n$ the conditions $x_{n+1}\leqq x_{n+2}$ and $\theta_{n+1}\leqq\theta_n$ alone are enough to ensure $\chi(z_n, z_{n+2})\geqq\varepsilon$. For the cases where $y_n<y_{n+1}$ it is equally evident, recalling the euclidean equation of the pseudo hyperbolic circle of radius $\varepsilon$ about $z_n$, that the worst case occurs for $\theta_{n+1}=\theta_n$ and $r_{n+1}=(r_n+\varepsilon)/(1+\varepsilon r_n)$. To finish the demonstration one can then, for example, calculate the point $w'_{n+1}=(r_n+\varepsilon)\cos\theta/(1+\varepsilon r_n)+i(r_n-\varepsilon)\cdot\sin\theta/(1-\varepsilon r_n)$ of this $\varepsilon$-circle about $z_n$ which lies directly below $w_{n+1}$. A direct calculation shows that $\chi(w_{n+1}, w'_{n+1})\to0$. Thus, once again, eventually $\chi(z_n, z_{n+2})\geqq\varepsilon$.

It remains to prove that the arclength of $\Gamma$ is a Carleson measure and this follows from a simple inequality. Namely, suppose $z=re^{i\theta}=x+iy$, $w=\rho e^{i\varphi}=u+iv$, $0<\varphi\leqq\theta$, $0<x\leqq u$ and $z$ and $w$ are close to 1. Then,

$$|z-w|\leqq 2(u-x)+2(\theta-\varphi).$$

To complete the proof fix $\theta$ and $h$ small and let $z_n=r_n e^{i\theta_n}=x_n+iy_n$, $n=1, \cdots, N$, be any finite subset of $\Gamma\cap S_{\theta,h}$ in the order for which the $x_n$'s are nondecreasing, the $\theta_n$'s nonincreasing. Then, from the inequality

$$\begin{aligned}\sum_{n=1}^{N-1}|z_n-z_{n+1}|&\leqq 2\sum_{n=1}^{N-1}(u_{n+1}-u_n)+2\sum_{n=1}^{N-1}(\theta_n-\theta_{n+1})\\&=2[(u_N-u_1)+(\theta_1-\theta_N)]\\&\leqq 2[(\text{diameter of } S_{\theta,h})+h]\\&\leqq 2[2h+h]=6h\,.\end{aligned}$$

This makes it clear that the total arclength of $\Gamma$ in $S_{\theta,h}$ is bounded above by $6h$ and shows that the arclength of $\Gamma$ is a Carleson measure.

Theorem 4.6 includes the case in [9] where the sequence $\{z_n\}$ is upper tangential with $r_n$ increasing and $\theta_n$ decreasing as well as the similar case in [7] where the sequence need not be tangential and the monotonicity need not be strict. In addition, it includes the answer to the question asked in [7] whether one obtains the same results for the case of $x_n$ nondecreasing and $y_n$ nonincreasing.

There is one final question we wish to answer. We have already pointed out that if a curve $\Gamma$ belongs to $\mathcal{C}$, then it ends at some point on the unit circle. Thus, from [8] the nontrivial homomorphisms in the closure of $\Gamma$ belong to the relative interior of the fiber of $\mathcal{M}$ lying above that point. It is formally possible at this point that some nontrivial homomorphisms in the relative interior of the fiber do not lie in the closure of a curve in $\mathcal{C}$. We show next that this possibility does not occur.

THEOREM 4.7. *Let $h$ be a nontrivial homomorphism in the relative interior of the fiber of $\mathcal{M}$ lying above* 1. *Then, $h$ is in the closure of some curve $\Gamma$ in $\mathcal{C}$.*

PROOF. By Hoffman's results $h$ lies in the closure of some interpolating sequence $\{z_n\}$. Without loss of generality we may assume this sequence (eventually) to lie above the radius to 1. Because $h$ is in the relative interior of the fiber there exists ([8]) a convex curve $\Gamma'$ which is tangent from above to the unit circle at 1 such that $\{z_n\}$ is (eventually) below $\Gamma'$.

We next construct a subsequence $\{w_n\}$ of the sequence $\{z_n\}$. Let $\varepsilon\in(0, 1)$ be such that $\{z_n\}$ is $\varepsilon$-separated. Given $z\in D$ define $\sigma(z)$ to be the $\varepsilon$-pseudo hyperbolic neighborhood of the radial segment in $D$ from $z$ to the unit circle. We assume that $\{z_n\}$ has been listed in non-decreasing order of modulus. Let

$\zeta_1=z_1$. Let $\zeta_2$ be the first point $z_n$ which does not belong to $\sigma(\zeta_1)$. Note that also $\zeta_1\notin\sigma(\zeta_2)$. Next, let $\zeta_3$ be the first point $z_n$ such that $\zeta_3\notin\sigma(\zeta_1)\cup\sigma(\zeta_2)$. This pattern continues inductively to produce a subsequence $\{\zeta_n\}$ of $\{z_n\}$ such that for each $n$, $z_n\in\sigma(\zeta_k)$ for some $k$ and such that for each $n$, $m$, $\zeta_n\notin\sigma(\zeta_m)$. We rearrange the sequence $\{\zeta_n\}$ in a non-increasing order of argument and obtain the desired subsequence $\{w_n\}$.

We now describe the construction of the curve $\Gamma$. For each $k$ let $w_k'$ denote the point of intersection of the radius through $w_k$ with $\Gamma'$. Beginning at $w_1'$ we traverse the radius toward $w_1$. Each time we arrive at a point of the same modulus as some $z_n$ in $\sigma(w_1)$ which lies to our right we traverse the segment to $z_n$ and back and then continue until we reach $w_1$. At this point we traverse the radius toward $w_1'$, adding excursions as before for points $z_n$ in $\sigma(w_1)$ to our right until we reach $w_1'$. From $w_1'$ we traverse $\Gamma'$ to $w_2'$. We traverse the radius from $w_2'$ to $w_2$ and back making excursions similar to those just described but only for points $z_n$ not already reached. Again we traverse $\Gamma'$ from $w_2'$ to $w_3'$. This scheme is continued and the resulting curve is the curve $\Gamma$ we seek.

It remains to show that the arclength of $\Gamma$ is a Carleson measure. Consider a region $S_{\theta,h}$ with $h$ sufficiently small. The part of $\Gamma$ in $S_{\theta,h}$ can be divided into three types. The first consists of the arcs of $\Gamma'$ in $S_{\theta,h}$. Because $\Gamma'$ is convex we see that their total length is no more than a constant times $h$. The second type consists of segments of radii in $S_{\theta,h}$ joining points $w_k'$ and $w_k$ for which $w_k\in S_{\theta,h}$ and, also, of the doubly traversed segment joining a point of such a radius to a point $z_n$ within a pseudo hyperbolic distance $\varepsilon$. For either of these (using Lemma 2.4 (b) for the latter) the total arclength added is no more than a constant times $1-|z_n|$. Using Garnett's theorem once more we see that the total contribution of this second type is no more than a constant times $h$. The third type consists of the portion in $S_{\theta,h}$ of segments of radii joining points $w_k'$ and $w_k$ for which $w_k\notin S_{\theta,h}$. By the construction of $\{w_n\}$ the points of intersection of these radii with the arc $|z|=1-h$, $|\arg z-\theta|<h/2$, form an $\varepsilon$-separated subset of this arc. Thus, there are at most a fixed number of these independent of $\theta$ and $h$. Each such segment has length no greater than $h$ and, thus, in total again contribute no more than a constant times $h$. From this we see that the arclength of $\Gamma$ is a Carleson measure and we are finished.

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