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Complexes and L-structures

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§ 0. Introduction.

The purpose of this paper is to study the simplicial complex K with Whitehead topology from the point of view of L-structures. It will be shown that the capacity of K to admit L-structures decreases as the dimension of K increases. As a consequence we know that there is a gap between the class of M_1-spaces and the class of weak L-spaces. Throughout the paper K is a simplicial complex with Whitehead topology and simplexes of K are so-called open ones. K^n denotes the n-section of K. As for terminology refer to the first author [3], [4] and [5].

§ 1. K with $\dim K\leqq 2$.

1.1. THEOREM. *If* $\dim K\leqq 1$, *then* K *is an* L-*space.*

PROOF. When $\dim K\leqq 0$, K is discrete and metrizable. Consider the case when $\dim K=1$. Let H be an arbitrary closed set of K. Let $\{s_\alpha : \alpha\in A\}$ be the set of 1-simplexes of K. Let U be an open set of K with

$$K^0-H\subset U\subset \bar{U}\subset K-H.$$

For each $\alpha\in A$, let $\mathcal{V}_\alpha$ be an approaching anti-cover of $(H\cap\bar{s}_\alpha)\cup\partial s_\alpha$ in $\bar{s}_\alpha$. Set

$$\mathcal{V}=(\cup\{\mathcal{V}_\alpha : \alpha\in A\})\cup\{U\}.$$

Then $\mathcal{V}$ is as can easily be seen an approaching anti-cover of H in K. That completes the proof.

1.2. THEOREM. *Let* K *be the* 2-*section of an infinite full complex. Then* K *is not an* L-*space.*

PROOF. Let s be a 1-simplex of K and $\{s_i : i=1, 2, \cdots\}$ a sequence of distinct 2-simplexes of K having s as their common face. Let p be an edge point of s and $\{p_i\}$ a sequence of points of s with $\lim p_i=p$. Let $\mathcal{V}$ be an arbitrary anti-cover of $\{p\}$. Choose $U_i\in\mathcal{V}$ with $p_i\in U_i$. Since $U_i\cap s_i\neq\emptyset$ for any i, we can pick a point $q_i\in U_i\cap s_i$ for each i. Set

$$Z=\{q_i : i=1, 2, \cdots\}.$$

Then Z is a closed set in K with $Z\cap\{p\}=\emptyset$. The inequalities

$$p\in \mathrm{Cl}\{p_i\}\subset \mathrm{Cl}\cup U_i\subset \mathrm{Cl}\,\mathcal{U}(Z)$$

show that $\mathcal{U}$ cannot be approaching to p. That completes the proof.

Since each Lašnev space is an L-space by the first author [**3**, Theorem 1.6], K in the above cannot be a Lašnev space. This fact answers Nagata [**6**, Problem 3] negatively.

§ 2. K with $\dim K<\infty$.

Before stating the next theorem, let us illustrate the lifting-up process. Let $\{s_\alpha: \alpha\in A_n\}$ be the set of n-simplexes of K. Let p_α be an arbitrary but fixed point of s_α. If $n\geqq 1$ and $\alpha\in A_n$, consider $\bar{s}_\alpha$ as an n-ball of radius 1 with the center p_α whose surface is $\partial s_\alpha=\bar{s}_\alpha-s_\alpha$. Let $s_\alpha(\varepsilon)$ be the open ball with the center p_α of radius ε, $0<\varepsilon<1$. Set

$$S_n(\varepsilon)=\cup\{s_\alpha(\varepsilon): \alpha\in A_n\}.$$

If G is a subset of ∂s_α, then $[G, p_\alpha]$ denotes the sum of all segments $\overline{p_\alpha q}$, $q\in G$. Set

$$G(\alpha, \varepsilon)=[G, p_\alpha]-\overline{s_\alpha(\varepsilon)},$$

$$G[\alpha, \varepsilon]=[G, p_\alpha]-s_\alpha(\varepsilon).$$

When G is a subset of K^{n-1}, set

$$G(n, \varepsilon)=\cup\{(G\cap\partial s_\alpha)(\alpha, \varepsilon): \alpha\in A_n\},$$

$$G[n, \varepsilon]=\cup\{(G\cap\partial s_\alpha)[\alpha, \varepsilon]: \alpha\in A_n\}.$$

When G is open or closed in K^{n-1}, G is lifted up to $G(n, \varepsilon)$ or $G[n, \varepsilon]$ which is open or closed in K^n respectively. Set

$$[G, p_\alpha)=[G, p_\alpha]-\{p_\alpha\},$$

$$G[n]=\cup\{[G\cap\partial s_\alpha, p_\alpha]: \alpha\in A_n\},$$

$$G(n)=\cup\{[G\cap\partial s_\alpha, p_\alpha): \alpha\in A_n\}.$$

When G is open or closed in K^{n-1}, $G(n)$ or $G[n]$ is open or closed in K^n respectively.

2.1. THEOREM. *If K is finite-dimensional, K is a free L-space.*

PROOF. Set $\dim K=m$. When $m\leqq 1$, K is an L-space by Theorem 1.1 and is of course a free L-space. Consider the case when $m\geqq 2$. For each $n\leqq m$ and each $\alpha\in A_n$ let $\mathcal{F}_\alpha$ be a countable network of s_α consisting of compact sets. Set

$$\mathfrak{F}_n=\cup\{\mathfrak{F}_\alpha: \alpha\in A_n\},$$

$$\mathfrak{F}=\cup\{\mathfrak{F}_n: n=0, 1, \cdots, m\}.$$

For each $F\in\mathfrak{F}_\alpha$, $\alpha\in A_n$, let $U_i(F)$, $i=1, 2, \cdots$, be open neighborhoods of F in $\bar{s}_\alpha$ such that

$$U_i(F)\supset \mathrm{Cl}\, U_{i+1}(F), \qquad i=1, 2, \cdots,$$

$$\mathrm{Cl}\, U_i(F)\cap\partial s_\alpha=\emptyset.$$

Let $\{a_i\}$ be a sequence with

$$1/4<a_1<a_2<\cdots, \lim a_i=1/3.$$

When $n<m$,

$$\{U_i(F)(n+1, a_i): i=1, 2, \cdots\}$$

forms a system of neighborhoods of $F[n+1, 1/3]$ in K^{n+1}. When $n+1<m$, $U_i(F)(n+1, a_i)$ is lifted up to $U_i(F)(n+1, a_i)(n+2, a_i)$ and $F[n+1, 1/3]$ is lifted up to $F[n+1, 1/3][n+2, 1/3]$. Continuing in this manner F is lifted up to

$$\hat{F}=F[n+1, 1/3][n+2, 1/3]\cdots[m, 1/3]$$

and $U_i(F)$ is lifted up to

$$U_i(\hat{F})=U_i(F)(n+1, a_i)(n+2, a_i)\cdots(m, a_i).$$

When $n=m$, set merely $\hat{F}=F$. Each $U_i(\hat{F})$ is an open set of K such that

$$\mathrm{Cl}\, U_{i+1}(\hat{F})\subset U_i(\hat{F}), \qquad i=1, 2, \cdots,$$

$$\hat{F}=\bigcap_{i=1}^{\infty} U_i(\hat{F}).$$

Set

$$\mathcal{V}_{\hat{F}}=\{K-\mathrm{Cl}\, U_2(\hat{F})\}\cup\{U_i(\hat{F})-\mathrm{Cl}\, U_{i+2}(\hat{F}): i=1, 2, \cdots\}.$$

Then $\mathcal{V}_{\hat{F}}$ is an anti-cover of $\hat{F}$ with respect to which $U_i(\hat{F})$ is canonical for $i=1, 2, \cdots$. Set

$$\hat{\mathfrak{F}}_n=\{\hat{F}: F\in\mathfrak{F}_n\},$$

$$\hat{\mathfrak{F}}=\{\hat{F}: F\in\mathfrak{F}\}.$$

Since each $\mathfrak{F}_n$ is σ-discrete in K, $\hat{\mathfrak{F}}_n$ is also σ-discrete in K and hence $\hat{\mathfrak{F}}$ is σ-discrete in K.

Let $\{b_i\}$ be a sequence with

$$b_1=1/3<b_2=1/2<b_3<\cdots, \lim b_i=1$$

and $\{c_i\}$ a sequence with

$$c_1=1/2>c_2=1/3>\cdots, \lim c_i=0.$$

Set

$$\mathcal{S}_j=\{S_j(c_i)-\mathrm{Cl}\,S_j(c_{i+2}):\ i=1,2,\cdots\}, \qquad j=1,\cdots,m.$$

Set

$$\mathcal{W}_\alpha=\{s_\alpha(b_2)\}\cup\{s_\alpha(b_{i+2})-\mathrm{Cl}\,s_\alpha(b_i):\ i=1,2,\cdots\},\ \alpha\in A_{n+1},$$

$$\mathcal{W}_{n+1}=\cup\{\mathcal{W}_\alpha:\ \alpha\in A_{n+1}\}.$$

If $n+2\leqq m$, set

$$\mathcal{W}_\alpha(n+2,1/3)=\{W(n+2,1/3):\ W\in\mathcal{W}_\alpha\},\ \alpha\in A_{n+1},$$

$$\mathcal{W}_\alpha(n+2)=\{W(n+2):\ W\in\mathcal{W}_\alpha\},\ \alpha\in A_{n+1}.$$

These types of abbreviations are used throughout in the following. If $n+2\leqq m$, set

$$\mathcal{T}_\alpha\{n+2\}=\mathcal{W}_\alpha(n+2)\wedge\mathcal{S}_{n+2},\quad \alpha\in A_{n+1},$$

$$\mathcal{U}_\alpha\{n+2\}=\mathcal{W}_\alpha(n+2,1/3),\quad \alpha\in A_{n+1},$$

$$\mathcal{W}_\alpha\{n+2\}=\mathcal{U}_\alpha\{n+2\}\cup\mathcal{T}_\alpha\{n+2\},\quad \alpha\in A_{n+1},$$

$$\mathcal{T}_{n+1}\{n+2\}=\cup\{\mathcal{T}_\alpha\{n+2\}:\ \alpha\in A_{n+1}\},$$

$$\mathcal{U}_{n+1}\{n+2\}=\cup\{\mathcal{U}_\alpha\{n+2\}:\ \alpha\in A_{n+1}\},$$

$$\mathcal{W}_{n+1}\{n+2\}=\cup\{\mathcal{W}_\alpha\{n+2\}:\ \alpha\in A_{n+1}\}.$$

If moreover $n+3\leqq m$, set further

$$\mathcal{T}_\alpha\{n+3\}=\mathcal{W}_\alpha\{n+2\}(n+3)\wedge\mathcal{S}_{n+3},\quad \alpha\in A_{n+1},$$

$$\mathcal{U}_\alpha\{n+3\}=\mathcal{W}_\alpha\{n+2\}(n+3,1/3),\quad \alpha\in A_{n+1},$$

$$\mathcal{W}_\alpha\{n+3\}=\mathcal{U}_\alpha\{n+3\}\cup\mathcal{T}_\alpha\{n+3\},\quad \alpha\in A_{n+1},$$

$$\mathcal{T}_{n+1}\{n+3\}=\cup\{\mathcal{T}_\alpha\{n+3\}:\ \alpha\in A_{n+1}\},$$

$$\mathcal{U}_{n+1}\{n+3\}=\cup\{\mathcal{U}_\alpha\{n+3\}:\ \alpha\in A_{n+1}\},$$

$$\mathcal{W}_{n+1}\{n+3\}=\cup\{\mathcal{W}_\alpha\{n+3\}:\ \alpha\in A_{n+1}\}.$$

Continuing in this manner we get at last

$$\mathcal{T}_\alpha\{m\}=\mathcal{W}_\alpha\{m-1\}(m)\wedge\mathcal{S}_m,\quad \alpha\in A_{n+1},$$

$$\mathcal{U}_\alpha\{m\}=\mathcal{W}_\alpha\{m-1\}(m,1/3),\quad \alpha\in A_{n+1},$$

$$\mathcal{D}_\alpha=\mathcal{W}_\alpha\{m\}=\mathcal{U}_\alpha\{m\}\cup\mathcal{T}_\alpha\{m\},\quad \alpha\in A_{n+1},$$

$$\mathcal{T}_{n+1}\{m\}=\cup\{\mathcal{T}_\alpha\{m\}:\ \alpha\in A_{n+1}\},$$

$$\mathcal{U}_{n+1}\{m\}=\cup\{\mathcal{U}_\alpha\{m\} : \alpha\in A_{n+1}\},$$

$$\mathcal{D}_n=\mathcal{W}_{n+1}\{m\}=\cup\{\mathcal{W}_\alpha\{m\} : \alpha\in A_{n+1}\}.$$

Then $\mathcal{D}_n$ is an open collection of K.

Let α and β be distinct elements of A_{n+1}. Since $\mathcal{W}_\alpha^\#=s_\alpha$, then $\mathcal{W}_\alpha^\#\cap\mathcal{W}_\beta^\#=s_\alpha\cap s_\beta=\emptyset$. Since $\mathcal{W}_\alpha\{n+2\}^\#=\mathcal{W}_\alpha(n+2)^\#=s_\alpha(n+2)$ and $s_\alpha(n+2)\cap s_\beta(n+2)=\emptyset$, then $\mathcal{W}_\alpha\{n+2\}^\#\cap\mathcal{W}_\beta\{n+2\}^\#=\emptyset$. At last $\mathcal{D}_\alpha^\#\cap\mathcal{D}_\beta^\#=\emptyset$. Set

$$\widetilde{K^n}=K^n[n+2][n+3]\cdots[m], \qquad n\leqq m-2,$$
$$\widetilde{K^{m-1}}=K^{m-1}.$$

Since

$$K^{n+2}=S_{n+1}(n+2)\cup K^n[n+2],$$
$$\cup\{\mathcal{W}_\alpha\{n+2\}^\# : \alpha\in A_{n+1}\}=S_{n+1}(n+2),$$

then $\mathcal{W}_{n+1}\{n+2\}=\cup\{\mathcal{W}_\alpha\{n+2\} : \alpha\in A_{n+1}\}$ is an anti-cover of $K^n[n+2]$ in K^{n+2}. This fact implies in turn that $\mathcal{W}_{n+1}\{n+3\}$ is an anti-cover of $K^n[n+2][n+3]$. At last we know that $\mathcal{D}_n$ is an anti-cover of $\widetilde{K^n}$.

Set

$$\mathcal{L}=\hat{\mathfrak{F}}\cup\left(\bigcup_{n=0}^{m-1}\widetilde{K^n}\right).$$

Then $\mathcal{L}$ is a σ-discrete closed cover of K. To prove that $\mathcal{L}$ with the anti-covers $\mathcal{V}_{\hat{F}}$ $(\hat{F}\in\hat{\mathfrak{F}})$ and $\mathcal{D}_n$ $(n=0, 1, \cdots, m-1)$ forms a free L-structure of K, let x be an arbitrary point of K and E an arbitrary open neighborhood of x in K. Let n be the number with $x\in K^n-K^{n-1}$. Let $\gamma\in A_n$ be the index with $x\in s_\gamma$ and F an element of $\mathfrak{F}_\gamma$ with $x\in F\subset E$. Let k be a number with

$$F\subset U_k(F)\subset \mathrm{Cl}\, U_k(F)\subset E.$$

When $n=m$, $\hat{F}=F$, $U_k(\hat{F})=U_k(F)$ and $U_k(\hat{F})$ is canonical.

Consider the case when $n<m$. Set

$$B_i=\{\alpha\in A_i : s_\alpha>s_\gamma\}, \qquad i=n+1, \cdots, m.$$

Then there exists a function $\varepsilon_{n+1} : B_{n+1}\to(1/2, 1)$ such that

$$(\mathrm{Cl}\, U_k(F))[n+1, \varepsilon_{n+1}(\alpha_{n+1})]\subset E, \ \alpha_{n+1}\in B_{n+1}.$$

When $n+1<m$, there exists one more function $\varepsilon_{n+2} : B_{n+2}\to(1/2, 1)$ such that

$$(\mathrm{Cl}\, U_k(F))[n+1, \varepsilon_{n+1}(\alpha_{n+1})][n+2, \varepsilon_{n+2}(\alpha_{n+2})]\subset E,$$
$$\alpha_{n+1}\in B_{n+1}, \ \alpha_{n+2}\in B_{n+2}.$$

Continuing this manner we obtain a sequence of functions:

$$\varepsilon_i:\ B_i \to (1/2,\ 1)\,, \qquad i=n+1,\ \cdots,\ m\,,$$

satisfying the condition:

$$(\mathrm{Cl}\, U_k(F))[n+1,\ \varepsilon_{n+1}(\alpha_{n+1})]\cdots[m,\ \varepsilon_m(\alpha_m)]\subset E\,,$$

$$\alpha_i \in B_i\ (i=n+1,\ \cdots,\ m)\,.$$

Set

$$C_j=\cup\{U_k(F)(n+1,\ \varepsilon_{n+1}(\alpha_{n+1}))\cdots(j,\ \varepsilon_j(\alpha_j)):\ \alpha_i\in B_i\ (i=n+1,\ \cdots,\ j)\}\,,$$

$$j=n+1,\ \cdots,\ m\,.$$

Then $C_m \subset E$.

For each j with $n+1\leqq j\leqq m$ and for each $\alpha\in B_j$ let $t(\alpha)$ be a number with

$$\varepsilon_j(\alpha)<b_{t(\alpha)}<1\,.$$

Set

$$P_j=\cup\{\mathrm{Cl}\, s_\alpha(b_{t(\alpha)}):\ \alpha\in B_j\}\,, \qquad j=n+1,\ \cdots,\ m\,,$$

$$\hat{P}_j=P_j[j+1,\ 1/4]\cdots[m,\ 1/4]\,, \qquad j=n+1,\ \cdots,\ m\,,$$

$$P_{jh}=\cup\{\mathrm{Cl}\, s_\alpha(b_{t(\alpha)+h}):\ \alpha\in B_j\}\,, \qquad j=n+1,\ \cdots,\ m,\ h=1,\ 2,\ \cdots\,,$$

$$Q_{jh}=\cup\{s_\alpha(b_{t(\alpha)+h}):\ \alpha\in B_j\}\,, \qquad j=n+1,\ \cdots,\ m,\ h=1,\ 2,\ \cdots\,.$$

Then $K-\hat{P}_j$ is a canonical neighborhood of $\widetilde{K^{j-1}}$ with respect to $\mathcal{D}_{j-1}$ for $j=n+1,\ \cdots,\ m$. We merely prove that $K-\hat{P}_{n+1}$ is a canonical neighborhood of $\widetilde{K^n}$ with respect to $\mathcal{D}_n$, since the rest is simple analogy. Set

$$\varphi(D)=\min\{i:\ D\cap K^i\neq\emptyset\}\,, \qquad D\in\mathcal{D}_n\,,$$

$$\mathcal{D}_{ni}=\{D\in\mathcal{D}_n:\ \varphi(D)\leqq i,\ D\cap\hat{P}_{n+1}\neq\emptyset\}\,, \qquad n+1\leqq i\leqq m\,.$$

Then

$$\mathcal{D}_{n,\,n+1}{}^{\#}=Q_{n+1,\,1}(n+2,\ 1/3)\cdots(m,\ 1/3)\,,$$

$$\mathcal{D}_{n,\,n+2}{}^{\#}=Q_{n+1,\,1}(n+2,\ c_3)(n+3,\ 1/3)\cdots(m,\ 1/3)\,,$$

....

$$\mathcal{D}_{nm}{}^{\#}=Q_{n+1,\,1}(n+2,\ c_3)\cdots(m,\ c_3)\,.$$

Since

$$\{D\in\mathcal{D}_n:\ D\cap\hat{P}_{n+1}\neq\emptyset\}=\mathcal{D}_{nm}\,,$$

$$\mathrm{Cl}(Q_{n+1,\,1}(n+2,\ c_3)\cdots(m,\ c_3))=P_{n+1,\,1}[n+2,\ c_3]\cdots[m,\ c_3]\,,$$

then

$$\mathrm{Cl}\,\mathcal{D}_n(K-\hat{P}_{n+1})=P_{n+1,\,1}[n+2,\ c_3]\cdots[m,\ c_3]\,.$$

Analogously we obtain, for an arbitrary positive integer r, the following equality:

$$\mathrm{Cl}\,\mathcal{D}_n^r(K-\hat{P}_{n+1})=P_{n+1,\,r}[n+2,\ c_{r+2}]\cdots[m,\ c_{r+2}]\,.$$

Since

$$P_{n+1,r}[n+2,\ c_{r+2}]\cdots[m,\ c_{r+2}]\cap\widetilde{K^n}=\emptyset\,,$$

then $K-\hat{P}_{n+1}$ is a canonical neighborhood of $\widetilde{K^n}$ with respect to $\mathfrak{D}^n$.

Last let us prove the inequality:

$$U_k(\hat{F})\cap\Big(\bigcap_{j=n+1}^{m}(K-\hat{P}_j)\Big)\subset E\,.$$

Since $\varepsilon_{n+1}(\alpha)>a_k>1/4$ $(\alpha\in B_{n+1})$, then

$$\begin{aligned}U_k(\hat{F})-\hat{P}_{n+1}&=U_k(F)(n+1,\ a_k)\cdots(m,\ a_k)-P_{n+1}[n+2,\ 1/4]\cdots[m,\ 1/4]\\&=(U_k(F)(n+1,\ a_k)-P_{n+1})(n+2,\ a_k)\cdots(m,\ a_k)\\&\subset C_{n+1}(n+2,\ a_k)\cdots(m,\ a_k)\,.\end{aligned}$$

Hence

$$\begin{aligned}U_k(\hat{F})-\hat{P}_{n+1}\cup\hat{P}_{n+2}&=(U_k(\hat{F})-\hat{P}_{n+1})-\hat{P}_{n+2}\\&\subset C_{n+1}(n+2,\ a_k)\cdots(m,\ a_k)-P_{n+2}[n+3,\ 1/4]\cdots[m,\ 1/4]\\&=(C_{n+1}(n+2,\ a_k)-P_{n+2})(n+3,\ a_k)\cdots(m,\ a_k)\\&\subset C_{n+2}(n+3,\ a_k)\cdots(m,\ a_k)\,.\end{aligned}$$

Continuing in this manner we get at last

$$U_k(\hat{F})-\sum_{j=n+1}^{m}\hat{P}_j\subset C_m\subset E\,.$$

Evidently

$$x\in\hat{F}\cap\Big(\bigcap_{j=n}^{m-1}\widetilde{K^j}\Big)\,.$$

Thus K is a free L-space and the proof is completed.

2.2. COROLLARY. *If X is a CW-complex with* $\dim X<\infty$, *then X is a free L-space.*

PROOF. Let D be an n-cell of X and $f:\bar{s}\to D$ a characteristic map of an n-simplex $\bar{s}$ to D (cf. Whitehead [8], p. 221). For a subset G of ∂D we can define for instance $G(n,\ \varepsilon)$ by:

$$G(n,\ \varepsilon)=f(f^{-1}(G)(n,\ \varepsilon))\,.$$

Thus lifting-up process can be applied for X and we can verify that X is a free L-space by analogous manner to the above theorem. That completes the proof.

§ 3. K **with** $\dim K=\infty$.

3.1. THEOREM. *If K is a full complex spanned by a countably infinite number of vertexes, then K is not a free L-space.*

PROOF. Assume that K admits a weak L-structure ($\mathfrak{F}$, $\mathcal{U}_F$ ($F\in\mathfrak{F}$)). Let p be an arbitrary vertex of K. Let $\{F_1, F_2, \cdots\}$ be the set of elements of $\mathfrak{F}$ which contain p. Set

$$\mathfrak{F}_n=\{F_1, \cdots, F_n\}.$$

Let $\{s_\alpha: \alpha\in A\}$ be the set of all simplexes of K with $p\in\bar{s}_\alpha$. Let π be the property for subsets of K as follows:

A subset T of K is said to have the property π if $T\cap\bar{s}_\alpha$ is a neighborhood of p for each $\alpha\in A$.

Let $\mathcal{H}_n$ be one of the maximal subcollections of $\mathfrak{F}_n$ such that $\mathcal{H}_n^\#$ does not have π. Set

$$H_n=\mathcal{H}_n^\#, \quad \mathcal{G}_n=\mathfrak{F}_n-\mathcal{H}_n,$$

$$V_n=\cap\{H_n\cup F: F\in\mathcal{G}_n\},$$

$$V_{n\alpha}=\text{Interior of } V_n\cap\bar{s}_\alpha \text{ in } \bar{s}_\alpha, \quad \alpha\in A.$$

Since $H_n\cup F$ has π for each $F\in\mathcal{G}_n$, V_n has π too. Hence $V_{n\alpha}$ is an open neighborhood of p in $\bar{s}_\alpha$ for each $\alpha\in A$. Let β be an index of A such that $H_n\cap\bar{s}_\beta$ is not a neighborhood of p in $\bar{s}_\beta$. Let $\{p_i\}$ be a sequence of distinct points of $\bar{s}_\beta-H_n$ with $\lim p_i=p$. For each i and each $F\in H_n$ there exists an element $U(i, F)\in\mathcal{U}_F$ with $p_i\in U(i, F)$. Set

$$U(i)=\cap\{U(i, F): F\in\mathcal{H}_n\}.$$

Then $U(i)$ is an open neighborhood of p_i. Let $\{\alpha_1, \alpha_2, \cdots\}$ be a sequence of indexes of A such that

$$s_\beta<s_{\alpha_i}, \qquad i=1, 2, \cdots,$$

$$n<\dim s_{\alpha_1}<\dim s_{\alpha_2}<\cdots.$$

Let $k(i)$ be a number such that

$$p_j\in V_{n\alpha_i}, \qquad j\geqq k(i),$$

$$k(1)<k(2)<\cdots.$$

Since $V_{n\alpha_i}\cap U(k(i))$ is an open neighborhood of $p_{k(i)}$ in $\bar{s}_{\alpha_i}$, then $s_{\alpha_i}\cap V_{n\alpha_i}\cap U(k(i))\neq\emptyset$. Let $Z_n=\{q_i\}$ be a sequence of points such that

$$q_i\in s_{\alpha_i}\cap V_{n\alpha_i}\cap U(k(i)), \qquad i=1, 2, \cdots.$$

Then $K-Z_n$ is an open neighborhood of p.

For each $F\in\mathcal{F}_n$ let $W(F)$ be a semi-canonical neighborhood of F. Set

$$W=\cap\{W(F): F\in\mathcal{F}_n\}.$$

We shall show that W cannot be contained in $K-Z_n$. Since $\{p_{k(i)}, q_i\}\subset U(k(i))$ and $U(k(i))$ refines $\mathcal{V}_F$ for each $F\in\mathcal{H}_n$, then $|Z_n-W(F)|<\infty$ for each $F\in\mathcal{H}_n$ by the same reason as in the proof of Theorem 1.2. Thus there exists a number m with

$$q_m\in\cap\{W(F): F\in\mathcal{H}_n\}.$$

Since $q_m\in V_{n\alpha_m}\subset V_n$ and

$$q_m\in U(k(m))=\cap\{U(k(m), F): F\in\mathcal{H}_n\}\subset\cap\{K-F: F\in\mathcal{H}_n\}$$
$$=K-\cup\{F: F\in\mathcal{H}_n\}=K-H_n,$$

then

$$q_m\in V_n-H_n=\cap\{H_n\cup F: F\in\mathcal{G}_n\}-H_n$$
$$=\cap\{F: F\in\mathcal{G}_n\}.$$

Thus

$$q_m\in\cap\{W(F): F\in\mathcal{G}_n\}$$

and hence

$$q_m\in\cap\{W(F): F\in\mathcal{H}_n\cup\mathcal{G}_n=\mathcal{F}_n\}=W.$$

W meets therefore with Z_n.

Set

$$Z=\cup\{Z_n: n=1, 2, \cdots\}.$$

Then Z is as can easily be seen closed in K and $K-Z$ is an open neighborhood of p. Let $\mathcal{F}'$ be an arbitrary finite subcollection of $\mathcal{F}$ and $W(F)$ be a semi-canonical neighborhood of F for each $F\in\mathcal{F}'$. Let t be a number with $\mathcal{F}'\subset\mathcal{F}_t$. Then as is shown above $\cap\{W(F): F\in\mathcal{F}'\}$ meets Z_t and $(\mathcal{F}, \mathcal{V}_F\,(F\in\mathcal{F}))$ cannot be a weak L-structure of K. That completes the proof.

Each K is an M_1-space by Ceder [**1**, Corollary 8.6]. Thus we know by this theorem that there is a gap between the class of M_1-spaces and the class of weak L-spaces.

3.2. COROLLARY. *There exists a countable M_1-space which is not a weak L-space but is metrizable except a singleton.*

PROOF. Let T be a countable subset of K in the above theorem such that $T\cap s$ is dense in s for each simplex s of K. Weaken the Whitehead topology of $T-\{p\}$ to a usual metric topology, while leave the neighborhood system of p unchanged. That T with this topology is the desired is verified by analogous argument to the above. That completes the proof.

This corollary should be compared to Gruenhage [**2**, Theorems 2 and 3]: An M_3-space which is countable or metrizable except a singleton is an M_1-space.

Let X_i, $i=1, 2, \cdots$, be metric spaces, BX_i the box product of them and p a point of BX_i. Let Ξ_p be the set of points in BX_i all but a finite number of whose coordinates are equal to those of p. San-ou [7, Corollary 3.3] proved that Ξ_p is an M_1-space. It is to be noted that, by a similar argument to Theorem 3.1, Ξ_p is not necessarily a weak L-space.

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