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On expansive homeomorphisms on manifolds

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1. Introduction.

X will be a metric space with a metric d. A homeomorphism f of X onto itself is expansive if there exists a positive number C (called expansive constant) such that for each pair (x, y) of distinct points of X, there is an integer n for which $d(f^n(x), f^n(y))>C$.

There is a question what manifolds admit such homeomorphisms. Several examples of existence and non-existence of expansive homeomorphisms are known. An open interval, a 1-sphere and a closed 2-disk do not admit expansive homeomorphisms (Bryant [**1**], Jakobsen and Utz [**2**]). An open $2n$-ball ($n\geqq 1$) and an r-dimensional torus ($r\geqq 2$) admit expansive homeomorphisms (Reddy [**3**]). In this paper, we prove the followings.

THEOREM 1. *Let M be a closed n-manifold ($n\geqq 1$), and J be an open interval. Then there exists an expansive homeomorphism of $M\times J$.*

THEOREM 2. *If M is a closed n-manifold ($n\geqq 1$), there exist an expansive homeomorphism of* $\mathrm{Int}(M^*\{point\})$. *Where P^*Q is the join of P and Q, and* $\mathrm{Int}\, M$ *is the interior of M.*

COROLLARY. *There exists an expansive homeomorphism of an open n-ball* ($n\geqq 2$).

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2. Proof of Theorem 1.

Let M be a closed n-manifold with a metric d. $J=(0, 2)$ and R^n be an open interval with a standard metric d_1 and an n-dimensional Euclidean space with a standard metric d_n, respectively. And put $U(x, \varepsilon)=\{y\in M \mid d(y, x)<\varepsilon\}$, $U_n(z, \delta)=\{y\in R^n \mid d_n(y, z)<\delta\}$. We define the metric ρ of $M\times J$ to be $d\times d_1$ (where $d\times d_1((x, t), (y, s))=d(x, y)+d_1(t, s)$ and $x, y\in M$ and $t, s\in J$), and I_k ($k\geqq 0$) to be $I_k=\left[\frac{1}{k+1}, \frac{1}{k}\right]$ ($k\in N$) and $I_0=[1, 2)$. Put $A_k=M\times I_k$.

First, we define several homeomorphisms of A_1. We will use these homeomorphisms for constructing an expansive homeomorphism of $M\times J$. For any element x of M, there is a neighborhood W_x which is homeomorphic to R^n. α_x is the homeomorphism from W_x to R^n. There is a positive number ε_x such that $cl(U(x, 3\varepsilon_x))\subset W_x$. Where $cl(Y)$ is the closure of Y. For this ε_x, there exist positive numbers ζ and ξ such that $U_n(\alpha_x(x), \zeta)\subset\alpha_x(U(x, \varepsilon_x))$ and $U_n(\alpha_x(x), \xi)\supset\alpha_x(U(x, 3\varepsilon_x))$. We denote $V_x=\alpha_x^{-1}(U_n(\alpha_x(x), \zeta)$ and $U_x=\alpha_x^{-1}(U_n(\alpha_x(x), \xi))$. Since $\{V_x\}$ is an open covering of M, we can choose a finite covering $\{V_{x_1}, V_{x_2}, \cdots, V_{x_m}\}$. We put $V_j=V_{x_j}$, $U_j=U_{x_j}$, $W_j=W_{x_j}$ and $\alpha_j=\alpha_{x_j}$.

Now, for any non-negative integer k, we define a finite open covering of M as follows. For any element x of M, there is some V_i such that $x\in V_i$. There are positive numbers s, t $(s<t)$ such that $\alpha_i(x)\in U_n(\alpha_i(x), s)\subsetneqq U_n(\alpha_i(x), t)\subset\alpha_i\Big(V_i\cap U\Big(x, \dfrac{1}{k+1}\Big)\Big)$. Put $O_x=\alpha_i^{-1}(U_n(\alpha_i(x), s))$ and $\tilde{O}_x=\alpha_i^{-1}(U_n(\alpha_i(x), t))$. Since $\{O_x\}$ is an open covering, we can choose a finite covering $\{O_{k,j}\}$ $1\leqq j\leqq\sigma(k)$ for some integer $\sigma(k)$. Put $\tilde{O}_{k,j}=\tilde{O}_x$ if $O_{k,j}=O_x$.

For each $O_{k,j}$, there is V_i that $\tilde{O}_{k,j}\subset V_i$. We fix one of them, namely V_p. Then we can define a homeomorphism, $\tilde{f}_{k,j}$, of M satisfying the following conditions,

1) $\tilde{f}_{k,j}|_{O_{k,j}}=$identity,

2) $\tilde{f}_{k,j}(\tilde{O}_{k,j})=U_p$,

3) $\tilde{f}_{k,j}|_{M-W_p}=$identity,

4) $\tilde{f}_{k,j}$ is isotopic to identity.

Now, we define a homeomorphism, $f_{k,j}$, of A_1 $(M\times I_1)$ that $f_{k,j}(x, t)=(\tilde{f}_{k,j}(x), t)$ $(x\in M, t\in I_1)$. For simplicity, we change the double suffix to single suffix. Put $f_i=f_{k,j}$ where $i=\sum_{q=0}^{k-1}\sigma(q)+j$. For each f_i, we define several homeomorphisms of A_1, f_i^-, f_i^0, f_i^+, f_i^{++}, as follows.

a) $f_i^-|_{M\times\{1\}}=$identity, $f_i^-|_{M\times\{1/2\}}=f_i|_{M\times\{1/2\}}$ and f_i^- is isotopic to identity.

b) $f_i^0=f_i\circ(f_i^-)^{-1}$.

c) $f_i^+\circ f_i^0\circ f_i^-|_{M\times\{1/2\}}=$identity, $f_i^+\circ f_i^0\circ f_i^-|_{M\times\{1\}}=f_i|_{M\times\{1\}}$ and f_i^+ is isotopic to identity.

d) $f_i^{++}=(f_i^-)^{-1}\circ(f_i^0)^{-1}\circ(f_i^+)^{-1}$.

Next we define homeomorphisms Z_j^i $(j\in \boldsymbol{N}\cup\{0\},\ 1\leqq i\leqq m)$. Put $D_1=\{x\in\boldsymbol{R}^n\,|\,d_n(x, 0)\leqq1\}$ and $D_2=\{x\in\boldsymbol{R}^n\,|\,d_n(x, 0)\leqq3\}$. For each integer j, we define a function $h_j(t)$ from $\Big[\dfrac{1}{2}, 1\Big]$ to R as follows, when $\dfrac{1}{2}+\dfrac{L-1}{8(j+1)}\leqq t\leqq\dfrac{1}{2}+\dfrac{L}{8(j+1)}$ (where L is a integer and $1\leqq L\leqq4(j+1)$)

$$h_j(t)=\begin{cases} 8(j+1)\Big(t-\Big(\frac{1}{2}+\frac{L-1}{8(j+1)}\Big)\Big) & \text{if } L\equiv 1 \pmod{4} \\ -8(j+1)\Big(t-\Big(\frac{1}{2}+\frac{L}{8(j+1)}\Big)\Big) & \text{if } L\equiv 2 \pmod{4} \\ -8(j+1)\Big(t-\Big(\frac{1}{2}+\frac{L-1}{8(j+1)}\Big)\Big) & \text{if } L\equiv 3 \pmod{4} \\ 8(j+1)\Big(t-\Big(\frac{1}{2}+\frac{L}{8(j+1)}\Big)\Big) & \text{if } L\equiv 0 \pmod{4}. \end{cases}$$

Let $Z_j: D_3\times I_1\rightarrow D_3$ be a function satisfying the following conditions. For $(x, t)=(x_1, x_2, \cdots, x_n, t)\in D_1\times I_1$, $Z_j(x, t)=(x_1+h_j(t), x_2, \cdots, x_n)$ and $Z_j|_{\partial(D_3\times I_1)}=$ identity and Z_j is homotopic to the projection from $D_3\times I_1$ to D_3. For each i $(1\leqq i\leqq m)$, there is a homeomorphism β_i from U_i to $\operatorname{Int} D_3$ and $\beta_i(V_i)=\operatorname{Int} D_1$. Where we can choose β_i satisfying that there exists a positive number δ_i for each $x, y\in U_i$ for which $\delta_i d_n(\beta_i(x), \beta_i(y))\leqq d(x, y)$. We define a homeomorphisms Z_j^i of A_1 $(1\leqq i\leqq m,\ j\in N\cup\{0\})$ as follows,

$$Z_j^i(x, t)=\begin{cases} (\beta_i^{-1}Z_j(\beta_i(x), t), t) & \text{if } x\in U_i \\ (x, t) & \text{if } x\notin U_i. \end{cases}$$

Let g_k be a homeomorphisms from A_1 to A_k (where k is positive integer) such that $g_k(x, t)=\Big(x, \frac{2t}{k(k+1)}+\frac{k-1}{k(k+1)}\Big)$. Then, we define a homeomorphism f of $M\times J$ by the following.

$$f|_{A_0}: A_0\longrightarrow A_0\cup A_1: (x, t)\longrightarrow\Big(x, \frac{3}{2}(t-2)+2\Big)$$

$$f|_{A_{jN+1}}=g_{jN+2}\circ f_j^-\circ(g_{jN+1})^{-1}$$

$$f|_{A_{jN+2}}=g_{jN+3}\circ f_j^0\circ(g_{jN+2})^{-1}$$

$$f|_{A_{jN+3}}=g_{jN+4}\circ f_j^+\circ(g_{jN+3})^{-1}$$

$$f|_{A_{jN+4}}=g_{jN+5}\circ f_j^{++}\circ(g_{jN+4})^{-1}$$

$$f|_{A_{jN+5}}=g_{jN+6}\circ(Z_j^1)\circ(g_{jN+5})^{-1}$$

$$f|_{A_{jN+6}}=g_{jN+7}\circ(Z_j^1)^{-1}\circ(g_{jN+6})^{-1}$$

$$\vdots$$

$$f|_{A_{jN+N-1}}=g_{jN+N}\circ Z_j^m\circ(g_{jN+N-1})^{-1}$$

$$f|_{A_{jN+N}}=g_{jN+N+1}\circ(Z_j^m)^{-1}\circ(g_{jN+N})^{-1}$$

where $N=4+2m$ and j is non-negative integer. Since $f|_{A_k}$ agrees $f|_{A_{k+1}}$ on $A_k\cap A_{k+1}$, f can be well defined on $M\times J$.

Now, we will show that this homeomorphism f is expansive. First, we define an expansive constant C. It is easy to check that there exists a positive constant C_0 such that for each t, s of I_1 and x of $\mathrm{Int}\, D_1$, there is an integer j for which $d_n(Z_j(x, t), Z_j(x, s)) > C_0$. We put $C = \frac{1}{2}\min\{\varepsilon_{x_1}, \varepsilon_{x_2}, \cdots, \varepsilon_{x_m}, \frac{1}{12}, \min(\delta_1, \cdots, \delta_m) \times C_0\}$.

We will show that each pair (x, t) and (y, s) of $M \times J$ (where $(x, t) \neq (y, s)$), there exists an integer L for which $\rho(f^L(x, t), f^L(y, s)) > C$. To do this, we need the following assertion.

ASSERTION.

(1) For each pair (x, y) of distinct points of M, there is some $O_{k,j}$ such that $x \in O_{k,i}$ and $y \notin \tilde{O}_{k,j}$.

(2) $f^{jN+2}(x, t) = g_{jN+3} f_j(x, t)$ for $(x, t) \in A_1$.

(3) $f^{jN+2i}(x, t) = g_{jN+2i+1}(x, t)$ for $(x, t) \in A_1$ $(2 \leqq i \leqq m+2)$.

(4) $f^{jN+2(i+1)+1}(x, t) = g_{jN+2(i+1)+2} Z_j^i(x, t)$ for $(x, t) \in A_1$ $(1 \leqq i \leqq m)$.

(5) $\tilde{\rho}(V_p \times I_1, U_p^c \times I_1) > C$, where $\tilde{\rho}(Y, Z) = \min\{\rho(y, z) \mid y \in Y, z \in Z\}$ and A^c is the complement of A in M $(1 \leqq p \leqq m)$.

(6) For any integers L and L', $\rho(g_L(x, t), g_{L'}(y, s)) \geqq d(x, y)$ (where $(x, t), (y, s) \in A_1$).

(2)-(6) are clear. We will show only (1). For $O_{k,j}$, there is $z \in M$ that $\tilde{O}_{k,j} \subset U\left(z, \frac{1}{k}\right)$ by definition. If both x and y contained in $\tilde{O}_{k,j}$, $d(x, y) \leqq d(x, z) + d(z, y) < \frac{1}{k} + \frac{1}{k} = \frac{2}{k}$. Since $\{O_{k,j}\}$ is an open covering of M, there is $O_{k,j}$ that $x \in O_{k,j}$ for any k. Specially, we choose k that greater than $\frac{2}{d(x, y)}$. Then, if $\tilde{O}_{k,j}$ which contains x contains y, $d(x, y) < \frac{2}{k} < d(x, y)$. This is a contradiction. (1) is established.

For any element $(x, t) \in M \times J$, there is some integer q that $f^q(x, t) \in A_1$. We may assume $(x, t) \in A_1$. And if (y, s) is contained in A_0, there is a positive integer r that $f^r(y, s) \in A_1$. Then $f^r(x, t) \in A_{r+1}$. Thus, we may assume $(y, s) \in A_k$ $(k \geqq 1)$.

Case 1; $(y, s) \in A_k$ $(k \geqq 3)$

$$\rho((x, t), (y, s)) \geqq \tilde{\rho}(A_1, A_k) = \frac{1}{6} > \frac{1}{12} \geqq C.$$

Case 2; $(y, s) \in A_1$

First, we prove the case $x \neq y$. By assertion (1), there is some $O_{k,j}$ such that $x \in O_{k,j}$ and $y \notin \tilde{O}_{k,j}$. We put $K = \sum_{q=0}^{k-1} \sigma(q) + j$. By assertion (2),

$$f^{NK+2}(x,\ t)=g_{NK+3}f_K(x,\ t)$$

$$f^{NK+2}(y,\ s)=g_{NK+3}f_K(y,\ s)\,.$$

Then, $f_K(x, t)$ is contained in $O_{k,j}\times I_1\subset V_p\times I_1$ and $f_K(y, s)$ is not contained in $U_p\times I_1$ (for some p). Thus,

$$\begin{aligned}&\rho(f^{KN+2}(x,\ t),\ f^{KN+2}(y,\ s))\\&\quad=\rho(g_{NK+3}f_K(x,\ t),\ g_{NK+3}f_K(y,\ s))\\&\quad\geqq\tilde{\rho}(g_{NK+3}(V_p\times I_1),\ g_{NK+3}(U_p^c\times I_1))\\&\quad=\tilde{\rho}(V_p\times I_1,\ U_p^c\times I_1)>C\,.\end{aligned}$$

Now, we may assume $x=y$, then $t\neq s$. There exists V_p which contains x. Then, there is some integer j for which $d_n(Z_j(\alpha_p(x),\ t),\ Z_j(\alpha_p(x),\ s))>C_0$. By assertion (4),

$$\begin{aligned}&\rho(f^{jN+2(p+1)+1}(x,\ t),\ f^{jN+2(p+1)+1}(x,\ s))\\&\quad=\rho(g_{jN+2(p+1)+2}Z_j^p(x,\ t),\ g_{jN+2(p+1)+2}Z_j^p(x,\ s))\\&\quad\geqq\rho(Z_j^p(x,\ t),\ Z_j^p(x,\ s))>\delta_p\times C_0\geqq C\,.\end{aligned}$$

Case 3; $(y, s)\in A_2$

Put $(y', s')=f^{-1}(y, s)\in A_1$. There is V_p which contains y'. Let y_j be an element of M that $(y_j, s')=Z_j^p(y', s')$. Since there are integers j and j' that $d(y_j, y_{j'})>2C$, there exists an integer j that $d(y_j, x)>C$. Then, $f^{jN+2(p+1)}(x, t)=g_{jN+2(p+1)+1}(x, t)$ and $f^{jN+2(p+1)}(y, s)=f^{jN+2(p+1)+1}(y', s')=g_{jN+2(p+1)+2}Z_j^p(y', s')=g_{jN+2(p+1)+2}(y_j, s')$. Thus,

$$\begin{aligned}&\rho(f^{jN+2(p+1)}(x,\ t),\ f^{jN+2(p+1)}(y,\ s))\\&\quad=\rho(g_{jN+2(p+1)+1}(x,\ t),\ g_{jN+2(p+1)+2}(y_j,\ s'))\\&\quad\geqq d(x,\ y_j)>C\,.\end{aligned}$$

This completes the proof.

3. Proof of Theorem 2 and the corollary.

We can see that $\mathrm{Int}(M^*\{p\})$ is $M\times(0, 2]/\sim$, where $(x, t)\sim(y, s)$ means $t=s=2$ or $(x, t)=(y, s)$. We consider the following diagram.

$$\begin{array}{ccc} M\times(0,2) & \xrightarrow{f} & M\times(0,2) \\ \downarrow i & & \downarrow i \\ M\times(0,2] & & M\times(0,2] \\ \downarrow \pi & & \downarrow \pi \\ M\times(0,2]/\sim & & M\times(0,2]/\sim \end{array}$$

where i is the injection and π is the natural projection. We define a homeomorphism g of $M\times(0, 2]/\sim$ as follows,

$$\begin{cases} g(\pi(x, 2))=\pi(x, 2) \\ g(\pi(x, t))=\pi ifi^{-1}(x, t) \quad \text{if } t\neq 2. \end{cases}$$

It is easy to check that g is expansive. Theorem 2 has proved.

To prove the corollary, we put $M=S^n$ (an n-sphere) in Theorem 2. Then, $\mathrm{Int}(M^*\{p\})$ is the open $(n+1)$-ball, this show that the corollary is established.

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