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On G-extensible regularity condition and Thom-Boardman singularities

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0. Introduction.

In [2], we have defined a G-extensible regularity condition on equivariant sections of differentiable G-fibre bundle P. In this paper, we only consider the case where P is a trivial G-fibre bundle as an application of Theorem 1.3 in [2].

We now formulate as follows: Let G be a compact Lie group. Let X, Y be smooth G-manifolds. Then the r-jet bundle $J^{r}(X, Y)$ is naturally a differentiable G-fibre bundle such that the action of G on $J^{r}(X, Y)$ is defined by $g(j_{x}^{r}f)=j_{gx}^{r}(gfg^{-1})$ where $g\in G$ and f is a germ of a map $X\rightarrow Y$ at $x\in X$. Let $J_{G}^{r}(X, Y)$ be the subspace of $J^{r}(X, Y)$ consisting of r-jets of "equivariant local maps" $X\rightarrow Y$. Then $J_{G}^{r}(X, Y)$ is a G-invariant subspace of $J^{r}(X, Y)$.

Now let $\Omega(X, Y)$ be an open G-subbundle of $J^{r}(X, Y)\rightarrow X$ invariant under the natural action by local equivariant diffeomorphism of X on $J^{r}(X, Y)$. Then $\Omega(X, Y)$ is called a *natural stable regularity condition.*

We shall say that a map $f: X\rightarrow Y$ is *Ω-regular* if $j^{r}f(X)\subset\Omega(X, Y)$.

DEFINITION 0.1. Let $\Omega(X, Y)$ be a natural stable regularity condition. We say that $\Omega(X, Y)$ is *G-extensible* if the following conditions hold:

There exists a natural stable regularity condition $\Omega'(X\times\boldsymbol{R}, Y)\subset J^{r}(X\times\boldsymbol{R}, Y)$ (where G acts on $\boldsymbol{R}$ trivially) such that

$$\begin{cases}\pi(i^{*}(\Omega'(X\times\boldsymbol{R}, Y)))=\Omega(X, Y)\\ \pi(i^{*}(\Omega'(X\times\boldsymbol{R}, Y)\cap J_{G}^{r}(X\times\boldsymbol{R}, Y)))=\Omega(X, Y)\cap J_{G}^{r}(X, Y),\end{cases}$$

where $\pi: i^{*}(J^{r}(X\times\boldsymbol{R}, Y))\rightarrow J^{r}(X, Y)$ is defined by $\pi(j_{(x,0)}^{r}f)=j_{x}^{r}fi$ for the canonical inclusion $i: X\hookrightarrow X\times\boldsymbol{R}$. (We call that $\Omega'(X\times\boldsymbol{R}, Y)$ is the *extension* of $\Omega(X, Y)$).

From [2], we have the following theorem.

THEOREM 0.2. *Let $C_{G\Omega}^{\infty}(X, Y)$ be the space of the Ω-regular equivariant maps $X\rightarrow Y$, with the C^{∞}-topology, and let $\Gamma_{G}^{0}(\Omega_{G}(X, Y))$ be the space of continuous equivariant sections of the map $\Omega(X, Y)\cap J_{G}^{r}(X, Y)\rightarrow X$ (with the compact-open topology). Then, if $\Omega(X, Y)$ is G-extensible,*

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$$j^r: C^\infty_{G\Omega}(X, Y) \longrightarrow \Gamma^0_G(\Omega_G(X, Y))$$

is a weak homotopy equivalence.

One of the example of Ω-regularity condition is given by notions of Thom-Boardman singularities.

Let $I=(i_1, \cdots, i_r)$ be a non-increasing sequence of r non-negative integers. We define $\Omega^I(X, Y)\subset J^r(X, Y)$ to be the union of Thom-Boardman singularities $\cup\{\Sigma^K | K\leqq I$ in lexicographic order$\}$. Then Ω^I-regularity is a natural stable regularity condition. (For the proof, see du Plessis [3] (1.4), and the fact that it is a G-subbundle is a trivial by definition).

We now define

$$m.f.d(Y)=\min\{\dim Y^H | H=G_y \text{ for some } y\in Y\},$$

where Y^H denote the fixed point set of H on Y and G_y the isotropy subgroup of $y\in Y$.

Then we have the following theorem.

THEOREM 0.3. *Let I be the r-sequence $(i_1, \cdots, i_r)$, $i_1\geqq \cdots \geqq i_r\geqq 0$. If $i_r>$ $\dim X-m.f.d(Y)-d^I$, then $\Omega^I(X, Y)$ is G-extensible, where $d^I=\sum_{s=1}^{r-1}\alpha_s$, and*

$$\alpha_s=\begin{cases} 1 & \text{if } i_s-i_{s+1}>1 \\ 0 & \text{otherwise.} \end{cases}$$

This result has been announced in [2] as Theorem 6.1. The structure of the proof is an equivariant generalization of du Plessis' method ([3]). For $r=1$, this result is an equivariant version of Feit's k-mersions theorem [1].

In section 1, we shall prove Theorem 0.3.

All manifold should satisfy the second countability axiom.

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1. Proof of Theorem 0.3.

Notations in this section are the same as those of du Plessis [3]. We shall show that, under the condition in Theorem 0.3, the extension may be taken as $\Omega^I(X\times \boldsymbol{R}, Y)$.

Since $m.f.d(Y)\leqq\dim Y$, if $i_r>\dim X-m.f.d(Y)-d^I$, then $\Omega^I(X, Y)$ is extensible (du Plessis [3], Theorem 2.7), and the extension is $\Omega^I(X\times \boldsymbol{R}, Y)$.

It remains to show that $\tilde{i}(\Omega^I(X\times \boldsymbol{R}, Y)\cap J^r_G(X\times \boldsymbol{R}, Y))=\Omega^I(X, Y)\cap J^r_G(X, Y)$. ($\tilde{i}: J^r(X\times \boldsymbol{R}, Y)\rightarrow J^r(X, Y)$ is defined by $\tilde{i}(j^r_{(x,p)}f)=j^r_x(fi_p)$, where $i_p: X\rightarrow X\times \boldsymbol{R}$ by $i_p(x)=(x, p)$).

Now, if f is a local equivariant map from an invariant open set in $X\times \boldsymbol{R}$

to Y, then fi_p is also equivariant. So it holds that $\tilde{\imath}(\Omega^I(X\times\boldsymbol{R}, Y)\cap J^r_G(X\times\boldsymbol{R}, Y))$ $\subset\Omega^I(X, Y)\cap J^r_G(X, Y)$ by Lemma (2.1) of [3].

For the proof of converse, we need the following two lemmas.

LEMMA 1.1. *Let* $y=j^r_x f\in\Sigma^I(X, Y)$, $I=(i_1, \cdots, i_r)$, *define* $d^I=\sum_{s=1}^{r-1}\alpha_s$, *where* $\alpha_s=1$ *if* $i_s-i_{s+1}>1$ *and* 0 *otherwise. Then*

(a) *If* $\dim Y-\dim X+i_r+d^I\geqq h$, *then there is a subspace* $W\subset E_y$ *of* $\dim W=h$ *such that*

$$u_s \operatorname{Hom}(K_s\circ\cdots\circ K_1, W)\cap d_{s+1}(K_s)=\{0\} \text{ at } y \text{ for any } s<r.$$

(b) *If* $m.f.\,d(Y)-\dim X+i_{r-1}+d^{i_1,\cdots,i_{r-1}}=h_{r-1}>0$ *and* $m.f.\,d(Y)-\dim X+i_r+d^I\leqq 0$, *then there is a subspace* $W\subset E_y$ *of* $\dim W=\dim Y-m.f.\,d(Y)+1$ *such that* $u_s \operatorname{Hom}(K_s\circ\cdots\circ K_1, W)\cap d_{s+1}(K_s)=\{0\}$ *at* y *for any* $s<r-1$ *and*

$$\dim(u_{r-1}\operatorname{Hom}(K_{r-1}\circ\cdots\circ K_1, W)\cap d_r(K_{r-1}))\leqq\begin{cases} i_{r-1}-i_r-h_{r-1} & (h_{r-1}>1)\\ i_{r-1}-i_r & (h_{r-1}=1).\end{cases}$$

We need the following two lemmas to prove Lemma 1.1.

SUBLEMMA 1.1.1. *Let* U, V, W *be vector spaces, and let* $b: U\to\operatorname{Hom}(V, W)$ *be a linear map of rank* r.

(a) *If* $r<\dim W$, *then there is a subspace* $A\subset W$ *of* $\dim A=\dim W-r$ *such that* $\operatorname{Im}(b)\cap\operatorname{Hom}(V, A)=\{0\}$.

(b) *If* $r\geqq\dim W$ *and there is a positive integer* s *with* $\dim W>s$, *then there exists a subspace* $A\subset W$ *of* $\dim A=s+1$ *such that*

$$\dim(\operatorname{Im}(b)\cap\operatorname{Hom}(V, A))\leqq r-\dim W+s+1.$$

PROOF. (a) See du Plessis [3], Lemma (3.1) (a).

(b) Since rank (b)$=r\geqq\dim W>s$, there is a subspace $U'\subset U$ of $\dim U'=\dim W-(s+1)$ such that $\operatorname{rank}(b|U')=\dim W-(s+1)$. Hence, by (a), there is an $(s+1)$-dimensional subspace $A\subset W$ such that $b(U')\cap\operatorname{Hom}(V, A)=\{0\}$. Thus $\dim(\operatorname{Im}(b)\cap\operatorname{Hom}(V, A))\leqq r-(\dim W-(s+1))$. Q. E. D.

SUBLEMMA 1.1.2. *Let* $b: K\to\operatorname{Hom}(K\otimes L, W)$ *be a linear map of rank* r *which is symmetric in* K.

(a) *If* $r\leqq\dim W$, *there is a subspace* $A\subset W$ *of*

$$\dim A=\begin{cases}\dim W-r+1 & (r>1)\\ \dim W-r & (r\leqq 1)\end{cases}$$

such that $\operatorname{Im}(b)\cap\operatorname{Hom}(K\otimes L, A)=\{0\}$.

(b) *If* $r\geqq\dim W$ *and there is a positive integer* s *with* $\dim W>s$, *there is a subspace* $A\subset W$ *of* $\dim A=s+1$ *such that*

$$\dim(\operatorname{Im}(b)\cap \operatorname{Hom}(K\otimes L,\ A))=\begin{cases} r-\dim W+s & (\dim W>s+1)\\ r-\dim W+1+s & (\dim W=s+1)\,. \end{cases}$$

PROOF. (a) See du Plessis [3], Lemma (3.2) (a).

(b) If $\dim W=s+1$, the result follows by Sublemma 1.1.1 (b). Now suppose $\dim W>s+1$. If there is a subspace $A\subset W$ of $\dim A=s+1$ such that $\operatorname{Im}(b)\cap \operatorname{Hom}(K\otimes L, A)=\{0\}$, we have the result. So suppose otherwise; then let t be a maximal integer such that there exists a subspace $B\subset W$ of $\dim B=t$ with $\operatorname{Im}(b)\cap \operatorname{Hom}(K\otimes L, B)=\{0\}$ $(s+1>t\geqq 0)$. For convenience, we choose a basis $\{b_i\}_{i=1}^{h}$ of W such that $\langle b_1, \cdots, b_t\rangle=B$. Then for each $w\in W-B$, $\operatorname{Im}(b)\cap \operatorname{Hom}(K\otimes L, B)\neq\{0\}$. Hence, there exists a $k\in K$ such that $\operatorname{Im}(b(k))\subset\langle w, B\rangle$ and $\operatorname{Im}(b(k))\not\subset B$. For b_i $(i>t)$, let $k_i\in K$ be such that $\operatorname{Im}(b(k_i))\subset\langle b_i, B\rangle$ and $\operatorname{Im}(b(k_i))\not\subset B$. Then $b(k_{t+1}), \cdots, b(k_h)$ is a linearly independent set.

For each pair $(k, k')\in K\times K$, we may regard $b(k)(k')$ as a linear map $L\to W$. Clearly $\operatorname{Im}(b(k_i)(k_j))\subset\langle b_i, B\rangle$ for each $j\geqq t+1$. But $b(k_i)(k_j)=b(k_j)(k_i)$, since b is symmetric, and so

$$\operatorname{Im}(b(k_i)(k_j))\subset\langle b_i, B\rangle\cap\langle b_j, B\rangle=B \quad \text{if} \quad i\neq j\,.$$

(i) Suppose $\operatorname{Im}(b(k_i)(k_i))\not\subset B$ $i=t+1, \cdots, h$; then $b(\langle k_{s+1}, \cdots, k_h\rangle)\cap \operatorname{Hom}(K\otimes L, \langle b_{t+1}+b_{t+2}, \cdots, b_{t+1}+b_{s+2}, B\rangle)=\{0\}$. (For if there is a $k=\sum_{j=s+1}^{h}\lambda_j k_j$ such that

$$\operatorname{Im}(b(k))\subset\langle b_{t+1}+b_{t+2}, \cdots, b_{t+1}+b_{s+2}, B\rangle\text{, then}$$

$$\operatorname{Im}(b(k)(k_i)-\lambda_i b(k_i)(k_i))\subset B \qquad \text{for each} \quad i\geqq s+1>t\,.$$

Hence $\operatorname{Im}(\lambda_i b(k_i)(k_i))\subset\langle b_{t+1}+b_{t+2}, \cdots, b_{t+1}+b_{s+2}\rangle\cap\langle b_i, B\rangle=B$. So, $\lambda_i=0$ for each $i\geqq s+1$ (i.e. $k=0$)).

Then

$$\dim(\operatorname{Im}(b)\cap \operatorname{Hom}(K\otimes L, \langle b_{t+1}+b_{t+2}, \cdots, b_{t+1}+b_{s+2}, B\rangle)\leqq r-\dim W+s\,.$$

(ii) Suppose $\operatorname{Im}(b(k_i)(k_i))\subset B$ for some $i\geqq t+1$. Let $k\in K$ be such that $\operatorname{Im}(b(k_i)(k))\not\subset B$, so that $k\notin\langle k_{s+1}, \cdots, k_h\rangle$ (for if $k=\sum_{i=t+1}^{h}\lambda_i k_i$, $\operatorname{Im}(b(k_i)(k))=\operatorname{Im}(b(k)(k_i))$ and $\operatorname{Im}(b(k)(k_i)-\lambda_i b(k_i)(k_i))\subset B$. So, $\operatorname{Im}(b(k_i)(k))\subset B$. It is a contradiction), and $b(k)$ is linearly independent of $b(k_{t+1}), \cdots, b(k_h)$.

We now suppose $i<s+1$, then

$$b(\langle k, k_{s+2}, k_{s+3}, \cdots, k_h\rangle)\cap \operatorname{Hom}(K\otimes L, \langle b_{t+1}, \cdots, b_{s+1}, B\rangle)=\{0\}\,.$$

(For if $\xi=\mu b(k)+\sum_{i=s+2}^{h}\alpha_i b(k_i)$ and $\operatorname{Im}(\xi)\subset\langle b_{t+1}, \cdots, b_{s+1}, B\rangle$, then $\operatorname{Im}(\xi(k_i)-\mu b(k)(k_i))\subset B$. Since $\operatorname{Im}(b(k)(k_i))\subset\langle b_i, B\rangle$, then $\operatorname{Im}(\mu b(k)(k_i))\subset\langle b_{t+1}, \cdots, b_{s+1}, B\rangle\cap\langle b_i, B\rangle=B$. Thus $\operatorname{Im}(\mu b(k)(k_i))\subset B$. But, by hypothesis $\operatorname{Im}(b(k)(k_i))\subset B$.

Then $\mu=0$, so that $\mathrm{Im}\,(\xi)=\mathrm{Im}\Bigl(\sum_{i=s+2}^{h}\alpha_i b(k_i)\Bigr)\subset\langle b_{t+1},\cdots,b_{s+1},B\rangle\cap\langle b_{s+2},\cdots,b_h\rangle=\{0\}$).

Alternatively, suppose $i\geqq s+1$. We may assume $i=s+1$ without loss of generality. Then, by the same argument in the case $i<s+1$, we have

$$b(\langle k,k_{s+1},k_{s+3},\cdots,k_h\rangle)\cap\mathrm{Hom}\,(K\otimes L,\langle b_{t+1},\cdots,b_s,b_{s+2},B\rangle)=\{0\}.$$

In either case, there is a subspace $A\subset W$ of $\dim A=s+1$ such that $\dim\,(\mathrm{Im}\,(b)\cap\mathrm{Hom}\,(K\otimes L,A))\leqq r-\dim W+s$.

This completes the proof. Q. E. D.

PROOF OF LEMMA 1.1. (a) See du Plessis [3], Lemma (2.6) (a).

(b) We note that

$$\dim Y-\dim X+i_{r-1}+d^{i_1,\cdots,i_{r-1}}=h_{r-1}+(\dim Y-m.f.\,d(Y)).$$

Hence, by hypothesis,

$$\dim Y-\dim X+i_{r-1}+d^{i_1,\cdots,i_{r-1}}>(\dim Y-m.f.\,d(Y)).$$

Now we let $s=\dim Y-m.f.\,d(Y)$, then (by (a)) there is a subspace $W_{r-1}\subset W$ of $\dim W_{r-1}=h_{r-1}+s$ such that

$$u_s\,\mathrm{Hom}\,(K_s\circ\cdots\circ K_1,W_{r-1})\cap d_{s+1}(K_s)=\{0\}\qquad\text{for any}\quad s<r-1.$$

Then $u_{r-1}|\mathrm{Hom}\,(K_{r-1}\circ\cdots\circ K_1,W_{r-1})$ is injective (see du Plessis [3], the proof of Lemma (2.6) (a)).

Define $L=d_r^{-1}(u_{r-1}\,\mathrm{Hom}\,(K_{r-1}\circ\cdots\circ K_1,W_{r-1}))\cap K_{r-1}$. Then $b'_{r-2}(L)\subset\mathrm{Hom}\,(K_{r-1}\circ\cdots\circ K_1,W_{r-1})$, (where b'_{r-2} is the bundle map given in Lemma (2.5) in [3]), so we have a map

$$b:L\longrightarrow\mathrm{Hom}\,(K_{r-1}\circ\cdots\circ K_1,W_{r-1})$$

such that for any subspace $W'\subset W_{r-1}$,

$$b(L)\cap\mathrm{Hom}\,(K_{r-1}\circ\cdots\circ K_1,W')\cong d_r(K_{r-1})\cap u_{r-1}\,\mathrm{Hom}\,(K_{r-1}\circ\cdots\circ K_1,W').$$

We now distinguish two cases.

(1) $L\subsetneqq K_{r-1}$; then $\mathrm{rank}\,(b)\leqq\mathrm{rank}\,(d_r|K_{r-1})-1=i_{r-1}-i_r-1$. We suppose that $\gamma=\mathrm{rank}\,(b)<\dim W_{r-1}=h_{r-1}+s$. In this case if $h_{r-1}=1$, then we have

$$\dim\,(\mathrm{Im}\,(b)\cap\mathrm{Hom}\,(K_{r-1}\circ\cdots\circ K_1,W_{r-1}))=\mathrm{rank}\,(b)\leqq i_{r-1}-i_r-1.$$

If $h_{r-1}>1$, (by Sublemma 1.1.1 (a)) there is a subspace $A\subset W$ of $\dim A=h_{r-1}+s-\gamma$ such that $\mathrm{Im}\,(b)\cap\mathrm{Hom}(K_{r-1}\circ\cdots\circ K_1,A)=\{0\}$. So if $h_{r-1}>\gamma$, then $\dim A=h_{r-1}+s-\gamma\geqq s+1$. If $\gamma\geqq h_{r-1}$, for some subspace $L'\subset L$ of $\dim L'=h_{r-1}-1$ such that $\mathrm{rank}\,(b)|L'=h_{r-1}-1<\dim W_{r-1}$. Hence, by sublemma 1.1.1 (a), there is a subspace $A\subset W_{r-1}$ of $\dim A=\dim W_{r-1}-(h_{r-1}-1)=s+1$ such that $b(L')\cap\mathrm{Hom}\,(K_{r-1}\circ\cdots\circ K_1,A)=\{0\}$. Thus $\dim\,(\mathrm{Im}\,(b)\cap\mathrm{Hom}\,(K_{r-1}\circ\cdots\circ K_1,A))\leqq\gamma-(h_{r-1}-1)\leqq i_{r-1}-i_r-1-h_{r-1}+1=i_{r-1}-i_r-h_{r-1}$.

Alternatively, suppose rank $(b) \geqq \dim W_{r-1} = h_{r-1} + s$. By Sublemma 1.1.1 (*b*), there is a $(s+1)$-dimensional subspace $W' \subset W_{r-1}$ such that

$$\dim(\operatorname{Im}(b) \cap \operatorname{Hom}(K_{r-1} \circ \cdots \circ K_1, W')) \leqq \operatorname{rank}(b) - \dim W_{r-1} + s$$
$$\leqq i_{r-1} - i_r - h_{r-1}.$$

(2) $L = K_{r-1}$; then $b: K_{r-1} \to \operatorname{Hom}(K_{r-1} \otimes (K_{r-2} \circ \cdots \circ K_1), W_{r-1})$ is a symmetric map of rank $(b) = \operatorname{rank}(d_r | K_{r-1}) = i_{r-1} - i_r$.

We now suppose that $\gamma = \operatorname{rank}(b) < \dim W_{r-1} = h_{r-1} + s$. In this case, if $h_{r-1} = 1$, then we have

$$\dim(\operatorname{Im}(b) \cap \operatorname{Hom}(K_{r-1} \circ \cdots \circ K_1, W_{r-1})) = \operatorname{rank}(b) \leqq i_{r-1} - i_r.$$

If $h_{r-1} > 1$, (by Sublemma 1.1.2 (a)) there is a subspace $A \subset W$ of

$$\dim A = \begin{cases} h_{r-1} + s - \gamma + 1 & (\gamma > 1) \\ h_{r-1} + s - \gamma & (\gamma \leqq 1) \end{cases}$$

such that $\operatorname{Im}(b) \cap \operatorname{Hom}(K_{r-1} \circ \cdots \circ K_1, A) = \{0\}$. So, if $\gamma \leqq 1$, then $\dim A = h_{r-1} + s - \gamma \geqq s + 1$. If $h_{r-1} \geqq \gamma \geqq 1$, then $\dim A = h_{r-1} - \gamma + s + 1 > s + 1$.

If $\gamma > h_{r-1}$, for some subspace $L' \subset K_{r-1}$ of $\dim L' = h_{r-1}$ such that rank $(b | L') = h_{r-1} < \dim W_{r-1}$. Hence, by Sublemma 1.1.2 (a), there is a subspace $A \subset W_{r-1}$ of $\dim A = \dim W_{r-1} - h_{r-1} + 1 = s + 1$ such that $b(L') \cap \operatorname{Hom}(K_{r-1} \circ \cdots \circ K_1, A) = \{0\}$. Thus,

$$\dim(\operatorname{Im}(b) \cap \operatorname{Hom}(K_{r-1} \circ \cdots \circ K_1, A)) \leqq \gamma - h_{r-1} = i_{r-1} - i_r - h_{r-1}.$$

Alternatively, suppose rank $(b) \geqq \dim W_{r-1}$. By Sublemma 1.1.2 (b), there is a $(s+1)$-dimensional subspace $W' \subset W_{r-1}$ such that

$$\dim(\operatorname{Im}(b) \cap \operatorname{Hom}(K_{r-1} \circ \cdots \circ K_1, W')) \leqq \begin{cases} i_{r-1} - \dim W_{r-1} + s \\ i_{r-1} - \dim W_{r-1} + 1 + s \end{cases}$$
$$\leqq \begin{cases} i_{r-1} - i - h_{r-1} - s + s & (\dim W_{r-1} > s + 1) \\ i_{r-1} - i_r - h_{r-1} - s + 1 + s & (\dim W_{r-1} = s + 1) \end{cases}$$
$$= \begin{cases} i_{r-1} - i_r - h_{r-1} & (h_{r-1} > 1) \\ i_{r-1} - i_r - h_{r-1} + 1 & (h_{r-1} = 1). \end{cases}$$

This completes the proof. Q.E.D.

LEMMA 1.2. *Let $\boldsymbol{R}^n$, $\boldsymbol{R}^p$ be Euclidean spaces on which G acts orthogonal. Let $f: (\boldsymbol{R}^n, 0) \to (\boldsymbol{R}^p, 0)$ be smooth equivariant map such that $j_0^r f \in \Sigma^I(\boldsymbol{R}^n, \boldsymbol{R}^p)$.*

If there is a local submersion $k: U \to \boldsymbol{R}^q$ (where $q < \dim(\boldsymbol{R}^p)^G$) of an invariant neighbourhood of $0 \in \boldsymbol{R}^p$ such that $j_0^r(kf) \in \Omega^I(\boldsymbol{R}^n, \boldsymbol{R}^q)$, then there is a local smooth equivariant map $\bar{F}: (\boldsymbol{R}^n \times \boldsymbol{R}, (0, 0)) \to (\boldsymbol{R}^p, 0)$ such that $j_{(0,0)}^r \bar{F} \in \Omega^I(\boldsymbol{R}^n \times \boldsymbol{R}, \boldsymbol{R}^p)$ and $j_0^r(\bar{F} i_0) = j_0^r f$.

PROOF. Since k is a submersion, there is an invariant neighbourhood U of $0 \in \boldsymbol{R}^p$ and a local diffeomorphism h of U onto a neighbourhood V of $0 \in \boldsymbol{R}^p$ such that the following diagram commutes:

$$\begin{array}{ccc} \boldsymbol{R}^p \supset U & \xrightarrow{\quad h \quad} & V \subset \boldsymbol{R}^p = \boldsymbol{R}^q \times \boldsymbol{R}^{p-q} \\ k \downarrow & \swarrow p & \\ \boldsymbol{R}^q & & \end{array}$$

(where p is a canonical projection).

Now U is an invariant neighbourhood and h is a diffeomorphism, then we introduce a G-action on V such that h is a G-equivariant diffeomorphism. We note that, in this action, $\dim V^G = \dim (\boldsymbol{R}^p)^G$. We explain $f' = hf = (f'_1, \cdots, f'_p)$ by coordinate functions f'_i. By the hypothesis $I(f_1, \cdots, f_p) = I(f'_1, \cdots, f'_p) \geqq I(f'_1, \cdots, f'_q)$, (where $I(f) = I(f_1, \cdots, f_p)$ denote the Boardman Symbol of $f = (f_1, \cdots, f_p)$).

1) The case $\boldsymbol{R}^q \times 0 \cap V \subset V^G$. If necessary, by changing the coordinate in $0 \times \boldsymbol{R}^{p-q}$, we may assume that $(q+1)$-th coordinate is contained in fixed point set V^G. Thus, in the representation $fh = f' = (f'_1, \cdots, f'_p)$, f'_{p+1} is G-invariant function.

We now define G-equivariant map:

$$F' : W \times (-\varepsilon\ \varepsilon) \longrightarrow V$$

by

$$F'(x, t) = (f'_1(x), \cdots, f'_q(x), f'_{q+1}(x) + t, f'_{q+2}(x), \cdots, f'_p(x))$$

for sufficiently small invariant neighbourhood $W \times (-\varepsilon\ \varepsilon)$ of $(0, 0) \in \boldsymbol{R}^n \times \boldsymbol{R}$.

By the definition, it is clear that $F'i = f'$. Let α be the diffeomorphism of $\boldsymbol{R}^n \times \boldsymbol{R}$ defined by $\alpha(x, t) = (x, t - f_{q+1}(x))$. Then $F'\alpha(x, t) = (f'_1(x), \cdots, f'_q(x), t, f'_{q+2}(x), \cdots, f'_p(x))$. Since $\alpha(0, 0) = (0, 0)$, $j^r_{(0,0)}F \in \Sigma^I(W \times (-\varepsilon\ \varepsilon), V)$ if and only if $j^r_{(0,0)}F'\alpha \in \Sigma^I(\boldsymbol{R}^n \times \boldsymbol{R}, \boldsymbol{R}^p)$. It follows from Lemma (2.2) of [3] that $j^r_{(0,0)}F'\alpha \in \Sigma^I(\boldsymbol{R}^n \times \boldsymbol{R}, \boldsymbol{R}^p)$ if and only if $j^r_0 f'' \in \Sigma^I(\boldsymbol{R}^n, \boldsymbol{R}^{p-1})$. (Where $f''(x) = (f'_1(x), \cdots, f'_q(x), f'_{q+2}(x), \cdots, f'_p(x))$. By the definition of Boardman Symbol, we have $I(f'_1, \cdots, f'_q) \geqq I(f'_1, \cdots, f'_q, f'_{q+2}, \cdots, f'_p) \geqq I(f'_1, \cdots, f'_p)$.

From the hypothesis of the theorem, we have $I(f'_1, \cdots, f'_q) = I(pf') = I(phf) = I(kf) = I(f)$. Hence, $j^r_0 f'' \in \Sigma^I(\boldsymbol{R}^n, \boldsymbol{R}^{p-1})$ if and only if $j^r_0 kf \in \Sigma^I(\boldsymbol{R}^n, \boldsymbol{R}^q)$.

We now define local equivariant map

$$\bar{F} : W \times (-\varepsilon\ \varepsilon) \longrightarrow U$$

by $\bar{F}(x, t) = h^{-1}F(x, t)$.

By the assumpsion, it is clear that $j^r_{(0,0)}\bar{F} \in \Omega^I(W \times (-\varepsilon\ \varepsilon), U)$ and $j^r_0(\bar{F}_i) = j^r_0(h^{-1}F_i) = j^r_0(h^{-1}f') = j^r_0 f$.

2) $\boldsymbol{R}^q \times 0 \cap V \not\subset V^G$. By the same technique as the case 1), we construct local equivariant map

$F: W\times(-\varepsilon\ \varepsilon)\longrightarrow U$ such that $j^r_{(0,0)}F\in\Omega^I(W\times(-\varepsilon\ \varepsilon), U)$ and $j^r_0(F_t)=j^r_0f$.
This completes the proof. Q. E. D.

PROOF OF THE THEOREM 0.3. It is enough to show that for each $y=j^r_xf\in\Omega^I(X, Y)$ there is a local submersion $k: U\to \boldsymbol{R}^q$ (where $q<\dim Y^{G_{f(x)}}$) of a $G_{f(x)}$-invariant neighbourhood U of $f(x)$ in Y such that $j^r_x(kf)\in\Omega^I(X, \boldsymbol{R}^q)$ (by Lemma 1.2 and differentiable slice theorem). This fact follows from Lemma (2.4) of [3] and Lemma 1.1 in exactly the same way that Theorem (2.7) of [3] follows from Lemma (2.4) and (2.6) of [3]. Q. E. D.

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