J. Math. Soc. Japan
Vol. 33, No. 3, 1981

Multiplicity one theorem and modular symbols[1)]

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(Received May 4, 1977)
(Revised Sept. 11, 1979)

Introduction.

1°. In this paper we shall investigate further (cf. [**9**]) on the periods of primitive forms (or the values at rational integer points in the critical strip of the Dirichlet series attached to primitive forms) from a different point of view, by generalizing the notion of modular symbols (cf. Birch [**2**], Manin [**11, 12**], Mazur [**13**]) to arbitrary levels and weights and using the successive convergents of rational numbers by the continued fractions.

2°. A beginning seems to be due to Shimura [**14**], who computed the ratios

$$\left(\int_0^{i\infty}\Delta(z)z\,dz : \int_0^{i\infty}\Delta(z)z^3dz : \int_0^{i\infty}\Delta(z)z^5dz : \int_0^{i\infty}\Delta(z)z^7dz : \int_0^{i\infty}\Delta(z)z^9dz\right)$$

and

$$\left(\int_0^{i\infty}\Delta(z)z^2dz : \int_0^{i\infty}\Delta(z)z^4dz : \int_0^{i\infty}\Delta(z)z^6dz : \int_0^{i\infty}\Delta(z)z^8dz\right)$$

explicitly where $\Delta(z)$ is the unique normalized cusp form of weight 12 on $SL(2, \boldsymbol{Z})$. Manin [**12**] generalized these results to eigenforms of arbitrary integral weights on $SL(2, \boldsymbol{Z})$ as follows. The ratios

$$\left(\int_0^{i\infty}f(z)z\,dz : \int_0^{i\infty}f(z)z^3dz : \int_0^{i\infty}f(z)z^5dz : \cdots : \int_0^{i\infty}f(z)z^{w-1}dz\right)$$

and

$$\text{(B)},\qquad \left(\int_0^{i\infty}f(z)dz : \int_0^{i\infty}f(z)z^2dz : \int_0^{i\infty}f(z)z^4dz : \cdots : \int_0^{i\infty}f(z)z^{w}dz\right)$$

are both rational over the field generated over $\boldsymbol{Q}$ by the Fourier coefficients of f at $z=i\infty$ if f is an eigenform of weight $w+2$ on $SL(2, \boldsymbol{Z})$. Furthermore he also showed that the Ramanujan type congruence for the Fourier coefficients of f is derived from the ratio (B). They were based on the Eichler-Shimura isomorphism for cusp forms. Recently Shimura [**16, 17**], using a totally different

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1) This paper forms a part of the author's doctorial thesis presented to Faculty of Science, University of Tokyo.

method associated with the convolution of Rankin type, has extended almost everything on the rationality to Neben type primitive forms of any level (in the sense of Atkin-Lehner [1], Miyake, Li and others). And we have given a new proof of that Theorem 1 (i), (ii) and (iii) of Shimura [17], in Hatada [8, 9] by combining the Eichler-Shimura isomorphism with the lemma of Shapiro. (See e. g. Lang [18] for details of the lemma of Shapiro.) In [9], we have also studied p-adic Hecke series, attached to the primitive forms, which take algebraic values. But there still remain some problems, along the older lines (cf. [12], [14]), left open in this field. The purpose of this paper is to solve one of them. Now we describe in 3° below, about what problem we shall study. In short, we shall give a concrete method for computing the ratios of certain periods of a primitive form of any level. We note that it relates to the Ramanujan type congruence for the Fourier coefficients of a primitive form.

3°. Let N and $w+1$ be positive integers, Γ be the Hecke's congruence subgroup $\Gamma_1(N)$ (resp. $\Gamma_0(N)$) and $S_{w+2}(\Gamma)$ be the space of cusp forms of weight $w+2$ with respect to Γ. Let dz_w be the $\boldsymbol{C}^{w+1}$ valued differential form ${}^t(dz, zdz, z^2dz, \cdots, z^w dz)$ on the complex upper half plane H. The special linear group $SL(2, \boldsymbol{R})$ acts on H to the left by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z)=(az+b)/(cz+d)$. Let $\rho_w : GL(2, \boldsymbol{Z}) \to GL(w+1, \boldsymbol{Z})$ be the representation given by $\rho_w(g)dz_w=(cz+d)^{w+2}(dz_w \circ g)$ for $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \boldsymbol{Z})$. Here $dz_w \circ g$ denotes the pull back of dz_w by g. Set $\eta_w = \operatorname{Ind}_{\Gamma \uparrow SL(2,\boldsymbol{Z})} (\rho_w | _\Gamma)$. Let $SL(2, \boldsymbol{Z})=\bigcup_{j=1}^{m} \Gamma g_j$ be the left coset decomposition. Set

$$\mathfrak{t}=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T=\rho_w(\mathfrak{t}), \quad \sigma_1=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2=\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix},$$

$$K=\begin{pmatrix} \rho_w(g_1) & & & & 0 \\ & \rho_w(g_2) & & & \\ & & \rho_w(g_3) & & \\ & & & \ddots & \\ 0 & & & & \rho_w(g_m) \end{pmatrix},$$

$$\mathfrak{D}(F)=\begin{pmatrix} (F(z)dz_w)\circ g_1 \\ (F(z)dz_w)\circ g_2 \\ (F(z)dz_w)\circ g_3 \\ \vdots \\ (F(z)dz_w)\circ g_m \end{pmatrix} \quad \text{and} \quad \mathfrak{D}(F)^{\varepsilon}=\begin{pmatrix} T(F(z)dz_w)\circ(\mathfrak{t}g_1\mathfrak{t}) \\ T(F(z)dz_w)\circ(\mathfrak{t}g_2\mathfrak{t}) \\ T(F(z)dz_w)\circ(\mathfrak{t}g_3\mathfrak{t}) \\ \vdots \\ T(F(z)dz_w)\circ(\mathfrak{t}g_m\mathfrak{t}) \end{pmatrix}$$

(for all $F \in S_{w+2}(\Gamma)$). We normalize η_w as $\eta_w(g)\mathfrak{D}(F)=\mathfrak{D}(F)\circ g$ for every $g \in SL(2, \boldsymbol{Z})$. Set $K^{-1}\mathfrak{D}(F)=D(F)$, $K^{-1}\mathfrak{D}(F)^{\varepsilon}=D(F)^{\varepsilon}$ and $\eta_w^*(g)=K^{-1}\eta_w(g)K$. Let

$T_{w+2}(n)$ be the usual Hecke operator acting on $S_{w+2}(\Gamma)$, defined by:

$$F|T_{w+2}(n)(z)=n^{(1/2)w}\sum_{ad=n}\sum_{b=0}^{d-1}F|_{w+2}\left[X_a\begin{pmatrix}a & \pm b\\ 0 & d\end{pmatrix}\right](z)$$

if $(n, N)=1$ $\qquad (SL(2, \boldsymbol{Z})\ni X_a\equiv\begin{pmatrix}a^{-1} & 0\\ 0 & a\end{pmatrix} \pmod N)$.

Let α be a positive rational number and

$$\alpha=b_n/d_n,\ b_{n-1}/d_{n-1}, \cdots, b_1/d_1,\ b_0/d_0=0/1,$$

be the successive convergents in irreducible form by the continued fraction of α. For each m with $1\leqq m\leqq n$, we have:

$$h_m=\begin{pmatrix}b_m & (-1)^{m-1}b_{m-1}\\ d_m & (-1)^{m-1}d_{m-1}\end{pmatrix}\in SL(2, \boldsymbol{Z}).$$

Using the technique of "Modular symbols" of Birch [2], Manin [**11**], [**12**] and Mazur [**13**] to $S_{w+2}(\Gamma)$, we obtain:

(0.0.1);
$$\int_0^\alpha F|_{w+2}[g](z)dz_w=\sum_{x=1}^n\int_{h_x(0)}^{h_x(i\infty)}F|_{w+2}[g](z)dz_w$$
$$=\sum_{x=1}^n\rho_w(h_x)\int_0^{i\infty}F|_{w+2}[gh_x](z)dz_w,$$

and

(0.0.2);
$$\int_0^{-\alpha}T\circ F|_{w+2}[{}^tg^t](z)dz_w=\sum_{x=1}^n\int_{{}^th_x{}^t(0)}^{{}^th_x{}^t(i\infty)}T\circ F|_{w+2}[{}^tg^t](z)dz_w$$
$$=\sum_{x=1}^n\rho_w(h_x)\int_0^{i\infty}T\circ F|_{w+2}[{}^tgh_x{}^t](z)dz_w$$

since $T^2=1$. Let $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in M_{2,2}(\boldsymbol{Z})$ with $\alpha\delta-\beta\gamma>0$. We have:

(0.0.3);
$$(\alpha\delta-\beta\gamma)^{(1/2)w}\int_0^{i\infty}F|_{w+2}\left[\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\right](z)z^kdz$$
$$=\int_{\beta/\delta}^{\alpha/\gamma}F(z)(\delta z-\beta)^k(\alpha-\gamma z)^{w-k}dz$$

for $k\in\boldsymbol{Z}$ with $0\leqq k\leqq w$. From (0.0.1), (0.0.2) and (0.0.3), we know that there exists a $m(w+1)\times m(w+1)$ integral matrix $M(n)=M_{w,\Gamma}(n)$ for each positive integer n with $(n, N)=1$ such that

$$\int_0^{i\infty} D(F|T_{w+2}(n))=M(n)\int_0^{i\infty} D(F) \quad \text{and such that}$$

$$\int_0^{i\infty} D(F|T_{w+2}(n))^{\varepsilon}=M(n)\int_0^{i\infty} D(F)^{\varepsilon}$$

for all $F\in S_{w+2}(\Gamma)$. Let J be the $(w+1)m\times(w+1)m$ integral matrix defined by $D(F)^{\varepsilon}=J\circ D(F)$ for all $F\in S_{w+2}(\Gamma)$. For each $g\in SL(2, \boldsymbol{Z})$ and each $F\in S_{w+2}(\Gamma)$, we have $\mathcal{D}(F)^{\varepsilon}\circ{}^t g^t=\eta_w(g)\mathcal{D}(F)^{\varepsilon}$, and

$$\int_{g(0)}^{g(i\infty)} D(F)\pm\int_{{}^t g^t(0)}^{{}^t g^t(i\infty)} D(F)^{\varepsilon}=\eta_w^*(g)\left(\int_0^{i\infty} D(F)\pm\int_0^{i\infty} D(F)^{\varepsilon}\right).$$

Let χ be a Dirichlet character mod N and $S_{w+2}(N, \chi)$ be the space

$$\left\{F\in S_{w+2}(\Gamma_1(N))\,\middle|\, F|_{w+2}\left[\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right]=\chi(d)F \quad \text{for all } \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_0(N)\right\}.$$

We want to study:

(0.1) PROBLEM. *Let f be a primitive form (or a common eigenfunction of all the Hecke operators) in $S_{w+2}(N, \chi)$ with $f(z)=\sum_{n=1}^{\infty} a_n \exp(2\pi\sqrt{-1}nz)$ $(a_1=1)$. Let $\boldsymbol{r}$ be a column vector in $\boldsymbol{C}^{(w+1)m}$.*

(i): $M(n)\boldsymbol{r}=a_n\boldsymbol{r}$ *for all the positive integers n with* $(n, N)=1$.

(ii): $J\boldsymbol{r}=\boldsymbol{r}$. (iii): $J\boldsymbol{r}=-\boldsymbol{r}$.

(iv): $(\eta_w^*(\sigma_1)+1)\boldsymbol{r}=\boldsymbol{0}$. (v): $(\eta_w^*(\sigma_2)^2+\eta_w^*(\sigma_2)+1)\boldsymbol{r}=\boldsymbol{0}$.

(vi): $\left(1-\eta_w^*\left(\begin{pmatrix} -1 & 0\\ 0 & -1\end{pmatrix}\right)\right)\boldsymbol{r}=\boldsymbol{0}$.

Let W^- (resp. W^+) be the space of the solutions of the system of the above equations (i), (ii), (iv), (v) *and* (vi) (*resp.* (i), (iii), (iv), (v) *and* (vi)) *over $\boldsymbol{C}$. Is it true that W^- (resp. W^+) is one dimensional over $\boldsymbol{C}$?*

$$(\dim W^{\pm}\geqq 1 \text{ since } W^-\ni\int_0^{i\infty}(D(f)+D(f)^{\varepsilon})\neq\boldsymbol{0} \text{ and } W^+\ni\int_0^{i\infty}(D(f)-D(f)^{\varepsilon})\neq\boldsymbol{0}.$$

The equation (vi) becomes trivial (viz. $\boldsymbol{0}=\boldsymbol{0}$) when $\Gamma=\Gamma_0(N)\ni -1$.)

Note that $M(n)=M_{w,\Gamma}(n)$ is not uniquely determined by n and $S_{w+2}(\Gamma)$.

(0.2) LEMMA. *Let p be a prime with $p\equiv 1$ (mod N). It is well known that*

$$\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\,\middle|\, ad-bc=p,\quad a-1\equiv b\equiv c\equiv 0 \ (\mathrm{mod}\ N)\right\}=\bigcup_{u=0}^{p}\Gamma(N)\alpha_u \ (disjoint)$$

where we put $\alpha_u=\begin{pmatrix} 1 & Nu\\ 0 & p\end{pmatrix}$ for $0\leqq u\leqq p-1$ and $\alpha_p=\begin{pmatrix} p & 0\\ 0 & 1\end{pmatrix}$. Then for each $g\in \Gamma(1)$, the map $\Gamma(N)\alpha_u\mapsto\Gamma(N)g^{-1}\alpha_u g$ induces a permutation on $\{0, 1, \cdots, p\}$. (Cf. Lemma 1.11 in [**9**].)

For primes p with $p\equiv 1 \pmod N$, choose $M(p)$ as follows. We compute:

$$\int_0^{i\infty} F|T_{w+2}(p)|_{w+2}[g_j](z)z^k dz = p^{w/2}\sum_{u=0}^{p}\int_0^{i\infty} F|_{w+2}[\alpha_u g_j](z)z^k dz$$

$$= p^{w/2}\sum_{u=0}^{p}\int_0^{i\infty} F|_{w+2}[g_j\alpha_u](z)z^k dz$$

$$= \sum_{v=0}^{p-1}\left(\int_0^{i\infty} - \int_0^{Nv/p}\right)F|_{w+2}[g_j](z)(pz-Nv)^k dz + p^{w-k}\int_0^{i\infty} F|_{w+2}[g_j](z)z^k dz$$

$$(F\in S_{w+2}(\Gamma)).$$

Using (0.0.1) (and (0.0.2)), expand this integration from 0 to Nv/p (resp. $-Nv/p$) into a $\boldsymbol{Z}$-linear combination of the fundamental periods

$$\left\{\int_0^{i\infty} F|_{w+2}[g](z)z^l dz \,\middle|\, g\in SL(2, \boldsymbol{Z}),\ l\in \boldsymbol{Z} \text{ with } 0\leqq l\leqq w\right\}.$$

In this way we have $M(p)=M_{w,\Gamma}(p)$ for primes p with $p\equiv 1 \pmod N$. Our result is:

(0.3) THEOREM. *Choose $M(p)=M_{w,\Gamma}(p)$ for primes $p\equiv 1 \pmod N$ as above. Let χ_0 be the trivial character* mod N. *Then both W^- and W^+ become one dimensional over $\boldsymbol{C}$ at least if f is either in $S_{w+2}(N, \chi_0)$ or in $S_2(q, \chi_0)$ where N and w are positive integers and q is a rational prime.* (*Here we put $N=q$ in the case of $w+2=2$.*)

(Note that the above theorem gives the concrete way for the computation of the ratio of the components of the vector $\mathrm{Re}\int_0^{i\infty} D(f)$ (resp. $\mathrm{Im}\int_0^{i\infty} D(f)$).)

This theorem is considered as a kind of "Multiplicity one theorem" on $Z_P^1(SL(2, \boldsymbol{Z}), \eta_w, \boldsymbol{R})$. First Shimura showed that W^- is one dimensional for $\Delta(z)\in S_{12}(SL(2, \boldsymbol{Z}))$. Next Manin showed that W^+ and W^- are both one dimensional for any eigenform in $S_{w+2}(SL(2, \boldsymbol{Z}))$ where $w+2$ is any weight. (Cf. [**12**].)

We obtain also:

(0.4) THEOREM. *Let f be a primitive form in $S_{w+2}(N, \chi_0)$ $(w+2>2)$ and L be the greatest common divisor of*

$$\left\{\mathrm{Im}\int_0^{i\infty} 2f|_{w+2}[g_j](z)z^k dz \Big/ \left(\sqrt{-1}\int_0^{i\infty} f(z)\,dz\right) \,\middle|\, 1\leqq k\leqq w-1,\ 1\leqq j\leqq m\right\}.$$

Then we obtain:

$$a_p\equiv 1+p^{w+1} \pmod L \qquad \textit{for all the primes } p \textit{ with } (p, N)=1.$$

$\left(\text{We have always } \int_0^{i\infty} f(z)dz\neq 0 \text{ if } w+2>2. \text{ Cf. (1.12) below.}\right)$

(0.5) THEOREM. *Let f be a primitive form in $S_2(q, \chi_0)$ with $\int_0^{i\infty} f(z)dz \neq 0$ and L be the greatest common divisor of*

$$\left\{\mathrm{Im}\int_0^{i\infty} 2f|_{w+2}[g_j](z)dz\Big/\left(\sqrt{-1}\int_0^{i\infty} f(z)dz\right)\right| \quad \Gamma_0(q)g_j \text{ is neither } \Gamma_0(q) \text{ nor } \Gamma_0(q)\sigma_1\right\}.$$

Then we obtain:

$$a_p \equiv 1+p \pmod{L} \qquad \text{for all the primes } p \text{ with } p \nmid 2q.$$

Main results of this paper were announced in a Proceedings of Japan Academy Note [7].

§ 1. Notations and preliminary results.

N: a positive integer. $w+1$: a positive integer.

$S_{w+2}^{\boldsymbol{R}}(\Gamma)$: the subspace of $S_{w+2}(\Gamma)$ whose elements have all their Fourier coefficients at $z=i\infty$ in the real numbers $\boldsymbol{R}$.

For every $g \in SL(2, \boldsymbol{Z})$, put $\bar{g}=\Gamma g \in \Gamma\backslash SL(2, \boldsymbol{Z})$.

$\boldsymbol{R}^{m(w+1)}$: the real vector space consisting of column vectors with the basis indexed by the pairs $\{(\bar{g}_j, u)\}$ which are the elements of the product set $\Gamma\backslash SL(2, \boldsymbol{Z}) \times ([0, w] \cap \boldsymbol{Z})$.

$H_{par}^1(\Gamma, \rho_w, \boldsymbol{R})$, $H_{par}^1(SL(2, \boldsymbol{Z}), \eta_w^*, \boldsymbol{R})$: the parabolic cohomology groups with coefficients in the $\boldsymbol{R}$.

ϕ: $S_{w+2}(\Gamma) \to H_{par}^1(\Gamma, \rho_w, \boldsymbol{R})$: the Shimura isomorphism.

Set $\Gamma=\Gamma_0(N)$ and $g_1=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

(1.1) LEMMA (cf. Hatada, Theorem 2.3 in [9], Theorem 1 in [8]). *The map $\Phi: S_{w+2}(\Gamma) \to H_{par}^1(SL(2, \boldsymbol{Z}), \eta_w^*, \boldsymbol{R})$, $F \mapsto$ the cohomology class of the cocycle $\left\{\sigma \in SL(2, \boldsymbol{Z}) \mapsto \mathrm{Re}\int_{z_0}^{\sigma z_0} D(F)\right\}$, is a surjective $\boldsymbol{R}$-linear isomorphism ($z_0 \in H$).*

(1.2) LEMMA. *The map*

$$\begin{array}{ccc} \Psi:\ S_{w+2}(\Gamma) & \longrightarrow & \boldsymbol{R}^{m(w+1)} \\ \Cup & & \Cup \\ F & \longrightarrow & \mathrm{Re}\displaystyle\int_0^{i\infty} D(F)(z) \end{array}$$

is an $\boldsymbol{R}$-linear embedding.

PROOF. Set $z_0=0$ in (1.1). Recall that σ_1 and σ_2 generate $SL(2, \boldsymbol{Z})$ and that $\sigma_1(0)=\sigma_2(0)=i\infty$. Hence we obtain:

$$\mathrm{Re}\int_0^{\sigma(0)} D(F)(z)=0 \quad \text{for all } \sigma\in SL(2, \boldsymbol{Z}) \quad \text{if } \mathrm{Re}\int_0^{i\infty} D(F)(z)=0\,.$$

Then from (1.1), we obtain (1.2).

(1.3) DEFINITIONS. $S^-(\Gamma)_w$ (resp.(1) $S^+(\Gamma)_w$, resp.(2) $S^*(\Gamma)_w$) is the $\boldsymbol{R}$-linear subspace of $\boldsymbol{R}^{m(w+1)}$ formed by all the vectors

$$\boldsymbol{r}=\{r(\bar{g}_j,\ k)\}_{1\leq j\leq m,\, 0\leq k\leq w}$$

satisfying the following system of equations $(A_{j,k})$, $(B_{j,k})$, $(C_{j,k})$ and $(D_{j,k})$ (resp.(1) $(A'_{j,k})$, $(B_{j,k})$, $(C_{j,k})$ and $(D_{j,k})$, resp.(2) $(B_{j,k})$, $(C_{j,k})$ and $(D_{j,k})$) for all the integers j and k with $1\leqq j\leqq m$ and $0\leqq k\leqq w$.

$(A_{j,k})$, $$r(\bar{g}_j,\ k)=(-1)^{k+1}r(\overline{tg_jt},\ k)$$

$(A'_{j,k})$, $$r(\bar{g}_j,\ k)=(-1)^{k}r(\overline{tg_jt},\ k)$$

$(B_{j,k})$, $$r(\bar{g}_j,\ k)+(-1)^{k}r(\overline{g_j\sigma_1},\ w-k)=0$$

$(C_{j,k})$, $$r(\bar{g}_j,\ k)+\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}r(\overline{g_j\sigma_2},\ w-k+l) +\sum_{l=0}^{w-k}\binom{w-k}{l}(-1)^{w-l}r(\overline{g_j\sigma_2^2},\ l)=0$$

$(D_{j,k})$, $$r(\bar{g}_j,\ k)=(-1)^{w}r(\overline{-g_j},\ k)\,.$$

Set $B=S^*(\Gamma)_w\cap\{\boldsymbol{r}\,|\,(\eta_w^*(\sigma_1)-1)\boldsymbol{r}'=(\eta_w^*(\sigma_2)-1)\boldsymbol{r}'=\boldsymbol{r},\ \boldsymbol{r}'\in\boldsymbol{R}^{m(w+1)}\}$ and $B^+=S^+(\Gamma)_w\cap B$ (resp. $B^-=S^-(\Gamma)_w\cap B$).

(1.4) DEFINITIONS. We define mappings $\phi^\pm$, ϕ, ξ, $\xi^\pm$ as follows.

$$\phi^-:\ S_{w+2}^{\boldsymbol{R}}(\Gamma)\longrightarrow S^-(\Gamma)_w/B^-\,,\qquad F\longmapsto \mathrm{Re}\int_0^{i\infty}D(F)\qquad (\mathrm{mod}\ B^-)\,.$$

$$\phi^+:\ \sqrt{-1}S_{w+2}^{\boldsymbol{R}}(\Gamma)\longrightarrow S^+(\Gamma)_w/B^+\,,\qquad F\longmapsto \mathrm{Re}\int_0^{i\infty}D(F)\qquad (\mathrm{mod}\ B^+)\,.$$

$$\phi:\ S_{w+2}(\Gamma)\longrightarrow S^*(\Gamma)_w/B\,,\qquad F\longmapsto \mathrm{Re}\int_0^{i\infty}D(F)\qquad (\mathrm{mod}\ B)\,.$$

$\xi:\ S^*(\Gamma)_w/B\longrightarrow H^1_{par}(SL(2,\ \boldsymbol{Z}),\ \eta_w^*,\ \boldsymbol{R})$

$\boldsymbol{r}$ (mod B) $\longmapsto$ the cohomology class of the cocycle

$$\{\sigma_1\longmapsto\boldsymbol{r} \text{ and } \sigma_2\longmapsto\boldsymbol{r}\}\,.$$

$$\xi^-:\ S^-(\Gamma)_w/B^-\longrightarrow H^1_{par}(SL(2,\ \boldsymbol{Z}),\ \eta_w^*,\ \boldsymbol{R})\cap\Phi(S_{w+2}^{\boldsymbol{R}}(\Gamma))\,,$$

$$\xi^+:\ S^+(\Gamma)_w/B^+\longrightarrow H^1_{par}(SL(2,\ \boldsymbol{Z}),\ \eta_w^*,\ \boldsymbol{R})\cap\Phi(\sqrt{-1}S_{w+2}^{\boldsymbol{R}}(\Gamma))\,,$$

$\boldsymbol{r}$ (mod $B^\pm$) $\longmapsto$ the cohomology class of the cocycle

$$\{\sigma_1 \longmapsto \boldsymbol{r} \text{ and } \sigma_2 \longmapsto \boldsymbol{r}\}.$$

(1.4.1), Well definedness of ξ. For each $\boldsymbol{r}$ in B, there exists a vector $\boldsymbol{r}'$ in $\boldsymbol{R}^{m(w+1)}$ such that $\boldsymbol{r}=(1-\eta_w^*(\sigma_1))\boldsymbol{r}'=(1-\eta_w^*(\sigma_2))\boldsymbol{r}'$. Hence $\xi(\boldsymbol{r})=\boldsymbol{0}=$the cohomology class of $\{\sigma\in SL(2, \boldsymbol{Z})\mapsto(1-\eta_w^*(\sigma))\boldsymbol{r}'\}$.

It is due to the following lemma that the image under $\xi^{\pm}$ of $S^{\pm}(\Gamma)_w/B^{\pm}$ coincides with $H^1_{par}(SL(2, \boldsymbol{Z}), \eta_w^*, \boldsymbol{R})\cap\Phi(\sqrt{-1}^{(1\pm1)/2}S^{\boldsymbol{R}}_{w+2}(\Gamma))$.

(1.5) LEMMA. *The composite map* $\xi\circ\psi$ *is the surjective isomorphism* Φ *given in* (1.1). *All the maps* ψ, $\psi^{\pm}$, ξ *and* $\xi^{\pm}$ *given in* (1.4) *are surjective isomorphisms.*

PROOF. Set $z_0=0$ in (1.1). By (1.1), (1.2) and (1.4), it is easy to see $\xi\circ\psi=\Phi$ and that ξ is injective. Hence ξ becomes an isomorphism and so does ψ. From $\xi\circ\psi=\Phi$, we have $\xi^{\pm}\circ\psi^{\pm}=\Phi^{\pm}$. Since $\Phi^{\pm}=\Phi|_{\sqrt{-1}^{(1\pm1)/2}S^{\boldsymbol{R}}_{w+2}(\Gamma)}$ and $\xi^{\pm}$ is injective, $\psi^{\pm}$ and $\xi^{\pm}$ become surjective isomorphisms.

(1.6) THE LEMMA OF SHAPIRO (See e.g. [18].) *The map* $sh: H^1(SL(2, \boldsymbol{Z}), \eta_w^*, R)\to H^1(\Gamma, \rho_w, R)$ *induced by the compatible maps,* $\Gamma\hookrightarrow SL(2, \boldsymbol{Z})$ *and the projection of* $R^{(w+1)m}$ *to the first* $(w+1)$ *components, is a surjective isomorphism. Here we set* $R=\boldsymbol{R}$ *or* $\boldsymbol{Z}$.

(1.7) LEMMA (Theorem 2.2 (ii) in Hatada [9], [8]). *The composite map* $sh\circ\Phi$ $(R=\boldsymbol{R})$ *is the Eichler-Shimura isomorphism* $S_{w+2}(\Gamma)\to H^1_{par}(\Gamma, \rho_w, \boldsymbol{R})$.

PROOF. This is a consequence of (1.1) and (1.6). Cf. [9].

From (1.6) and (1.7), we obtain:

(1.8) LEMMA (Theorem 2.2 (iii) in Hatada [9], [8]). *The map sh in* (1.6) *induces the surjective isomorphism*

$$H^1_{par}(SL(2, \boldsymbol{Z}), \eta_w^*, \boldsymbol{R}) \longrightarrow H^1_{par}(\Gamma, \rho_w, \boldsymbol{R}).$$

Furthermore we have proved in Theorem 0.1 in Hatada [9] that *the map sh in* (1.6) *induces the surjective isomorphism*

$$H^1_{par}(SL(2, \boldsymbol{Z}), \eta_w^*, \boldsymbol{Z}) \longrightarrow H^1_{par}(\Gamma, \rho_w, \boldsymbol{Z}). \qquad \text{(Also cf. [8].)}$$

We need the following well known results.

(1.9) LEMMA (Manin [11]). *There exists a bijection between the right* $SL(2, \boldsymbol{Z})$ *sets, given by*

$$\begin{array}{ccc} \Gamma_0(N)\backslash SL(2, \boldsymbol{Z}) & \longrightarrow & \boldsymbol{P}^1(\boldsymbol{Z}/N\boldsymbol{Z}) \\ \Cup & & \Cup \\ \Gamma_0(N)\begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & (c \pmod N : d \pmod N). \end{array}$$

(1.10) The field $\boldsymbol{Q}(a_1, a_2, a_3, \cdots)$ is a totally real algebraic number field for any Haupt type primitive form $f(z)=\sum_{n=1}^{\infty} a_n \exp(2\pi\sqrt{-1}nz)$ $(a_1=1)$. (See e.g. Shimura [15].)

(1.11) Let λ_p be an eigenvalue of the Hecke operator $T_{w+2}(p)$ (for a prime p with $p \nmid N$) acting on $S_{w+2}(\Gamma_0(N))$. Then we have $|\lambda_p/(1+p^{w+1})| \to 0$ when $p \to +\infty$ for any Archimedean absolute value $|\cdot|$. (For the more precise result, see [**3**].)

(1.12) REMARK. We sketch a proof of $\int_0^{i\infty} f(z)dz \neq 0$ if $w+2>2$ in Theorem 0.4. From $(B_{1,0})$ in (1.3), we obtain $\int_0^{i\infty} f(z)dz = -\int_0^{i\infty} f|_{w+2}[\sigma_1](z)z^w dz$. Since $\sigma_1\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -N & 0 \end{pmatrix}$, $\int_0^{i\infty} f(z)dz = -N^{w/2}\int_0^{i\infty} f|_{w+2}\left[\begin{pmatrix} 0 & 1 \\ -N & 0 \end{pmatrix}\right](z)z^w dz$. It is well known that the zeta function attached to $f|_{w+2}\left[\begin{pmatrix} 0 & 1 \\ -N & 0 \end{pmatrix}\right]$ have no zeros on $\{z \in \boldsymbol{C} \mid \operatorname{Re} z \geqq (w+3)/2\}$. Note $w+1 \geqq (w+3)/2$ if even $w \geqq 1$.

§ 2. Proof of Theorem 0.3 in case of $w+2>2$.

Let $N \geqq 1$ and $w+2>2$ be integers and Γ be $\Gamma_0(N)$. First we prove 10 lemmas.

(2.1) LEMMA. *Let* $\boldsymbol{r} = \{r(\bar{g}_j, k)\}_{1 \leqq j \leqq m,\, 0 \leqq k \leqq w}$ *be an element of* B. *Then*

$$r(\bar{g}_j, k) = 0$$

for all $j \in \boldsymbol{Z}$ *with* $1 \leqq j \leqq m$ *and* $k \in \boldsymbol{Z}$ *with* $1 \leqq k \leqq w-1$.

PROOF. By the definition of B, there exists a vector $\boldsymbol{r}_1 \in \boldsymbol{R}^{m(w+1)}$ such that $\boldsymbol{r} = (1-\eta_w^*(\sigma_1))\boldsymbol{r}_1$ and $\eta_w^*(\sigma_2^{-1}\sigma_1)\boldsymbol{r}_1 = \boldsymbol{r}_1$. For each g_j, there exists a positive integer n_j with $n_j \leqq m$ such that $\Gamma_0(N)g_j(\sigma_2^{-1}\sigma_1)^{n_j} = \Gamma_0(N)g_j$. Note that $(\sigma_2^{-1}\sigma_1)^{n_j} = \begin{pmatrix} 1 & 0 \\ n_j & 1 \end{pmatrix}$. Compute:

$$\rho_w\left(\begin{pmatrix} 1 & 0 \\ n_j & 1 \end{pmatrix}\right)^t(r_1(\bar{g}_j, 0), r_1(\bar{g}_j, 1), r_1(\bar{g}_j, 2), \cdots, r_1(\bar{g}_j, w))$$
$$= {}^t(r_1(\bar{g}_j, 0), r_1(\bar{g}_j, 1), r_1(\bar{g}_j, 2), \cdots, r_1(\bar{g}_j, w)).$$

Then we obtain:

$$r_1(\bar{g}_j, k) = 0 \qquad \text{for all } k \in \boldsymbol{Z} \text{ with } 1 \leqq k \leqq w.$$

Computing $\eta_w^*(\sigma_1)$ and $(1-\eta_w^*(\sigma_1))\boldsymbol{r}_1$, we obtain (2.1).

(2.2) LEMMA. *Let* p *be a prime with* $p \equiv 1 \pmod N$ *and* T_p *be the set of integral matrices:*

$$T_p = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2,2}(\boldsymbol{Z}) \,\middle|\, ad-bc=p,\ \ a-1 \equiv b \equiv c \equiv 0 \pmod N\right\}.$$

Then for any $g \in SL(2, \boldsymbol{Z})$, *we have* $gT_pg^{-1} = T_p$. (Cf. Lemma 1.11 in [**9**].)

PROOF. For any $h \in T_p$, we have $\det(ghg^{-1}) = \det(h) = p$ and $ghg^{-1} \pmod N$

$\equiv g\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}g^{-1}\equiv 1 \pmod N$, q. e. d.

(2.3) LEMMA. *Let p be a prime with $p\equiv 1 \pmod N$ and g be an element of $SL(2, \boldsymbol{Z})$. Then we have:*

(2.3.1), $$\int_0^{i\infty} F|T_{w+2}(p)|_{w+2}[g](z)dz$$

$$=(p+p^w)\int_0^{i\infty} F|_{w+2}[g](z)dz-\sum_{v=1}^{p-1}\int_0^{Nv/p} F|_{w+2}[g](z)dz$$

for all $F\in S_{w+2}(\Gamma_0(N))$.

PROOF. Set $\alpha_v=\begin{pmatrix}1 & Nv\\ 0 & p\end{pmatrix}$ for $v\in\boldsymbol{Z}$ with $0\leqq v\leqq p-1$ and $\alpha_p=\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}$. Then we have:

$$\textit{the left side of (2.3.1)}=p^{w/2}\sum_{v=0}^{p}\int_0^{i\infty} F|_{w+2}[\alpha_v g](z)\,.$$

By (2.2),

$$\textit{the left side of (2.3.1)}=p^{w/2}\sum_{v=0}^{p}\int_0^{i\infty} F|_{w+2}[g\alpha_v](z)$$

$$=p^w\int_0^{i\infty} F|_{w+2}[g](z)dz+\sum_{v=0}^{p-1}\int_{Nv/p}^{i\infty} F|_{w+2}[g](z)dz$$

$$=(p+p^w)\int_0^{i\infty} F|_{w+2}[g](z)dz-\sum_{v=1}^{p-1}\int_0^{Nv/p} F|_{w+2}[g](z)dz\,,$$ q. e. d.

(2.4) LEMMA. *Let $\boldsymbol{r}=\{r(\bar g_j, k)\}_{1\leqq j\leqq m,\, 0\leqq k\leqq w}$ be an element of $S^*(\Gamma_0(N))_w$. Then we have:*

$$-r(\overline{g\sigma_2^2\sigma_1}, 0)+r(\bar g, 0)$$
$$=-r(\overline{g\sigma_2}, 1)+r(\overline{g\sigma_2^2}, w-1)+\sum_{x=1}^{w-1}\binom{w-1}{x}(-1)^{w-1-x}r(\bar g, x)$$

for all $g\in SL(2, \boldsymbol{Z})$.

PROOF. Let F be a form in $S_{w+2}(\Gamma_0(N))$. Then we have

(2.4.1), $$\left(\int_0^{i\infty}+\int_{\sigma_2(0)}^{\sigma_2(i\infty)}+\int_{\sigma_2^2(0)}^{\sigma_2^2(i\infty)}\right)F|_{w+2}[g\sigma_2](z)zdz=0\,.$$

Changing the variable of the integrations in (2.4.1), we have:

(2.4.2), $$\int_0^{i\infty} F|_{w+2}[g\sigma_2](z)zdz+\int_0^{i\infty} F|_{w+2}[g\sigma_2^2](z)(z-1)z^{w-1}dz$$

$$-\int_0^{i\infty} F|_{w+2}[g](z-1)^{w-1}dz=0\,.$$

By the definitions of η_w^* and $S^*(\Gamma_0(N))_w$, (2.4.2) proves (2.4).

(2.4.3) REMARK. For each $F\in S_{w+2}(\Gamma_0(N))$, the vector $\mathrm{Re}\int_0^{i\infty}D(F)$ is in $S^*(\Gamma_0(N))_w$ and satisfies the formula in (2.4).

Set $\tau=\sigma_2^{-1}\sigma_1=-\sigma_2^2\sigma_1$ below.

(2.5) LEMMA. *Let g be an element of $SL(2, \boldsymbol{Z})$ and v be a rational integer. Then there exist rational integers $E(\bar{g}, v, j, l)$ for all the pairs (j, l) in $([1, m]\cap\boldsymbol{Z})\times([0, w]\cap\boldsymbol{Z})$ such that*

$$r(\overline{g\tau^v}, 0)=r(\bar{g}, 0)+\sum_{j=1}^{m}\sum_{l=1}^{w-1}E(\bar{g}, v, j, l)r(\bar{g}_j, l)$$

for all the vectors $\boldsymbol{r}=\{r(\bar{g}_j, k)\}_{1\leq j\leq m, 0\leq k\leq w}$ in $S^*(\Gamma_0(N))_w$.

PROOF. We can prove (2.5) by the recursive argument on v using (2.4),

q. e. d.

(2.6) LEMMA. *Let $h=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ and $h'=\begin{pmatrix} a' & b' \\ c' & d'\end{pmatrix}$ be two elements of $SL(2, \boldsymbol{Z})$ with $b=b'$ and $d=d'$. Then there exists a rational integer e such that*

$$\Gamma_0(N)h\tau^e=\Gamma_0(N)h'$$

where we put $\tau=\begin{pmatrix} 1 & 0 \\ 1 & 1\end{pmatrix}$.

PROOF. *Case* 1 *of* $(d, N)=1$. Use the bijection given in (1.9). There exist rational integers c_1 and c_1' such that

$$(c:d)=(c_1:1) \quad\text{and}\quad (c':d')=(c_1':1) \qquad \text{in } \boldsymbol{P}^1(\boldsymbol{Z}/N\boldsymbol{Z}).$$

Put $e=c_1'-c_1$. Then we have

$$(c_1:1)\tau^e=(c_1+e:1)=(c_1':1).$$

(2.6) is proved in case 1.

Case 2 *of* $(d, N)\neq 1$. Let $d_0>1$ be the greatest common divisor of d and N (namely $d_0|d$, $d_0|N$ and $(d_0^{-1}d, N)=1$). Since h and h' are in $SL(2, \boldsymbol{Z})$, we have

$$-bc\equiv 1 \pmod{d_0} \quad\text{and}\quad -bc'\equiv 1 \pmod{d_0}.$$

Hence

$$-b(c-c')\equiv 0 \pmod{d_0} \quad\text{and}\quad c\equiv c' \pmod{d_0}$$

since $(b, d)=(b, d_0)=1$. Put $x=(c'-c)/d_0\in\boldsymbol{Z}$. There exists an integer k' such that $(d/d_0)^{k'}\equiv 1 \pmod{N}$. Put $k=k'-1$ and $y=(d/d_0)^k x$. Note that

$$(c:d)=(c(d/d_0)^k:d_0) \qquad \text{in } \boldsymbol{P}^1(\boldsymbol{Z}/N\boldsymbol{Z}),$$

$$(c':d)=(c'(d/d_0)^k:d_0) \qquad \text{in } \boldsymbol{P}^1(\boldsymbol{Z}/N\boldsymbol{Z})$$

and that

$$c'(d/d_0)^k=c(d/d_0)^k+d_0x(d/d_0)^k.$$

By the bijection in (1.9), we have

$$(c(d/d_0)^k : d_0)\tau^y = (c'(d/d_0)^k : d_0).$$

Hence $(c : d)\tau^y = (c' : d')$, namely $\Gamma_0(N)h\tau^y = \Gamma_0(N)h'$, q. e. d.

Now let

$$SL(2, \boldsymbol{Z}) = \bigcup_{j=1}^{m_0} \Gamma_0(N)G_j\langle\tau\rangle$$

be the disjoint union with respect to the action of $\Gamma_0(N)$ and the infinite cyclic group $\langle\tau\rangle$ to the left and to the right respectively on $SL(2, \boldsymbol{Z})$. Here $m_0 = \#(\Gamma_0(N)\backslash SL(2, \boldsymbol{Z})/\langle\tau\rangle)$. We set $G_1 = g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Let p be a prime with $p \nmid N$ and v be an integer with $(v, p) = 1$. Let

$$Nv/p = b_n/d_n,\ b_{n-1}/d_{n-1},\ \cdots,\ b_1/d_1,\ b_0/d_0 = 0/1$$

be the successive convergents of Nv/p in irreducible form by the continued fraction of Nv/p. It is well known that each

$$h_t = \begin{pmatrix} b_t & (-1)^{t-1}b_{t-1} \\ d_t & (-1)^{t-1}d_{t-1} \end{pmatrix} \qquad (1 \leqq t \leqq n)$$

is an element of $SL(2, \boldsymbol{Z})$. We put $H_t = \pm h_t$ such that $H_1 = h_1$, $H_t\begin{pmatrix} 1 \\ 0 \end{pmatrix} = H_{t+1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for all $t \in \boldsymbol{Z}$ with $1 \leqq t \leqq n-1$. Then we write

$$H_t = \begin{pmatrix} B_t & B_{t-1} \\ D_t & D_{t-1} \end{pmatrix} \qquad \text{for all } t \in \boldsymbol{Z} \text{ with } 1 \leqq t \leqq n.$$

It is easy to see

$$(2.7.1), \qquad \int_0^{Nv/p} F(z)dz = \sum_{t=1}^{n} \int_{H_t(0)}^{H_t(i\infty)} F(z)dz$$

$$= \sum_{t=1}^{n} \int_0^{i\infty} F|_{w+2}[H_t](z)(D_t z + D_{t-1})^w dz$$

for all $F \in S_{w+2}(\Gamma_0(N))$.

Using the formulae

(i) the binomial expansions of $(D_t z + D_{t-1})^w$ for $t \in \boldsymbol{Z}$ with $1 \leqq t \leqq n$, and

$$\text{(ii)} \qquad \int_0^{i\infty} F|_{w+2}[g](z)dz + \int_0^{i\infty} F|_{w+2}[g\sigma_1](z)z^w dz = 0$$

for all $F \in S_{w+2}(\Gamma_0(N))$ and $g \in SL(2, \boldsymbol{Z})$,
we can simplify the right side of (2.7.1) as follows.

(2.7) LEMMA. *There exist rational integers* $a_{s,l}(Nv/p)$ *with* $1\leqq s\leqq m$ *and* $1\leqq l\leqq w-1$ *such that*

the right side of (2.7.1)

$$=(1-p^{w})\int_{0}^{i\infty}F(z)dz+\sum_{s=1}^{m}\sum_{l=1}^{w-1}a_{s,l}(Nv/p)\int_{0}^{i\infty}F|_{w+2}[g_{s}](z)z^{l}dz$$

for all $F\in S_{w+2}(\Gamma_{0}(N))$. *(Note that* l *is neither* 0 *nor* w *and that* $\{a_{s,l}(Nv/p)\}_{s,l}$ *are explicitly computable by the elements* $\{H_{t}\}_{t=1}^{n}$.)

PROOF. We compute as follows.

$$(2.7.2),\qquad \int_{0}^{i\infty}F(z)|_{w+2}[H_{t}](z)(D_{t}z+D_{t-1})^{w}dz$$

$$=-D_{t}^{w}\int_{0}^{i\infty}F|_{w+2}[H_{t}\sigma_{1}](z)dz+D_{t-1}^{w}\int_{0}^{i\infty}F|_{w+2}[H_{t}](z)dz$$

$$+\sum_{l=1}^{w-1}\binom{w}{l}D_{t}^{l}D_{t-1}^{w-l}\int_{0}^{i\infty}F|_{w+2}[H_{t}](z)z^{l}dz\,.$$

$$(2.7.3),\qquad \sum_{t=1}^{n}\int_{0}^{i\infty}F|_{w+2}[H_{t}](z)(D_{t}z+D_{t-1})^{w}dz$$

$$=\sum_{t=1}^{n-1}\left(D_{t}^{w}\int_{0}^{i\infty}F|_{w+2}[H_{t+1}](z)-D_{t}^{w}\int_{0}^{i\infty}F|_{w+2}[H_{t}\sigma_{1}](z)dz\right)$$

$$-D_{n}^{w}\int_{0}^{i\infty}F|_{w+2}[H_{n}\sigma_{1}](z)dz$$

$$+D_{0}^{w}\int_{0}^{i\infty}F|_{w+2}[H_{1}](z)dz$$

$$+\sum_{t=1}^{n}\sum_{l=1}^{w-1}\binom{w}{l}D_{t}^{l}D_{t-1}^{w-l}\int_{0}^{i\infty}F|_{w+2}[H_{t}](z)z^{l}dz$$

for all $F\in S_{w+2}(\Gamma_{0}(N))$.

In case 1 *of* $1\leqq t\leqq n-1$. It is easy to see that H_{t+1} and $H_{t}\sigma_{1}$ satisfy the condition in Lemma 2.6 (for $h=H_{t+1}$ and $h'=-H_{t}\sigma_{1}$). Namely $\Gamma_{0}(N)H_{t+1}\langle\tau\rangle=\Gamma_{0}(N)H_{t}\sigma_{1}\langle\tau\rangle$. Then apply Lemma 2.5. Then we see that there exist rational integers $\{b_{t,s,l}\}$ such that

$$(2.7.4),\qquad D_{t}^{w}\left(\int_{0}^{i\infty}F|_{w+2}[H_{t+1}](z)dz-\int_{0}^{i\infty}F|_{w+2}[H_{t}\sigma_{1}](z)\,dz\right)$$

$$=\sum_{s=1}^{m}\sum_{l=1}^{w-1}b_{t,s,l}\int_{0}^{i\infty}F|_{w+2}[g_{s}](z)z^{l}dz$$

for all $F\in S_{w+2}(\Gamma_{0}(N))$.

In case 2 *of* $t=1$. Since $D_0=1$ and $H_1=\begin{pmatrix} * & 0 \\ * & 1 \end{pmatrix}$, we have $\Gamma_0(N)H_1\langle\tau\rangle=\Gamma_0(N)\langle\tau\rangle$. Apply Lemma 2.5. We see that there exist rational integers $\{b_{1,s,l}\}$ such that

(2.7.5), $$D_0^w\int_0^{i\infty}F|_{w+2}[H_1](z)dz-\int_0^{i\infty}F(z)dz$$

$$=\sum_{s=1}^{m}\sum_{l=1}^{w-1}b_{1,s,l}\int_0^{i\infty}F|_{w+2}[g_s](z)z^l dz$$

for all $F\in S_{w+2}(\Gamma_0(N))$.

In case 3 *of* $t=n$. Note that $D_n=\pm p$. Since $(p, N)=1$, we obtain that there exists some $x\in \boldsymbol{Z}$ such that

$$(\pm D_{n-1}: D_n)=(\pm D_{n-1}: p)=(x:1) \qquad \text{in } \boldsymbol{P}^1(\boldsymbol{Z}/N\boldsymbol{Z}).$$

Namely

$$\Gamma_0(N)H_n\sigma_1=\Gamma_0(N)\tau^x .$$

Hence

$$\Gamma_0(N)H_n\sigma_1\langle\tau\rangle=\Gamma_0(N)\langle\tau\rangle .$$

Apply Lemma 2.5. We see that there exist rational integers $\{b_{n,s,l}\}$ such that

(2.7.6), $$-p^w\int_0^{i\infty}F|_{w+2}[H_n\sigma_1](z)\,dz+p^w\int_0^{i\infty}F(z)dz$$

$$=\sum_{s=1}^{m}\sum_{l=1}^{w-1}b_{n,s,l}\int_0^{i\infty}F|_{w+2}[g_s](z)z^l dz$$

for all $F\in S_{w+2}(\Gamma_0(N))$. (2.7.3), (2.7.4), (2.7.5) and (2.7.6) prove (2.7), q. e. d.

We recall

$$SL(2, \boldsymbol{Z})=\bigcup_{j=1}^{m_0}\Gamma_0(N)G_j\langle\tau\rangle \qquad \text{(disjoint)}.$$

It is easy to see

(2.8.1), $$\int_0^{Nv/p}F|_{w+2}[G_j](z)dz=\sum_{t=1}^{n}\int_{H_t(0)}^{H_t(i\infty)}F|_{w+2}[G_j](z)dz$$

$$=\sum_{t=1}^{n}\int_0^{i\infty}F|_{w+2}[G_jH_t](z)(D_tz+D_{t-1})^w dz$$

for all $F\in S_{w+2}(\Gamma_0(N))$ and $\{G_j\}_{j=1}^{m_0}$.

Let p be a prime with $p\equiv 1 \pmod N$ and G_j be a representative of a double coset $\Gamma_0(N)G_j\langle\tau\rangle\subset SL(2, \boldsymbol{Z})$. Using the formulae:

(i), the binomial expansions of $(D_tz+D_{t-1})^w$ for $t\in \boldsymbol{Z}$ with $1\leqq t\leqq n$, and (ii), $\int_0^{i\infty} F|_{w+2}[g](z)dz+\int_0^{i\infty} F|_{w+2}[g\sigma_1](z)z^w dz=0$

for all $F\in S_{w+2}(\Gamma_0(N))$ and $g\in SL(2, \boldsymbol{Z})$,
we can simplify the right side of (2.8.1) as follows.

(2.8) LEMMA. *There exist rational integers* $a_{s,l}^{(j)}(Nv/p)$ *with* $1\leqq s\leqq m$ *and* $1\leqq l\leqq w-1$ *such that*

the right side of (2.8.1)

$$=(1-p^w)\int_0^{i\infty} F|_{w+2}[G_j](z)dz+\sum_{s=1}^{m}\sum_{l=1}^{w-1} a_{s,l}^{(j)}(Nv/p)\int_0^{i\infty} F|_{w+2}[g_s](z)\, z^l dz$$

for all $F\in S_{w+2}(\Gamma_0(N))$. (*Note that* l *is neither* 0 *nor* w). *Hence* $\{a_{s,l}^{(j)}(Nv/p)\}$ *are explicitly computable by the elements* $\{H_t\}_{t=1}^{n}$ *and* $\{G_j\}_{j=1}^{m_0}$.

PROOF. In the same way of (2.7.2) and (2.7.3), we obtain:

$$\text{(2.8.2),}\qquad \sum_{t=1}^{n}\int_0^{i\infty} F|_{w+2}[G_jH_t](z)(D_t z+D_{t-1})^w dz$$

$$=\sum_{t=1}^{n-1}\Big(D_t^w\int_0^{i\infty} F|_{w+2}[G_jH_{t+1}](z)dz-D_t^w\int_0^{i\infty} F|_{w+2}[G_jH_t\sigma_1](z)dz\Big)$$

$$-D_n^w\int_0^{i\infty} F|_{w+2}[G_jH_n\sigma_1](z)dz+D_0^w\int_0^{i\infty} F|_{w+2}[G_jH_1](z)dz$$

$$+\sum_{t=1}^{n}\sum_{l=1}^{w-1}\binom{w}{l}D_t^l D_{t-1}^{w-l}\int_0^{i\infty} F|_{w+2}[g_jH_t](z)z^l dz$$

for all $F\in S_{w+2}(\Gamma_0(N))$.

In case 1 *of* $1\leqq t\leqq n-1$. It is easy to see that G_jH_{t+1} and $-G_jH_t\sigma_1$ satisfy the condition of Lemma 2.6 for $h=G_jH_{t+1}$ and $h'=-G_jH_t\sigma_1$. Hence $\Gamma_0(N)G_jH_{t+1}\langle\tau\rangle =\Gamma_0(N)G_jH_t\sigma_1\langle\tau\rangle$. Apply Lemma 2.5. Then we see that there exist rational integers $\{b_{t,s,l}^{(j)}\}$ such that

$$\text{(2.8.3),}\qquad D_t^w\Big(\int_0^{i\infty} F|_{w+2}[G_jH_{t+1}](z)dz-\int_0^{i\infty} F|_{w+2}[G_jH_t\sigma_1](z)dz\Big)$$

$$=\sum_{s=1}^{m}\sum_{l=1}^{w-1} b_{t,s,l}^{(j)}\int_0^{i\infty} F|_{w+2}[g_s](z)\, z^l dz$$

for all $F\in S_{w+2}(\Gamma_0(N))$.

In case 2 *of* $t=1$. Since $D_0=1$ and $H_1=\begin{pmatrix} * & 0\\ * & 1\end{pmatrix}$, we have $\Gamma_0(N)G_jH_1\langle\tau\rangle= \Gamma_0(N)G_j\langle\tau\rangle$ by Lemma 2.6. Apply Lemma 2.5. We see that there exist rational integers $\{b_{1,s,l}^{(j)}\}$ such that

$$(2.8.4),\qquad D_0^w\int_0^{i\infty}F|_{w+2}[G_jH_1](z)dz-\int_0^{i\infty}F|_{w+2}[G_j](z)dz$$

$$=\sum_{s=1}^{m}\sum_{l=1}^{w-1}b_{1,s,l}^{(j)}\int_0^{i\infty}F|_{w+2}[g_s](z)z^l dz$$

for all $F\in S_{w+2}(\Gamma_0(N))$.

In case 3 *of* $t=n$. By Lemma 2.7, we may assume that

$$\Gamma_0(N)G_j\langle\tau\rangle\neq\Gamma_0(N)\langle\tau\rangle\,.$$

By the bijection given in Lemma 1.9, $\Gamma_0(N)G_j$ corresponds to $(c_j:e_j)\in\boldsymbol{P}^1(\boldsymbol{Z}/N\boldsymbol{Z})$ for some $c_j\in\boldsymbol{Z}$ and $e_j\in\boldsymbol{Z}$ with $e_j|N$ and $e_j\geqq 2$. We prove:

$$(2.8.5),\qquad \Gamma_0(N)G_jH_n\sigma_1\langle\tau\rangle=\Gamma_0(N)G_j\langle\tau\rangle\,.$$

PROOF OF (2.8.5). We use the bijection in Lemma 1.9. Then

$$(c_j:e_j)\begin{pmatrix}\pm N & B_{n-1}\\ \pm p & D_{n-1}\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}=(c_jB_{n-1}+e_jD_{n-1}:\mp c_jN\mp e_jp)$$

$$=(c_jB_{n-1}+e_jD_{n-1}:\mp e_jp)\in\boldsymbol{P}^1(\boldsymbol{Z}/N\boldsymbol{Z})\,.$$

Since $p\equiv 1\ (\mathrm{mod}\,N)$, $\mp B_{n-1}\equiv 1\ (\mathrm{mod}\,N)$ and

$$(c_jB_{n-1}+e_jD_{n-1}:\mp e_jp)=(\mp c_j+e_jD_{n-1}:\mp e_j)$$

$$=(c_j\mp e_jD_{n-1}:e_j)=(c_j:e_j)\begin{pmatrix}1 & 0\\ \mp D_{n-1} & 1\end{pmatrix}.$$

Hence $\Gamma_0(N)G_jH_n\sigma_1=\Gamma_0(N)G_j\tau^{\pm D_{n-1}}$. (2.8.5) is proved.

From (2.8.5) and Lemma 2.5, we obtain that there exist rational integers $\{b_{n,s,l}^{(j)}\}$ such that

$$-p^w\Big(\int_0^{i\infty}F|_{w+2}[G_jH_n\sigma_1](z)\,dz-\int_0^{i\infty}F|_{w+2}[G_j](z)dz\Big)$$

$$=\sum_{s=1}^{m}\sum_{l=1}^{w-1}b_{n,s,l}^{(j)}\int_0^{i\infty}F|_{w+2}[g_s](z)z^l dz$$

for all $F\in S_{w+2}(\Gamma_0(N))$. (2.8.2), (2.8.3), (2.8.4) and (2.8.6) prove (2.8), q.e.d.

(2.9) LEMMA. *Let p be a prime with $p\nmid N$. Then there exist even integers $\{b(s,l,p)\}_{1\leqq s\leqq m,1\leqq l\leqq w-1}$ such that*

$$\int_0^{i\infty}F|T_{w+2}(p)(z)dz=(1+p^{w+1})\int_0^{i\infty}F(z)dz+\sum_{s=1}^{m}\sum_{l=1}^{w-1}b(s,l,p)\int_0^{i\infty}F|_{w+2}[g_s](z)z^l dz$$

for all $F\in S_{w+2}(\Gamma_0(N))$.

PROOF. By the linearity, we may assume $F\in\sqrt{-1}S_{w+2}^{\boldsymbol{R}}(\Gamma_0(N))$. Then

$$F|T_{w+2}(p)\in\sqrt{-1}S_{w+2}^{R}(\Gamma_0(N)) \quad\text{and}\quad \overline{\int_0^{Nv/p}F(z)dz}=\int_0^{-Nv/p}F(z)dz\,.$$

Let p be odd.

$$\begin{aligned}\int_0^{i\infty}F|T_{w+2}(p)(z)dz&=p^{w/2}\sum_{v=0}^{p}\int_0^{i\infty}F|_{w+2}[\alpha_v](z)dz\\&=p^w\int_0^{i\infty}F(z)dz+\sum_{v=-(p-1)/2}^{(p-1)/2}\int_{Nv/p}^{i\infty}F(z)dz\\&=(p+p^w)\int_0^{i\infty}F(z)dz-2\sum_{v=1}^{(p-1)/2}\int_0^{Nv/p}F(z)dz\end{aligned}$$

where we put $\alpha_v=\begin{pmatrix}1 & Nv\\ 0 & p\end{pmatrix}$ for v with $0\leqq v\leqq(p-1)/2$

$$\alpha_v=\begin{pmatrix}1 & N((p-1)/2-v)\\ 0 & p\end{pmatrix}\quad\text{for } v \text{ with } (p+1)/2\leqq v\leqq p-1$$

$$\alpha_p=\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}.$$

Apply Lemma 2.7. We obtain

$$\begin{aligned}\int_0^{i\infty}F|T_{w+2}(p)(z)dz&=(p+p^w)\int_0^{i\infty}F(z)dz-2\sum_{v=1}^{(p-1)/2}(1-p^w)\int_0^{i\infty}F(z)dz\\&\quad-2\sum_{v=1}^{(p-1)/2}\sum_{s=1}^{m}\sum_{l=1}^{w-1}a_{s,l}(Nv/p)\int_0^{i\infty}F|_{w+2}[g_s](z)z^l dz\\&=(1+p^{w+1})\int_0^{i\infty}F|_{w+2}[g_s](z)z^l dz\\&\quad-2\sum_{v=1}^{(p-1)/2}\sum_{s=1}^{m}\sum_{l=1}^{w-1}a_{s,l}(Nv/p)\int_0^{i\infty}F|_{w+2}[g_s](z)z^l dz\end{aligned}$$

for all $F\in\sqrt{-1}S_{w+2}^{R}(\Gamma_0(N))$ (and hence for all $F\in S_{w+2}(\Gamma_0(N))$). For the case of $p=2$, it is proved in a similar way (cf. Hatada [**6**]).

(2.10) LEMMA. *Let p be a prime with $p\equiv 1 \pmod N$ and G_j be a representative of a double coset $\Gamma_0(N)G_j\langle\tau\rangle\subset SL(2,\boldsymbol{Z})$. Then there exist rational integers $\{b_j(s,l,p)\}_{1\leq s\leq m,\,1\leq l\leq w-1}$ such that*

$$\begin{aligned}&\int_0^{i\infty}F|T_{w+2}(p)|_{w+2}[G_j](z)dz\\&\quad=(1+p^{w+1})\int_0^{i\infty}F|_{w+2}[G_j](z)dz+\sum_{s=1}^{m}\sum_{l=1}^{w-1}b_j(s,l,p)\int_0^{i\infty}F|_{w+2}[g_s](z)z^l dz\end{aligned}$$

for all $F\in S_{w+2}(\Gamma_0(N))$.

PROOF. From (2.3.1) and Lemma 2.8, we obtain

$$\int_0^{i\infty} F|T_{w+2}(p)|_{w+2}[G_j](z)dz$$

$$=(p+p^w)\int_0^{i\infty} F|_{w+2}[G_j](z)dz+(1-p)(1-p^w)\int_0^{i\infty} F|_{w+2}[G_j](z)dz$$

$$-\sum_{v=1}^{p-1}\sum_{s=1}^{m}\sum_{l=1}^{w-1} a_{s,l}^{(j)}(Nv/p)\int_0^{i\infty} F|_{w+2}[g_s](z)z^l dz$$

$$=(1+p^{w+1})\int_0^{i\infty} F|_{w+2}[G_j](z)dz-\sum_{v=1}^{p-1}\sum_{s=1}^{m}\sum_{l=1}^{w-1} a_{s,l}^{(j)}(Nv/p)\int_0^{i\infty} F|_{w+2}[g_s](z)z^l dz,$$

q. e. d.

(2.11) FINAL STEP OF PROOF OF THEOREM 0.3 IN CASE OF $w+2>2$.

Let $f(z)=\sum_{n=1}^{\infty} a_n \exp(2\pi\sqrt{-1}nz)$ $(a_1=1)$ be a primitive form in $S_{w+2}(\Gamma_0(N))$. Let $\boldsymbol{r}_1=\{r_1(\bar{g}_s, l)\}_{1\leq s\leq m, 0\leq l\leq w}$ (resp. $\boldsymbol{r}_2=\{r_2(\bar{g}_s, l)\}_{1\leq s\leq m, 0\leq l\leq w}$) be a column vector in $S^-(\Gamma_0(N))_w$ (resp. $S^+(\Gamma_0(N))_w)\subset\boldsymbol{R}^{m(w+1)}$ such that

$$M(n)\boldsymbol{r}_1=a_n\boldsymbol{r}_1 \qquad (\text{resp. } M(n)\boldsymbol{r}_2=a_n\boldsymbol{r}_2)$$

for all positive $n\in\boldsymbol{Z}$ with $(n, N)=1$. By Lemmas 1.5 and 2.1, there exists a cusp form f_1 (resp. f_2) in $S_{w+2}^{\boldsymbol{R}}(\Gamma_0(N))$ (resp. $\sqrt{-1}S_{w+2}^{\boldsymbol{R}}(\Gamma_0(N))$) such that

(2.11.0), $$\mathrm{Re}\int_0^{i\infty} f_1|_{w+2}[g_s](z)z^l dz=r_1(g_s, l)$$

$$\left(\text{resp. } \mathrm{Re}\int_0^{i\infty} f_2|_{w+2}[g_s](z)z^l dz=r_2(\bar{g}_s, l)\right)$$

for all $s\in\boldsymbol{Z}$ with $1\leqq s\leqq m$ and all $l\in\boldsymbol{Z}$ with $1\leqq l\leqq w-1$. Let $SL(2, \boldsymbol{Z})=\bigcup_{j=1}^{m_0}\Gamma_0(N)G_j\langle\tau\rangle$, and p be a prime with $p\equiv 1 \pmod N$. Then we shall show first:

(2.11.1), $$\mathrm{Re}\left\{\int_0^{i\infty} f_1|T_{w+2}(p)|_{w+2}[G_j](z)dz-(1+p^{w+1})\int_0^{i\infty} f_1|_{w+2}[G_j](z)dz\right\}$$

$$=(a_p-1-p^{w+1})r_1(\overline{G}_j, 0)$$

$$\left(\text{resp. } \mathrm{Re}\left\{\int_0^{i\infty} f_2|T_{w+2}(p)|_{w+2}[G_j](z)dz-(1+p^{w+1})\int_0^{i\infty} f_2|[G_j](z)dz\right\}\right.$$

$$\left.=(a_p-1-p^{w+1})r_2(\overline{G}_j, 0)\right).$$

PROOF OF (2.11.1). Apply Lemma 2.10. Then we obtain:

$$\textit{the left side of } (2.11.1) = \sum_{s=1}^{m} \sum_{l=1}^{w-1} b_j(s, l, p) \operatorname{Re} \int_0^{i\infty} f_1 |_{w+2} [g_s](z) z^l dz$$

$$= \sum_{s=1}^{m} \sum_{l=1}^{w-1} b_j(s, l, p) r_1(\bar{g}_s, l) \qquad \text{from } (2.11.0).$$

Since $M(p)\boldsymbol{r}_1 = a_p \boldsymbol{r}_1$, we obtain from Lemma 2.10,

$$\sum_{s=1}^{m} \sum_{l=1}^{w-1} b_j(s, l, p) r_1(\bar{g}_s, l) = (a_p - 1 - p^{w+1}) r_1(\bar{G}_j, 0).$$

Hence (2.11.1) is proved for f_1. In the same way, (2.11.1) is also proved for f_2,

q. e. d.

Now divide both the sides of (2.11.1) by $1+p^{w+1}$. Since $S_{w+2}(\Gamma_0(N))$ is a semi-simple $\boldsymbol{C}[\{T_{w+2}(n) \mid n>0,\ n \in \boldsymbol{Z},\ (n, N)=1\}]$-module (cf. Shimura [**15**]), we obtain from (1.11) that

$$\left| (1+p^{w+1})^{-1} \operatorname{Re} \int_0^{i\infty} f_1 | T_{w+2}(p) |_{w+2} [G_j](z) dz \right| \longrightarrow 0$$

$$\left(\text{resp. } \left| (1+p^{w+1})^{-1} \operatorname{Re} \int_0^{i\infty} f_2 | T_{w+2}(p) |_{w+2} [G_j](z) dz \right| \longrightarrow 0 \right)$$

when $p \to +\infty$ with $p \equiv 1 \pmod N$. Hence we obtain from (1.11) and (2.11.1):

$$(2.11.2), \qquad \operatorname{Re} \int_0^{i\infty} f_1 |_{w+2} [G_j](z) dz = r_1(\bar{G}_j, 0)$$

$$\left(\text{resp. } \operatorname{Re} \int_0^{i\infty} f_2 |_{w+2} [G_j](z) dz = r_2(\bar{G}_j, 0) \right)$$

for all $j \in \boldsymbol{Z}$ with $1 \leqq j \leqq m_0$. Now apply Lemma 2.5. Then we have:

$$r_1(\overline{G_j \tau^v}, 0) = r_1(\bar{G}_j, 0) + \sum_{s=1}^{m} \sum_{l=1}^{w-1} E(\bar{G}_j, v, s, l) r_1(\bar{g}_s, l)$$

and

$$\operatorname{Re} \int_0^{i\infty} f_1 |_{w+2} [G_j \tau^v](z) dz = \operatorname{Re} \int_0^{i\infty} f_1 |_{w+2} [G_j](z) dz$$

$$+ \sum_{s=1}^{m} \sum_{l=1}^{w-1} E(\bar{G}_j, v, s, l) \operatorname{Re} \int_0^{i\infty} f_1 |_{w+2} [g_s](z) z^l dz.$$

Since we have proved already that the above right sides are equal to each other, we obtain:

$$(2.11.3)_1, \qquad r_1(\overline{G_j \tau^v}, 0) = \operatorname{Re} \int_0^{i\infty} f_1 |_{w+2} [G_j \tau^v](z) dz$$

for all $v \in \boldsymbol{Z}$. (In the same way, we obtain:

(2.11.3)$_2$, $$r_2(\overline{G_j\tau^v}, 0)=\mathrm{Re}\int_0^{i\infty} f_2|_{w+2}[G_j\tau^v](z)dz$$

for all $v\in \boldsymbol{Z}$). Note that the vectors $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\mathrm{Re}\int_0^{i\infty} D(f_1)$ and $\mathrm{Re}\int_0^{i\infty} D(f_2)$ satisfy the formula $(B_{j,k})$ in (1.3) Definition. Hence from (2.11.0), (2.11.2) and (2.11.3), we obtain $\boldsymbol{r}_1=\mathrm{Re}\int_0^{i\infty} D(f_1)$ and $\boldsymbol{r}_2=\mathrm{Re}\int_0^{i\infty} D(f_2)$. (Namely

$$r_1(\bar{g}_s, l)=\mathrm{Re}\int_0^{i\infty} f_1|_{w+2}[g_s](z)z^l dz$$

and

$$r_2(\bar{g}_s, l)=\mathrm{Re}\int_0^{i\infty} f_2|_{w+2}[g_s](z)z^l dz \qquad \text{for all } s\in \boldsymbol{Z}$$

with $1\leqq s\leqq m$ and $l\in \boldsymbol{Z}$ with $0\leqq l\leqq w$.) We have assumed

$$M(n)\boldsymbol{r}_1=a_n\boldsymbol{r}_1 \qquad (\text{resp. } M(n)\boldsymbol{r}_2=a_n\boldsymbol{r}_2)$$

for all the rational integers $n>0$ with $(n, N)=1$. This implies that

(2.11.4), The form f_1 (resp. f_2) is an eigenform with $f_1|T_{w+2}(n)=a_n f_1$ (resp. $f_2|T_{w+2}(n)=a_n f_2$) for all the integers $n>0$ with $(n, N)=1$.

PROOF OF (2.11.4). We have

$$M(n)\boldsymbol{r}_1=\mathrm{Re}\left(M(n)\int_0^{i\infty} D(f_1)\right)=\mathrm{Re}\left(\int_0^{i\infty} D(f_1|T_{w+2}(n))\right)$$

$$=a_n\boldsymbol{r}_1=\mathrm{Re}\left(\int_0^{i\infty} D(a_n f_1)\right)$$

since a_n is real. In the same way, we have,

$$\mathrm{Re}\left(\int_0^{i\infty} D(f_1|T_{w+2}(n))\right)=\mathrm{Re}\left(\int_0^{i\infty} D(a_n f_2)\right).$$

By Lemma 1.5, ϕ^- (resp. ϕ^+) is injective. Hence $f_1|T_{w+2}(n)=a_n f_1$ (resp. $f_2|T_{w+2}(n)=a_n f_2$). (2.11.4) is proved.

By multiplicity one theorem (cf. [1]) there exists a unique $c_1\in\boldsymbol{R}$ (resp. $c_2\in\boldsymbol{R}$) such that $f_1=c_1 f$ (resp. $f_2=\sqrt{-1}c_2 f$). Hence we obtain:

$$\boldsymbol{r}_1=c_1\,\mathrm{Re}\left(\int_0^{i\infty} D(f)\right) \quad \text{and} \quad \boldsymbol{r}_2=c_2\,\mathrm{Im}\left(\int_0^{i\infty} D(f)\right).$$

Hence W^- (resp. W^+) is one dimensional and spanned by the vector $\mathrm{Re}\left(\int_0^{i\infty} D(f)\right)$ $\left(\text{resp. } \mathrm{Im}\left(\int_0^{i\infty} D(f)\right)\right)$ over $\boldsymbol{R}$, q.e.d.

§ 3. Proof of Theorem 0.3 in case of $w+2=2$ and a prime $N=q$.

Let N be a prime and $w+2$ be 2. Then $m=N+1$.

(3.1) LEMMA. *Let $\boldsymbol{r}=\{r(\bar{g}_j, 0)\}_{1\leq j\leq m}$ be an element of B. Then $r(\bar{g}_j, 0)=0$ for all g_j with $\Gamma_0(N)g_j\neq\Gamma_0(N)$ and $\Gamma_0(N)g_j\neq\Gamma_0(N)\sigma_1$.*

PROOF. Set $g_t=\begin{pmatrix}1 & 0\\ t-1 & 1\end{pmatrix}$ for $1\leqq t\leqq N$ and $g_N=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}=\sigma_1$. Then from the definition of B, B is spanned by the vectors $(1-\eta_w^*(\sigma_1))\boldsymbol{r}^{(1)}$ and $(1-\eta_w^*(\sigma_1))\boldsymbol{r}^{(2)}$ defined as follows.

$$\boldsymbol{r}^{(1)}=\{r^{(1)}(\bar{g}_j, 0)\}_{1\leq j\leq m} \quad \text{with} \quad r^{(1)}(\bar{g}_1, 0)=r^{(1)}(\bar{g}_j, 0)=1$$

for all j with $1\leqq j\leqq N$, and $r^{(1)}(\bar{g}_p, 0)=0$, and

$$\boldsymbol{r}^{(2)}=\{r^{(2)}(\bar{g}_j, 0)\}_{1\leq j\leq m} \quad \text{with} \quad r^{(2)}(\bar{g}_1, 0)=r^{(2)}(\bar{g}_j, 0)=0$$

for all j with $1\leqq j\leqq N$ and $r^{(2)}(\bar{g}_p, 0)=1$.
Note that $\overline{g_j\sigma_1}\in\{\bar{g}_2, \bar{g}_3, \cdots, \bar{g}_{N-1}\}$ for any j with $2\leqq j\leqq N-1$ and that $\overline{g_1\sigma_1}=\bar{g}_N$ ($\overline{g_N\sigma_1}=\bar{g}_1$). Hence (3.1) is proved.

Note that Lemmas 2.2 and 2.3 hold in the case of $w+2=2$. Let p be a prime with $p\nmid N$, v be an integer with $(v, p)=1$ and $Nv/p=b_n/d_n$, $b_{n-1}/d_{n-1}, \cdots$, b_1/d_1, $b_0/d_0=0/1$ be the successive convergents of Nv/p by the continued fraction. We use the same notations h_t and H_t for $t\in\boldsymbol{Z}$ with $1\leqq t\leqq n$ as in § 2. (2.7.1) holds also in the case of $w+2=2$.

(3.2) LEMMA. *Let $w=0$. Using only the formulae $\{B_{j,0}\}_{1\leq j\leq N+1}$ in* (1.3) *Definition, we can simplify the right side of* (2.7.1) *as follows (when $w+2=2$). There exist rational integers $a_s(Nv/p)$ with $2\leqq s\leqq N$ such that*

$$\textit{the right side of } (2.7.1)=\sum_{s=2}^{N} a_s(Nv/p)\int_0^{i\infty} F|_2[g_s](z)dz$$

for all $F\in S_2(\Gamma_0(N))$. (Note that $s\neq 1$ and $s\neq N+1$.)

PROOF. It is sufficient to consider those H_t such as $N|D_t$ or $N|D_{t-1}$. Since $p\nmid N$, we have $p\nmid D_n$. Let t be an integer with $1\leqq t\leqq n-1$ and $N|D_t$. Then we have

$$\begin{aligned}&\int_0^{i\infty} F|_2[H_t](z)dz+\int_0^{i\infty} F|_2[H_{t+1}](z)dz\\ &=\int_0^{i\infty} F(z)dz+\int_0^{i\infty} F|_2[\sigma_1](z)dz\\ &=0, \qquad\qquad \text{q. e. d.}\end{aligned}$$

(3.3) LEMMA. *Let p be a prime with $p\nmid 2N$. Then there exist even integers $\{b(s, p)\}_{2\leq s\leq N}$ such that*

$$\int_0^{i\infty} F|T_2(p)(z)dz=(1+p)\int_0^{i\infty} F(z)dz+\sum_{s=2}^{N} b(s,\ p)\int_0^{i\infty} F|_2[g_s](z)dz$$

for all $F\in S_2(\Gamma_0(N))$.

PROOF. By the linearity, we may assume $F\in\sqrt{-1}S_2^{\boldsymbol{R}}(\Gamma_0(N))$. In the same way as in Lemma 2.9, we have, for odd primes p,

$$\int_0^{i\infty} F|T_2(p)(z)dz=(1+p)\int_0^{i\infty} F(z)dz-2\sum_{v=1}^{(p-1)/2}\int_0^{Nv/p} F(z)dz\,.$$

Apply Lemma 3.2. Then we have:

$$\int_0^{i\infty} F|T_2(p)(z)dz=(1+p)\int_0^{i\infty} F(z)dz-2\sum_{s=2}^{N}\sum_{v=1}^{(p-1)/2} a_s(Nv/p)\int_0^{i\infty} F|_2[g_s](z)dz$$

for all $F\in\sqrt{-1}S_2^{\boldsymbol{R}}(\Gamma_0(N))$ (and hence for all $F\in S_2(\Gamma_0(N))$).

(3.4) FINAL STEP OF THE PROOF OF THEOREM 0.3 IN CASE OF $w+2=2$ AND A PRIME N.

Let $f(z)=\sum_{n=1}^{\infty} a_n \exp(2\pi\sqrt{-1}nz)$ $(a_1=1)$ be a primitive form in $S_2(\Gamma_0(N))$. Let $\boldsymbol{r}_1=\{r_1(\bar{g}_s,\ 0)\}_{1\leq s\leq N+1}$ (resp. $\boldsymbol{r}_2=\{r_2(\bar{g}_s,\ 0)\}_{1\leq s\leq N+1}$) be a column vector in $S^-(\Gamma_0(N))_0$ (resp. $S^+(\Gamma_0(N))_0)\subset\boldsymbol{R}^{N+1}$ with $M(n)\boldsymbol{r}_1=a_n\boldsymbol{r}_1$ (resp. $M(n)\boldsymbol{r}_2=a_n\boldsymbol{r}_2$) for all the positive integers n with $(n,\ N)=1$. By Lemmas 1.5 and 3.1, there exists a cusp form f_1 (resp. f_2) in $S_2^{\boldsymbol{R}}(\Gamma_0(N))$ (resp. $\sqrt{-1}S_2^{\boldsymbol{R}}(\Gamma_0(N))$) such that

(3.4.0), $$\mathrm{Re}\int_0^{i\infty} f_1|_2[g_s](z)dz=r_1(\bar{g}_s,\ 0)$$

$$\left(\text{resp. } \mathrm{Re}\int_0^{i\infty} f_2|_2[g_s](z)dz=r_2(\bar{g}_s,\ 0)\right)$$

for all the integers $s\in\boldsymbol{Z}$ with $2\leqq s\leqq N$. Let p be a prime with $(p,\ 2N)=1$. We shall show first:

(3.4.1), $$\mathrm{Re}\left\{\int_0^{i\infty} f_1|T_2(p)(z)dz-(1+p)\int_0^{i\infty} f_1(z)dz\right\}$$
$$=(a_p-1-p)r_1(\bar{g}_1,\ 0)$$
$$\left(\text{resp. } \mathrm{Re}\left\{\int_0^{i\infty} f_2|T_2(p)(z)dz-(1+p)\int_0^{i\infty} f_2(z)dz\right\}\right.$$
$$\left.=(a_p-1-p)r_2(\bar{g}_1,\ 0)\right).$$

PROOF OF (3.4.1). Apply Lemma 3.3. Then we have:

$$\textit{the left side of } (3.4.1)=\sum_{s=2}^{N} b(s,\ p)\,\mathrm{Re}\int_0^{i\infty} f_1|_2[g_s](z)dz$$

$$=\sum_{s=2}^{N} b(s,\ p) r_1(\bar{g}_s,\ 0)\,.$$

Since $M(p)\boldsymbol{r}_1=a_p\boldsymbol{r}_1$, we obtain from Lemma 3.2 that

$$\sum_{s=2}^{N} b(s,\ p) r_1(\bar{g}_s,\ 0)=(a_p-1-p) r_1(\bar{g}_1,\ 0)\,.$$

Hence (3.4.1) is proved for f_1. In the same way, (3.4.1) is proved for f_2, q.e.d.

Since $S_2(\Gamma_0(N))$ is a semi-simple $\boldsymbol{C}[\{T_2(n)\,|\,n>0,\ n\in\boldsymbol{Z},\ (n,\ N)=1\}]$-module, we obtain also from (1.11) that

$$\left|(1+p)^{-1}\operatorname{Re}\int_0^{i\infty} f_1\,|\,T_2(p)(z)dz\right|\longrightarrow 0$$

$$\left(\text{resp.}\quad \left|(1+p)^{-1}\operatorname{Re}\int_0^{i\infty} f_2\,|\,T_2(p)(z)dz\right|\longrightarrow 0\right)$$

when $p\rightarrow+\infty$. Hence we obtain from (3.4.1),

$$\operatorname{Re}\int_0^{i\infty} f_1(z)dz=r_1(\bar{g}_1,\ 0) \tag{3.4.2}$$

$$\left(\text{resp.}\quad \operatorname{Re}\int_0^{i\infty} f_2(z)dz=r_2(\bar{g}_1,\ 0)\right).$$

The vectors $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\operatorname{Re}\int_0^{i\infty} D(f_1)$ and $\operatorname{Re}\int_0^{i\infty} D(f_2)$ satisfy the formulae $\{(B_{j,0})\}$ in (1.3) Definition. Hence (3.4.0) and (3.4.2) assert that

$$\boldsymbol{r}_1=\operatorname{Re}\int_0^{i\infty} D(f_1)\quad\text{and}\quad \boldsymbol{r}_2=\operatorname{Re}\int_0^{i\infty} D(f_2)\,.$$

We have assumed that

$$M(n)\boldsymbol{r}_1=a_n\boldsymbol{r}_1\qquad (\text{resp.}\ M(n)\boldsymbol{r}_2=a_n\boldsymbol{r}_2)$$

for all $n\in\boldsymbol{Z}$ with $(n,\ N)=1$. By the same argument as in § 2, we obtain

$$f_1\,|\,T_2(n)=a_n f_1\qquad (\text{resp.}\ f_2\,|\,T_2(n)=a_n f_2)$$

for all $n\in\boldsymbol{Z}$ with $(n,\ N)=1$. By multiplicity one theorem there exists a unique $c_1\in\boldsymbol{R}$ (resp. $c_2\in\boldsymbol{R}$) such that $f_1=c_1 f$ (resp. $f_2=\sqrt{-1}c_2 f$). Hence we obtain:

$$\boldsymbol{r}_1=c_1\operatorname{Re}\left(\int_0^{i\infty} D(f)\right)\quad\text{and}\quad \boldsymbol{r}_2=c_2\operatorname{Im}\left(\int_0^{i\infty} D(f)\right).$$

Hence W^- (resp. W^+) is one dimensional over $\boldsymbol{R}$, q. e. d.

§4. Applications of Theorem 0.3.

(4.1) PROOFS OF THEOREMS 0.4 AND 0.5.

By Theorem 0.3, the ratio of the components of $\mathrm{Im}\left(\int_0^{i\infty} D(f)\right)$ is obtained by finite steps as is shown in §2 (resp. §3). Then Theorem 0.4 (resp. 0.5) is a direct consequence of Lemma 2.9 (resp. Lemma 3.3).

For the explicit computations of the ratio of the components of $\mathrm{Re}\int_0^{i\infty} D(f)$ $\left(\text{resp. } \mathrm{Im}\int_0^{i\infty} D(f)\right)$, the following proposition is useful (or saves much efforts).

(4.2) PROPOSITION (Hatada [**10**]). *If N is a square free integer,* 4 *or* 8, $B^{-}=\{0\}$ *for* $\Gamma=\Gamma_0(N)$.

(4.3) EXAMPLES. The following are obtained by the computation by hand.

Example 1. Let $f(z)=\sum_{n=1}^{\infty} a_n \exp(2\pi\sqrt{-1}nz)$ be the unique primitive form in $S_8(\Gamma_0(2))$ $(a_1=1)$. Set $g_1=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $g_2=\begin{pmatrix}1&0\\1&1\end{pmatrix}$, $\boldsymbol{r}_1=\mathrm{Re}\left(\int_0^{i\infty} D(f)\right)$ and $\boldsymbol{r}_2=\mathrm{Im}\left(\int_0^{i\infty} D(f)\right)$. Then we have:

$$(r_1(\bar{g}_1, 1) : r_1(\bar{g}_1, 3) : r_1(\bar{g}_1, 5) : r_1(\bar{g}_2, 1) : r_1(\bar{g}_2, 3)) = (8 : -3 : 2 : -10 : 6)$$

and

$$(r_2(\bar{g}_1, 0) : r_2(\bar{g}_1, 2) : r_2(\bar{g}_1, 4) : r_2(\bar{g}_1, 6) : r_2(\bar{g}_2, 0) : r_2(\bar{g}_2, 2))$$
$$=(120 : -34 : 17 : -15 : -135 : 51).$$

G.C.M. of $\{-34, 17, 51\}=17$. Hence we have

$$a_p \equiv 1+p^7 \pmod{17} \qquad \text{for all the odd primes } p.$$

Example 2. Let $f(z)=\sum_{n=1}^{\infty} a_n \exp(2\pi\sqrt{-1}nz)$ be the unique primitive form in $S_{10}(\Gamma_0(2))$. Set $g_1=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $g_2=\begin{pmatrix}1&0\\1&1\end{pmatrix}$, $\boldsymbol{r}_1=\mathrm{Re}\left(\int_0^{i\infty} D(f)\right)$ and $\boldsymbol{r}_2=\mathrm{Im}\left(\int_0^{i\infty} D(f)\right)$. Then we have:

$$(r_1(\bar{g}_1, 1) : r_1(\bar{g}_1, 3) : r_1(\bar{g}_1, 5) : r_1(\bar{g}_1, 7) : r_1(\bar{g}_2, 1) : r_1(\bar{g}_2, 3))$$
$$=(-8 : 2 : -1 : 1 : 7 : -1)$$

and

$$(r_2(\bar{g}_1, 0) : r_2(\bar{g}_1, 2) : r_2(\bar{g}_1, 4) : r_2(\bar{g}_1, 6) : r_2(\bar{g}_1, 8) : r_2(\bar{g}_2, 0) : r_2(\bar{g}_2, 2) : r_2(\bar{g}_2, 4))$$
$$=(-3360 : 620 : -217 : 155 : -210 : 3150 : -465 : 0).$$

G.C.M. of $\{620, -217, 155, -465\}=31$. Hence we obtain:

$$a_p \equiv 1+p^9 \pmod{31} \qquad \text{for all the odd primes } p.$$

Example 3. Let $f(z)$ be the unique primitive form in $S_6(\Gamma_0(4))$. Set $\boldsymbol{r}_2=\mathrm{Im}\left(\int_0^{i\infty} D(f)\right)$, $g_1=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $g_2=\begin{pmatrix}1&0\\1&1\end{pmatrix}$ and $g_3=\begin{pmatrix}1&0\\2&1\end{pmatrix}$. We have:

$$\begin{aligned}&(r_2(\bar{g}_1, 0) : r_2(\bar{g}_1, 2) : r_2(\bar{g}_1, 4) : r_2(\bar{g}_3, 0) : r_2(\bar{g}_3, 2) : r_2(\bar{g}_3, 4) :\\ &\quad r_2(\bar{g}_2, 0) : r_2(\bar{g}_2, 1) : r_2(\bar{g}_2, 2))\\ &\quad =(-48:8:-3:48:-8:3:0:-24:0).\end{aligned}$$

G.C.M. of $\{8, -8, -24, 0\}=8$. 8 divides 48.

Example 4. Let $f(z)=\sum_{n=1}^{\infty} a_n \exp(2\pi\sqrt{-1}nz)$ $(a_1=1)$ be the unique primitive form in $S_4(\Gamma_0(6))$. Set $\boldsymbol{r}_2=\mathrm{Im}\int_0^{i\infty} D(f)$, $g_1=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $g_2=\begin{pmatrix}1&0\\3&1\end{pmatrix}$, $g_3=\begin{pmatrix}2&1\\3&2\end{pmatrix}$, $g_4=\begin{pmatrix}1&0\\1&1\end{pmatrix}$ and $g_5=\begin{pmatrix}1&0\\2&1\end{pmatrix}$. Then we have:

$$\begin{aligned}&(r_2(\bar{g}_1, 0) : r_2(\bar{g}_2, 0) : r_2(\bar{g}_2, 2) : r_2(\bar{g}_3, 2) : r_2(\bar{g}_4, 0) : r_2(\bar{g}_5, 0) : r_2(\bar{g}_5, 1))\\ &\quad =(-3:0:0:1:-3:-1:-1).\end{aligned}$$

Note that 3 is odd. Hence we obtain:

$$a_p \equiv 0 \pmod 2 \qquad \text{for all the primes } p\neq 2, 3.$$

(4.4) Examples 4-6 in [7] are obtained by our Theorems 0.3 and 0.5 (cf. Theorems 7.9 and 8.3 computations of the table in Manin [**11**]).

(4.5) REMARK. Let $f(z)=\sum_{n=1}^{\infty} a_n \exp(2\pi\sqrt{-1}nz)$ be a primitive form in $S_{w+2}(\Gamma_0(N))$. If the algebraic number field $\boldsymbol{Q}(a_1, a_2, a_3, \cdots)$ is an extension of a small degree over $\boldsymbol{Q}$, for example, 1, 2, 3 or 4, the ratio of the components of the vector $\mathrm{Im}\int_0^{i\infty} D(f)$ is computed easily.

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Added in proof.

From proposition 4.2, we obtain $\dim_C W^- = 1$ also in the case of $w+2=2$ and N: a square free integer.