J. Math. Soc. Japan
Vol. 33, No. 3, 1981

Differentiability of solutions of some unilateral problem of parabolic type

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(Received June 4, 1979)
(Revised July 30, 1979)

Let us begin with the following simple example of a parabolic unilateral problem

$$\begin{aligned} &\partial u/\partial t-\Delta u\geqq 0, \quad u\geqq \Psi \\ &(\partial u/\partial t-\Delta u)(u-\Psi)=0 \end{aligned} \quad \text{in} \quad \Omega\times(0, T] \tag{0.1}$$

$$u=0 \quad \text{on} \quad \Gamma\times(0, T] \tag{0.2}$$

$$u(x, 0)=u_0(x)\geqq \Psi(x) \quad \text{in} \quad \Omega. \tag{0.3}$$

Here Ω is a domain in R^N with sufficiently smooth boundary Γ, and Ψ is a function such that $\Psi\in W^{2,p}(\Omega)$ and $\Psi|_\Gamma\leqq 0$. We wish to make p small; however, assume

$$1<p<2<p^*=pN/(N-p). \tag{0.4}$$

In view of Sobolev's imbedding theorem it follows that

$$W^{2,p}(\Omega)\subset H^1(\Omega)\subset L^{p'}(\Omega), \qquad p'=p/(p-1). \tag{0.5}$$

Let L_q be the realization of $-\Delta$ in $L^q(\Omega)$ under the Dirichlet boundary condition, and M_q be the multivalued mapping defined by

$$D(M_q)=\{u\in L^q(\Omega): u\geqq \Psi \text{ a.e. in } \Omega\}, \tag{0.6}$$

$$\begin{aligned} M_q u=\{g\in L^q(\Omega): g\leqq 0 \text{ a.e. in } \Omega, \\ g(x)=0 \quad \text{if} \quad u(x)>\Psi(x)\}. \end{aligned} \tag{0.7}$$

Note that $M_2=\partial I_K$ where I_K is the indicatrix of the closed convex set $K=D(M_2)$. The problem (0.1)-(0.3) is formulated in $L^p(\Omega)$ as

$$du(t)/dt+(L_p+M_p)u(t)\ni 0 \tag{0.8}$$

$$u(0)=u_0. \tag{0.9}$$

This work was partly supported by Grant-in-Aid for Scientific Research 454029.

It can be shown that L_p+M_p is m-accretive, and hence we can apply a result of M. G. Crandall and T. M. Liggett [6] to construct the solution $u(t)$ of (0.8), (0.9) in some sense by an exponential formula. We are interested in the differentiability of this solution with respect to t assuming only $\Psi\leqq u_0\in L^p(\Omega)$ or $u_0\in\overline{D(L_p+M_p)}$ for the initial value u_0. With the aid of a comparison theorem we can show $u(t)\in L^2(\Omega)$ for $t>0$. Hence noting that $\Psi\in H^1(\Omega)$ in view of (0.5) we may consider $u(t)$ as the solution of

$$du(t)/dt+\partial\phi(u(t))\ni 0 \tag{0.10}$$

in $(0, T]$, where $\phi: L^2(\Omega)\mapsto[0, \infty]$ is the convex function

$$\phi(u)=\begin{cases}\dfrac{1}{2}\displaystyle\int_\Omega|\nabla u|^2dx & \text{if} \quad \Psi\leqq u\in H_0^1(\Omega)\,, \\ \infty & \text{otherwise}\,.\end{cases}$$

Thus we may apply a general result on the subdifferential of a convex function to establish the differentiability of $u(t)$ in $L^2(\Omega)$. With the aid of another application of a comparison theorem we can show that $du(t)/dt\in L^r(\Omega)$ for any $r>2$, if $t>0$. We note $L_2+M_2\subsetneqq\partial\phi$ in general under our hypothesis as the following counter example shows. Suppose Ψ is such that $\Psi\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ $=D(L_p)$ and $0\leqq-\Delta\Psi\notin L^2(\Omega)$. Let v be an arbitrary element of $D(\phi)$. Then $v-\Psi\in L^{p'}(\Omega)$ by virtue of (0.5). Hence with the aid of an integration by part

$$0\leqq(-\Delta\Psi,\ v-\Psi)=(\nabla\Psi,\ \nabla v-\nabla\Psi)\leqq\phi(v)-\phi(\Psi)\,,$$

which implies $\Psi\in D(\partial\phi)$. However $\Psi\notin D(L_2+M_2)=D(L_2)\cap D(M_2)$ since $\Delta\Psi\notin L^2(\Omega)$.

In this paper we consider the more general problem

$$\left.\begin{array}{l}\partial u/\partial t+\mathcal{L}u\geqq f,\ u\geqq\Psi \\ (\partial u/\partial t+\mathcal{L}u-f)(u-\Psi)=0\end{array}\right\} \quad \text{in} \quad \Omega\times(0, T] \tag{0.11}$$

$$-\partial u/\partial n\in\beta(x, u) \qquad \text{on} \quad \Gamma\times(0, T] \tag{0.12}$$

$$u(x, 0)=u_0(x) \qquad \text{in} \quad \Omega\,. \tag{0.13}$$

Here Ω is not assumed to be bounded. $\mathcal{L}$ is a not necessarily symmetric linear elliptic operator of second order, and $\partial/\partial n$ is the differentiation in the outward conormal direction with respect to $\mathcal{L}$. $\beta(x, \cdot)$ is a maximal monotone graph in R^2 with $0\in\beta(x, 0)$ for each fixed $x\in\Gamma$. Ψ is a function such that

$$\Psi\in W^{2,p}(\Omega),\ \partial\Psi/\partial n+\beta^-(x, \Psi)\leqq 0 \qquad \text{on} \quad \Gamma \tag{0.14}$$

with p satisfying (0.4). $\beta^-(x, r)$, which will be defined later, is roughly speaking $\min\beta(x, r)$.

First we formulate the elliptic boundary value problem

$$\mathcal{L}u=f \text{ in } \Omega, \ -\partial u/\partial n\in\beta(x, u) \quad \text{on} \quad \Gamma \tag{0.15}$$

in $L^2(\Omega)$ as some variational problem. With the aid of a result of H. Brézis [2] the problem thus formulated is expressed as $L_2u=f$ with some single-valued m-accretive operator L_2 in $L^2(\Omega)$. Since $(1+\lambda L_2)^{-1}$ is a contraction for $\lambda>0$ also in L^q norm, $1\leqq q<\infty$, an m-accretive operator L_q in $L^q(\Omega)$ is defined as the smallest closed extension of the operator with graph $G(L_2)\cap(L^q(\Omega)\times L^q(\Omega))$, where $G(L_2)$ is the graph of L_2. Thus for $1\leqq q<\infty$ the problem (0.15) is formulated in $L^q(\Omega)$ as $L_qu=f$. Following the idea of B. D. Calvert and C. P. Gupta [5] it is shown that $D(L_q)\subset W^{1,q}(\Omega)$ for $1<q\leqq 2$, which will be used frequently in the subsequent argument.

In addition to (0.14) we assume also

$$\Psi\in W^{1,1}(\Omega), \qquad \mathcal{L}\Psi\in L^1(\Omega).$$

Then it is shown that $A_q=L_q+M_q$ is m-accretive in $L^q(\Omega)$ for $1\leqq q\leqq p$, where M_q is the mapping defined by (0.6) and (0.7). For $p<q\leqq 2$ A_q is defined as the m-accretive extension of L_q+M_q. If $f\in W^{1,1}(0, T; L^q(\Omega))$ and $\Psi\leqq u_0\in L^q(\Omega)$, the problem (0.11)-(0.13) is expressed as

$$du(t)/dt+A_qu(t)\ni f(t), \qquad 0<t\leqq T,$$

$$u(0)=u_0.$$

With the aid of Theorem 5.1 of M. G. Crandall and A. Pazy [7] it is possible to construct the solution of this problem by an exponential formula. Suppose further $f\in W^{1,1}(0, T; L^q(\Omega)\cap L^r(\Omega))$ for $1\leqq q\leqq 2\leqq r$. Then by a comparison theorem it follows that $u(t)\in L^2(\Omega)$ for $t>0$. Instead of (0.10) we have

$$du(t)/dt+Au(t)\ni f(t) \tag{0.16}$$

this time where A is the mapping defined by $Au=(Lu+\partial\phi(u))\cap L^2(\Omega)$, L is the linear isomorphism from $H^1(\Omega)$ onto $H^1(\Omega)^*$ associated with $\mathcal{L}$ and ϕ is some proper convex function on $H^1(\Omega)$ associated with β and Ψ. It will be shown that the solution of (0.16) constructed by the exponential formula is differentiable a.e. (Theorem 6.1). As in the problem (0.1)-(0.3) we can show that $du(t)/dt \in L^r(\Omega)$ for $t>0$ with the aid of a comparison theorem following F. J. Massey, III [11] and L. C. Evans [8], [9]. The main theorem of the present paper is Theorem 7.1. Related results are found in the above papers of Massey and Evans. In [11] the equation of the form

$$\partial u/\partial t+\mathcal{L}u+\beta(u)\ni f \tag{0.17}$$

is studied, and in [8], [9] various types of problems including (0.17) are investigated.

The result of this paper was announced in [**13**] and [**14**].

§ 1. Assumptions and notations.

All functions considered in this paper are real valued.

Let Ω be a not necessarily bounded domain in R^N. We assume that the boundary Γ of Ω is uniformly regular of class C^2 and locally regular of class C^4 in the sense of F.E. Browder [3]. $W^{m,p}(\Omega)$ denotes the usual Sobolev space and $H^m(\Omega)=W^{m,2}(\Omega)$. The norm of $W^{m,p}(\Omega)$ is denoted by $\| \ \|_{m,p}$ and that of $L^p(\Omega)$ is simply by $\| \ \|_p$ if there is no fear of confusion. $W^{1-1/p,p}(\Gamma)$ is the set of the boundary values of functions belonging to $W^{1,p}(\Omega)$. $W^{1-1/p,p}(\Gamma)$ is a Banach space with norm

$$[h]_{1-1/p,p}=\inf\{\|u\|_{1,p}: u\in W^{1,p}(\Omega),\ u=h \text{ on } \Gamma\}\ .$$

We denote by $\rightarrow$ strong convergence and by $\rightharpoonup$ weak convergence. For a mapping A multivalued in general $D(A)$, $R(A)$ and $G(A)$ stand for its domain, range and graph respectively.

Let

$$a(u, v)=\int_\Omega\left(\sum_{i,j=1}^N a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}+\sum_{i=1}^N b_i\frac{\partial u}{\partial x_i}v+cuv\right)dx \tag{1.1}$$

be a bilinear form defined in $H^1(\Omega)\times H^1(\Omega)$. The coefficients a_{ij}, b_i are bounded and continuous in $\bar{\Omega}$ together with first derivatives and c is bounded and measurable in Ω. $\{a_{ij}(x)\}$ is uniformly positive definite in Ω, i.e. for some positive constant δ

$$\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j\geqq\delta|\xi|^2, \qquad x\in\Omega,\quad \xi\in R^N\ . \tag{1.2}$$

We assume that there exists a positive constant α such that

$$c\geqq\alpha\,, \qquad c-\sum_{i=1}^N \partial b_i/\partial x_i\geqq\alpha \quad \text{a.e. in } \Omega\,. \tag{1.3}$$

We denote by $\mathcal{L}$ the linear differential operator associated with the bilinear form (1.1):

$$\mathcal{L}=-\sum_{i,j=1}^N \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial}{\partial x_i}\right)+\sum_{i=1}^N b_i\frac{\partial}{\partial x_i}+c\,.$$

The conormal derivative with respect to $\mathcal{L}$ is denoted by

$$\partial/\partial n=\sum_{i,j=1}^N a_{ij}\nu_j\partial/\partial x_j$$

where $\nu=(\nu_1, \cdots, \nu_N)$ is the outward normal vector to Γ.

Let $j(x, r)$ be a function defined on $\Gamma\times R$ such that for each fixed $x\in\Gamma$ $j(x, r)$ is a proper convex lower semicontinuous function of r such that

$$j(x, r)\geqq j(x, 0)=0. \tag{1.4}$$

We denote by $\beta(x, \cdot)=\partial j(x, \cdot)$ the subdifferential of $j(x, r)$ with respect to r. As for the regularity with respect to x we assume that for each $t\in R$ and $\lambda>0$ $(1+\lambda\beta(x, \cdot))^{-1}(t)$ is a measurable function of x (cf. B. D. Calvert and C. P. Gupta [5]). Unless $j(x, r)=\infty$ for $r\neq 0$ (namely the boundary condition is of Dirichlet type), we assume that

$$\sum_{i=1}^{N} b_i\nu_i\geqq 0 \qquad \text{on } \Gamma. \tag{1.5}$$

Let $\Psi(x)$ be a function satisfying

$$\Psi\in W^{2,p}(\Omega), \tag{1.6}$$

$$\Psi\in W^{1,1}(\Omega), \qquad \mathcal{L}\Psi\in L^1(\Omega), \tag{1.7}$$

$$\partial\Psi(x)/\partial n+\beta^-(x, \Psi(x))\leqq 0 \qquad x\in\Gamma \tag{1.8}$$

where p is an exponent satisfying

$$1<p<2<p^*=Np/(N-p) \tag{1.9}$$

and

$$\beta^-(x, r)=\begin{cases} \min\{z: z\in\beta(x, r)\} & \text{if } r\in D(\beta(x, \cdot)), \\ \infty & \text{if } r\notin D(\beta(x, \cdot)) \text{ and } r\geqq\sup D(\beta(x, \cdot)), \\ -\infty & \text{if } r\notin D(\beta(x, \cdot)) \text{ and } r\leqq\inf D(\beta(x, \cdot)) \end{cases}$$

(cf. p. 55 of H. Brézis [2]).

§ 2. Preliminaries (1).

In this section we collect some preliminary results mainly due to H. Brézis [2] and B. D. Calvert and C. P. Gupta [5] concerning the boundary value problem $\mathcal{L}u=f$ in Ω, $-\partial u/\partial n\in\beta(u)$ on Γ. Here $\beta(u)$ stands for the (multivalued in general) function $x\mapsto\beta(x, u(x))$. In our case the proofs are simpler than those of the corresponding results of [5] since $\mathcal{L}$ is linear and we can use the Yosida approximation of $\beta(x, \cdot)$ according to Proposition 2.1 below.

Let $a(u, v)$ be a bilinear form (1.1) such that

$$a(u, u)\geqq c_0\|u\|_{1,2}^2, \qquad u\in H^1(\Omega) \tag{2.1}$$

for some $c_0>0$. It will be shown in Lemma 2.2 that such a constant c_0 exists under our hypothesis. Let Φ be a proper convex, lower semicontinuous convex function defined in $L^2(\Gamma)$ such that $\Phi\not\equiv\infty$ on $H^{1/2}(\Gamma)$. Then it is known that for any $f\in L^2(\Omega)$ there exists a unique solution $u\in H^1(\Omega)$, $\Phi(u|_\Gamma)<\infty$, of the inequality

$$a(u,\ v-u)+\Phi(v|_\Gamma)-\Phi(u|_\Gamma)\geqq(f,\ v-u)\,,\qquad v\in H^1(\Omega)\,. \tag{2.2}$$

Furthermore the solution is characterized by

$$\mathcal{L}u=f \quad \text{in } \Omega \text{ in the distribution sense}, \tag{2.3}$$

$$-\partial u/\partial n\in\partial\tilde{\phi}(u|_\Gamma) \tag{2.4}$$

where $\tilde{\phi}$ is the restriction of ϕ to $H^{1/2}(\Gamma)$ (cf. Theorem 1.7 of [2]).

For $\varepsilon>0$ let

$$\Phi_\varepsilon(u)=\frac{1}{2\varepsilon}\int_\Gamma(u-J_\varepsilon u)^2d\Gamma+\Phi(J_\varepsilon u)$$

be the Yosida approximation of Φ where $J_\varepsilon=(1+\varepsilon\partial\Phi)^{-1}$.

PROPOSITION 2.1. *For $f\in L^2(\Omega)$ let $u_\varepsilon\in H^1(\Omega)$ be the solution of the inequality*

$$a(u_\varepsilon,\ v-u_\varepsilon)+\Phi_\varepsilon(v|_\Gamma)-\Phi_\varepsilon(u_\varepsilon|_\Gamma)\geqq(f,\ v-u_\varepsilon)\,,\qquad v\in H^1(\Omega)\,.$$

Then

$$-\partial u_\varepsilon/\partial n=\Phi_\varepsilon'(u_\varepsilon|_\Gamma)\in L^2(\Gamma)\,. \tag{2.5}$$

As $\varepsilon\to0$ u_ε converges to the solution u of (2.2) *in the strong topology of $H^1(\Omega)$ and*

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\int_\Gamma(u_\varepsilon-J_\varepsilon u_\varepsilon)^2d\Gamma=0\,. \tag{2.6}$$

PROOF. This proposition was proved by H. Brézis (Theorem 1.8 of [2]) under the assumption that Ω is bounded. In case Ω is unbounded the proof is essentially unchanged and hence we only sketch it. If we put

$$\tilde{a}(u,\ v)=(a(u,\ v)+a(v,\ u))/2\,,$$

then $\tilde{a}(u,\ u)=a(u,\ u)\geqq c_0\|u\|_{1,2}^2$. Hence $a(u,\ u)^{1/2}=\tilde{a}(u,\ u)^{1/2}$ may be considered as a norm of $H^1(\Omega)$. As in Theorem 1.8 of [2], $a(u_\varepsilon,\ u_\varepsilon)$ and $\varepsilon^{-1}\int_\Gamma(u_\varepsilon-J_\varepsilon u_\varepsilon)^2d\Gamma$ are bounded as $\varepsilon\to0$. If $u_{\varepsilon_n}\rightharpoonup u^*$ in $H^1(\Omega)$, then $u_{\varepsilon_n}|_\Gamma\rightharpoonup u^*|_\Gamma$ in $H^{1/2}(\Gamma)$ and $J_{\varepsilon_n}u_{\varepsilon_n}|_\Gamma\rightharpoonup u^*|_\Gamma$ in $L^2(\Gamma)$. Letting $\varepsilon=\varepsilon_n\to0$ in

$$a(u_\varepsilon,\ v)+\Phi_\varepsilon(v|_\Gamma)\geqq(f,\ v-u_\varepsilon)+a(u_\varepsilon,\ u_\varepsilon)+\Phi(J_\varepsilon u_\varepsilon|_\Gamma)$$

we get

$$a(u^*,\ v)+\Phi(v|_\Gamma)\geqq(f,\ v-u^*)+\limsup a(u_{\varepsilon_n},\ u_{\varepsilon_n})+\Phi(u^*|_\Gamma)\,.$$

Hence $\Phi(u^*|_\Gamma)<\infty$ and

$$a(u^*, u^*) \leqq \liminf a(u_{\varepsilon_n}, u_{\varepsilon_n}) \leqq \limsup a(u_{\varepsilon_n}, u_{\varepsilon_n})$$
$$\leqq a(u^*, v) + \Phi(v|_\Gamma) - \Phi(u^*|_\Gamma) - (f, v-u^*) .$$

Letting $v=u^*$ we get $a(u^*, u^*)=\lim a(u_{\varepsilon_n}, u_{\varepsilon_n})$, and hence $u_{\varepsilon_n}\to u^*$ in $H^1(\Omega)$. It is easily seen that u^* is a solution of (2.2), and hence $u^*=u$ and $u_\varepsilon \to u$ in $H^1(\Omega)$. (2.5) is established in Theorem 1.8 of [2]. The proof of (2.6) is easy and is omitted.

LEMMA 2.1. *Suppose χ is a uniformly Lipschitz continuous increasing function in R such that $\chi(0)=0$. Then for any $u\in H^1(\Omega)$*

$$a(u, \chi(u)) \geqq \alpha(u, \chi(u)) . \tag{2.7}$$

PROOF. Let ζ be the indefinite integral of χ such that $\zeta(0)=0$. (2.7) is easily established by noting

$$\partial u/\partial x_i \cdot \chi(u) = \partial \zeta(u)/\partial x_i , \qquad u\chi(u) \geqq \zeta(u)$$

and using (1.2), (1.3), (1.5).

LEMMA 2.2. *For any $u\in H^1(\Omega)$*

$$a(u, u) \geqq \min \{\delta, \alpha\} \|u\|_{1,2}^2 . \tag{2.8}$$

PROOF. (2.8) is clear from the proof of Lemma 2.1.

In what follows $\Phi : L^2(\Gamma) \mapsto [0, \infty]$ denotes the function

$$\Phi(u) = \begin{cases} \int_\Gamma j(x, u(x))d\Gamma & \text{if} \quad j(u)\in L^1(\Gamma) \\ \infty & \text{otherwise} \end{cases} \tag{2.9}$$

where $j(u)$ is the function $j(x, u(x))$. By the proof of Lemma 3.1 of [5] $j(x, u(x))$ is measurable for $u\in L^2(\Gamma)$ and Φ is proper convex, lower semicontinuous on $L^2(\Gamma)$.

DEFINITION 2.1. L_2 is the operator with domain and range contained in $L^2(\Omega)$ such that $L_2 u=f$ if $f\in L^2(\Omega)$, $u\in H^1(\Omega)$, $\Phi(u|_\Gamma)<\infty$ and (2.2) holds.

Note that L_2 is single valued since (2.3) holds if $L_2 u=f$. It is known that the following proposition holds.

PROPOSITION 2.2 *L_2 is m-accretive and $R(L_2)=L^2(\Omega)$.*

For the Yosida approximation Φ_ε of Φ we have

$$\partial\Phi_\varepsilon(u)(x) = \beta_\varepsilon(x, u(x)) \tag{2.10}$$

where $\beta_\varepsilon(x, \cdot)$ is the Yosida approximation of $\beta(x, \cdot)$. To simplify the notation we write $\beta_\varepsilon(u)$ to denote the function $\beta_\varepsilon(x, u(x))$.

We denote by $L_{2,\varepsilon}$ the operator defined as L_2 with Φ_ε in place of Φ in Definition 2.1. If $L_{2,\varepsilon} u_\varepsilon = f$, then

$$-\partial u_\varepsilon/\partial n=\beta_\varepsilon(u_\varepsilon)\in L^2(\Gamma) \tag{2.11}$$

(Theorem 1.8 of [2]) and in view of Proposition 2.1 $u_\varepsilon\to u$ in $H^1(\Omega)$ where u is the solution of $L_2u=f$.

DEFINITION 2.2. For $1\leqq q<\infty$ the operator L_q with domain and range contained in $L^q(\Omega)$ is defined by

$$G(L_q)=\text{the closure of } G(L_2)\cap(L^q(\Omega)\times L^q(\Omega)) \quad \text{in} \quad L^q(\Omega)\times L^q(\Omega).$$

LEMMA 2.3. *Let χ be a uniformly Lipschitz continuous increasing function in R such that $\chi(0)=0$. Then for any $u, v\in D(L_2)$*

$$(L_2u-L_2v, \chi(u-v))\geqq a(u-v, \chi(u-v)). \tag{2.12}$$

PROOF. (2.12) is easily established by approximating u, v by the solutions of $L_{2,\varepsilon}u_\varepsilon=L_2u$, $L_{2,\varepsilon}v_\varepsilon=L_2v$, and noting (2.11).

LEMMA 2.4. *Suppose $1\leqq q<\infty$, $\lambda>0$, $f, g\in L^2(\Omega)\cap L^q(\Omega)$,*

$$u+\lambda L_2u=f, \qquad v+\lambda L_2v=g. \tag{2.13}$$

Then $u, v\in L^q(\Omega)$ and

$$(1+\lambda\alpha)\|u-v\|_q\leqq\|f-g\|_q. \tag{2.14}$$

PROOF. First consider the case $1<q<2$. Let

$$\chi_n(t)=\begin{cases} |t|^{q-2}t & \text{if } |t|\geqq 1/n, \\ n^{2-q}t & \text{if } |t|<1/n. \end{cases}$$

In view of Lemmas 2.1 and 2.3

$$\alpha(u-v, \chi_n(u-v))\leqq(L_2u-L_2v, \chi_n(u-v)). \tag{2.15}$$

It follows from (2.13) and (2.15) that

$$(1+\lambda\alpha)(u-v, \chi_n(u-v))\leqq(f-g, \chi_n(u-v)). \tag{2.16}$$

Applying Hölder's inequality to the right side of (2.16) and noting for $q'=q/(q-1)$

$$\begin{aligned}&\int_\Omega|\chi_n(u-v)|^{q'}dx\\ &\leqq\int_{|u-v|\geqq 1/n}|u-v|^q dx+n^{2-q}\int_{|u-v|<1/n}(u-v)^2dx\\ &=\int_\Omega(u-v)\chi_n(u-v)dx,\end{aligned}$$

we get

$$(1+\lambda\alpha)\left\{\int_{|u-v|\geqq 1/n}|u-v|^q dx\right\}^{1/q} \tag{2.17}$$

$$\leqq(1+\lambda\alpha)\left\{\int_{\Omega}(u-v)\chi_n(u-v)dx\right\}^{1/q}\leqq\|f-g\|_q .$$

Letting $n\to\infty$ in (2.17) we see that $u-v\in L^q(\Omega)$ and (2.14) holds. Applying (2.14) for $v=g=0$ we get $u\in L^q(\Omega)$. Other cases are handled analogously.

In what follows we write for $q>1$

$$F_q(r)=|r|^{q-2}r \tag{2.18}$$

and

$$\text{sign}^0 r=\begin{cases} 1 & \text{if } r>0 \\ 0 & \text{if } r=0 \\ -1 & \text{if } r<0 . \end{cases} \tag{2.19}$$

PROPOSITION 2.3. *For* $1\leqq q<\infty$ L_q *is m-accretive and* $R(L_q)=L^q(\Omega)$. *For* $u,\ v\in D(L_q)$

$$\alpha\|u-v\|_q^q\leqq(L_qu-L_qv,\ F_q(u-v)) \qquad \textit{if} \quad q>1 , \tag{2.20}$$

$$\alpha\|u-v\|_1\leqq(L_1u-L_1v,\ \text{sign}^0(u-v)) \qquad \textit{if} \quad q=1 , \tag{2.21}$$

$$\alpha\|u\|_q\leqq\|L_qu\|_q \qquad \textit{for} \quad q\geqq 1 . \tag{2.22}$$

PROOF. The first part of the proposition is an easy consequence of Lemma 2.4. Letting $n\to\infty$ in (2.15) we get (2.20) and (2.21) for $u,\ v\in D(L_2)\cap L^q(\Omega)$, $L_2u,\ L_2v\in L^q(\Omega)$. For general $u,\ v\in D(L_q)$ these two inequalities are established by approximating $u,\ v$ according to the definition of L_q. Letting $v=0$ in (2.20), (2.21) we obtain (2.22).

LEMMA 2.5. *If* $1<q<2$, *then* $D(L_q)\subset W^{1,q}(\Omega)$ *and there exists a constant* c_q *such that for any* $u,\ v\in D(L_q)$

$$(L_qu-L_qv,\ F_q(u-v))\geqq c_q\|u-v\|_{1,q}^q . \tag{2.23}$$

For the proof of this lemma we refer to Proposition 3.2 of [**5**].

§ 3. Preliminaries (2).

Let Ψ and p be such that (1.6), (1.7), (1.8) and (1.9) hold. In view of Sobolev's imbedding theorem

$$W^{2,p}(\Omega)\subset H^1(\Omega)\subset L^{p'}(\Omega) , \qquad p'=p/(p-1) . \tag{3.1}$$

Let P be the operator defined by

$$(Pu)(x)=\max\{u(x),\ \Psi(x)\} .$$

Then

$$u-Pu=-(\Psi-u)^+ , \tag{3.2}$$

$$F_q(u-Pu)=-((\Psi-u)^+)^{q-1} \tag{3.3}$$

(recall (2.18) for the definition of F_q).

LEMMA 3.1. *If $\phi(r)$ is a uniformly Lipschitz continuous function which vanishes for $r<0$, then for any $u\in D(L_2)$*

$$(L_2u,\ \phi(\Psi-u))\leqq(\mathcal{L}\Psi,\ \phi(\Psi-u))\,. \tag{3.4}$$

PROOF. First note that by (3.1) $\Psi-u$ belongs to $L^2(\Omega)\cap L^{p'}(\Omega)$, and so does $\phi(\Psi-u)$ if $u\in D(L_2)$, and hence both sides of (3.4) are meaningful. Let u_ε be such that $L_{2,\varepsilon}u_\varepsilon=L_2u$. Then by Proposition 2.1

$$-\partial u_\varepsilon/\partial n=\beta_\varepsilon(u_\varepsilon)\in L^2(\Gamma)\,, \tag{3.5}$$

$$u_\varepsilon\to u \quad \text{in} \quad H^1(\Omega)\,, \tag{3.6}$$

$$\varepsilon\int_\Gamma \beta_\varepsilon(u_\varepsilon)^2 d\Gamma \longrightarrow 0\,. \tag{3.7}$$

In view of (3.1) $\Psi-u_\varepsilon\in H^1(\Omega)$, $\phi(\Psi-u_\varepsilon)\in H^1(\Omega)$. By Sobolev's imbedding theorem

$$\partial\Psi/\partial n\in W^{1-1/p,p}(\Gamma)\subset L^{p(N-1)/(N-p)}(\Gamma)\,, \tag{3.8}$$

$$\phi(\Psi-u_\varepsilon)|_\Gamma\in W^{1/2,2}(\Gamma)\subset L^{2(N-1)/(N-2)}(\Gamma)\,. \tag{3.9}$$

Since

$$\frac{N-p}{p(N-1)}+\frac{N-2}{2(N-1)}<1\,, \qquad \frac{1}{p}+\frac{1}{2}>1$$

there exist exponents q, r such that

$$\frac{N-p}{p(N-1)}\leqq\frac{1}{q}\leqq\frac{1}{p}\,, \qquad \frac{N-2}{2(N-1)}\leqq\frac{1}{r}\leqq\frac{1}{2}\,, \qquad \frac{1}{q}+\frac{1}{r}=1\,. \tag{3.10}$$

In view of (3.8), (3.9), (3.10)

$$\partial\Psi/\partial n\in L^q(\Gamma),\ \phi(\Psi-u_\varepsilon)|_\Gamma\in L^r(\Gamma)$$

which implies

$$\partial\Psi/\partial n\cdot\phi(\Psi-u_\varepsilon)|_\Gamma\in L^1(\Gamma)\,. \tag{3.11}$$

Therefore

$$(\mathcal{L}(\Psi-u_\varepsilon),\ \phi(\Psi-u_\varepsilon))=-\int_\Gamma\Big(\frac{\partial\psi}{\partial n}-\frac{\partial u_\varepsilon}{\partial n}\Big)\phi(\Psi-u_\varepsilon)d\Gamma$$
$$+a(\Psi-u_\varepsilon,\ \phi(\Psi-u_\varepsilon))\,.$$

Hence

$$(L_{2,\varepsilon}u_\varepsilon,\ \phi(\Psi-u_\varepsilon))=(\mathcal{L}u_\varepsilon,\ \phi(\Psi-u_\varepsilon))$$
$$=(\mathcal{L}\Psi,\ \phi(\Psi-u_\varepsilon))-(\mathcal{L}(\Psi-u_\varepsilon),\ \phi(\Psi-u_\varepsilon)) \tag{3.12}$$

$$=(\mathcal{L}\Psi,\ \phi(\Psi-u_\varepsilon))+\int_\Gamma\Big(\frac{\partial\Psi}{\partial n}-\frac{\partial u_\varepsilon}{\partial n}\Big)\phi(\Psi-u_\varepsilon)d\Gamma$$
$$-a(\Psi-u_\varepsilon,\ \phi(\Psi-u_\varepsilon)).$$

By (3.1) and (3.6) $\Psi-u_\varepsilon\to\Psi-u$ in $L^2(\Omega)\cap L^{p'}(\Omega)$ as $\varepsilon\to 0$, and hence $\phi(\Psi-u_\varepsilon)\to\phi(\Psi-u)$ in $L^2(\Omega)\cap L^{p'}(\Omega)$. Thus

$$\lim_{\varepsilon\to 0}(L_{2,\varepsilon}u_\varepsilon,\ \phi(\Psi-u_\varepsilon))=(L_2u,\ \phi(\Psi-u)), \tag{3.13}$$

$$\lim_{\varepsilon\to 0}(\mathcal{L}\Psi,\ \phi(\Psi-u_\varepsilon))=(\mathcal{L}\Psi,\ \phi(\Psi-u)). \tag{3.14}$$

As for the boundary integral in (3.12)

$$\int_\Gamma\Big(\frac{\partial\Psi}{\partial n}-\frac{\partial u_\varepsilon}{\partial n}\Big)\phi(\Psi-u_\varepsilon)d\Gamma$$
$$=\int_\Gamma\Big(\frac{\partial\Psi}{\partial n}-\frac{\partial u_\varepsilon}{\partial n}\Big)\phi(\Psi-(1+\varepsilon\beta)^{-1}u_\varepsilon)d\Gamma \tag{3.15}$$
$$+\int_\Gamma\Big(\frac{\partial\Psi}{\partial n}-\frac{\partial u_\varepsilon}{\partial n}\Big)(\phi(\Psi-u_\varepsilon)-\phi(\Psi-(1+\varepsilon\beta)^{-1}u_\varepsilon))d\Gamma.$$

If $\phi(\Psi(x)-(1+\varepsilon\beta)^{-1}u_\varepsilon(x))\neq 0$, $x\in\Gamma$, then $\Psi(x)>(1+\varepsilon\beta(x,\cdot))^{-1}u_\varepsilon(x)$, which implies $\beta^-(x,\Psi(x))\geqq\beta_\varepsilon(x,u_\varepsilon(x))$. Hence in view of (1.8) and (3.5) $\partial\Psi(x)/\partial n\leqq\partial u_\varepsilon(x)/\partial n$. Consequently

$$\int_\Gamma\Big(\frac{\partial\Psi}{\partial n}-\frac{\partial u_\varepsilon}{\partial n}\Big)\phi(\Psi-(1+\varepsilon\beta)^{-1}u_\varepsilon)d\Gamma\leqq 0. \tag{3.16}$$

For some constant C

$$|\phi(\Psi-u_\varepsilon)-\phi(\Psi-(1+\varepsilon\beta)^{-1}u_\varepsilon)|$$
$$\leqq C|u_\varepsilon-(1+\varepsilon\beta)^{-1}u_\varepsilon|=C\varepsilon|\beta_\varepsilon(u_\varepsilon)|,$$

and hence by (3.7)

$$\Big|\int_\Gamma\frac{\partial u_\varepsilon}{\partial n}(\phi(\Psi-u_\varepsilon)-\phi(\Psi-(1+\varepsilon\beta)^{-1}u_\varepsilon))d\Gamma\Big|$$
$$\leqq C\varepsilon\int_\Gamma\beta_\varepsilon(u_\varepsilon)^2d\Gamma\longrightarrow 0 \tag{3.17}$$

as $\varepsilon\to 0$. If $p\geqq 2N/(N+1)$, then

$$\partial\Psi/\partial n\in W^{1-1/p,p}(\Gamma)\subset L^2(\Gamma).$$

Consequently as $\varepsilon\to 0$

$$\Big|\int_\Gamma\frac{\partial\Psi}{\partial n}(\phi(\Psi-u_\varepsilon)-\phi(\Psi-(1+\varepsilon\beta)^{-1}u_\varepsilon))d\Gamma\Big|$$

$$\leqq C\varepsilon\int_\Gamma\left|\frac{\partial\Psi}{\partial n}\right||\beta_\varepsilon(u_\varepsilon)|d\Gamma \tag{3.18}$$

$$\leqq C\varepsilon\left\{\int_\Gamma\left(\frac{\partial\Psi}{\partial n}\right)^2 d\Gamma\right\}^{1/2}\left\{\int_\Gamma\beta_\varepsilon(u_\varepsilon)^2 d\Gamma\right\}^{1/2}\longrightarrow 0\,.$$

If $p<2N/(N+1)$, we put $\theta=N+2-2N/p$. Then $0<\theta<1$ and

$$\frac{N-p}{p(N-1)}+\frac{\theta}{2}+\frac{(N-2)(1-\theta)}{2(N-1)}=1\,. \tag{3.19}$$

Noting $|\beta_\varepsilon(u_\varepsilon)|\leqq\varepsilon^{-1}|u_\varepsilon|$,

$$\left|\int_\Gamma\frac{\partial\Psi}{\partial n}(\phi(\Psi-u_\varepsilon)-\phi(\Psi-(1+\varepsilon\beta)^{-1}u_\varepsilon))d\Gamma\right|$$

$$\leqq C\varepsilon\int_\Gamma\left|\frac{\partial\Psi}{\partial n}\right||\beta_\varepsilon(u_\varepsilon)|d\Gamma \tag{3.20}$$

$$\leqq C\varepsilon^\theta\int_\Gamma\left|\frac{\partial\Psi}{\partial n}\right||\beta_\varepsilon(u_\varepsilon)|^\theta|u_\varepsilon|^{1-\theta}d\Gamma\,.$$

By (3.6) $u_\varepsilon|_\Gamma$ is bounded in $H^{1/2}(\Gamma)\subset L^{2(N-1)/(N-2)}(\Gamma)$. Hence by (3.8), (3.19) the final member of (3.20) does not exceed

$$C\varepsilon^\theta\left\|\frac{\partial\Psi}{\partial n}\right\|_{L^{p(N-1)/(N-p)}(\Gamma)}\|\beta_\varepsilon(u_\varepsilon)\|^\theta_{L^2(\Gamma)}\|u_\varepsilon\|^{1-\theta}_{L^{2(N-1)/(N-2)}(\Gamma)}\,,$$

which tends to 0 as $\varepsilon\to 0$. Combining this with (3.15), (3.16), (3.17) we obtain

$$\limsup_{\varepsilon\to 0}\int_\Gamma\left(\frac{\partial\Psi}{\partial n}-\frac{\partial u_\varepsilon}{\partial n}\right)\phi(\Psi-u_\varepsilon)d\Gamma\leqq 0\,. \tag{3.21}$$

By Lemma 2.1

$$a(\Psi-u_\varepsilon,\ \phi(\Psi-u_\varepsilon))\geqq 0\,. \tag{3.22}$$

(3.4) follows from (3.12), (3.13), (3.14), (3.21) and (3.22).

LEMMA 3.2. *For* $u\in D(L_q)$, $1<q\leqq p$,

$$(L_q u,\ F_q(u-Pu))\geqq(\mathcal{L}\Psi,\ F_q(u-Pu))\,. \tag{3.23}$$

PROOF. Let ϕ_n be the function defined by

$$\phi_n(r)=\begin{cases} r^{q-1} & \text{if } r\geqq 1/n\,,\\ n^{2-q}r & \text{if } 0<r<1/n\,,\\ 0 & \text{if } r\leqq 0\,,\end{cases} \tag{3.24}$$

and $u_m\in D(L_2)\cap L^q(\Omega)$ be such that $L_2u_m\in L^q(\Omega)$, $u_m\to u$, $L_2u_m\to L_qu$ in $L^q(\Omega)$. By Lemma 3.1

$$(L_2 u_m,\ \phi_n(\Psi - u_m)) \leqq (\mathcal{L}\Psi,\ \phi_n(\Psi - u_m)). \tag{3.25}$$

Letting $n \to \infty$ in (3.25) we get

$$(L_2 u_m,\ F_q(u_m - Pu_m)) \geqq (\mathcal{L}\Psi,\ F_q(u_m - Pu_m)). \tag{3.26}$$

By Lemma 1.1 of [5] there exists a constant K such that

$$\|F_q u - F_q v\|_{q'} \leqq K\|u - v\|_q^{q-1}, \qquad q' = q/(q-1) \tag{3.27}$$

for $u,\ v \in L^q(\Omega)$. Hence letting $m \to \infty$ in (3.26) we get the desired result.

Let sign_+^0 be the function defined by

$$\mathrm{sign}_+^0 r = \begin{cases} 1 & \text{if } r > 0, \\ 0 & \text{if } r \leqq 0. \end{cases}$$

LEMMA 3.3. *For* $u \in D(L_1)$

$$(L_1 u,\ \mathrm{sign}_+^0(\Psi - u)) \leqq (\mathcal{L}\Psi,\ \mathrm{sign}_+^0(\Psi - u)). \tag{3.28}$$

PROOF. Let ϕ_n be the function such that

$$\phi_n(r) = \begin{cases} 1 & \text{if } r \geqq 1/n, \\ nr & \text{if } 0 < r < 1/n, \\ 0 & \text{if } r \leqq 0, \end{cases}$$

and $u_m \in D(L_2) \cap L^1(\Omega)$ be such that $L_2 u_m \in L^1(\Omega)$, $u_m \to u$, $L_2 u_m \to L_1 u$ in $L^1(\Omega)$, $u_m(x) \to u(x)$ a.e. in Ω. By Lemma 3.1

$$(L_2 u_m,\ \phi_n(\Psi - u_m)) \leqq (\mathcal{L}\Psi,\ \phi_n(\Psi - u_m)). \tag{3.29}$$

Now,

$$\begin{aligned}(L_2 u_m,\ &\phi_n(\Psi - u_m)) - (L_1 u,\ \phi_n(\Psi - u)) \\ &= (L_2 u_m - L_1 u,\ \phi_n(\Psi - u_m)) \\ &\quad + (L_1 u,\ \phi_n(\Psi - u_m) - \phi_n(\Psi - u)).\end{aligned} \tag{3.30}$$

It is obvious that the first term on the right of (3.30) tends to 0 as $m \to \infty$. The integrand of the second term is bounded by $2|L_1 u|$ in absolute value and converges to 0 a.e. as $m \to \infty$. Hence as $m \to \infty$

$$(L_2 u_m,\ \phi_n(\Psi - u_m)) \longrightarrow (L_1 u,\ \phi_n(\Psi - u)). \tag{3.31}$$

Similarly we see that the right side of (3.29) tends to $(\mathcal{L}\Psi,\ \phi_n(\Psi - u))$ as $m \to \infty$. Hence

$$(L_1 u,\ \phi_n(\Psi - u)) \leqq (\mathcal{L}\Psi,\ \phi_n(\Psi - u)).$$

Finally letting $n \to \infty$ we get (3.28).

§ 4. Elliptic unilateral problem in $L^q(\Omega)$, $1\leqq q\leqq 2$.

Let M_q be the multivalued mapping defined by

$$D(M_q)=\{u\in L^q(\Omega): u\geqq \Psi \quad \text{a.e. in } \Omega\}\,,$$

$$M_q u=\{g\in L^q(\Omega): g\leqq 0 \quad \text{a.e.}, \ g(x)=0 \text{ if } u(x)>\Psi(x)\}\,.$$

$D(M_q)$ is not empty for $1\leqq q\leqq p^*$ since $\Psi\in L^q(\Omega)$ for these values of q. For $\lambda>0$ and $u\in L^q(\Omega)$, $1\leqq q\leqq p^*$

$$Pu=(1+\lambda M_q)^{-1}u\,. \tag{4.1}$$

DEFINITION 4.1. For $1\leqq q\leqq p^*$ the operator A_q is defined as follows:

(i) $A_q=L_q+M_q$ for $1\leqq q\leqq p$,

(ii) for $p<q\leqq p^*$

$G(A_q)=$the closure of $G(A_p)\cap(L^q(\Omega)\times L^q(\Omega))$ in $L^q(\Omega)\times L^q(\Omega)$.

For $\lambda>0$ denote by $M_{q,\lambda}$ the Yosida approximation of M_q. By (4.1)

$$M_{q,\lambda}u=(u-Pu)/\lambda\,. \tag{4.2}$$

PROPOSITION 4.1. *For $1<q\leqq p$ A_q is m-accretive and $R(A_q)=L^q(\Omega)$.*

PROOF. It is easy to show that A_q is accretive. For $f\in L^q(\Omega)$, $\lambda>0$, u_λ be the solution of

$$L_q u_\lambda+M_{q,\lambda}u_\lambda=f\,. \tag{4.3}$$

u_λ is the fixed point of the mapping

$$u\longmapsto (1+\lambda L_q)^{-1}(f+Pu) \tag{4.4}$$

which is a strict contraction from $L^q(\Omega)$ to itself in view of Lemma 2.4 and Proposition 2.3. Forming the scalar product of (4.3) and $F_q(u_\lambda-Pu_\lambda)$, and noting (4.2), we get

$$(L_q u_\lambda, F_q(u_\lambda-Pu_\lambda))+\|u_\lambda-Pu_\lambda\|_q^q/\lambda=(f, F_q(u_\lambda-Pu_\lambda))\,. \tag{4.5}$$

By Lemma 3.2

$$\begin{aligned}(L_q u_\lambda, F_q(u_\lambda-Pu_\lambda))&\geqq(\mathcal{L}\Psi, F_q(u_\lambda-Pu_\lambda))\\ &\geqq((\mathcal{L}\Psi)^+, F_q(u_\lambda-Pu_\lambda))\geqq-\|(\mathcal{L}\Psi)^+\|_q\|u_\lambda-Pu_\lambda\|_q^{q-1}\,.\end{aligned} \tag{4.6}$$

From (4.5) and (4.6) it follows that

$$\|u_\lambda-Pu_\lambda\|_q^q/\lambda\leqq(\|f\|_q+\|(\mathcal{L}\Psi)^+\|_q)\|u_\lambda-Pu_\lambda\|_q^{q-1}$$

which implies

$$\|M_{q,\lambda}u_\lambda\|_q \leqq \|f\|_q + \|(\mathcal{L}\Psi)^+\|_q, \tag{4.7}$$

$$\|L_q u_\lambda\|_q \leqq 2\|f\|_q + \|(\mathcal{L}\Psi)^+\|_q. \tag{4.8}$$

Write (4.3) with λ, $\mu>0$, take the difference, multiply by $F_q(u_\lambda-u_\mu)$ and integrate over Ω. This yields

$$(L_q u_\lambda - L_q u_\mu,\ F_q(u_\lambda-u_\mu)) + (M_{q,\lambda}u_\lambda - M_{q,\mu}u_\mu,\ F_q(u_\lambda-u_\mu)) = 0. \tag{4.9}$$

By Lemma 2.5, (4.9), and the accretiveness of M_q

$$\begin{aligned} &c_q\|u_\lambda-u_\mu\|_{1,q}^q \\ &\qquad + (M_{q,\lambda}u_\lambda - M_{q,\mu}u_\mu,\ F_q(u_\lambda-u_\mu) - F_q(Pu_\lambda - Pu_\mu)) \leqq 0. \end{aligned} \tag{4.10}$$

Applying Hölder's inequality and (3.27) we get

$$\begin{aligned} &c_q\|u_\lambda-u_\mu\|_{1,q}^q \\ &\qquad \leqq K\|M_{q,\lambda}u_\lambda - M_{q,\mu}u_\mu\|_q \|\lambda M_{q,\lambda}u_\lambda - \mu M_{q,\mu}u_\mu\|_q^{q-1}. \end{aligned} \tag{4.11}$$

In view of (4.7) the right side of (4.11) goes to 0 as λ, $\mu\to 0$. Hence there exists an element u of $W^{1,q}(\Omega)$ such that

$$u_\lambda \to u \quad \text{in} \quad W^{1,q}(\Omega). \tag{4.12}$$

From (4.8) and the demiclosedness of L_q it follows that $L_q u_\lambda \rightharpoonup L_q u$ in $L^q(\Omega)$, and also $M_{q,\lambda}u_\lambda \rightharpoonup f - L_q u$ in $L^q(\Omega)$. By (4.7), (4.12) $Pu_\lambda \to u$ in $L^q(\Omega)$. By (4.1) $M_{q,\lambda}u_\lambda \in M_q P u_\lambda$. Hence $f - L_q u \in M_q u$, or $f\in A_q u$. Since $f\in L^q(\Omega)$ is arbitrary, A_q is surjective. From (2.20) it follows that $(1+\lambda\alpha)\|u-\hat{u}\|_q \leqq \|f-\hat{f}\|_q$ if $f\in(1+\lambda A_q)u$, $\hat{f}\in(1+\lambda A_q)\hat{u}$, $\lambda>0$. Hence A_q is m-accretive.

LEMMA 4.1. *If* $f\in L^q(\Omega)\cap L^r(\Omega)$, $q\geqq 1$, $r\geqq 1$, *then for any* $\lambda>0$

$$(1+\lambda L_q)^{-1}f = (1+\lambda L_r)^{-1}f. \tag{4.13}$$

PROOF. The conclusion follows easily from the definition of L_q, L_r, and Lemma 2.4.

PROPOSITION 4.2. *For* $p<q\leqq p^*$ A_q *is m-accretive.*

PROOF. Let f, $\hat{f}\in L^p(\Omega)\cap L^q(\Omega)$ and $\varepsilon>0$. Put

$$u=(1+\varepsilon A_p)^{-1}f, \qquad \hat{u}=(1+\varepsilon A_p)^{-1}\hat{f}.$$

By Sobolev's imbedding theorem

$$W^{1,p}(\Omega)\subset L^q(\Omega). \tag{4.14}$$

In view of Lemma 2.5 and (4.14) u, $\hat{u}\in L^q(\Omega)$. Let u_λ, $\hat{u}_\lambda$ be the solutions of

$$(1+\varepsilon L_p+\varepsilon M_{p,\lambda})u_\lambda = f, \qquad (1+\varepsilon L_p+\varepsilon M_{p,\lambda})\hat{u}_\lambda = \hat{f}. \tag{4.15}$$

u_λ is the fixed point of the strictly contractive mapping

$$Tv=\left(1+\frac{\lambda\varepsilon}{\lambda+\varepsilon}L_p\right)^{-1}\frac{\lambda f+\varepsilon Pv}{\lambda+\varepsilon}$$

from $L^p(\Omega)$ to itself. Since $f,\ \Psi\in L^p(\Omega)\cap L^q(\Omega)$

$$(\lambda f+\varepsilon Pv)/(\lambda+\varepsilon)\in L^p(\Omega)\cap L^q(\Omega)$$

if $v\in L^p(\Omega)\cap L^q(\Omega)$. Hence by Lemma 4.1 T is also a strict contraction from $L^p(\Omega)\cap L^q(\Omega)$ to itself. Consequently (4.15) may be rewritten as

$$(1+\varepsilon L_q+\varepsilon M_{q,\lambda})u_\lambda=f,\qquad (1+\varepsilon L_q+\varepsilon M_{q,\lambda})\hat{u}_\lambda=\hat{f}. \tag{4.16}$$

Since L_q and $M_{q,\lambda}$ are accretive

$$\|u_\lambda-\hat{u}_\lambda\|_q\leqq\|f-\hat{f}\|_q. \tag{4.17}$$

Since $u_\lambda-\hat{u}_\lambda\rightarrow u-\hat{u}$ in $W^{1,p}(\Omega)\subset L^q(\Omega)$ by the proof of Proposition 4.1 we get from (4.17)

$$\|u-\hat{u}\|_q\leqq\|f-\hat{f}\|_q. \tag{4.18}$$

Once this is established for $f,\hat{f}\in L^p(\Omega)\cap L^q(\Omega)$ the remaining part of the proof is accomplished in the usual obvious manner.

PROPOSITION 4.3. *A_1 is m-accretive and $R(A_1)=L^1(\Omega)$. If for $f,\hat{f}\in L^1(\Omega)$*

$$A_1u+g=f,\quad g\in M_1u,\quad A_1\hat{u}+\hat{g}=\hat{f},\quad \hat{g}\in M_1\hat{u},$$

then

$$\alpha\|u-\hat{u}\|_1+\|g-\hat{g}\|_1\leqq\|f-\hat{f}\|_1. \tag{4.19}$$

PROOF. As is easily seen

$$(g-\hat{g},\ \mathrm{sign}^0(u-\hat{u}))\geqq 0$$

if $g\in M_1u$ and $\hat{g}\in M_1\hat{u}$. Combining this with (2.21) the accretivity of A_1 follows. Suppose $f\in L^1(\Omega)\cap L^p(\Omega)$. Let u_λ be the solution of

$$L_1u_\lambda+M_{1,\lambda}u_\lambda=f. \tag{4.20}$$

u_λ is the fixed point of the mapping

$$Tv=(1+\lambda L_1)^{-1}(\lambda f+Pv).$$

In view of Proposition 2.3, Lemma 4.1 and the fact $f,\ \Psi\in L^1(\Omega)\cap L^p(\Omega)$ T is a strict contraction from $L^1(\Omega)\cap L^p(\Omega)$ to itself. Hence (4.20) may be rewritten as

$$L_pu_\lambda+M_{p,\lambda}u_\lambda=f. \tag{4.21}$$

Multiply both sides of (4.20) by $\mathrm{sign}_+^0(\Psi-u_\lambda)$ and integrate over Ω. Noting (3.2) we get

$$(L_1u_\lambda, \operatorname{sign}_+^0(\Psi-u_\lambda))-\|M_{1,\lambda}u_\lambda\|_1=(f, \operatorname{sign}_+^0(\Psi-u_\lambda)).$$

Using Lemma 3.3

$$\|M_{1,\lambda}u_\lambda\|_1\leqq-(f, \operatorname{sign}_+^0(\Psi-u_\lambda))+(\mathcal{L}\Psi, \operatorname{sign}_+^0(\Psi-u_\lambda))$$
$$\leqq\|f\|_1+\|(\mathcal{L}\Psi)^+\|_1. \tag{4.22}$$

From (2.22), (4.20), (4.22) it follows that

$$\alpha\|u_\lambda\|_1\leqq\|L_1u_\lambda\|_1\leqq 2\|f\|_1+\|(\mathcal{L}\Psi)^+\|_1. \tag{4.23}$$

By the proof of Proposition 4.1 $u_\lambda\to u$ in $W^{1,p}(\Omega)$, $L_pu_\lambda\rightharpoonup L_pu$, $M_{p,\lambda}u_\lambda\rightharpoonup g=f-L_pu$ in $L^p(\Omega)$ and $g\in M_pu$. Since u_λ, $L_pu_\lambda=L_1u_\lambda$, $M_{p,\lambda}u_\lambda=M_{1,\lambda}u_\lambda$ are bounded in $L^1(\Omega)$ in view of (4.22), (4.23), u, L_pu, g all belong to $L^1(\Omega)$. Since

$$u+L_pu=f-g+u\in L^1(\Omega)\cap L^p(\Omega)$$

if follows from Lemma 4.1 that

$$u=(1+L_p)^{-1}(f-g+u)=(1+L_1)^{-1}(f-g+u),$$

or

$$L_1u+g=f, \qquad g\in M_1u.$$

Thus we have proved $R(A_1)\supset L^1(\Omega)\cap L^p(\Omega)$.

Suppose next $f, \hat{f}\in L^1(\Omega)\cap L^p(\Omega)$ and

$$L_1u+g=f, \quad g\in M_1u, \quad L_1\hat{u}+\hat{g}=\hat{f}, \quad \hat{g}\in M_1\hat{u}.$$

In view of Proposition 2.3 u, $\hat{u}$ are uniquely determined by $f, \hat{f}$. Let u_λ, $\hat{u}_\lambda$ be the solutions of

$$L_1u_\lambda+M_{1,\lambda}u_\lambda=f, \qquad L_1\hat{u}_\lambda+M_{1,\lambda}\hat{u}_\lambda=\hat{f}. \tag{4.24}$$

Then by the above argument $u_\lambda\to u$, $\hat{u}_\lambda\to\hat{u}$ in $W^{1,p}(\Omega)$, $M_{1,\lambda}u_\lambda\rightharpoonup g$, $M_{1,\lambda}\hat{u}_\lambda\rightharpoonup\hat{g}$ in $L^p(\Omega)$. Multiplying both sides of

$$L_1u_\lambda-L_1\hat{u}_\lambda+M_{1,\lambda}u_\lambda-M_{1,\lambda}\hat{u}_\lambda=f-\hat{f}$$

by $\operatorname{sign}^0(u_\lambda-\hat{u}_\lambda)$ and noting

$$((u_\lambda-Pu_\lambda)-(\hat{u}_\lambda-P\hat{u}_\lambda), \operatorname{sign}^0(u_\lambda-\hat{u}_\lambda))$$
$$=\|(u_\lambda-Pu_\lambda)-(\hat{u}_\lambda-P\hat{u}_\lambda)\|_1$$

we get

$$(L_1u_\lambda-L_1\hat{u}_\lambda, \operatorname{sign}^0(u_\lambda-\hat{u}_\lambda))+\|M_{1,\lambda}u_\lambda-M_{1,\lambda}\hat{u}_\lambda\|_1$$
$$=(f-\hat{f}, \operatorname{sign}^0(u_\lambda-\hat{u}_\lambda)).$$

From this equality and Proposition 2.3 it follows that

$$\alpha\|u_\lambda-\hat{u}_\lambda\|_1+\|M_{1,\lambda}u_\lambda-M_{1,\lambda}\hat{u}_\lambda\|_1\leqq\|f-\hat{f}\|_1. \tag{4.25}$$

In view of Fatou's lemma

$$\|u-\hat{u}\|_1 \leqq \liminf \|u_\lambda - \hat{u}_\lambda\|_1 . \tag{4.26}$$

Let $\Omega_r = \{x : x \in \Omega, |x| < r\}$ for $r > 0$. Then $M_{1,\lambda} u_\lambda \rightharpoonup g$, $M_{1,\lambda} \hat{u}_\lambda \rightharpoonup \hat{g}$ in $L^1(\Omega_r)$. Hence

$$\int_{\Omega_r} |g-\hat{g}| \, dx \leqq \liminf \int_{\Omega_r} |M_{1,\lambda} u_\lambda - M_{1,\lambda} \hat{u}_\lambda| \, dx$$
$$\leqq \liminf \|M_{1,\lambda} u_\lambda - M_{1,\lambda} \hat{u}_\lambda\|_1 .$$

Since $r > 0$ is arbitrary

$$\|g-\hat{g}\|_1 \leqq \liminf \|M_{1,\lambda} u_\lambda - M_{1,\lambda} \hat{u}_\lambda\|_1 . \tag{4.27}$$

From (4.25), (4.26) and (4.27) it follows that

$$\alpha \|u-\hat{u}\|_1 + \|g-\hat{g}\|_1 \leqq \|f-\hat{f}\|_1 . \tag{4.28}$$

Finally suppose $f, \hat{f}$ are arbitrary elements of $L^1(\Omega)$. Let $\{f_n\}$, $\{\hat{f}_n\}$ be sequences of $L^1(\Omega) \cap L^p(\Omega)$ tending to $f, \hat{f}$ respectively in $L^1(\Omega)$, and u_n, $\hat{u}_n$, g_n, $\hat{g}_n$ be such that

$$L_1 u_n + g_n = f_n , \quad g_n \in M_1 u_n , \quad L_1 \hat{u}_n + \hat{g}_n = \hat{f}_n , \quad \hat{g}_n \in M_1 \hat{u}_n .$$

An application of (4.28) yields the existence of the elements u, $g \in L^1(\Omega)$ such that $u_n \to u$, $g_n \to g$ in $L^1(\Omega)$. Replacing by a subsequence if necessary we may assume $u_n(x) \to u(x)$, $g_n(x) \to g(x)$ at almost every $x \in \Omega$. Since $g_n \leqq 0$ a.e. in Ω the same is true of g. If $u(x) > \Psi(x)$, then $u_n(x) > \Psi(x)$ if n is sufficiently large, and hence $g_n(x) = 0$ for these values of n, which implies $g(x) = 0$. Consequently we have proved $g \in M_1 u$. Since L_1 is a closed operator $u \in D(L_1)$ and $L_1 u + g = f$. Hence we have established $R(A_1) = L^1(\Omega)$. Letting $n \to \infty$ in

$$\alpha \|u_n - \hat{u}_n\|_1 + \|g_n - \hat{g}_n\|_1 \leqq \|f_n - \hat{f}_n\|_1$$

we obtain (4.19).

LEMMA 4.2. $j(\Psi^+|_\Gamma) \in L^1(\Gamma)$.

PROOF. By (1.4) and (1.8)

$$0 \leqq j(x, \Psi^+(x)) \leqq \beta^-(x, \Psi^+(x)) \Psi^+(x) \leqq -\partial \Psi(x)/\partial n \cdot \Psi^+(x) . \tag{4.29}$$

By the assumption and Sobolev's imbedding theorem

$$\frac{\partial \Psi}{\partial n} \in L^{p(N-1)/(N-p)}(\Gamma) , \qquad \Psi^+|_\Gamma \in H^{1/2}(\Gamma) \subset L^{2(N-1)/(N-2)}(\Gamma) .$$

If we choose q and r so that (3.10) holds, then $\partial \Psi/\partial n \in L^q(\Gamma)$, $\Psi^+|_\Gamma \in L^r(\Gamma)$, and hence $\partial \Psi/\partial n \cdot \Psi^+|_\Gamma \in L^1(\Gamma)$. Combining this and (4.29) we get the desired result.

Let $\phi : L^2(\Omega) \to [0, \infty]$ be the function defined by

$$\phi(u)=\begin{cases}\dfrac{1}{2}\displaystyle\int_\Omega\Big(\sum_{i,j=1}^N a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}+\alpha u^2\Big)dx+\int_\Gamma j(u|_\Gamma)d\Gamma \\ \quad \text{if } \Psi\leqq u\in H^1(\Omega),\ j(u|_\Gamma)\in L^1(\Gamma), \\ \infty \quad \text{otherwise}.\end{cases} \tag{4.30}$$

In view of Lemma 4.2 $\Psi^+|_\Gamma\in D(\phi)$, and hence ϕ is proper convex. Let B be the linear differential operator

$$B=\sum_{i=1}^N b_i\frac{\partial}{\partial x_i}+c-\alpha.$$

LEMMA 4.3. *Let $f\in L^2(\Omega)$, $u\in D(\phi)$. Then $f\in\partial\phi(u)+Bu$ if and only if*

$$a(u, v-u)+\Phi(v|_\Gamma)-\Phi(u|_\Gamma)\geqq(f, v-u) \tag{4.31}$$

for every v satisfying $\Psi\leqq v\in H^1(\Omega)$, $j(v|_\Gamma)\in L^1(\Gamma)$, where Φ is the function defined by (2.9). *$\partial\phi+B$ is demiclosed.*

PROOF. The proof of the first part is straightforward. The demiclosedness of $\partial\Psi+B$ is verified without difficulty with the aid of the first part of the lemma and noting that $a(u, u)^{1/2}$ is a norm of $H^1(\Omega)$.

By (1.9) and Proposition 4.2 the mapping A_2 is defined and m-accretive in $L^2(\Omega)$.

LEMMA 4.4. $A_2=\partial\phi+B$.

PROOF. Suppose first that $f\in A_p u$, f, $u\in L^2(\Omega)$. Let u_λ be the solution of

$$L_p u_\lambda+M_{p,\lambda}u_\lambda=f=L_2 u_\lambda+M_{2,\lambda}u_\lambda, \tag{4.32}$$

where we used Lemma 4.1 as in the proof of Proposition 4.2. Let $v\in D(\phi)$. From (4.32) and the definition of L_2 it follows that

$$a(u_\lambda, v-u_\lambda)+\Phi(v|_\Gamma)-\Phi(u_\lambda|_\Gamma)\geqq(f-M_{2,\lambda}u_\lambda, v-u_\lambda). \tag{4.33}$$

If $u_\lambda(x)-Pu_\lambda(x)<0$ at some point x, then $u_\lambda(x)<\Psi(x)\leqq v(x)$ there. Consequently $M_{2,\lambda}u_\lambda\cdot(v-u_\lambda)\leqq 0$ a.e. Hence from (4.33) it follows that

$$a(u_\lambda, v-u_\lambda)+\Phi(v|_\Gamma)-\Phi(u_\lambda|_\Gamma)\geqq(f, v-u_\lambda). \tag{4.34}$$

By the proof of Proposition 4.1 $u_\lambda\to u$ in $W^{1,p}(\Omega)\subset L^2(\Omega)$. It is easily shown that u satisfies (4.31). The remaining part of the proof is omitted.

LEMMA 4.5. *Suppose $f\in L^1(\Omega)\cap L^2(\Omega)$. Then for $\varepsilon>0$, $1\leqq q\leqq 2$*

$$(1+\varepsilon A_1)^{-1}f=(1+\varepsilon A_q)^{-1}f=(1+\varepsilon A_2)^{-1}f. \tag{4.35}$$

PROOF. In case $p<q\leqq 2$ (4.35) is an immediate consequence of the definition of A_q. In case $1\leqq q\leqq p$ (4.35) is easily established with the aid of Proposition 4.1 and 4.3.

REMARK. It follows from Lemma 4.5 that if $f\in L^{q}(\Omega)\cap L^{r}(\Omega)$, $1\leqq q<r\leqq 2$, then $(1+\varepsilon A_{q})^{-1}f=(1+\varepsilon A_{r})^{-1}f$ for $\varepsilon>0$.

PROPOSITION 4.4. *For* $1\leqq q\leqq 2$

$$\overline{D(A_{q})}=\{u\in L^{q}(\Omega): u\geqq\Psi \text{ a.e.}\} \tag{4.36}$$

where the left side of (4.36) *is the closure of* $D(A_{q})$ *in* $L^{q}(\Omega)$.

PROOF. It is obvious that the left side of (4.36) is contained in the right side.

(i) We first prove (4.36) for $1<q\leqq p$. Let $\Psi\leqq u\in L^{q}(\Omega)$. We set

$$u_{n}=(1+n^{-1}L_{q})^{-1}(u+n^{-1}\mathcal{L}\Psi).$$

Then

$$\Psi-u_{n}+n^{-1}\mathcal{L}\Psi-n^{-1}L_{q}u_{n}=\Psi-u\leqq 0. \tag{4.37}$$

Form the inner product of (4.37) and $((\Psi-u_{n})^{+})^{q-1}$. This yields

$$\|(\Psi-u_{n})^{+}\|_{q}^{q}+n^{-1}(\mathcal{L}\Psi-L_{q}u_{n}, ((\Psi-u_{n})^{+})^{q-1})\leqq 0. \tag{4.38}$$

By Lemma 3.2 and (3.3)

$$(L_{q}u_{n}, ((\Psi-u_{n})^{+})^{q-1})\leqq(\mathcal{L}\Psi, ((\Psi-u_{n})^{+})^{q-1}). \tag{4.39}$$

Combining (4.38) and (4.39) we get $\Psi\leqq u_{n}$. Hence $u_{n}\in D(L_{q})\cap D(M_{q})=D(A_{q})$. Since $C_{0}^{2}(\Omega)\subset D(L_{q})$, $D(L_{q})$ is dense in $L^{q}(\Omega)$. Hence $\|v-(1+n^{-1}L_{q})^{-1}v\|_{q}\to 0$ as $n\to\infty$ for any $v\in L^{q}(\Omega)$. Thus it follows easily that $u_{n}\to u$ in $L^{q}(\Omega)$, and hence $u\in\overline{D(A_{q})}$.

(ii) In case $q=1$ the proof is almost identical with that of (i). Form the inner product of (4.37) and $\text{sign}_{+}^{0}(\Psi-u_{n})$, and use Lemma 3.3.

(iii) In this step we consider the case $q=2$. Noting Lemma 4.4 and $\overline{D(\partial\phi+B)}=\overline{D(\partial\phi)}=\overline{D(\phi)}$ it suffices to show

$$\overline{D(\phi)}\supset\{u\in L^{2}(\Omega): u\geqq\Psi \text{ a.e.}\}. \tag{4.40}$$

Let χ be a smooth function such that $\chi(0)=0$, $\chi(t)>0$ for $t>0$, $\chi(t)=1$ for $t\geqq 1$ and $0\leqq\chi(t)\leqq 1$ for all $t\geqq 0$. Set $\rho(x)=\text{dist}(x, \partial\Omega)$ and $\chi_{n}(t)=\chi(n\rho(x))$. Then $\chi_{n}\in C^{2}(\bar{\Omega})$ if n is sufficiently large. Let u be an arbitrary element such that $\Psi\leqq u\in L^{2}(\Omega)$. Let v_{n} be a sequence in $H^{1}(\Omega)$ such that $v_{n}\to u$ in $L^{2}(\Omega)$ and $w_{n}(x)=\max\{v_{n}(x), \Psi(x)\}$. Then $\Psi\leqq w_{n}\in H^{1}(\Omega)$ and

$$\begin{aligned}\int_{\Omega}(u-w_{n})^{2}dx&=\int_{\Omega}(\max\{u, \Psi\}-\max\{v_{n}, \Psi\})^{2}dx\\&\leqq\int_{\Omega}(u-v_{n})^{2}dx\longrightarrow 0 \quad \text{as} \quad n\to\infty.\end{aligned} \tag{4.41}$$

Put $u_{n}=(1-\chi_{n})\Psi^{+}+\chi_{n}w_{n}$. Then $\Psi\leqq u_{n}\in H^{1}(\Omega)$ and $j(u_{n}|_{\Gamma})=j(\Psi^{+}|_{\Gamma})\in L^{1}(\Gamma)$

by Lemma 4.2. Hence $u_n \in D(\phi)$. Now,

$$\int_\Omega (u-u_n)^2 dx = \int_{\rho<1/n} (u-u_n)^2 dx + \int_{\rho \geqq 1/n} (u-w_n)^2 dx . \tag{4.42}$$

By (4.41) the second term on the right of (4.42) tends to 0 as $n \to \infty$, while as for the first term

$$\int_{\rho<1/n} (u-u_n)^2 dx = \int_{\rho<1/n} (u-\Psi^+ - \chi_n(w_n-\Psi^+))^2 dx$$

$$\leqq 2\int_{\rho<1/n} (u-\Psi^+)^2 dx + 2\int_{\rho<1/n} (w_n-\Psi^+)^2 dx \longrightarrow 0$$

since

$$\int_{\rho<1/n} w_n^2 dx \leqq 2\int_\Omega (w_n-u)^2 dx + 2\int_{\rho<1/n} u^2 dx .$$

Hence $u_n \to u$ in $L^2(\Omega)$ which implies $u \in \overline{D(\phi)}$.

(iv) In the final step we consider the case $p<q<2$. Suppose that $\Psi \leqq u \in L^q(\Omega)$. If we define

$$u_n(x) = \begin{cases} u(x) & \text{if } |x| \leqq n, \ u(x) \leqq n \\ \Psi(x) & \text{otherwise}, \end{cases} \tag{4.43}$$

then $\Psi \leqq u_n \in L^1(\Omega) \cap L^2(\Omega)$, and

$$\int_\Omega |u-u_n|^q dx \leqq \int_{|x|>n} |u-\Psi|^q dx + \int_{u>n} |u-\Psi|^q dx \longrightarrow 0$$

as $n \to \infty$. Thus it suffices to show that any element u satisfying $\Psi \leqq u \in L^1(\Omega) \cap L^2(\Omega)$ belongs to $\overline{D(A_q)}$. Let

$$u_n = (1+n^{-1}A_p)^{-1}u = (1+n^{-1}A_2)^{-1}u = (1+n^{-1}A_q)^{-1}u .$$

Here we recall Lemma 4.5. By (i) and (iii) $u \in \overline{D(A_p)} \cap \overline{D(A_2)}$. Hence as $n \to \infty$, $u_n \to u$ in $L^p(\Omega) \cap L^2(\Omega) \subset L^q(\Omega)$. Since $u_n \in D(A_q)$ it follows that $u \in \overline{D(A_q)}$.

REMARK. From the proof of (iv) of Proposition 4.4 it follows that for $\Psi \leqq u \in L^q(\Omega)$ there exists a sequence $\{u_n\} \subset D(A_p) \cap L^q(\Omega)$ such that $u_n \to u$ in $L^q(\Omega)$.

§ 5. L^2-estimate of solutions.

For $f \in W^{1,1}(0, T; L^q(\Omega))$, $1 \leqq q \leqq 2$, $u_0 \in \overline{D(A_q)}$ and $0 \leqq s \leqq t \leqq T$, set

$$U_q(t, s; f)u_0 = \lim_{n\to\infty} \prod_{i=1}^n \left\{1 + \frac{t-s}{n}\left(A_q - f\left(s+\frac{i}{n}(t-s)\right)\right)\right\}^{-1} u_0 . \tag{5.1}$$

The convergence of the right side of (5.1) was established by M. G. Crandall and A. Pazy (Theorem 5.1 of [7]). If $1<q\leqq 2$ and $u_0\in D(A_q)$, then $u(t)=U_q(t, 0; f)u_0$ is the unique strong solution of

$$du(t)/dt+A_q u(t)\ni f(t), \qquad 0\leqq t\leqq T, \tag{5.2}$$

$$u(0)=u_0, \tag{5.3}$$

i.e. $u(t)$ is an absolutely continuous (actually Lipschitz continuous) function in $[0, T]$ with values in $L^q(\Omega)$, $u(t)\in D(A_q)$ and (5.2) holds a.e. in $[0, T]$, and (5.3) holds.

LEMMA 5.1. *If* $u\in D(L_p)$, $0\leqq v\in W^{2,p}(\Omega)$, $\partial v/\partial n=(\partial\Psi/\partial n)^+$ *on* Γ, *then*

$$(L_p u-\mathcal{L}v, ((u-v)^+)^{p-1})\geqq 0. \tag{5.4}$$

PROOF. First we note $(\partial\Psi/\partial n)^+\in W^{1-1/p,p}(\Gamma)$. Let us begin with the case $u\in D(L_2)\cap L^p(\Omega)$, $L_2u\in L^p(\Omega)$. Let ϕ_n be the function defined by (3.24) with p in place of q. Let u_ε be the solution of $L_{2,\varepsilon}u_\varepsilon=L_2u$. Noting $v\in H^1(\Omega)$ and (2.11)

$$\begin{aligned}(L_2u, \phi_n(u_\varepsilon-v))&=(L_{2,\varepsilon}u_\varepsilon, \phi_n(u_\varepsilon-v))\\&=\int_\Gamma \beta_\varepsilon(u_\varepsilon)\phi_n(u_\varepsilon-v)d\Gamma+a(u_\varepsilon, \phi_n(u_\varepsilon-v)).\end{aligned} \tag{5.5}$$

If $u_\varepsilon(x)>v(x)$ at some point x, then $u_\varepsilon(x)>0$, which implies $\beta_\varepsilon(x, u_\varepsilon(x))\geqq 0$. Hence

$$\int_\Gamma \beta_\varepsilon(u_\varepsilon)\phi_n(u_\varepsilon-v)d\Gamma\geqq 0.$$

Combining this and (5.5)

$$(L_2u, \phi_n(u_\varepsilon-v))\geqq a(u_\varepsilon, \phi_n(u_\varepsilon-v)). \tag{5.6}$$

Repeating the arguments running from (3.8) to (3.12) and using the hypothesis we get

$$\begin{aligned}(\mathcal{L}v, \phi_n(u_\varepsilon-v))&=-\int_\Gamma\left(\frac{\partial\Psi}{\partial n}\right)^+\phi_n(u_\varepsilon-v)d\Gamma+a(v, \phi_n(u_\varepsilon-v))\\&\leqq a(v, \phi_n(u_\varepsilon-v)).\end{aligned} \tag{5.7}$$

Combining (5.6) and (5.7), and using Lemma 2.1

$$\begin{aligned}(L_2u-\mathcal{L}v, \phi_n(u_\varepsilon-v))&\geqq a(u_\varepsilon-v, \phi_n(u_\varepsilon-v))\\&\geqq \alpha(u_\varepsilon-v, \phi_n(u_\varepsilon-v))\geqq 0.\end{aligned} \tag{5.8}$$

In view of Proposition 2.1 and (3.1) $u_\varepsilon-v\to u-v$ in $L^{p'}(\Omega)$ as $\varepsilon\to 0$, and so $\phi_n(u_\varepsilon-v)\to\phi_n(u-v)$ in $L^{p'}(\Omega)$. Hence first letting $\varepsilon\to 0$ and then $n\to\infty$ in (5.8) we get (5.4) in this special case. The conclusion in the general case is easily obtained by noting

$$|(r^+)^{p-1}-(s^+)^{p-1}|\leqq K|r-s|^{p-1}, \qquad r\geqq 0, \quad s\geqq 0.$$

Let $G_q(t)$, $1\leqq q<\infty$, be the semigroup generated by the realization of $-\mathcal{L}$ under the Neumann boundary condition $\partial u/\partial n=0$ on Γ. $G_q(t)$ is an integral operator with kernel $G(t, x, y)$ satisfying

$$0\leqq G(t, x, y)\leqq Ct^{-N/2}H(t, x-y), \tag{5.9}$$

$$|(\partial/\partial x_i)G(t, x, y)|\leqq Ct^{-(N+1)/2}H(t, x-y), \tag{5.10}$$

$$|(\partial/\partial t)G(t, x, y)|\leqq Ct^{-N/2-1}H(t, x-y), \tag{5.11}$$

where $H(t, x)=\exp(-c|x|^2/t)$, and C and c are some positive constants. Part of the above estimates were established in [12]. $G(t, x, y)$ does not depend on q, and we write simply $G(t)$ instead of $G_q(t)$.

LEMMA 5.2. *Let $f\in W^{1,1}(0, T; L^p(\Omega))$ and $\Psi^+\leqq v_0\in L^q(\Omega)$. Let v be such that*

$$v\in C([0, T]; L^p(\Omega))\cap C((0, T]; W^{2,p}(\Omega)), \tag{5.12}$$

$$\partial v/\partial t+\mathcal{L}v=f^++(\mathcal{L}\Psi)^+ \quad \textit{in} \quad \Omega\times(0, T), \tag{5.13}$$

$$\partial v/\partial n=(\partial\Psi/\partial n)^+ \quad \textit{on} \quad \Gamma\times(0, T), \tag{5.14}$$

$$v(x, 0)=v_0(x) \quad \textit{in} \quad \Omega. \tag{5.15}$$

Then $v(x, t)\geqq\Psi^+(x)$ a.e. in $\Omega\times(0, T)$.

PROOF. The conclusion is easily established by integrating by part in

$$(\partial v/\partial t+\mathcal{L}v, v^-)\leqq 0, \qquad ((\partial/\partial t+\mathcal{L})(\Psi-v), (\Psi-v)^+)\leqq 0,$$

where $v^-=\min\{v, 0\}$. Here we note $u(t)\in H^1(\Omega)\subset L^{p'}(\Omega)$ for $t>0$.

LEMMA 5.3. *Let u be the strong solution of* (5.2) *and* (5.3) *with $f\in W^{1,1}(0, T; L^p(\Omega))$, $u_0\in D(A_p)$ and $q=p$. Let v be the function satisfying* (5.12)-(5.15) *with v_0 replaced by u_0^+. Then*

$$\Psi\leqq u\leqq v \quad \textit{a.e. in } \Omega\times(0, T). \tag{5.16}$$

PROOF. Let g be such that

$$du(t)/dt+L_pu(t)+g(t)=f(t), \qquad g(t)\in M_pu(t) \ \text{ a.e.} \tag{5.17}$$

Then

$$\partial(u-v)/\partial t+L_pu-\mathcal{L}v+g=f-f^+-(\mathcal{L}\Psi)^+\leqq 0. \tag{5.18}$$

Hence

$$\begin{aligned}&(\partial(u-v)/\partial t, ((u-v)^+)^{p-1})\\ &\quad+(L_pu-\mathcal{L}v, ((u-v)^+)^{p-1})+(g, ((u-v)^+)^{p-1})\leqq 0.\end{aligned} \tag{5.19}$$

In view of Lemma 5.2 $v\geqq 0$ a.e. in $\Omega\times(0, T)$, and hence Lemma 5.1 implies

$$(L_p u - \mathcal{L} v, ((u-v)^+)^{p-1}) \geqq 0 . \tag{5.20}$$

If $u > v$ somewhere, then by Lemma 5.2 $u > \Psi$ and hence $g=0$ there. Consequently

$$(g, ((u-v)^+)^{p-1}) = 0 . \tag{5.21}$$

Combining (5.19), (5.20) and (5.21) we get

$$\|(u(t)-v(t))^+\|_p \leqq \|(u_0 - u_0^+)^+\|_p = 0 . \tag{5.22}$$

Thus we conclude $u \leqq v$ a.e. in $\Omega \times (0, T)$. $\Psi \leqq u$ is clear since $u(t) \in D(A_p)$ for every $t \in [0, T]$, and the proof of the lemma is complete.

PROPOSITION 5.1. *Suppose that* $f \in W^{1,1}(0, T; L^q(\Omega))$, $1 \leqq q \leqq 2$, *and* $\Psi \leqq u_0 \in L^q(\Omega)$. *Let* $u(t) = U_q(t, 0; f) u_0$ *and* v *be the solution of* (5.13), (5.14), (5.15) *with* u_0^+ *in place of* v_0. *Then*

$$\Psi \leqq u \leqq v \quad a.e. \text{ in } \Omega \times (0, T) . \tag{5.23}$$

PROOF. Let $f_n \in W^{1,1}(0, T; L^q(\Omega) \cap L^p(\Omega))$ and $u_{0n} \in D(A_p) \cap L^q(\Omega)$ be such that $f_n \to f$, $u_{0n} \to u_0$ in $W^{1,1}(0, T; L^q(\Omega))$, $L^q(\Omega)$ respectively. Here we recall the remark after Proposition 4.4. Let

$$u_n(t) = U_q(t, 0; f_n) u_{0n} = U_p(t, 0; f_n) u_{0n}$$

where the second equality is due to the remark after Lemma 4.5, and v_n be the solution of (5.13), (5.14), (5.15) with f_n^+, u_{0n}^+ in place of f^+, v_0 respectively. Then for each fixed $t > 0$

$$v_n(t) - v(t)$$
$$= G(t)(u_{0n}^+ - u_0^+) + \int_0^t G(t-s)(f_n^+(s) - f^+(s)) ds \longrightarrow 0$$

in $L^q(\Omega)$ as $n \to \infty$. In view of Lemma 5.3

$$\Psi \leqq u_n \leqq v_n \quad \text{a.e. in } \Omega \times (0, T) . \tag{5.24}$$

By the fact that $U_q(t, 0; f_n)$ is a contraction and Theorem 4.1 of M. G. Crandall and A. Pazy [7] $u_n(t) \to u(t)$ in $L^q(\Omega)$. Going to the limit in (5.24) we conclude (5.23).

Let w be the solution of the boundary value problem

$$w = 0 \quad \text{in } \Omega , \qquad \frac{\partial w}{\partial n} = \left(\frac{\partial \Psi}{\partial n} \right)^+ \quad \text{on } \Gamma .$$

In view of the a priori estimate of the elliptic boundary value problem

$$\|w\|_{2,p} \leqq C \left[\left(\frac{\partial \Psi}{\partial n} \right)^+ \right]_{1-1/p, p} . \tag{5.25}$$

The function v in Proposition 5.1 is expressed as

$$v(t)=w-G(t)w+G(t)u_0^+ \\ +\int_0^t G(t-s)(f^+(s)+(\mathcal{L}\Psi)^+)ds\,. \tag{5.26}$$

In order to estimate the right side of (5.26) we use the following lemma, a proof of which is found in Lemma 2.6.1 of [**10**].

LEMMA 5.4. *Let $G(x, y)$ be a kernel which is measurable in $X\times Y$ where X and Y are open subsets of R^N. Suppose*

$$\int_X |G(x, y)|^q dx \leqq K^q \qquad \text{for all} \quad y\in Y,$$

and

$$\int_Y |G(x, y)|^q dy \leqq K^q \qquad \text{for all} \quad x\in X.$$

Let $1\leqq p, q, r\leqq\infty$, $1/r=1/p+1/q-1$, and set

$$(Gf)(x)=\int_Y G(x, y)f(y)dy$$

for $f\in L^p(Y)$. Then $\|Gf\|_r\leqq K\|f\|_p$.

Suppose $f\in W^{1,1}(0, T; L^q(\Omega)\cap L^2(\Omega))$, $1\leqq q\leqq 2$, and $\Psi\leqq u_0\in L^q(\Omega)$. In view of (3.1) and (5.25)

$$\|w-G(t)w\|_2\leqq 2\|w\|_2\leqq C\Big[\Big(\frac{\partial\Psi}{\partial n}\Big)^+\Big]_{1-1/p,p}. \tag{5.27}$$

(5.9) implies

$$\int_\Omega G(t, x, y)^q dx\leqq Ct^{N(1-q)/2}, \tag{5.28}$$

$$\int_\Omega G(t, x, y)^q dy\leqq Ct^{N(1-q)/2} \tag{5.29}$$

with some constant C. Hence with the aid of Lemma 5.4

$$\|G(t)u_0^+\|_2\leqq Ct^{N(2^{-1}-q^{-1})/2}\|u_0^+\|_q, \tag{5.30}$$

$$\Big\|\int_0^t G(t-s)(\mathcal{L}\Psi)^+ds\Big\|_2\leqq Ct^{N(2^{-1}-p^{-1})/2+1}\|(\mathcal{L}\Psi)^+\|_p. \tag{5.31}$$

We used $N(2^{-1}-p^{-1})/2+1>1/2>0$ in the derivation of (5.31). Hence $v(t)\in L^2(\Omega)$ if $t>0$ and

$$\|v(t)\|_2\leqq C\Big[\Big(\frac{\partial\Psi}{\partial n}\Big)^+\Big]_{1-1/p,p}+Ct^{N(2^{-1}-q^{-1})/2}\|u_0^+\|_q \tag{5.32}$$

$$+\int_0^t \|f^+(s)\|_2 ds + Ct^{N(2^{-1}-p^{-1})/2+1}\|(\mathcal{L}\Psi)^+\|_p .$$

In view of Proposition 5.1 $u(t)\in L^2(\Omega)$ if $t>0$ and

$$\|u(t)\|_2 \leqq \|\Psi\|_2 + \text{the right side of (5.32)}. \tag{5.33}$$

Furthermore applying Proposition 4.4 and noting the remark after Lemma 4.5 we conclude that for $0<\tau\leqq t\leqq T$

$$u(t)=U_2(t, \tau ; f)u(\tau). \tag{5.34}$$

§6. Differential equations in Hilbert space.

In order to derive the differentiability of the right side of (5.32) and establish some estimates of the derivative we investigate a certain differential equation in Hilbert space in this section.

Let H and V be Hilbert spaces such that $V\subset H$ algebraically and topologically, and V is dense in H. The norm and inner product of H are denoted by $|\ \ |$ and $(\ ,\ )$ respectively, and those of V are by $\|\ \ \|$ and $((\ ,\ ))$. Identifying H with its dual we consider $V\subset H\subset V^*$. The norm of V^* is denoted by $\|\ \ \|_*$. The pairing between V and V^* is also denoted by $(\ ,\ )$.

Let $a(u, v)$ be a bilinear form defined on $V\times V$ such that for some positive constants C and α

$$|a(u, v)|\leqq C\|u\|\|v\|, \qquad a(u, u)\geqq\alpha\|u\|^2. \tag{6.1}$$

The associated linear operator is denoted by L:

$$a(u, v)=(Lu, v) \qquad \text{for} \quad u, v\in V. \tag{6.2}$$

L is a bounded operator from V onto V^*. L is also considered as an operator from $L^2(0, T ; V)$ to $L^2(0, T ; V^*)$ by $(Lu)(t)=Lu(t)$.

Let ϕ be a proper convex, lower semicontinuous function defined on V. Let Φ be a convex function on $L^2(0, T ; V)$ defined by

$$\Phi(u)=\begin{cases} \int_0^T \phi(u(t))dt & \text{if} \quad \phi(u)\in L^1(0, T), \\ \infty & \text{otherwise}. \end{cases} \tag{6.3}$$

Following H. Brézis [2] we say $f\in M_{u_0}(u)$ for a fixed $u_0\in H$ if $u\in D(\Phi)$, $f\in L^2(0, T ; V^*)$ and

$$\int_0^T (v', v-u)dt+\Phi(v)-\Phi(u)\geqq\int_0^T (f, v-u)dt-\frac{1}{2}|v(0)-u_0|^2$$

for each $v \in D(\Phi)$, $v' \in L^2(0, T; V^*)$ where $v' = dv/dt$. Let A be the mapping defined by

$$Au = (Lu + \partial\Phi(u)) \cap H. \tag{6.4}$$

By Theorem 2 of F. E. Browder [4] $L + \partial\phi$ is maximal monotone in $V \times V^*$, and so is A in $H \times H$. Furthermore by Theorem 4 of [4] $R(L + \partial\Phi) = V^*$, and hence $R(A) = H$.

For $f \in W^{1,1}(0, T; H)$ and $u_0 \in \overline{D(A)}$ we set

$$U(t, 0; f)u_0 = \lim_{n\to\infty} \prod_{i=1}^{n} \left\{1 + \frac{t}{n}\left(A - f\left(\frac{i}{n}t\right)\right)\right\}^{-1} u_0. \tag{6.5}$$

The main result of this section is as follows (cf. Theorem 3.2 of [11]).

THEOREM 6.1. *Suppose $f \in W^{1,1}(0, T; H)$ and $u_0 \in \overline{D(A)}$. Then $u(t) = U(t, 0; f)u_0$ is the strong solution of*

$$du(t)/dt + Au(t) \ni f(t), \tag{6.6}$$

$$u(0) = u_0, \tag{6.7}$$

and there exists a constant K such that

$$|tD^+u(t)| \leqq K(|u_0 - v| + t|A^\circ v| + \int_0^t |f(s)|\,ds + \int_0^t |sf'(s) + f(s)|\,ds) \tag{6.8}$$

where D^+ is the right derivative, A° is the minimal cross section of A, and v is an arbitrary element of $D(A)$.

LEMMA 6.1. *If $u_0 \in D(A)$, then $u(t) = U(t, 0; f)u_0$ is a function belonging to $L^2(0, T; V)$ and satisfies the variational inequality*

$$\int_0^T (v' + Lu, v - u)dt + \Phi(v) - \Phi(u) \geqq \int_0^T (f, v - u)dt - \frac{1}{2}|v(0) - u_0|^2 \tag{6.9}$$

for all $v \in D(\Phi)$, $v' \in L^2(0, T; V^*)$, i.e. $f - Lu \in M_{u_0}(u)$.

PROOF. Under the hypothesis of the lemma $u(t)$ is the strong solution of (6.6), (6,7). Let $M = L + \partial\Phi$. M^{-1} is an everywhere defined single valued mapping on V^* to V satisfying a uniform Lipschitz condition. Hence $u(t) = M^{-1}(f(t) - u'(t))$ is a measurable function with values in V. Let $h \in V^*$ and γ be such that

$$\phi(u) \geqq (h, u) + \gamma \tag{6.10}$$

for any $u \in V$. Let $v \in D(\phi)$. Then

$$\phi(v)\geqq(f(t)-u'(t)-Lu(t),\ v-u(t))+\phi(u(t))$$

$$\geqq(f(t),\ v-u(t))+\frac{1}{2}\frac{d}{dt}|u(t)-v|^2-a(u(t),\ v)$$

$$+a(u(t),\ u(t))+(h,\ u(t))+\gamma$$

$$\geqq(f(t),\ v-u(t))+\frac{1}{2}\frac{d}{dt}|u(t)-v|^2+\alpha\|u(t)\|^2$$

$$-C\|u(t)\|\|v\|-\|h\|_*\|u(t)\|+\gamma.$$

Hence $u\in L^2(0,\ T\ ;\ V)$. Let $v\in D(\Phi)$, $v'\in L^2(0,\ T\ ;\ V^*)$. Then

$$\phi(v(t))\geqq(f(t)-u'(t)-Lu(t),\ v(t)-u(t))+\phi(u(t))$$

$$=(f(t),\ v(t)-u(t))+(v'(t)-u'(t),\ v(t)-u(t))$$

$$-(v'(t),\ v(t)-u(t))-(Lu(t),\ v(t)-u(t))+\phi(u(t)),$$

which implies $u\in D(\Phi)$. Integrating this inequality over $[0,\ T]$ we get (6.9).

Let Λ be the operator defined by

$$((u,\ v))=(\Lambda u,\ v)\qquad \text{for}\quad u,\ v\in V. \tag{6.11}$$

Λ is a linear bounded operator from V onto V^*, and $\|\Lambda u\|_*=\|u\|$ for any $u\in V$. Since $\Lambda^{-1}\partial\phi$ is the subdifferential of ϕ when V is identified with V^* by Riesz' theorem,

$$\phi_\varepsilon(u)=\frac{1}{2\varepsilon}\|u-J_\varepsilon u\|^2+\phi(J_\varepsilon u) \tag{6.12}$$

is the Yosida approximation of ϕ, where

$$J_\varepsilon=(1+\varepsilon\Lambda^{-1}\partial\phi)^{-1}. \tag{6.13}$$

We denote by Φ_ε the function defined by (6.3) with ϕ_ε in place of ϕ. Set

$$A_\varepsilon u=(Lu+\partial\phi_\varepsilon(u))\cap H. \tag{6.14}$$

The operator defined by (6.5) with A replaced by A_ε is denoted by $U_\varepsilon(t,\ 0\ ;f)$.

LEMMA 6.2. *Let* $u(t)=U(t,\ 0\ ;f)u_0$, $u_\varepsilon(t)=U_\varepsilon(t,\ 0\ ;f)u_{0\varepsilon}$, $u_0\in D(A)$, $u_{0\varepsilon}\in D(A_\varepsilon)$. *If* $u_{0\varepsilon}\to u_0$ *in* H, *then* $u_\varepsilon\to u$ *in* $L^2(0,\ T\ ;\ V)$.

PROOF. Let $v\in D(\Phi)$, $v'\in L^2(0,\ T\ ;\ V^*)$. In view of Lemma 6.1

$$\Phi_\varepsilon(v)\geqq\int_0^T(f,\ v-u_\varepsilon)dt-\frac{1}{2}|v(0)-u_{0\varepsilon}|^2$$
$$-\int_0^T(v'+Lu_\varepsilon,\ v-u_\varepsilon)dt+\Phi_\varepsilon(u_\varepsilon). \tag{6.15}$$

In view of (6.12) and (6.10)

$$\begin{aligned}\Phi_\varepsilon(u_\varepsilon)&=\frac{1}{2\varepsilon}\int_0^T\|u_\varepsilon-J_\varepsilon u_\varepsilon\|^2dt+\int_0^T\phi(J_\varepsilon u_\varepsilon)dt\\ &\geqq\frac{1}{2\varepsilon}\int_0^T\|u_\varepsilon-J_\varepsilon u_\varepsilon\|^2dt+\int_0^T(h,\ u_\varepsilon)dt \qquad (6.16)\\ &-\|h\|_*\int_0^T\|J_\varepsilon u_\varepsilon-u_\varepsilon\|dt+T\gamma.\end{aligned}$$

Combining (6.15) and (6.16)

$$\begin{aligned}\Phi_\varepsilon(v)\geqq&\int_0^T(f,\ v-u_\varepsilon)dt-\frac{1}{2}|v(0)-u_{0\varepsilon}|^2-\int_0^T(v',\ v-u_\varepsilon)dt\\ &-\int_0^Ta(u_\varepsilon,\ v)dt+\int_0^Ta(u_\varepsilon,\ u_\varepsilon)dt+\frac{1}{2\varepsilon}\int_0^T\|u_\varepsilon-J_\varepsilon u_\varepsilon\|^2dt\\ &+\int_0^T(h,\ u_\varepsilon)dt-\|h\|_*\int_0^T\|J_\varepsilon u_\varepsilon-u_\varepsilon\|dt+T\gamma.\end{aligned}$$

Hence $\int_0^T\|u_\varepsilon\|^2dt$ and $\varepsilon^{-1}\int_0^T\|u_\varepsilon-J_\varepsilon u_\varepsilon\|^2dt$ is bounded as $\varepsilon\to0$. Let $\{u_{\varepsilon_n}\}$ be a subsequence such that $u_{\varepsilon_n}\rightharpoonup u^*$ in $L^2(0,\ T;\ V)$. Then $J_{\varepsilon_n}u_{\varepsilon_n}\rightharpoonup u^*$ in $L^2(0,\ T;\ V)$. Letting $\varepsilon=\varepsilon_n\to0$ in

$$\begin{aligned}\Phi_\varepsilon(v)\geqq&\int_0^T(f,\ v-u_\varepsilon)dt-\frac{1}{2}|v(0)-u_{0\varepsilon}|^2-\int_0^T(v',\ v-u_\varepsilon)dt\\ &-\int_0^T(Lu_\varepsilon,\ v)dt+\int_0^Ta(u_\varepsilon,\ u_\varepsilon)dt+\Phi(J_\varepsilon u_\varepsilon)\end{aligned}\qquad(6.17)$$

we get

$$\begin{aligned}\Phi(v)\geqq&\int_0^T(f,\ v-u^*)dt-\frac{1}{2}|v(0)-u_0|^2-\int_0^T(v',\ v-u^*)dt\\ &-\int_0^T(Lu^*,\ v)dt+\int_0^Ta(u^*,\ u^*)dt+\Phi(u^*),\end{aligned}$$

or $f-Lu^*\in M_{u_0}(u^*)$. Here we used that $a(u,\ u)^{1/2}$ is a norm of V as was indicated in the proof of Proposition 2.1. By Lemma 6.1 $f-Lu\in M_{u_0}(u)$. By virtue of Theorem II.3 of H. Brézis [2] $u^*\in C([0,\ T];H)$ and

$$\frac{1}{2}|u(t)-u^*(t)|^2\leqq-\int_0^Ta(u-u^*,\ u-u^*)ds\leqq0,$$

which implies $u=u^*$ and $u_\varepsilon\rightharpoonup u$ in $L^2(0,\ T;\ V)$. Noting $u'\in L^\infty(0,\ T;H)\subset L^2(0,\ T;\ V^*)$ and $u\in D(\Phi)$ let $\varepsilon\to0$ in (6.17) with $v=u$. Then we get

$$\int_0^Ta(u,\ u)dt\geqq\limsup_{\varepsilon\to0}\int_0^Ta(u_\varepsilon,\ u_\varepsilon)dt.$$

Thus we conclude $u_\varepsilon \to u$ in $L^2(0, T; V)$ since $\left\{\int_0^T a(u, u)dt\right\}^{1/2}$ is a norm of $L^2(0, T; V)$.

The following lemma is proved in a routine manner and the proof is omitted.

LEMMA 6.3. *Let* $u_\varepsilon=(1+\varepsilon A_\varepsilon)^{-1}u_0$ *for* $u_0 \in D(A)$. *The* $u_\varepsilon \to u_0$ *in* H *as* $\varepsilon \to 0$.

PROOF OF THEOREM 6.1. It suffices to show the theorem in the special case

$$\min_u \phi(u)=\phi(0)=0 \tag{6.18}$$

since the general case is easily reduced to this case. Hence in what follows we assume (6.18). Next suppose that the theorem was established when $u_0 \in D(A)$ and $f \in W^{1,2}(0, T; H)$. For $u_0 \in \overline{D(A)}$ and $f \in W^{1,1}(0, T; H)$ let $u_{0j} \in D(A)$ and $f_j \in W^{1,2}(0, T; H)$ be such that $u_{0j} \to u_0$ in H and $f_j \to f$ in $W^{1,1}(0, T; H)$. Then with the aid of Theorem 4.1 of [7] $U(t, 0; f_j)u_{0j} \to U(t, 0; f)u_0$ in $C([0, T]; H)$. Hence (6.8) for $u(t)=U(t, 0; f)u_0$ follows. Finally by virtue of Lemmas 6.2 and 6.3 it suffices to prove the theorem for A_ε in place of A with constant K in (6.8) independent of ε. Thus in what follows we assume (6.18), $u_0 \in D(A_\varepsilon)$ and $f \in W^{1,2}(0, T; H)$, and set $u(t)=U_\varepsilon(t, 0; f)u_0$.

Form the scalar product of

$$u'+Lu+\partial\phi_\varepsilon(u)=f \tag{6.19}$$

and u. This and

$$0 \leqq \phi_\varepsilon(u) \leqq (\partial\phi_\varepsilon(u), u)$$

yield

$$\frac{1}{2}\frac{d}{dt}|u|^2+(Lu, u)+\phi_\varepsilon(u) \leqq (f, u).$$

Integrating (6.20) over $[0, t]$ and noting

$$|u(t)| \leqq |u_0|+\int_0^t |f(s)|ds$$

we get

$$\begin{aligned} &\frac{1}{2}|u(t)|^2+\int_0^t (Lu, u)ds+\int_0^t \phi_\varepsilon(u)ds \\ &\leqq \frac{1}{2}\left(|u_0|+\int_0^t |f|ds\right)^2. \end{aligned} \tag{6.21}$$

Set for $h>0$

$$u_h(t)=\frac{1}{h}(u(t+h)-u(t)), \qquad f_h(t)=\frac{1}{h}(f(t+h)-f(t)).$$

It follows from (6.19) and the monotonicity of $\partial\phi_\varepsilon$ that

$$\frac{1}{2}\frac{d}{dt}|u_h|^2+(Lu_h, u_h)\leqq(f_h, u_h).$$

Using (6.1) and the Schwarz inequality we get

$$\frac{d}{dt}|u_h|^2+\alpha\|u_h\|^2\leqq\frac{1}{\alpha}\|f_h\|_*^2.$$

Integrating this inequality over $[0, T-h]$

$$|u_h(T-h)|^2+\alpha\int_0^{T-h}\|u_h\|^2dt\leqq|u_h(0)|^2+\frac{1}{\alpha}\int_0^{T-h}\|f_h\|_*^2dt.$$

Since $u(t)$ is a Lipschitz continuous function with values in H on $[0, T]$, the right side of the inequality just obtained is bounded as $h\to 0$. Hence $u'\in L^2(0, T; V)$. Since

$$\|\partial\phi_\varepsilon(u(t))-\partial\phi_\varepsilon(u(s))\|_*\leqq\varepsilon^{-1}\|u(t)-u(s)\|$$

$\partial\phi_\varepsilon(u(t))$ is absolutely continuous and $(\partial\phi_\varepsilon(u))'\in L^2(0, T; V^*)$. Hence $u''\in L^2(0, T; V^*)$ and

$$u''+Lu'+(\partial\phi_\varepsilon(u))'=f'. \tag{6.22}$$

Multiplying both sides of (6.22) by t

$$\frac{d}{dt}(tu')-u'+tLu'+t\frac{d}{dt}\partial\phi_\varepsilon(u)=tf'. \tag{6.23}$$

Forming the scalar product of tu' and (6.23), noting

$$((d/dt)(\partial\phi_\varepsilon(u(t)), u'(t))\leqq 0$$

in view of the monotonicity of $\partial\phi_\varepsilon$, and integrating over $[0, t]$, we get

$$\begin{aligned}\frac{1}{2}|tu'(t)|^2-\int_0^t s|u'|^2ds+\int_0^t(sLu', su')ds\\ \leqq\int_0^t(sf', su')ds.\end{aligned} \tag{6.24}$$

Note here that $u'\in C([0, T]; H)$ since $u'\in L^2(0, T; V)$ and $u''\in L^2(0, T; V^*)$. Since ϕ_ε is Fréchet differentiable, $\phi_\varepsilon(u)$ is absolutely continuous, and as is easily seen at a Lebesgue point of $u'\in L^2(0, T; V)$

$$(d/dt)\phi_\varepsilon(u(t))=(\partial\phi_\varepsilon(u(t)), u'(t)).$$

Consequently multiplying both sides of (6.19) by tu' and integrating the equality thus obtained over $[0, t]$ we get

$$\int_0^t s|u'|^2ds+\int_0^t(Lu, su')ds \tag{6.25}$$

$$\leqq \int_0^t \phi_\varepsilon(u)ds + \int_0^t (f,\ su')ds\,.$$

Combining (6.24) and (6.25)

$$\frac{1}{2}|tu'(t)|^2 + \int_0^t (sLu' + Lu,\ su')ds \leqq \int_0^t \phi_\varepsilon(u)ds + \int_0^t (f + sf',\ su')ds\,. \tag{6.26}$$

Noting

$$(Lv + Lu,\ v) \geqq -\left(\frac{C}{2\alpha}\right)^2 (Lu,\ u)$$

for $u,\ v \in V$, we get from (6.26)

$$\frac{1}{2}|tu'(t)|^2 \leqq \left(\frac{C}{2\alpha}\right)^2 \int_0^t (Lu,\ u)ds + \int_0^t \phi_\varepsilon(u)ds + \int_0^t (f + sf',\ su')ds\,. \tag{6.27}$$

Combining (6.27) and (6.21) we obtain

$$\frac{1}{2}|tu'(t)|^2 \leqq \frac{1}{2}\max\left\{\left(\frac{C}{2\alpha}\right)^2,\ 1\right\}\left(|u_0| + \int_0^t |f|\,ds\right)^2 + \int_0^t |f + sf'|\,s\,|u'|\,ds\,. \tag{6.28}$$

Applying the following lemma to (6.28) we complete the proof.

LEMMA 6.4. *Let σ be a real valued continuous function on* $[0, T]$ *and m be a nonnegative integrable function on* $[0, T]$. *Let a be a nonnegative increasing function on* $[0, T]$. *If*

$$\sigma(t)^2 \leqq a(t)^2 + 2\int_0^t m(s)\sigma(s)ds$$

in $[0, T]$, *then*

$$|\sigma(t)| \leqq a(t) + \int_0^t m(s)ds\,.$$

This lemma is proved in p. 157 of H. Brézis [1] when a is constant. The case where a is increasing is easily reduced to the case a is constant.

§7. Final result.

The goal of this paper is the following theorem.

THEOREM 7.1. *Suppose that* $\Psi \leqq u_0 \in L^q(\Omega)$ *and* $f \in W^{1,1}(0, T; L^q(\Omega) \cap L^r(\Omega))$, $1 \leqq q \leqq 2 \leqq r$. *Then*

$$u(t)=\lim_{n\to\infty} \prod_{i=1}^{n}\Big(1+\frac{t}{u}\Big(A_q-f\Big(\frac{i}{n}t\Big)\Big)\Big)^{-1} u_0\,,$$

which exists and is continuous in $[0, T]$ *with* $u(0)=u_0$ *in the strong topology of* $L^q(\Omega)$, *is a strong solution of*

$$du(t)/dt+A_2u(t)\ni f(t)$$

in $(0, T]$. *The right derivative* $D^+u(t)$, *which exists at every* $t>0$ *in the strong topology of* $L^2(\Omega)$, *belongs to* $L^r(\Omega)$ *and the following inequality holds:*

$$\begin{aligned}\|D^+u(t)\|_r \leqq C\{t^{-\beta-1}\Big(\|\Psi\|_2+\|v\|_2+t\|A_2^\circ v\|_2+\Big[\Big(\frac{\partial\Psi}{\partial n}\Big)^+\Big]_{1-1/p,\,p}\Big)\\ +t^{-\gamma-1}\|u_0\|_q+t^{-\delta}\|(\mathcal{L}\Psi)^+\|_p+t^{-\beta-1}\int_0^t\|f(s)\|_2ds \qquad (7.1)\\ +t^{-\beta-1}\int_0^t s\|f'(s)\|_2ds+\int_0^t\|f'(s)\|_r ds\}\end{aligned}$$

where v *is an arbitrary element of* $D(A_2)$, $A_2^\circ v$ *is the element of* A_2v *of the minimal norm, and* $\beta=N(2^{-1}-r^{-1})/2$, $\gamma=N(q^{-1}-r^{-1})/2$, $\delta=N(p^{-1}-r^{-1})/2$.

Let $a(u, v)$ again be the bilinear form defined by (1.1), and ϕ be the convex function on $H^1(\Omega)$ defined by either

$$\phi(u)=\begin{cases}\displaystyle\int_\Gamma j(x, u(x))d\Gamma & \text{if}\quad \Psi\leqq u\in H^1(\Omega)\,,\quad j(u|_\Gamma)\in L^1(\Gamma)\,,\\ \infty & \text{otherwise}\end{cases} \qquad (7.2)$$

or

$$\begin{aligned}&\phi(u)=\phi_1(u)+\phi_2(u)\,,\\ &\phi_1(u)=\Phi(u|_\Gamma)\,,\qquad \phi_2(u)=\frac{1}{2\lambda}\|u-Pu\|_2^2\,,\qquad \lambda>0\end{aligned} \qquad (7.3)$$

where Φ is the function defined by (2.9). The effective domain $D(\phi)$ of ϕ defined by (7.1) is not empty since $\Psi^+\in D(\phi)$ in view of Lemma 4.2.

The following lemma is easily established and the proof is omitted.

LEMMA 7.1. *Let* A *be the mapping defined by*

$$Au=(Lu+\partial\phi(u))\cap L^2(\Omega)\,.$$

If ϕ *is the function defined by* (7.2), *then* $A=A_2$. *If* ϕ *is defined by* (7.3), *then*

$A=L_2+M_{2,\lambda}$.

In view of (5.32) and Lemma 7.1 we can apply Theorem 6.1 to $u(t)=U_q(t, 0; f)u_0$ in $t>\tau>0$ taking $H=L^2(\Omega)$ and $V=H^1(\Omega)$. It follows that $u(t)$ is differentiable in $L^2(\Omega)$ a. e. in $(0, T]$, and

$$\|(t-\tau)D^+u(t)\|_2 \leqq K\Big\{\|u(\tau)-v\|_2+(t-\tau)\|A_2^\circ v\|_2 + \int_0^t \|f(\sigma)\|_2 d\sigma+\int_\tau^t \|\sigma f'(\sigma)+f(\sigma)\|_2 d\sigma\Big\} \tag{7.4}$$

for $t>\tau>0$ and $v\in D(A_2)$.

REMARK. If we use the expression $A_2=\partial\phi+B$ and consider B as a perturbation to $\partial\phi$, we get an estimate analogous to (7.8), but with $1+\sqrt{t-\tau}$ as a factor in the right hand side.

LEMMA 7.2. *If $p<q\leqq 2$, then for any $f\in L^q(\Omega)$ and $\varepsilon>0$*

$$(1+\varepsilon(L_q+M_{q,\lambda}))^{-1}f \longrightarrow (1+\varepsilon A_q)^{-1}f$$

in $L^q(\Omega)$ as $\lambda\to 0$.

PROOF. If $f\in L^p(\Omega)\cap L^q(\Omega)$ it follows from the proof of Proposition 4.1 that

$$(1+\varepsilon(L_q+M_{q,\lambda}))^{-1}f=(1+\varepsilon(L_p+M_{p,\lambda}))^{-1}f \longrightarrow (1+\varepsilon A_p)^{-1}f=(1+\varepsilon A_q)^{-1}f$$

in $W^{1,p}(\Omega)\subset L^q(\Omega)$. The conclusion in the general case follows easily from that in this special case.

For $\lambda>0$, $t\geqq s>\tau>0$ let

$$u_\lambda(t)=\lim_{n\to\infty}\prod_{i=1}^n\Big\{1+\frac{t-s}{n}\Big(L_2+M_{2,\lambda}-f\Big(s+\frac{i}{n}(t-s)\Big)\Big)\Big\}^{-1}u(s).$$

In view of Theorem 6.1 and Lemma 7.1 $u_\lambda(t)$ is the strong solution of

$$du_\lambda/dt+L_2u_\lambda+M_{2,\lambda}u_\lambda=f, \qquad s<t\leqq T, \tag{7.5}$$

$$u_\lambda(s)=u(s). \tag{7.6}$$

By virtue of Theorem 4.1 of [7] and Lemma 7.2

$$u_\lambda \to u \qquad \text{in} \quad C([s, T]; L^2(\Omega)) \tag{7.7}$$

as $\lambda\to 0$.

LEMMA 7.3. *For w, $\hat{w}\in D(L_2)$ and $0\leqq v\in H^2(\Omega)$, $\partial v/\partial n=0$ on Γ,*

$$(L_2w-L_2\hat{w}-\mathcal{L}v, (w-\hat{w}-v)^+)\geqq 0. \tag{7.8}$$

PROOF. (7.8) is shown by approximating w, $\hat{w}$ by the solutions w_ε, $\hat{w}_\varepsilon$ of $L_{2,\varepsilon}w_\varepsilon=L_2w$, $L_{2,\varepsilon}\hat{w}_\varepsilon=L_2\hat{w}$, and noting

$$(\mathcal{L}v, (w-\hat{w}-v)^+)=a(v, (w-\hat{w}-v)^+).$$

Now we follow the argument of [8], [9], [11] to show $u'(t)\in L^r(\Omega)$ for $t>0$. For $h>0$ let $v_\pm$ be the solution of

$$\begin{aligned} &\partial v_\pm/\partial t+\mathcal{L}v_\pm=(f(x,\ t+h)-f(x,\ t))^\pm && \text{in} \quad \Omega\times(s,\ T),\\ &\partial v_\pm/\partial n=0 && \text{on} \quad \Gamma\times(s,\ T),\\ &v_\pm(x,\ s)=(u(x,\ s+h)-u(x,\ s))^\pm && \text{in} \quad \Omega. \end{aligned}$$

For $h>0$, $\lambda>0$ let v_λ be the solution of

$$\begin{aligned} &\partial v_\lambda/\partial t+\mathcal{L}v_\lambda=(f(x,\ t+h)-f(x,\ t))^+ && \text{in} \quad \Omega\times(s,\ T),\\ &\partial v_\lambda/\partial n=0 && \text{on} \quad \Gamma\times(s,\ T),\\ &v_\lambda(x,\ s)=(u_\lambda(x,\ s+h)-u_\lambda(x,\ s))^+ && \text{in} \quad \Omega. \end{aligned}$$

$v_\pm$ is expressed as

$$\begin{aligned} v_\pm(t)=&G(t-s)(u(s+h)-u(s))^\pm\\ &+\int_s^t G(t-\sigma)(f(\sigma+h)-f(\sigma))^\pm d\sigma. \end{aligned} \tag{7.9}$$

By (5.9) $v_+\geqq 0$, $v_-\leqq 0$ a.e. in $\Omega\times(s,\ T)$. Similarly $v_\lambda\geqq 0$ a.e. in $\Omega\times(s,\ T)$. By (7.7)

$$v_\lambda \longrightarrow v_+ \quad \text{in} \quad C([s,\ T]\,;\ L^2(\Omega)) \tag{7.10}$$

as $\lambda\to 0$. Set $u_{\lambda,h}(t)=u_\lambda(t+h)-u_\lambda(t)$. With the aid of Lemma 7.3

$$(L_2u_\lambda(t+h)-L_2u_\lambda(t)-\mathcal{L}v_\lambda(t),\ (u_{\lambda,h}(t)-v_\lambda(t))^+)\geqq 0. \tag{7.11}$$

If $u_{\lambda,h}(x,\ t)-v_\lambda(x,\ t)>0$ at some point $(x,\ t)$, then $u_\lambda(x,\ t+h)>u_\lambda(x,\ t)$ since $v_\lambda\geqq 0$, and so $M_{2,\lambda}u_\lambda(x,\ t+h)\geqq M_{2,\lambda}u_\lambda(x,\ t)$ there as is easily seen by (4.2). Hence

$$(M_{2,\lambda}u_\lambda(t+h)-M_{2,\lambda}u_\lambda(t),\ (u_{\lambda,h}(t)-v_\lambda(t))^+)\geqq 0. \tag{7.12}$$

In view of (7.10) and (7.11)

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|(u_{\lambda,h}-v_\lambda)^+\|_2^2&=(u'_{\lambda,h}-v'_\lambda,\ (u_{\lambda,h}-v_\lambda)^+)\\ &=(u'_\lambda(t+h)-u'_\lambda(t)-v'_\lambda(t),\ (u_{\lambda,h}(t)-v_\lambda(t))^+)\\ &\leqq(f(t+h)-f(t)-(f(t+h)-f(t))^+,\ (u_{\lambda,h}(t)-v_\lambda(t))^+)\leqq 0. \end{aligned}$$

Hence

$$\|(u_{\lambda,h}(t)-v_\lambda(t))^+\|_2\leqq\|(u_{\lambda,h}(s)-u_{\lambda,h}(s)^+)^+\|_2=0,$$

which implies $u_{\lambda,h}\leqq v_\lambda$. Letting $\lambda\to 0$

$$u(t+h)-u(t)\leqq v_+(t) \tag{7.13}$$

in view of (7.7) and (7.9). Analogously we can show

$$v_-(t)\leqq u(t+h)-u(t). \tag{7.14}$$

With the aid of Lemma 5.4, (5.28), (5.29), (7.9), (7.13) and (7.14) we get

$$\|(u(t+h)-u(t))/h\|_r\leqq(\|v_+(t)\|_r+\|v_-(t)\|_r)/h$$
$$\leqq C(t-s)^{N(r^{-1}-2^{-1})/2}\|(u(s+h)-u(s))/h\|_2$$
$$+\int_s^t\|(f(\sigma+h)-f(\sigma))/h\|_r d\sigma.$$

Letting $h\to 0$

$$\|D^+u(t)\|_r\leqq C(t-s)^{N(r^{-1}-2^{-1})/2}\|D^+u(s)\|_2 \tag{7.15}$$
$$+\int_s^t\|f'(\sigma)\|_r d\sigma.$$

Combining (5.33) with $t/3$ in place of t, (7.4) with $2t/3$ and $3/t$ in place of t and τ respectively, and (7.15) with $s=2t/3$, we obtain (7.1).

Bibliography

[1] H. Brézis, Opérateurs maximaux monotones et semigroupes de contractions dans les espaces Hilbert, Math. Studies, North Holland Math. Studies, 5, North-Holland, Amsterdam, 1973.

[2] H. Brézis, Problémes unilatéraux, J. Math. Pures Appl., **51** (1972), 1-168.

[3] F.E. Browder, On the spectral theory of elliptic differential operators I, Math. Ann., **142** (1961), 22-130.

[4] F.E. Browder, Nonlinear maximal monotone operators in Banach space, Math. Ann., **175** (1968), 89-113.

[5] B.D. Calvert and C.P. Gupta, Nonlinear elliptic boundary value problems in L^p-spaces and sums of ranges of accretive operators, Nonlinear Anal., **2** (1978), 1-26.

[6] M.G. Crandall and T.M. Liggett, Generation of semi-groups of nonlinear transformations on general Banach spaces, Amer. J. Math., **93** (1971), 265-298.

[7] M.G. Crandall and A. Pazy, Nonlinear evolution equations in Banach spaces, Israel J. Math., **11** (1972), 56-94.

[8] L.C. Evans, Differentiability of a nonlinear semigroup in L^1, J. Math. Anal. Appl., **60** (1977), 703-715.

[9] L.C. Evans, Regularity properties for the heat equation subject to nonlinear boundary constraints, Nonlinear Anal., **1** (1977), 593-602.

[10] N. Kerzman, Hölder and L^p estimates for solutions of $\bar{\partial}u=f$ in strongly pseudoconvex domains, Comm. Pure Appl. Math., **24** (1971), 289-300.

[11] F.J. Massey, III, Semilinear parabolic equations with L^1 initial data, Indiana Univ. Math. J., **26** (1977), 399-412.

[12] H. Tanabe, On Green's functions of elliptic and parabolic boundary value problems, Proc. Japan Acad., **48** (1972), 709-711.

[13] H. Tanabe, On some unilateral problem of elliptic and parabolic type, Proc. Japan

Acad., **54** A (1978), 194-196.
[14] H. Tanabe, Remarks on the differentiability of solutions of some semilinear parabolic equations, Proc. Japan Acad., **54** A (1978), 314-317.

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