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Recurrence properties of Lotka-Volterra models with random fluctuations

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1. Introduction and statement of results.

A number of authors ([6], [14], [15]) have recently considered the (Ito) stochastic equation

$$
\begin{aligned}
(1.1) \qquad dX_i(t) &= X_i(t)\{a_{i1}dW_1 + a_{i2}dW_2\} \\
&\quad + X_i(t)\{k_i - b_{i1}X_1^{\theta_1}(t) - b_{i2}X_2^{\theta_2}(t)\}\,dt\,, \qquad i=1, 2
\end{aligned}
$$

on the first quadrant $Q \equiv \{X_1>0,\ X_2>0\}$. Here a_{ij}, b_{ij}, k_i and θ_i are constants which satisfy

$$
(1.2) \qquad a_{11}a_{22} - a_{12}a_{21} \neq 0\,, \qquad \theta_i > 0\,.
$$

The interest in these equations arises from their interpretation as a description of a system of two competing species or a predator-prey model in a randomly varying environment. In this interpretation $X_i(t)$ represents the amount of species i present at time t. Turelli [15] gives a thorough discussion of the validity of this interpretation. Even though there are difficulties in justifying (1.1) as the correct model for competing species in a randomly varying environment, it is believed that its solution behaves similar to real systems as far as absorption and explosion is concerned. In this paper we take (1.1) for granted and discuss the question of recurrence or transience of the system.

Specifically, we introduce

$$
\xi'_M = \inf\{t \geqq 0 : |X(t)| \geqq M\}\,,
$$

$$
\xi''_M = \inf\{t \geqq 0 : X_1(t) \leqq M^{-1} \text{ or } X_2(t) \leqq M^{-1}\}\,,
$$

$$
\xi' = \lim_{M\to\infty} \xi'_M\,, \qquad \xi'' = \lim_{M\to\infty} \xi''_M\,.
$$

ξ' and ξ'' are called the explosion time, respectively absorption time. For $X(0) \in Q$ fixed the solution of (1.1) is unique up till time $\xi' \wedge \xi''$ ([5] Theorem

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5.2.1, [13], Ch. 3.2). We give necessary and sufficient conditions for[2)]

(1.3) $$P^x\{\xi'\wedge\xi''<\infty \text{ or } \lim_{t\to\infty} X_i(t)=0 \text{ or } \infty \text{ for some } i\}=0.$$

It is known (see also beginning of sect. 2) that under condition (1.2) the probability in (1.3) is independent of x, and that if (1.3) holds, then for all $x\in Q$ and non-empty open sets $U\subset Q$

(1.4) $$P^x\{\exists \text{ arbitrarily large } t \text{ for which } X(t)\in U\}=1.$$

When (1.3) and (1.4) hold we call the X-process *recurrent*, and *transient* otherwise. As is well known, recurrence is not sufficient to guarantee the existence of an invariant probability distribution for X. When (1.3) holds, the process may still be only *null recurrent*, i.e. it is still possible that for every $M>0$

(1.5) $$\lim_{t\to\infty} P^x\{M^{-1}\leqq X_i(t)\leqq M \text{ for } i=1 \text{ and } i=2\}=0.$$

(Again if (1.5) holds for one $x\in Q$ it holds for all $x\in Q$ (see section 2)). Even though $X(\cdot)$ will return infinitely often to any open set of Q in this case, we surely do not want to say that the "two species coexist" if (1.5) prevails. It is, however, reasonable to talk about this if for every $\varepsilon>0$ there exists an $M<\infty$ such that

(1.6) $$\liminf_{t\to\infty} P^x\{\xi'\wedge\xi''>t \text{ and } M^{-1}\leqq X_i(t)\leqq M \text{ for } i=1 \text{ and } 2\}\geqq 1-\varepsilon.$$

Again, (see sect. 2) under (1.2), if (1.6) holds for all $\varepsilon>0$ for some $x\in Q$, then it holds for all $x\in Q$. If this is the case we call the X-process *positive recurrent*. It is known that then the X-process has an invariant probability measure μ; moreover the probability in (1.6) actually has a limit as $t\to\infty$ which equals $\mu([M^{-1}, M]\times[M^{-1}, M])$, independently of x (see [10], Theorems 4.4.1 and 4.7.1).

To state our theorems we introduce the coefficients

(1.7) $$b_i=k_i-\frac{1}{2}(a_{i1}^2+a_{i2}^2).$$

THEOREM 1. *Assume that* (1.2) *holds, as well as*

(1.8) $$b_i>0, \quad b_{ii}>0, \quad i=1, 2.$$

a) *Let* b_{12} *and* $b_{21}>0$. *Then*

(1.9) $b_{22}b_1-b_{12}b_2>0$ *and* $b_{11}b_2-b_{21}b_1>0 \Rightarrow X$ *is positive recurrent,*

(1.10) $b_{22}b_1-b_{12}b_2\geqq 0$ *and* $b_{11}b_2-b_{21}b_1=0 \Rightarrow X$ *is null recurrent,*

(1.11) $b_{22}b_1-b_{12}b_2<0$ *or* $b_{11}b_2-b_{21}b_1<0 \Rightarrow X$ *is transient.*

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2) $P^x\{\ \}$ denotes the probability measure governing paths starting at $X(0)=x$. $a\wedge b=\min\{a, b\}$.

b) *Let* $b_{12} \leqq 0 \leqq b_{21}$. *Then*

$$b_{11}b_2 - b_{21}b_1 > 0 \Rightarrow X \text{ is positive recurrent,} \tag{1.12}$$

$$b_{11}b_2 - b_{21}b_1 = 0 \Rightarrow X \text{ is null recurrent,} \tag{1.13}$$

$$b_{11}b_2 - b_{21}b_1 < 0 \Rightarrow X \text{ is transient.} \tag{1.14}$$

c) *Let* $b_{12} < 0$ *and* $b_{21} < 0$. *Then*

$$b_{11}b_{22} - b_{12}b_{21} > 0 \Rightarrow X \text{ is positive recurrent,} \tag{1.15}$$

$$b_{11}b_{22} - b_{12}b_{21} \leqq 0 \Rightarrow X \text{ is transient.} \tag{1.16}$$

THEOREM 2. *Assume that* (1.2) *holds as well as*

$$b_{ii} > 0, \quad i=1, 2 \quad \text{and} \quad b_1 < 0 < b_2 . \tag{1.17}$$

a) *If* $b_{12} \geqq 0$ *then* X *is transient.*
b) *Let* $b_{12} < 0 \leqq b_{21}$. *Then*

$$b_{22}b_1 - b_{12}b_2 > 0 \Rightarrow X \text{ is positive recurrent,} \tag{1.18}$$

$$b_{22}b_1 - b_{12}b_2 = 0 \Rightarrow X \text{ is null recurrent,} \tag{1.19}$$

$$b_{22}b_1 - b_{12}b_2 < 0 \Rightarrow X \text{ is transient.} \tag{1.20}$$

c) *Let* $b_{12} < 0$ *and* $b_{21} < 0$. *Then*

$$b_{11}b_{22} - b_{12}b_{21} > 0 \text{ and } b_{22}b_1 - b_{12}b_2 > 0 \Rightarrow X \text{ is positive recurrent,} \tag{1.21}$$

$$b_{11}b_{22} - b_{12}b_{21} \geqq 0 \text{ and } b_{22}b_1 - b_{12}b_2 = 0 \Rightarrow X \text{ is null recurrent,} \tag{1.22}$$

$$b_{11}b_{22} - b_{12}b_{21} \leqq 0 \text{ or } b_{22}b_1 - b_{12}b_2 < 0, \text{ but not} \tag{1.23}$$

$$b_{11}b_{22} - b_{12}b_{21} = b_{22}b_1 - b_{12}b_2 = 0 \Rightarrow X \text{ is transient.}$$

THEOREM 3. *Assume that* (1.2) *holds as well as*

$$b_{ii} > 0 \quad \text{and} \quad b_i < 0 , \qquad i=1, 2 . \tag{1.24}$$

Then X *is transient.*

REMARKS. (i) Theorems 1-3 cover all cases with $b_i \neq 0$, $b_{ii} > 0$. The cases which are not given explicitly follow by interchanging the role of the indices 1 and 2.

(ii) For $b_{ij} > 0$ Gillespie and Turelli [**6**] already conjectured the necessary and sufficient conditions for positive recurrence and proved sufficiency of their condition. Theorem 1 a) confirms their conjecture; also parts b) and c) are in agreement with the method suggested in [**6**] if in [**6**] one adds conditions to prevent the process to escape to ∞. The case of Gillespie and Turelli with $b_{ij} > 0$ corresponds to a competition model. The above theorems also give the recurrence classification when $b_{12} \leqq 0 \leqq b_{21}$ or vice versa (a predator-prey model) or $b_{12}<0$ and $b_{21}<0$ (sometimes viewed as a model for symbiosis). As we see from Theorem 2, if we allow $b_{12}<0$ then it is possible to have positive recurrence even though $b_1<0$. This is in contrast to the situation of the two species competition example (2) of [6].[3)]

(iii) It is remarkable that the criteria in Theorems 1-3 are independent of θ_1 and θ_2. It seems likely that in some of the cases where positive recurrence is proved by means of a Lyapounov function one can even replace $X_i^{\theta_i}$ by more general positive functions which tend to ∞ as $X_i\to\infty$ at a suitable rate.

We shall only prove representative parts of Theorem 1. All remaining cases are similar to one of the explicitly treated cases. Most difficult are (1.18) and (1.21) whose proof is similar to that of (1.12) in sect. 3.

2. Proof of Theorem 1 (with the exception of (1.12)).

As in [6] we make a logarithmic transformation. Ito's formula shows that $Y_i(t)\equiv\theta_i \log X_i(t)$ satisfies

$$\begin{aligned} dY_i(t) &= \theta_i\{a_{i1}dW_1(t)+a_{i2}dW_2(t)\}+\theta_i\{b_i-b_{i1}e^{Y_1(t)}-b_{i2}e^{Y_2(t)}\}\,dt \\ &= \sigma_i dW_{2+i}(t)+\{B_i-B_{i1}e^{Y_1(t)}-B_{i2}e^{Y_2(t)}\}\,dt\,, \end{aligned} \tag{2.1}$$

where

$$B_i=\theta_i b_i\,,\qquad B_{ij}=\theta_i b_{ij}\,,$$

$$\sigma_i^2=\theta_i^2\{a_{i1}^2+a_{i2}^2\}>0 \qquad \text{and}$$

$$W_{2+i}(t)=\frac{\theta_i}{\sigma_i}\{a_{i1}W_1(t)+a_{i2}W_2(t)\}\,. \tag{2.2}$$

$W_{2+i}(t)$ is a one-dimensional Brownian motion for $i=1, 2$, but W_3 and W_4 are not independent, in general. Note that the criteria in Theorem 1 are unchanged when b_i, b_{ij} are changed into B_i, B_{ij}. $Y(\cdot)$ is a diffusion in the whole plane with generator

$$L=\frac{1}{2}\sum_{i,j=1}^{2}A_{ij}\frac{\partial^2}{\partial y_i\partial y_j}+\sum_{i=1}^{2}\{B_i-B_{i1}e^{y_1}-B_{i2}e^{y_2}\}\frac{\partial}{\partial y_i}\,, \tag{2.3}$$

where

$$A_{ij}=\theta_i\theta_j\{a_{i1}a_{j1}+a_{i2}a_{j2}\}\,. \tag{2.4}$$

Denote the explosion time of Y by

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3) After completion of this paper we learned of the paper "Some basic properties of stochastic population models" by M. Barra et al., pp. 155-164 in Systems Theory in Immunology, Lecture Notes in Biomathematics, vol. 32, Springer Verlag, 1978. This paper deals with case b) of Theorem 2 and Theorem 3.

$$\zeta \equiv \lim_{M\to\infty} \inf\{t \geqq 0 : |Y(t)| \geqq M\}.$$

Then one easily sees that

$$\{\xi' \wedge \xi'' < \infty \text{ or } \lim_{t\to\infty} X_i(t) = 0 \text{ or } \infty \text{ for some } i\}$$
$$= \{\zeta < \infty \text{ or } \lim_{t\to\infty} |Y(t)| = \infty\}.$$

Moreover, since the matrix A in (2.4) is constant and strictly positive definite (by (1.2))

$$P^y\{\zeta < \infty \text{ or } \lim_{t\to\infty} |Y(t)| = \infty\} \tag{2.5}$$

can take on only the values 0 or 1 and its value is independent of y (see [**1**], Theorem 3.2). Thus X is transient if and only if Y is transient in the sense that (2.5) equals 1 for some (and hence all) y. Similarly X is recurrent if (2.5) equals 0 for some y. Finally X is positive recurrent if and only if for all $\varepsilon > 0$ there exists an M such that

$$\liminf_{t\to\infty} P^y\{\zeta > t \text{ and } |Y(t)| \leqq M\} \geqq 1-\varepsilon. \tag{2.6}$$

Again, this will hold or fail for all y simultaneously. In fact this condition is equivalent to the finiteness of the expected first hitting time by Y of K, for any compact set K and any starting point $Y(0)$. (See [**10**], Theorem 4.7.1, remark on top of p. 172 and Lemma 4.2.2; also [**1**], Theorem 3.2 and final remark.)

In view of the above remarks it suffices to prove (1.9)-(1.16) with b_i, b_{ij} replaced by B_i, B_{ij} and X by Y. From now on we shall only discuss the Y process. *Also* $B_i > 0$ *throughout, as in Theorem* 1.

PROOF OF (1.9) AND (1.15). As in [**6**], [**14**], we obtain positive recurrence in these cases by constructing a suitable positive supermartingale for Y. In particular we take

$$V(y_1, y_2) = Cy_1^2 - 2y_1y_2 + Dy_2^2, \tag{2.7}$$

with

$$C > 0, \quad D > 0, \quad CD > 1.$$

Then $V \geqq 0$ and $V(y_1, y_2) \to \infty$ as $|y| \to \infty$, and

$$LV(y) = \Gamma_0 + 2y_1\{\Gamma_1 + \Gamma_{11}e^{y_1} + \Gamma_{12}e^{y_2}\} + 2y_2\{\Gamma_2 + \Gamma_{21}e^{y_1} + \Gamma_{22}e^{y_2}\} \tag{2.8}$$

with

$$\Gamma_0 = CA_{11} - 2A_{12} + DA_{22},$$
$$\Gamma_1 = B_1C - B_2, \qquad \Gamma_2 = B_2D - B_1,$$
$$\Gamma_{11} = B_{21} - B_{11}C, \qquad \Gamma_{22} = B_{12} - B_{22}D,$$

$$\Gamma_{12}=B_{22}-B_{12}C\,,\qquad \Gamma_{21}=B_{11}-B_{21}D\,.$$

If $LV(y)\leqq -1$ for y outside some compact set, then by pp. 115, 116 and Theorem 3.7.1 of [**10**] Y (and hence X) will be positive recurrent. Now assume (as in (1.9))

$$\text{(2.9)}\qquad \begin{gathered} B_{12}>0\,,\qquad B_{21}>0\,,\\ B_{22}B_1-B_{12}B_2>0 \quad\text{and}\quad B_{11}B_2-B_{21}B_1>0\,. \end{gathered}$$

In this case we take

$$C=\frac{B_{22}}{B_{12}}>\frac{B_2}{B_1},\qquad D=\frac{B_{11}}{B_{21}}>\frac{B_1}{B_2}\,.$$

One easily checks that for this C and D (2.9) implies

$$CD>1,\quad \Gamma_i>0,\quad \Gamma_{ii}<0,\quad i=1,2 \quad\text{and}\quad \Gamma_{12}=\Gamma_{21}=0\,.$$

This immediately implies $LV(y)\to-\infty$, as $|y|\to\infty$, and hence positive recurrence.

Next assume the hypotheses of (1.15). Specifically

$$\text{(2.10)}\qquad B_{12}<0,\quad B_{21}<0 \quad\text{and}\quad B_{11}B_{22}-B_{12}B_{21}>0\,.$$

We now take C and D such that[4]

$$C>\frac{B_2}{B_1}\vee 1,\quad D>\frac{B_1}{B_2}\vee 1,\quad\text{and}\quad -\frac{B_{21}}{B_{11}}<\frac{C-1}{D-1}<\frac{B_{22}}{-B_{12}}\,.$$

(This is possible by (2.10).) Again we have $CD>1$ as well as

$$\text{(2.11)}\qquad \Gamma_i>0,\quad \Gamma_{ii}<0,\quad \Gamma_{1i}+\Gamma_{2i}<0,\quad i=1,2 \quad\text{and}\quad \Gamma_{12}>0,\quad \Gamma_{21}>0\,.$$

If $y_1\to-\infty$ and y_2 remains bounded above, then the principal contribution to $LV(y)$ is

$$2y_1\{\Gamma_1+\Gamma_{12}e^{y_2}\}+2y_2\Gamma_2$$

and this tends to $-\infty$, since $\Gamma_i>0$ and $\Gamma_{12}>0$. If $y_1\to-\infty$ and $y_2\to+\infty$ then we have

$$y_1\{\Gamma_1+\Gamma_{11}e^{y_1}+\Gamma_{12}e^{y_2}\}\sim\Gamma_{12}y_1e^{y_2}\to-\infty$$

as above; but also, since $\Gamma_{22}<0$,

$$y_2\{\Gamma_2+\Gamma_{21}e^{y_1}+\Gamma_{22}e^{y_2}\}\sim\Gamma_{22}y_2e^{y_2}\to-\infty\,.$$

Similarly $LV(y)\to-\infty$ if $y_1\to\infty$ and y_2 remains bounded. Lastly consider the case $y_1\to\infty$, $y_2\geqq y_1$. If also $y_2-y_1\to\infty$, then

$$\frac{1}{2}LV(y)\sim(\Gamma_{12}y_1+\Gamma_{22}y_2)e^{y_2}\leqq(\Gamma_{12}+\Gamma_{22})y_2e^{y_2}\to-\infty\,.$$

If on the other hand y_2-y_1 remains bounded, then by (2.8) and (2.11)

4) $a\vee b=\max\{a,b\}$.

$$LV(y)=2\{\Gamma_{22}+\Gamma_{12}+o(1)\}y_2e^{y_2}+2\{\Gamma_{11}+\Gamma_{21}\}y_1e^{y_1}$$
$$+2(y_2-y_1)(\Gamma_{21}e^{y_1}-\Gamma_{12}e^{y_2})$$
$$\leqq 2\{\Gamma_{22}+\Gamma_{12}+o(1)\}y_2e^{y_2}\to-\infty .$$

Since y_1 and y_2 play completely symmetric roles we again obtain $LV(y)\to-\infty$ as $|y|\to\infty$ and hence (1.15) holds.

Lastly we observe that positive recurrence is proved easily whenever $B_{12}=B_{21}=0$ by taking $V(y)=y_1^2+y_2^2$. However, we cannot prove (1.12) in general by this method and the full proof of (1.12) will be postponed till the next section.

All further proofs rely on the following

COMPARISON LEMMA. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{\mathcal{F}_t\}_{t\geq 0}$ a right continuous increasing family of sub σ-fields of $\mathcal{F}$ such that $\mathcal{F}_0$ contains all subsets of P-null sets. Let $\sigma(\cdot, \cdot): [0, \infty)\times \boldsymbol{R}\to[0, \infty)$ be a measurable function which satisfies the uniform Lipschitz condition*

$$(2.12)\qquad |\sigma(t, x')-\sigma(t, x'')|\leqq K|x'-x''|,\ x', x''\in \boldsymbol{R},\ t\geqq 0 .$$

Furthermore, let $\{W(t, \omega)\}_{t\geq 0}$ be an $\mathcal{F}_t$-Brownian motion such that $W(0)=0$ a.s., and let $\beta_i(\cdot, \cdot, \cdot): [0, \infty)\times\boldsymbol{R}\times\Omega\to\boldsymbol{R}$, $i=1, 2$, be two functions with the following properties:

(2.13) *$(t, x)\longrightarrow\beta_i(t, x, \omega)$ is continuous for almost all ω,*

(2.14) *For each $t\geqq 0$ $(x, \omega)\longrightarrow\beta_i(t, x, \omega)$ is $\mathcal{B}\times\mathcal{F}_t$ measurable, where $\mathcal{B}$ is the Borel field of $\boldsymbol{R}$.*

Finally, let $\{Z_i(t, \omega)\}_{t\geq 0}$, $i=1, 2$, be two $\mathcal{F}_t$ adapted real valued continuous processes which satisfy

$$(2.15)\qquad Z_i(t, \omega)-Z_i(0, \omega)=\int_0^t\sigma(s, Z_i(s, \omega))dW(s, \omega)+\int_0^t\beta_i(s, Z_i(s, \omega), \omega)ds,\quad t\leqq T ,$$

for some stopping time T. If

$$(2.16)\qquad Z_1(0)\leqq Z_2(0)\qquad w.p. 1 ,$$

then with probability 1

$$(2.17)\qquad Z_1(t)\leqq Z_2(t)\qquad \text{for all } 0\leqq t\leqq S ,$$

where

$$(2.18)\qquad S=T\wedge\inf\{t\geqq 0: \sup_x[\beta_1(t, x)-\beta_2(t, x)]\geqq 0\} .$$

If there exist constants $K(M)<\infty$ such that at least one of the β_i, $i=1$ or 2, *also satisfies*

$$|\beta_i(t, x', \omega)-\beta_i(t, x'', \omega)|\leqq K(M)|x'-x''| \quad \text{for all } |x'|, |x''|\leqq M,\ t\leqq T, \tag{2.19}$$

then S may be replaced by

$$S'=T\wedge\inf\{t\geqq 0:\ \sup_x[\beta_1(t, x)-\beta_2(t, x)]>0\}. \tag{2.20}$$

This lemma is a simple variant of Theorem 1.1 in [8]. No significant change in the proof is necessary, except that (1.10) of [8] should be proved conditionally on $S>0$, and similarly one now proves that $P\{\theta<S\}=0$, rather than $P\{\theta<\infty\}=0$ as on pp. 621, 622 of [8]. We also note that if (2.19) holds we have "pathwise uniqueness" even when β_i is replaced by $\beta_i+\varepsilon$ in the sense that for each $\varepsilon\in\boldsymbol{R}$, $z\in\boldsymbol{R}$ there exists on Ω a non-anticipating continuous solution Z_i^ε to

$$Z_i^\varepsilon(t, \omega)=z+\int_0^t\sigma(s, Z_i^\varepsilon(s, \omega))dW(s, \omega) +\int_0^t\{\beta_i(s, Z_i^\varepsilon(s, \omega), \omega)+\varepsilon\}\,ds$$

for

$$t\leqq T^\varepsilon\equiv T\wedge(\text{explosion time of } Z_i^\varepsilon).$$

Moreover, if $\tilde{Z}_i^\varepsilon$ is another solution to this equation, then

$$P\{Z_i^\varepsilon(t)=\tilde{Z}_i^\varepsilon(t) \text{ for all } t\leqq T^\varepsilon\}=1.$$

The proof of existence and uniqueness of Z_i^ε under the Lipschitz conditions (2.12) and (2.19) is the standard one (cf. [5], Ch. 5.1, 2, [13], Ch. 3.2, 3). In all our applications of the comparison lemma (2.19) will be satisfied and we use (2.17) with S replaced by S'.

We now turn to the

PROOF OF (1.11) AND (1.14). By symmetry we may restrict ourselves to the case where

$$B_{11}B_2-B_{21}B_1<0.$$

By (1.8) this forces

$$B_{21}>0. \tag{2.21}$$

Now take $0<\tilde{B}_1<B_1$ such that

$$B_{11}B_2-B_{21}\tilde{B}_1<0 \tag{2.22}$$

and $\Delta=\log((B_1-\tilde{B}_1)|B_{12}|^{-1})$, so that

$$B_1-B_{12}e^z\geqq\tilde{B}_1 \quad \text{on } \{z\leqq\Delta\}. \tag{2.23}$$

Next let $Y_1^{(1)}(t)=Y_1^{(1)}(t\,;y_1)$ be the solution of

$$dY_1^{(1)}(t)=\sigma_1 dW_3(t)+\{\tilde{B}_1-B_{11}\exp Y_1^{(1)}(t)\}\,dt\,,$$

$$Y_1^{(1)}(0)=y_1\,,$$

and set

$$Y_2^{(1)}(t)=y_2+\sigma_2 W_4(t)+\int_0^t\{B_2-B_{21}\exp Y_1^{(1)}(s)\}\,ds\,. \tag{2.24}$$

$Y_1^{(1)}$ is positive recurrent; in fact as y_1 tends to $+\infty$, the drift vector $\tilde{B}_1-B_{11}\exp y_1$ tends to $-\infty$, and as $y_1\to-\infty$, the drift vector tends to $\tilde{B}_1>0$. (One can also easily check $\tilde{L}\tilde{V}(y_1)\leqq-1$ for sufficiently large $|y_1|$, for $\tilde{V}(y_1)=y_1^2$ and $\tilde{L}$ the generator of $Y_1^{(1)}$.) Thus $(Y_1^{(1)}, Y_2^{(1)})$ does not explode and is defined for all time. Now denote the solution of (2.1) which starts at $(Y_1(0), Y_2(0))=(y_1, y_2)$ by $(Y_1(t\,;y_1, y_2), Y_2(t\,;y_1, y_2))$ and take

$$T=\inf\{t: Y_2(t\,;y_1, y_2)\geqq\Delta\}\wedge\zeta\,.$$

Then by the comparison lemma and (2.23)

$$Y_1(t\,;y_1, y_2)\geqq Y_1^{(1)}(t) \qquad \text{on } \{t<T\}\,.$$

Again by the comparison lemma (use (2.21))

$$Y_2(t\,;y_1, y_2)\leqq Y_2^{(1)}(t) \qquad \text{on } \{t<T\}\,. \tag{2.25}$$

We now show that for y_2 sufficiently small

$$P\{\zeta<\infty \text{ or } (\sup_t Y_2^{(1)}(t)<\Delta \text{ and } \lim_{t\to\infty} Y_2^{(1)}(t)=-\infty)\}>0\,, \tag{2.26}$$

which together with (2.25) will imply (1.11) and (1.14). In turn, (2.26) will be immediate if we prove for fixed y_1

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\{B_2-B_{21}\exp Y_1^{(1)}(s)\}\,ds<0\,, \tag{2.27}$$

because the left hand side of (2.27) is independent of y_2 and, by (2.24), (2.25)

$$\frac{1}{t}Y_2(t\,;y_1, y_2)\leqq\frac{1}{t}Y_2^{(1)}(t)\leqq o(1)+\frac{1}{t}\int_0^t\{B_2-B_{21}\exp Y_1^{(1)}(s)\}\,ds\,,$$

$$t\to\infty\,.$$

Finally, (2.27) follows from the ergodic theorem. Indeed $Y_1^{(1)}(t)$ has the stationary probability measure ([2], problem 16.11.18, [11], Theorem 4.4)

$$m(dx)=\frac{1}{C}e^{B(x)}dx\,, \tag{2.28}$$

where

$$B(x)=\frac{2}{\sigma_1^2}\int_0^x \{\tilde{B}_1-B_{11}e^u\}\,du=\frac{2}{\sigma_1^2}\tilde{B}_1 x-\frac{2}{\sigma_1^2}B_{11}(e^x-1), \tag{2.29}$$

and

$$C=\int_{-\infty}^{\infty} e^{B(x)}\,dx .$$

Thus, by the ergodic theorem ([**10**], Theorem 4.5.1, [**11**], Theorem 4.8, [**12**] Theorem 5.1) the left hand side of (2.27) equals

$$\int_{-\infty}^{+\infty}\{B_2-B_{21}e^x\}\,m(dx)=\{B_2-B_{21}\tilde{B}_1(B_{11})^{-1}\}<0 \qquad \text{(by (2.22))}.$$

This proves (2.27) and hence (1.11) and (1.14).

The proof of (1.16) is quite easy because under its hypotheses

$$\begin{aligned} d\{-B_{21}Y_1(t)+B_{11}Y_2(t)\} &= -\sigma_1 B_{21}\,dW_3(t)+\sigma_2 B_{11}\,dW_4(t)\\ &\quad +\{-B_1B_{21}+B_2B_{11}+(B_{21}B_{12}-B_{11}B_{22})e^{Y_2(t)}\}\,dt \end{aligned}$$

which has a drift coefficient

$$\geqq -B_1B_{21}+B_2B_{11}>B_2B_{11}>0 .$$

Thus $B_{11}Y_2(t)-B_{21}Y_1(t)$ grows at least linearly (e.g., by the comparison lemma).

Next we indicate how to *prove* (1.10) *and* (1.13). The basic idea is in [7]. For the time being assume only

$$B_{11}B_2-B_{21}B_1=0 ,$$

and consequently (2.21). Set

$$U(t)=B_1Y_1(t)+B_2Y_2(t),$$

$$V(t)=B_2Y_1(t)-B_1Y_2(t).$$

Then

$$\begin{aligned} dU(t) &= \bar{\sigma}_1\,d\overline{W}_3(t)+\Big\{\bar{B}_1-\bar{B}_{11}\exp\Big(\frac{B_1U(t)+B_2V(t)}{B_1^2+B_2^2}\Big)\\ &\qquad -\bar{B}_{12}\exp\Big(\frac{B_2U(t)-B_1V(t)}{B_1^2+B_2^2}\Big)\Big\}\,dt , \end{aligned}$$

and

$$dV(t)=\bar{\sigma}_2\,d\overline{W}_4(t)+\bar{B}_{22}\exp\Big(\frac{B_2U(t)-B_1V(t)}{B_1^2+B_2^2}\Big)\,dt ,$$

where $\overline{W}_3$, $\overline{W}_4$ are suitable Brownian motions and

$$\bar{B}_1=B_1^2+B_2^2>0 ,$$

$\bar{B}_{11}=B_1B_{11}+B_2B_{21}>0$,

$\bar{B}_{12}=B_1B_{12}+B_2B_{22}$,

$\bar{B}_{22}=B_1B_{22}-B_2B_{12}\geqq 0$ (by the hypothesis of (1.10), respectively $B_{12}\leqq 0$ in (1.13)),

$\bar{\sigma}_1>0, \quad \bar{\sigma}_2>0$.

The drift vector of the U-component at the point (u, v) equals

$$\bar{b}_1(u, v)=\bar{B}_1-\bar{B}_{11}\exp\left(\frac{B_1u+B_2v}{B_1^2+B_2^2}\right)-\bar{B}_{12}\exp\left(\frac{B_2u-B_1v}{B_1^2+B_2^2}\right),$$

so that for a suitable constant $\Delta>0$

$$\bar{b}_1(u, v)\geqq\frac{1}{2}\bar{B}_1>0 \quad \text{on } \left\{(u, v): u\leqq-\frac{3}{2}\frac{B_2}{B_1}v,\ v\geqq\Delta\right\}, \tag{2.30}$$

and

$$\bar{b}_1(u, v)\leqq-\frac{1}{2}\bar{B}_{11}<0 \quad \text{on } \left\{(u, v): 0\leqq u\leqq\frac{B_1}{2B_2}v,\ v\geqq\Delta\right\}. \tag{2.31}$$

On the other hand, the drift of the V-component,

$$\bar{b}_2(u, v)=\bar{B}_{22}\exp\left(\frac{B_2u-B_1v}{B_1^2+B_2^2}\right)\leqq\bar{B}_{22}\exp\left(-\frac{1}{2}B_1(B_1^2+B_2^2)^{-1}v\right) \quad \text{on } \left\{u\leqq\frac{B_1}{2B_2}v\right\}. \tag{2.32}$$

Now consider the line segments

$$I_k=\left\{(u, v): v=2^k,\ -\frac{7}{4}\frac{B_2}{B_1}2^k\leqq u\leqq\frac{3}{8}\frac{B_1}{B_2}2^k\right\},$$

$$J_k=\left\{(u, v): v=2^k,\ -\frac{3}{2}\frac{B_2}{B_1}2^k\leqq u\leqq\frac{1}{4}\frac{B_1}{B_2}2^k\right\}.$$

Then $J_k\subset I_k$. Denote the endpoints of I_k by p_k and q_k and those of J_k by p_k' and q_k' and let L_k' (M_k') be the line through the points p_{k+1} and p_k' (respectively q_{k+1} and q_k'). L_k' has the equation

$$u+2\frac{B_2}{B_1}v=2^{k-1}\frac{B_2}{B_1}. \tag{2.33}$$

Finally we consider the line L_k (M_k) through p_k (q_k) parallel to L_k' (respectively M_k') and the trapezoid R_k, bounded by I_{k+1}, J_k, L_k' and M_k', and define the stopping time

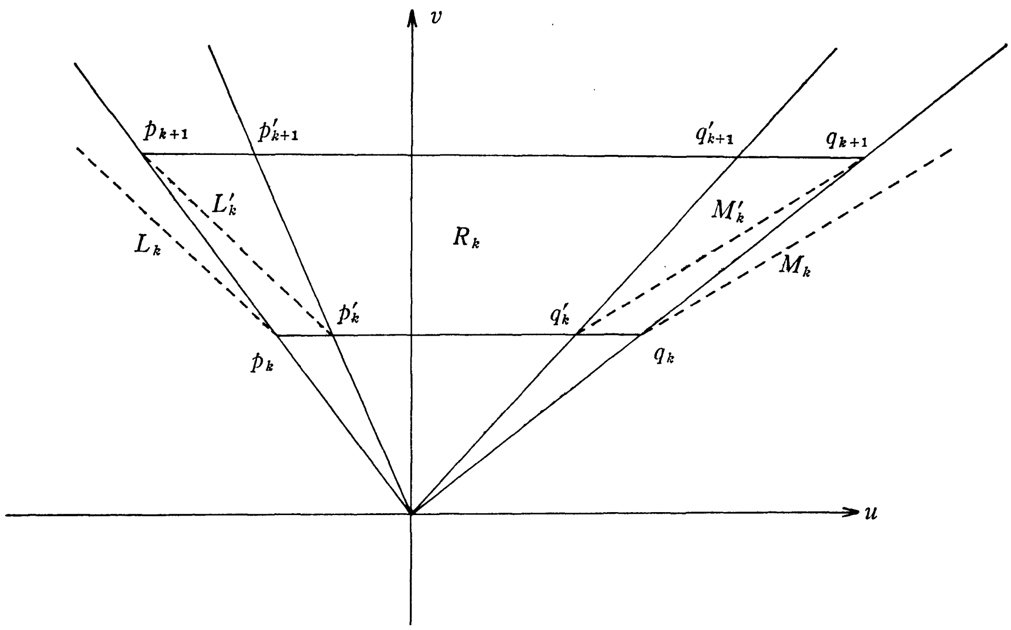

Fig. 1.

$$\tau_k=\inf\{t\geqq 0:\ V(t)=2^k\}.$$

We then have the following estimate:

LEMMA 1. *There exists a constant* k_0 *such that for* $k\geqq k_0$ *and* $(u_0, v_0)\in R_k$

$$P^{(u_0,v_0)}\{\tau_k<\infty \text{ and } (U(\tau_k),\ V(\tau_k))\in I_k\}\geqq 1-4(\bar{\sigma}_2)^{-1}k^{-3/2}. \tag{2.34}$$

PROOF. Let

$$\sigma_k=\inf\left\{t:U(t)\geqq \frac{B_1}{2B_2}V(t)\right\}.$$

Then for $(U(0),\ V(0))\in R_k$, $\sigma_k\geqq$first crossing time of M_k. Also, by (2.32)

$$\bar{b}_2(U(s),\ V(s))\leqq \bar{B}_{22}\exp\left(-\frac{1}{2}B_1(B_1^2+B_2^2)^{-1}2^k\right)\qquad \text{for } s\leqq \sigma_k\wedge\tau_k.$$

Consequently, for k large, uniformly in $(u_0,\ v_0)\in R_k$

$$\begin{aligned}(2.35)\qquad & P^{(u_0,v_0)}\{\sigma_k\wedge\tau_k\geqq k^3 2^{2k}\}\\ &\leqq P^{(u_0,v_0)}\Big\{v_0+\bar{\sigma}_2\overline{W}_4(t)+k^3 2^k\bar{B}_{22}\exp\left(-\frac{1}{2}B_1(B_1^2+B_2^2)^{-1}2^k\right)\geqq 2^k\\ &\qquad\qquad \text{for all } t\leqq k^3 2^{2k}\Big\}\\ &\leqq P\left\{\inf_{t\leq k^3 2^{2k}}\overline{W}_4(t)\geqq \frac{1}{\bar{\sigma}_2}(2^k-v_0)-1\geqq -\frac{2^k}{\bar{\sigma}_2}-1\right\}\\ &\leqq 3(\bar{\sigma}_2)^{-1}k^{-3/2}.\end{aligned}$$

(2.35) shows that $\sigma_k \wedge \tau_k$ will not exceed $k^3 2^{2k}$ with high probability. We next indicate how to prove that $(U(t), V(t))$ only has a small probability to cross the lines L_k or M_k before $\tau_k \wedge k^3 2^{2k}$. More precisely, we claim:

$$(2.36) \qquad P^{(u_0, v_0)}\{(U(t), V(t)) \text{ crosses } L_k \text{ before } \tau_k \wedge k^3 2^{2k}\} \leqq 2^{1-k}, \ (u_0, v_0) \in R_k .$$

To prove (2.36) note that L_k has the equation

$$u + \frac{2B_2}{B_1} v = 2^{k-2} \frac{B_2}{B_1}$$

(compare (2.33)) and that for $(u_0, v_0) \in R_k$

$$u_0 + \frac{2B_2}{B_1} v_0 \geqq 2^{k-1} \frac{B_2}{B_1} .$$

Thus in order to hit L_k $U(t)+(2B_2/B_1)V(t)$ has to decrease at least $2^{k-2}B_2/B_1$. However, the drift coefficient of $U(t)+(2B_2/B_1)V(t)$ at (u, v) equals

$$\bar{b}_1(u, v) + \frac{2B_2}{B_1} \bar{b}_2(u, v) \geqq \frac{1}{2} \bar{B}_1 > 0$$

for $v \geqq 2^k$, (u, v) between L_k' and L_k (by (2.30) and (2.32)). Thus, in order to move from R_k to L_k the diffusion $U(t)+(2B_2/B_1)V(t)$ has to decrease by at least $2^{k-2}B_2/B_1$, while it passes through a region where its drift is at least $\bar{B}_1/2$. (2.36) therefore follows immediately from the following

SUBLEMMA. *Let $Z(t)$ be a nonanticipating continuous solution of*

$$dZ(t) = \sigma dW(t) + \beta(t, Z(t))dt ,$$

where $\sigma > 0$ is constant, $W(t)$ is a Brownian motion and $\beta(s, z)$ a continuous function of $s, z \in [0, \infty) \times \boldsymbol{R}$ satisfying

$$\beta(s, z) \geqq \beta_0 > 0 \qquad on \ \{z \leqq 0\} .$$

Then for each $B > 0$ there exists a constant k_1 such that for all $z \geqq 0$ and $k \geqq k_1$

$$P^z\{Z(t) \text{ enters } (-\infty, -B2^k] \text{ before time } k^3 2^{2k}\} \leqq 2^{-k} + 2k^6 2^{4k} e^{-\delta 2^k}$$

where $\delta = 2B\beta_0\sigma^{-2}$.

We do not give the relatively simple details of the proof of this sublemma. It can be proved by looking at the successive excursions of the Z-process from $-c$ to $\{0, -B2^k\}$ and back to $-c$, for some $c > 0$, The probability that Z hits $-B2^k$ during the first $k^6 2^{4k}$ excursions is at most $2k^6 2^{4k} e^{-\delta 2^k}$. On the other hand, the probability that the first $k^6 2^{4k}$ excursions take less time than $k^3 2^{2k}$

is at most 2^{-k} (e.g., by Chebyshev's inequality). Compare [7], Lemma 2. Alternatively one can prove the sublemma by comparing $Z(t)$ with a diffusion on $(-\infty, 0]$ with 0 as reflecting boundary and generator $(1/2)\sigma^2(d^2/dx^2)+\beta_0(d/dx)$ on $(-\infty, 0)$. For this diffusion one can make sufficiently explicit estimates to prove the sublemma.

The same proof works if one replaces L_k in (2.36) by M_k. This together with (2.35) implies (2.34) because, if $(U(t), V(t))$ does not cross L_k nor M_k before $\tau_k \wedge k^3 2^{2k}$, but $\sigma_k \wedge \tau_k \leqq k^3 2^{2k}$, then necessarily

$$\tau_k \leqq k^3 2^{2k} \wedge (\text{first crossing time of } L_k \cup M_k) \quad \text{and} \quad (U(\tau_k), V(\tau_k)) \in I_k . \quad \square$$

Now notice that the sector

$$C=\left\{(u, v) : v>0, -\frac{3B_2}{2B_1} v \leqq u \leqq \frac{B_1}{4B_2} v\right\} \subset \bigcup_{k \geqq 0} R_k .$$

Exactly as in [9] (proof of Theorem 2 from (2.60) on) or [7], Theorem 1, one can now derive from Lemma 1 that for any $(u_0, v_0) \in C$

$$P^{(u_0, v_0)}\{(U(t), V(t)) \text{ enters } \bigcup_{0 \leqq k \leqq k_2} R_k \text{ at some finite time}\}$$

$$\geqq 1-\sum_{k \geqq k_2} 4(\bar{\sigma}_2)^{-1} k^{-3/2} \geqq \frac{1}{2}$$

for a suitable $k_2 \geqq k_0$. In other words the probability of entering the compact set

$$R=\bigcup_{k \leqq k_2} R_k$$

is at least 1/2 from any point in C, and by the strong Markov property, this probability will be at least 1/4 from any starting point if we can prove

$$P\{(U(t), V(t)) \in C \text{ for some finite time } | Y(0)=y\} \geqq \frac{1}{2} \tag{2.37}$$

for all y. Thus, recurrence of Y (and X) has been reduced to (2.37), which we shall now prove.

First we choose $\Delta_1, \Delta_2>0$ such that

$$B_2-B_{21}e^{\Delta_1} \leqq -1, \quad \Delta_2>B_1 B_2^{-1} \Delta_1 \tag{2.38}$$

(recall that (2.21) holds). Then on the half line

$$H_0=\{(y_1, y_2) : y_1=\Delta_1, y_2 \leqq -\Delta_2\},$$

$$u \equiv B_1 y_1+B_2 y_2 \quad \text{and} \quad v \equiv B_2 y_1-B_1 y_2$$

satisfy

$$v>0, \quad -B_2 B_1^{-1} v \leqq u<0$$

and hence $(u, v)\in C$. The same argument shows that for each M there exists a $K(M)$ such that

$$H(M)\equiv\{(y_1, y_2): |y_1|\leqq M,\ y_2\leqq -K(M)\}\subset C.$$

Thus, it suffices to prove

$$P^y\{Y(t) \text{ hits } H_0\cup\bigcup_{M=1}^{\infty}H(M) \text{ at some finite time}\}\geqq\frac{1}{2}. \tag{2.39}$$

From here on the proofs of (1.10) and (1.13) differ slightly. First consider (1.13), i.e., assume $B_{12}\leqq 0$. We now define $(Y_1^{(2)}(t\,;y_1),\ Y_2^{(2)}(t\,;y_2))$ as the solution of

$$dY_i^{(2)}(t\,;y_i)=\sigma_i dW_{2+i}(t)+\{B_i-B_{ii}\exp Y_i^{(2)}(t\,;y_i)\}\,dt, \tag{2.40}$$
$$Y_i^{(2)}(0\,;y_i)=y_i,\qquad i=1, 2.$$

As with $Y^{(1)}$ this does not explode, and since $B_{12}\leqq 0<B_{21}$ the comparison lemma shows

$$Y_1(t\,;y_1, y_2)\geqq Y_1^{(2)}(t\,;y_1),\quad Y_2(t\,;y_1, y_2)\leqq Y_2^{(2)}(t\,;y_2),\quad t<\zeta, \tag{2.41}$$

where as before $Y(t\,;y_1, y_2)$ is the solution of (2.1) with $Y_i(0\,;y_1, y_2)=y_i$. We already know that $Y^{(2)}$ is recurrent so that $Y^{(2)}$ enters the set

$$G=\{(z_1, z_2): z_1\geqq\Delta_1,\ z_2\leqq -2\Delta_2\}$$

with probability one at some finite time, say T_1. By (2.41) the first entrance time of G by Y, call it T_2, must satisfy $T_2\leqq T_1$ unless $\zeta\leqq T_1$. We set

$$T_3=\inf\{t: Y(t)\in H_0\cup\cup H(M)\}$$

and begin by proving

$$P^y\{\zeta\leqq T_1\wedge T_2\wedge T_3\}=0. \tag{2.42}$$

(2.42) can be seen as follows. By the recurrence of $Y^{(2)}$ and (2.41) (we suppress y_1, y_2 in the notation in most of the remaining proof)

$$\inf_{t<\zeta\wedge T_1} Y_1(t)\geqq\inf_{t\leqq T_1} Y_1^{(2)}(t)>-\infty$$

and similarly

$$\sup_{t<\zeta\wedge T_1} Y_2(t)\leqq\sup_{t\leqq T_1} Y_2^{(2)}(t)<\infty.$$

Thus $\zeta\leqq T_1\wedge T_2\wedge T_3$ is possible only if for some random $M<\infty$

$$\liminf_{t\uparrow\zeta} Y_2(t)=-\infty,\qquad \inf_{t<\zeta} Y_1(t)\geqq -M \tag{2.43}$$

or

(2.44) $$\limsup_{t\uparrow\zeta} Y_1(t)=+\infty\,, \qquad \sup_{t<\zeta} Y_2(t)\leqq M\,.$$

(2.43) is inconsistent with $\zeta\leqq T_2\wedge T_3$ because on $\{t<\zeta\wedge T_2,\ Y_2(t)\leqq -2\Delta_2\}$ one must have $Y_1(t)<\Delta_1$, so that $\zeta\leqq T_2\wedge T_3$ together with (2.43) implies

$$-M\leqq Y_1(t)<\Delta_1 \quad\text{and}\quad Y_2(t)<-K(M\vee\Delta_1)\,,$$
$$\text{or}\quad Y(t)\in H(M\vee\Delta_1)$$

for some $t<T_3$ which is impossible. This only leaves (2.44). However, (2.44) and $\zeta\leqq T_1\wedge T_2\wedge T_3$ can occur only if

(2.45) $$Y_1(t)\leqq y_1+\sigma_1 W_3(t)+\{B_1-B_{12}e^M\}\,t\,, \qquad t<\zeta\,.$$

Together with the law of iterated logarithm (2.45) would imply $\zeta=\infty$. This contradicts $\zeta\leqq T_1\wedge T_2\wedge T_3$ since we know $T_1<\infty$. Hence (2.42) follows.

From (2.42) and the lines preceding it, we conclude that $T_2\wedge T_3\leqq T_1<\infty$ a.s. (Otherwise $T_3>T_1$ and $T_2>T_1$, and the latter implies $\zeta\leqq T_1$ hence $\zeta\leqq T_1\wedge T_2\wedge T_3$.) Thus Y enters $G\cup H_0\cup\cup H(M)$ at some finite time. Once Y enters $H_0\cup\cup H(M)$ the event in (2.39) occurs. Thus, by the strong Markov property it suffices to prove (2.39) for $y\in G$ only.

To complete the proof for $y\in G$ we once again apply the comparison lemma. Set

$$T_4=\inf\{t\geqq 0:\ Y_1(t)\leqq\Delta_1 \text{ or } Y_2(t)\geqq 0\}$$

and let $Y^{(3)}$ be the solution of

(2.46) $$dY_1^{(3)}(t)=\sigma_1 dW_3(t)+\{B_1-B_{12}-B_{11}\exp Y_1^{(3)}(t)\}\,dt\,,$$
$$Y_1^{(3)}(0)=y_1,\quad Y_2^{(3)}(t)=y_2+\sigma_2 W_4(t)-t\,.$$

Again $Y_1^{(3)}$ is positive recurrent and does not explode, and by the comparison lemma we have for $(y_1, y_2)\in G$ and $0\leqq t\leqq T_4$, $t<\zeta$,

(2.47) $$Y_1(t\,;\,y_1,\,y_2)\leqq Y_1^{(3)}(t),\quad Y_2(t\,;\,y_1,\,y_2)\leqq Y_2^{(3)}(t)$$

(recall $B_{12}\leqq 0$ and (2.38)). Thus, if

$$T_5=\inf\{t\geqq 0:\ Y_1^{(3)}=\Delta_1\}\,,$$

then

$$T_4\leqq T_5<\infty \quad\text{or}\quad \zeta\leqq T_5 \quad \text{w.p.1.}$$

Just as with (2.43) one shows that $\zeta<T_3\wedge T_4$ has probability zero. Consequently

$$P^y\{T_3\wedge T_4\leqq T_5<\infty\}=1\,, \qquad y\in G\,.$$

On $\{T_3<\infty\}$ the event in (2.39) occurs and on $\{T_4\leqq T_5<\infty=T_3\}$

$$Y_2(T_4)\leqq \sup_{t\geqq 0} Y_2^{(3)}\leqq y_2+\sup_{t\geqq 0}\{\sigma_2 W_4(t)-t\}\,.$$

Thus, if $\varDelta_2$ is chosen so large that

$$P\{\sup_{t\geqq 0}\{\sigma_2 W_4(t)-t\}\leqq\varDelta_2\}\geqq\frac{1}{2},$$

then for $y\in G$ (and hence $y_2\leqq -2\varDelta_2$) we have

$$P^y\{T_3<\infty \text{ or } T_4<\infty \text{ and } Y_2(T_4)\leqq -\varDelta_2\}\geqq\frac{1}{2}.$$

This proves (2.39), since $T_4<\infty$ and $Y_2(T_4)\leqq -\varDelta_2$ means $Y(T_4)\in\{\varDelta_1\}\times(-\infty, -\varDelta_2]=H_0$. This proves the recurrence in case (1.13).

In the case (1.10) we merely have to redefine $Y_1^{(2)}(t)$ as the solution of $Y_1^{(2)}(0)=y_1$,

$$dY_1^{(2)}(t)=\sigma_1 dW_3(t)+\{B_1-B_{11}\exp Y_1^{(2)}(t)-B_{12}\exp Y_2^{(2)}(t)\}\,dt,$$

while $Y_2^{(2)}(t)$ is as in (2.40). With B_{12}, $B_{21}>0$ (2.41) remains unchanged. This new $Y^{(2)}$ is recurrent by (1.12), (1.13) with the indices 1 and 2 interchanged. The only other change needed is that B_1-B_{12} in (2.46) should be replaced by B_1 if $B_{12}>0$.

This proves that Y is recurrent in the cases (1.10) and (1.13). The fact that we must have null recurrence and not positive recurrence follows from the fact that

$$V(t)\geqq V(0)+\bar\sigma_2\overline{W}_4(t) \qquad (\text{since } \bar B_{22}\geqq 0).$$

3. Proof of (1.12).

Since the case $B_{12}=B_{21}=0$ was already treated (just before the comparison lemma) we restrict ourselves to the case where

$$B_{12}\leqq 0<B_{21},\quad B_{11}B_2-B_{21}B_1>0. \tag{3.1}$$

First we choose

$$\varDelta_1=\log B_{21}^{-1}(B_2+4\sigma_2^2), \tag{3.2}$$

and $\varDelta_2\leqq -1$ so small that (3.34) below holds. K_1, K_2, $\cdots$ will be various constants which depend on the B_i, B_{ij} and $\varDelta_i$, but whose particular value is unimportant[5]. We introduce the vertical line

$$L_1=\{(y_1,\ y_2):\ y_1=\varDelta_1\}$$

and the horizontal line

$$L_2=\{(y_1,\ y_2):\ y_2=\varDelta_2\},$$

and their respective hitting times

5) Note that (3.34) does not involve the K_i, so that $\varDelta_2$ is determined first as a function of the B_i, B_{ij} only, and then the K_i as functions of the B_i, B_{ij}, $\varDelta_1$ and $\varDelta_2$.

$$\tau_i=\inf\{t: Y(t)\in L_i\}.$$

($\tau_i=\infty$ if Y does not hit L_i before ζ). The proof of (1.12) rests on Lemmas 2-5 below; these lemmas become more intuitive by looking at the behavior of the driftvector $b(y_1, y_2)=(b_1(y_1, y_2), b_2(y_1, y_2))$ where

$$b_i(y_1, y_2)=B_i-B_{i1}e^{y_1}-B_{i2}e^{y_2}.$$

Fig. 2 gives the vector b at some typical points. b is vertical (horizontal) on the curve C_1, obtained by setting $b_2=0$ (respectively C_2, obtained by setting $b_1=0$). These curves are indicated in the figure (recall $B_{12}\leqq 0<B_{21}$). Throughout $|y|$ denotes $(y_1^2+y_2^2)^{1/2}$.

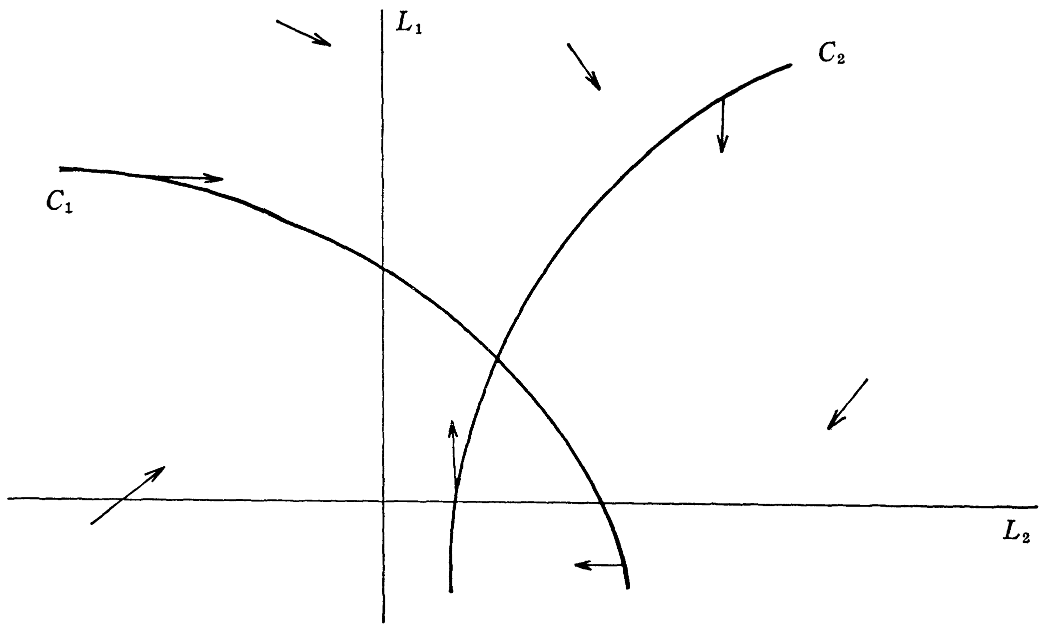

Fig. 2.

LEMMA 2. *For some* $0<K_1, K_2, K_3<\infty$ *and all* $y_2\leqq\Delta_2$

$$E^{(\Delta_1, y_2)}\tau_2\leqq K_1(1+|y_2|^2) \tag{3.3}$$

and

$$P^{(\Delta_1, y_2)}\{|Y_1(\tau_2)|\geqq x\}\leqq K_2e^{-K_3x}, \qquad x\geqq 0. \tag{3.4}$$

This lemma pretty much tells us that Y will hit some compact set in a time which is not too large if the initial point is on the half line $H_1=L_1\cap\{(y_1, y_2): y_2\leqq\Delta_2\}$. The following lemmas will serve to show that Y will hit H_1 and to provide estimates of the place where H_1 is hit. First we consider the case where the initial point is to the left of L_1.

LEMMA 3. *For some* $0<K_4, K_5<\infty$ *and all* $y_1\leqq\Delta_1$, $y_2\in\boldsymbol{R}$

(3.5) $$E^{(y_1, y_2)}\tau_1^3 \leqq K_4(1+|y_1|^3)$$

and

(3.6) $$E^{(y_1, y_2)}|Y_2(\tau_1)|^3 \leqq K_5(1+|y|^3).$$

The last two lemmas are very similar and deal with initial points to the right of L_1 and above, respectively below L_2. However, Lemma 4 deals with τ_2 and Lemma 5 with τ_1.

LEMMA 4. *For some* $0<K_6$, $K_7<\infty$ *and all* $y_1 \geqq \Delta_1$, $y_2 \geqq \Delta_2$

(3.7) $$E^{(y_1, y_2)}\tau_2^3 \leqq K_6(1+|y|^3)$$

and

(3.8) $$E^{(y_1, y_2)}|Y_1(\tau_2)|^3 \leqq K_7(1+|y|^3).$$

Moreover, for any $\varepsilon>0$ *there exists a* Γ *such that for all* $y_1 \geqq \Delta_1$, $y_2 \geqq \Delta_2$

(3.9) $$P^{(y_1, y_2)}\{Y_1(\tau_2) \geqq -\Gamma\} \geqq 1-\varepsilon.$$

LEMMA 5. *For some* $0<K_8$, $K_9<\infty$ *and all* $y_1 \geqq \Delta_1$, $y_2 \leqq \Delta_2$

(3.10) $$E^{(y_1, y_2)}\tau_1^3 \leqq K_8(1+|y|^3)$$

and

(3.11) $$E^{(y_1, y_2)}|Y_2(\tau_1)|^3 \leqq K_9(1+|y|^3).$$

Moreover, for any $\varepsilon>0$ *there exists a* Γ *such that for all* $y_1 \geqq \Delta_1$, $y_2 \leqq \Delta_2$

(3.12) $$P^{(y_1, y_2)}\{Y_2(\tau_1) \leqq \Gamma\} \geqq 1-\varepsilon.$$

Before proving these lemmas we show how they imply (1.12). Choose ε such that

(3.13) $$\{1-(1-\varepsilon)^3\}^{1/12} < K_{11}^{-1},$$

where

(3.14) $$K_{11} = 8\max\{2K_2K_3^{-3}, K_5, K_7, K_9\} + 8\{1+|\Delta_1|^3+|\Delta_2|^3\}.$$

Next, fix Γ such that (3.9) and (3.12) hold as well as

(3.15) $$K_2 e^{-K_3\Gamma} \leqq \varepsilon, \quad \Gamma \geqq |\Delta_1|+|\Delta_2|.$$

Define

$$\rho_0 = 0,$$

$$\rho_1 = \begin{cases} \inf\{t \geqq \tau_2 : Y(t) \in L_1\} & \text{if } Y_1(0)>\Delta_1,\ Y_2(0)>\Delta_2, \\ \tau_1 & \text{otherwise}, \end{cases}$$

$$\rho_{2n+2} = \inf\{t \geqq \rho_{2n+1} : Y(t) \in L_2\},$$

$$\rho_{2n+3} = \inf\{t \geqq \rho_{2n+2} : Y(t) \in L_1\}.$$

Also set

$$N=\inf\{n: Y_i(\rho_n)\in[-\Gamma, +\Gamma] \quad \text{for } i=1, 2\}.$$

The Y process is positive recurrent if (see [10], Ch. 4.3)

$$(3.16) \qquad E^y\{\rho_N\}<\infty \qquad \text{for all } y\in \boldsymbol{R}^2.$$

Here we set $\rho_N=\infty$ whenever $\rho_k=\infty$ for some $k<N$ or $N=\infty$. Note first that $\tau_1<\infty$ w.p.1 if $y_1\leqq\Delta_1$ or $y_1\geqq\Delta_1$, $y_2\leqq\Delta_2$, by virtue of (3.5) and (3.10). But also if $y_1>\Delta_1$, $y_2>\Delta_2$, then $\rho_1=\tau_2+\tau_1\cdot\theta_{\tau_2}<\infty$ w.p.1 by (3.7) and (3.5) or (3.10) again (since $Y(\tau_2)\in L_2$). Thus $\rho_1<\infty$ w.p.1 for each initial point and a similar argument shows that $\rho_n<\infty$ a.s. for all n. Next we show that $N<\infty$ a.s. In fact we claim that

$$(3.17) \qquad P^y\{N\geqq n\}\leqq\{1-(1-\varepsilon)^3\}^{[n/4]-1}, \qquad y\in\boldsymbol{R}^2.$$

(3.17) will be immediate from the strong Markov property once we prove

$$(3.18) \qquad P^y\{N\leqq 4\}\geqq(1-\varepsilon)^3, \qquad y\in\boldsymbol{R}^2,$$

because (for $n>4$)

$$P^y\{N\geqq n\}\leqq E^y\{P^{Y(\rho_4)}\{N\geqq n-4\}\ ;\ N>4\}.$$

We prove (3.18) only for $y_1\leqq\Delta_1$. For brevity denote the set $[-\Gamma, +\Gamma]\times[-\Gamma, +\Gamma]$ by A. Then, for $y_1\leqq\Delta_1$

$$(3.19) \qquad \begin{aligned} P^y\{N\leqq 4\}\geqq & P^y\{|Y_2(\tau_1)|\leqq\Gamma\} \\ & +E^y\{P^{Y(\tau_1)}\{Y(\tau_2)\in A\}\ ;\ Y_2(\tau_1)<-\Gamma\} \\ & +P^y\{Y_2(\tau_1)>\Gamma \text{ and } N\leqq 4\}. \end{aligned}$$

The second term in the right hand side of (3.19) is at least

$$(3.20) \qquad \begin{aligned} & P^y\{Y_2(\tau_1)<-\Gamma\}\inf_{y_2\leq\Delta_2} P^{(\Delta_1, y_2)}\{|Y_1(\tau_2)|\leqq\Gamma\} \\ & \geqq P^y\{Y_2(\tau_1)<-\Gamma\}(1-\varepsilon), \end{aligned}$$

by virtue of $Y_1(\tau_1)=\Delta_1$, (3.4) and (3.15). Similarly the third term is at least

$$(3.21) \qquad \begin{aligned} & P^y\{Y_2(\tau_1)>\Gamma,\ Y_1(\rho_2)>-\Gamma, \text{ and } N\leqq 4\} \\ & \geqq P^y\{Y_2(\tau_1)>\Gamma,\ |Y_1(\rho_2)|\leqq\Gamma\} \\ & \quad +E^y\{P^{Y(\rho_2)}\{N\leqq 2\}\ ;\ Y_2(\tau_1)>\Gamma,\ Y_1(\rho_2)>\Gamma\} \\ & \geqq P^y\{Y_2(\tau_1)>\Gamma,\ |Y_1(\rho_2)|\leqq\Gamma\} \\ & \quad +P^y\{Y_2(\tau_1)>\Gamma,\ Y_1(\rho_2)>\Gamma\}\inf_{y_1\geq\Delta_1} P^{(y_1, \Delta_2)}\{N\leqq 2\} \end{aligned}$$

$$\geqq E^y\{P^{Y(\tau_1)}\{Y_1(\tau_2)>-\Gamma\}\ ;\ Y_2(\tau_1)>\Gamma\}\inf_{y_1\geqq\Delta_1}P^{(y_1,\Delta_2)}\{N\leqq 2\}$$

$$\geqq P^y\{Y_2(\tau_1)>\Gamma\}(1-\varepsilon)\inf_{y_1\geqq\Delta_1}P^{(y_1,\Delta_2)}\{N\leqq 2\}$$

(by (3.9)). In turn, for $y_1\geqq\Delta_1$

(3.22)
$$P^{(y_1,\Delta_2)}\{N\leqq 2\}\geqq P^{(y_1,\Delta_2)}\{|Y_2(\tau_1)|\leqq\Gamma\}$$
$$+E^{(y_1,\Delta_2)}\{P^{Y(\tau_1)}\{|Y_1(\tau_2)|\leqq\Gamma\}\ ;\ Y_2(\tau_1)<-\Gamma\}$$
$$\geqq(1-\varepsilon)P^{(y_1,\Delta_2)}\{Y_2(\tau_1)<\Gamma\}\quad\text{(by (3.4) and (3.15))}$$
$$\geqq(1-\varepsilon)^2\quad\text{(by (3.12))}.$$

Substitution of (3.20)-(3.22) into (3.19) gives (3.18).

One now quickly derives (3.16). Write

(3.23)
$$E^y\{\rho_N\}=\sum_{n=0}^{\infty}E^y\{\rho_{n+1}-\rho_n\ ;\ N>n\}.$$

Now observe, that for n odd

(3.24)
$$E^y\{\rho_{n+1}-\rho_n\ ;\ N>n\}=E^y\{E^{Y(\rho_n)}\{\tau_2\}\ ;\ N>n\}$$
$$\leqq E^y\{K_1(1+|Y(\rho_n)|^2)+K_6^{1/3}(1+|Y(\rho_n)|^3)^{1/3}\ ;\ N>n\}$$

(by (3.3) and (3.7))

$$\leqq K_{10}E^y\{(1+|Y(\rho_n)|^3)^{2/3}\ ;\ N>n\},$$

where

(3.25)
$$K_{10}=\max\{2K_1+K_6^{1/3},\ K_4^{1/3}+K_8^{1/3}\}.$$

(3.5) and (3.10) show that (3.24) is also valid for any even $n\neq 0$. Thus, by (3.23)

$$E^y\{\rho_N\}\leqq E^y\{\rho_1\}+K_{10}\sum_{n=1}^{\infty}E^y\{(1+|Y(\rho_n)|^3)^{2/3}\ ;\ N>n\}$$

$$\leqq E^y\{\rho_1\}+K_{10}\sum_{n=1}^{\infty}(E^y\{1+|Y(\rho_n)|^3\})^{2/3}\cdot(P^y\{N>n\})^{1/3}$$

$$\leqq E^y\{\rho_1\}+K_{10}\sum_{n=1}^{\infty}(E^y\{1+|Y(\rho_n)|^3\})^{2/3}\cdot\{1-(1-\varepsilon)^3\}^{n/12-2}.$$

Finally, by (3.6) and (3.11) for odd $n>1$

(3.26)
$$E^y\{1+|Y(\rho_n)|^3\}=E^y\{E^{Y(\rho_{n-1})}\{1+|Y(\tau_1)|^3\}\}$$
$$\leqq K_{11}E^y\{1+|Y(\rho_{n-1})|^3\}.$$

Again (3.26) also holds for even $n\geqq 2$ (by (3.4) and (3.8)) so that

$$E^y\{1+|Y(\rho_n)|^3\}\leqq K_{11}^{n-1}(1+E^y|Y(\rho_1)|^3).$$

Similar arguments show

$$E^y\{1+|Y(\rho_1)|^3\}\leqq K_{11}^2(1+|y|^3) \quad \text{and} \quad E^y\{\rho_1\}<\infty .$$

Consequently (see (3.13))

$$E^y\{\rho_N\}\leqq E^y\{\rho_1\}+K_{10}(1+|y|^3)\sum_{n=1}^{\infty}K_{11}^{n+1}\{1-(1-\varepsilon)^3\}^{n/12-2}<\infty .$$

We have now reduced (3.16) and (1.12) to Lemmas 2–5. In preparation of their proofs we give the following

LEMMA 6. *Let $Z(t)$ be the nonanticipating continuous solution of*

$$dZ(t)=\sigma dW(t)+\beta(Z(t))dt,\ Z(0)=z ,$$

where $\sigma>0$ is constant, W a Brownian motion and $\beta(\cdot)$ a function on $\boldsymbol{R}$ which satisfies the following conditions:

(i) *For each M there exists a $K(M)<\infty$ such that*

$$|\beta(z')-\beta(z'')|\leqq K(M)|z'-z''|,\ |z'|,\ |z''|\leqq M,$$

(ii) *For some $M_0<\infty$ and $\beta_0>0$*

$$\beta(z)\geqq\beta_0>0 \qquad for\ z\leqq -M_0,$$

and

$$\beta(z)\leqq -\beta_0<0 \qquad for\ z\geqq M_0.$$

Let Δ be fixed and set

$$\tau=\inf\{t\geqq 0:\ Z(t)=\Delta\}.$$

Then for some $\lambda>0$, K_{12}, $K_{13}<\infty$ and all z

$$E^z\exp\{\lambda\sup_{t\leqq\tau}Z(t)\}<\infty , \tag{3.27}$$

$$E^z\exp\{-\lambda\inf_{t\leqq\tau}Z(t)\}<\infty , \tag{3.28}$$

and

$$E^z e^{\lambda\tau}\leqq K_{12}e^{K_{13}\lambda|z|} . \tag{3.29}$$

PROOF. (3.28) will follow from (3.27) by replacing Z by $-Z$. Also $\sup_{t\leqq\tau}Z(t)\leqq\Delta$ for $Z(0)=z\leqq\Delta$ so that (3.27) only needs proof for $z\geqq\Delta$. The same is true for (3.29), again because Z and $-Z$ satisfy identical hypotheses. From now on we take $z\geqq\Delta$. Without loss of generality we take $M_0\geqq|\Delta|$. Finally we take $\tilde\beta$ Lipschitz continuous on compact sets and such that $\tilde\beta(x)\geqq\beta(x)$ for $x<M_0$, $\tilde\beta(x)=-\beta_0$ for $x\geqq M_0$. We denote by $\tilde Z(t)$ the solution of

$$d\tilde Z(t)=\sigma dW(t)+\tilde\beta(\tilde Z(t))dt,\quad \tilde Z(0)=z$$

and set

$$\tilde\tau=\inf\{t:\ \tilde Z(t)=\Delta\}.$$

By the comparison lemma, $Z(t)\leqq\tilde{Z}(t)$ for all t, and hence if $Z(0)=z\geqq\Delta$

$$\tau\leqq\tilde{\tau},\quad \sup_{t\leq\tau} Z(t)\leqq\sup_{t\leq\tilde{\tau}}\tilde{Z}(t).$$

Thus, it suffices to prove (3.27) and (3.29) for $\tilde{Z}$ and $\tilde{\tau}$ instead of Z and τ. For $\tilde{Z}$ one can now compute various quantities explicitly. E. g., the scale function s of $\tilde{Z}$ is given by

$$s(x)=\int_0^x \exp-\tilde{B}(y)dy,\quad \tilde{B}(y)=\frac{2}{\sigma^2}\int_0^y\tilde{\beta}(s)ds$$

(cf. [11], p. 13, [2], Proposition 16.78). Thus

$$s(x)\sim\text{constant }\exp\frac{2}{\sigma^2}\beta_0 x,\qquad x\to\infty.$$

Consequently (see [2], Theorem 16.27)

$$P^z\{\sup_{t\leq\tilde{\tau}}\tilde{Z}(t)\geqq r\}=P^z\{\tilde{Z}(\cdot)\text{ hits } r \text{ before } \Delta\}$$

$$=\frac{s(2)-s(\Delta)}{s(r)-s(\Delta)}=O\left(\exp-\frac{2}{\sigma^2}\beta_0 r\right),\qquad r\to\infty.$$

This proves (3.27).

To prove (3.29), set

$$\rho_0=\inf\{t:\ \tilde{Z}(t)=M_0+1 \text{ or } \tilde{Z}(t)=\Delta\},$$

$$\rho_{2n+1}=\inf\{t>\rho_{2n}:\ \tilde{Z}(t)=M_0\},$$

$$\rho_{2n+2}=\inf\{t>\rho_{2n+1}:\ \tilde{Z}(t)=M_0+1 \text{ or } \tilde{Z}(t)=\Delta\}.$$

Also let

$$L=\inf\{n:\ \tilde{Z}(\rho_{2n})=\Delta\}.$$

Then $\tilde{\tau}\leqq\rho_{2L}$ and

$$E^z e^{\lambda\tilde{\tau}}\leqq E^z e^{\lambda\rho_0}+\sum_{n=0}^{\infty}E^z\{e^{\lambda\rho_{2n+2}}-e^{\lambda\rho_{2n}};\ L>n\}. \tag{3.30}$$

First we estimate $E^z\exp\lambda\rho_0$. For $\Delta\leqq z\leqq M_0+1$ and $\lambda>0$ sufficiently small this term is finite because $\sigma^2>0$ ([2], Proof of Lemma 16.25, [10], p. 132).

For $z>M_0+1$, ρ_0 is just the first hitting time of M_0+1. Since $\tilde{\beta}(x)=-\beta_0$ is constant on $[M_0+1, \infty)$ the $\tilde{Z}$ process is a Brownian motion with constant negative drift up till ρ_0 and for $\lambda<(2\sigma^2)^{-1}\beta_0^2$

$$\varphi(z)\equiv E^z e^{\lambda\rho_0}=\exp\frac{1}{\sigma^2}(\beta_0-\sqrt{\beta_0^2-2\lambda\sigma^2})(z-M_0-1),\quad z>M_0+1. \tag{3.31}$$

((3.31)$=\lim_{r\to\infty}\varphi(z;r)$ where $\varphi(z;r)$ is the solution of

$$\left(\frac{1}{2}\sigma^2\frac{d^2}{dz^2}-\beta_0\frac{d}{dz}+\lambda\right)\varphi(z\,;\,r)=0\,, \qquad M_0+1<z<r\,,$$

which equals 1 at $z=M_0+1$ and at $z=r$; $\varphi(z\,;\,r)=E^z\exp\lambda$ (first hitting time of M_0+1 or r).)

This takes care of the first term in the right hand side of (3.30) and we now estimate the infinite series. We have

$$E^z\{e^{\lambda\rho_{2n+2}}-e^{\lambda\rho_{2n}}\,;\ L>n\}\leqq(E^z\{e^{2\lambda\rho_{2n+2}}\,;\ L>n\}\,P^z\{L>n\})^{1/2}. \tag{3.32}$$

Also

$$P^z\{L>n+1\,|\,L>n\}\leqq P^{Z(\rho_{2n+1})}\{\tilde{Z}\text{ hits }M_0+1\text{ before }\Delta\}$$
$$=P^{M_0}(\tilde{Z}\text{ hits }M_0+1\text{ before }\Delta\}=\theta\,,$$

for some $\theta<1$ (see [**2**], Theorems 16.27 and 16.28.) Thus

$$P^z\{L>n\}\leqq\theta^{n-1}.$$

Next,

$$E^z\{e^{2\lambda\rho_{2n+2}}\,;\ L>n\}=E^z\{e^{2\lambda\rho_{2n}}E^{Z(\rho_{2n})}\{e^{2\lambda\rho_2}\}\ ;\ L>n\}\,,$$

and if we can show that on $\{L>n\}$ for some $\lambda>0$

$$E^{Z(\rho_{2n})}\{e^{2\lambda\rho_2}\}\leqq\theta^{-1/4}\,, \tag{3.33}$$

then by iteration

$$E^z\{e^{2\lambda\rho_{2n+2}}\,;\ L>n\}\leqq\theta^{-n/4-1}E^ze^{2\lambda\rho_0}\,,$$

and by (3.30)-(3.32) we will obtain

$$E^ze^{\lambda\tilde{\tau}}\leqq E^ze^{\lambda\rho_0}+\{E^ze^{2\lambda\rho_0}\}^{1/2}\sum_{n=1}^{\infty}\theta^{-n/8+n/2-1}\leqq K_{12}e^{K_{13}\lambda|z|}$$

as desired. However (3.33) is easy now. Indeed $Z(\rho_{2n})=M_0+1$ on $\{L>n\}$ so that the left hand side of (3.33) becomes

$$E^{M_0+1}\{e^{2\lambda\rho_1}E^{M_0}\{e^{2\lambda\rho_0}\}\}=E^{M_0}\{e^{2\lambda\rho_0}\}\,E^{M_0+1}\{e^{2\lambda\rho_1}\}$$
$$=E^{M_0}\{e^{2\lambda\rho_0}\}\exp\frac{1}{\sigma^2}(\beta_0-\sqrt{\beta_0^2-2\lambda\sigma^2})\,.$$

The last equality is proved exactly as (3.31), because $\tilde{Z}$ is a Brownian motion with constant drift on (M_0,∞). We already saw that $E^{M_0}\exp2\lambda\rho_0<\infty$ for some $\lambda>0$. By the dominated convergence theorem it tends to 1 as $\lambda\downarrow0$. Thus also the left hand side of (3.33) tends to 1 as $\lambda\downarrow0$ and (3.33) and (3.29) have been proved. □

PROOF OF LEMMA 2. Take $V(y)$ as in (2.7) with

$$C=\frac{B_2+1}{B_1}>\frac{B_{21}}{B_{11}}\quad\text{and}\quad D=\frac{B_{11}}{B_{21}}>\frac{B_1}{B_2}$$

(cf. (3.1)). Then $CD>1$ and the constants Γ in (2.8) satisfy

$$\Gamma_0=\frac{B_2+1}{B_1}A_{11}-2A_{12}+\frac{B_{11}}{B_{21}}A_{22}\,,$$

$$\Gamma_1=1,\ \Gamma_2>0,\ \Gamma_{11}<0\,,$$

$$\Gamma_{22}=B_{12}-B_{22}B_{11}(B_{21})^{-1}<0 \quad \text{(by (3.1))}\,,$$

$$\Gamma_{12}\geqq B_{22}>0,\ \Gamma_{21}=0\,.$$

It follows that for $\Delta_2\leqq -1$

$$\sup_{\substack{y_1\in R\\ y_2\leqq\Delta_2}} y_1\{\Gamma_1+\Gamma_{11}e^{y_1}+\Gamma_{12}e^{y_2}\}$$

$$\leqq \sup_{y_1\geqq 0} y_1\{\Gamma_1+\Gamma_{12}+\Gamma_{11}e^{y_1}\}+\sup_{y_1\leqq 0} y_1\{\Gamma_1+\Gamma_{11}e^{y_1}\}<\infty\,,$$

and we can choose Δ_2 so small that on $y_2\leqq\Delta_2$ (cf. (2.8))

(3.34) $$LV(y)\leqq\Gamma_0+2\sup_{y_1} y_1\{\Gamma_1+\Gamma_{12}+\Gamma_{11}e^{y_1}\}+2\Delta_2\{\Gamma_2+\Gamma_{22}e^{\Delta_2}\}\leqq -1\,.$$

In particular $V(Y_{t\wedge\tau_2\wedge\zeta})$ is a positive supermartingale and

$$P^y\{\sup_{t\leqq\tau_2\wedge\zeta} V(Y_t)\geqq M\}\leqq\frac{V(y)}{M}\to 0, \qquad M\to\infty\,.$$

In other words $\zeta>\tau_2$ w. p. 1 and exactly as in Theorem 3.7.1 of [**10**] we now obtain for $y_2\leqq\Delta_2$

$$E^y\{\tau_2\}\leqq V(y)\leqq K_1|y|^2$$

for suitable K_1. This proves (3.3).

To prove (3.4) we define

$$\pi_0=0\,, \qquad \text{and for } n\geqq 0$$

$$\pi_{2n+1}=\tau_2\wedge\inf\{t>\pi_{2n}:\ Y_1(t)=\Delta_1-1\}\,,$$

$$\pi_{2n+2}=\tau_2\wedge\inf\{t>\pi_{2n+1}:\ Y_1(t)=\Delta_1\}$$

and write for $\lambda>0$, $y_2<\Delta_2$,

(3.35) $$E^{(\Delta_1,y_2)}e^{\lambda|Y_1(\tau_2)|}=\sum_{n=0}^{\infty}E^{(\Delta_1,y_2)}\{e^{\lambda|Y_1(\tau_2)|};\ \pi_{2n}<\tau_2\leqq\pi_{2n+2}\}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}E^{(\Delta_1,y_2)}\{E^{Y(\pi_{2n})}\{e^{\lambda|Y_1(\tau_2)|};\ \tau_2\leqq\pi_2\}\,;$$

$$\pi_{2n}<\tau_2,\ \Delta_2-k-1\leqq Y_2(\pi_{2n})<\Delta_2-k\}\,.$$

Now for $n\neq 0$ $Y(\pi_{2n})=(\Delta_1, z)$ for some z, and by Schwarz' inequality we have

(3.36)
$$(E^{(\Delta_1,z)}\{e^{\lambda|Y_1(\tau_2)|};\ \tau_2\leqq\pi_2\})^2 \leqq E^{(\Delta_1,z)}\{e^{2\lambda|Y_1(\tau_2)|};\ \tau_2\leqq\pi_2\}\, P^{(\Delta_1,z)}\{\tau_2\leqq\pi_2\}$$

The first factor in the right hand side equals

(3.37)
$$E^{(\Delta_1,z)}\{e^{2\lambda|Y_1(\tau_2)|};\ \tau_2\leqq\pi_1\} + E^{(\Delta_1,z)}\{E^{Y(\pi_1)}\{e^{2\lambda|Y_1(\tau_2)|};\ \tau_2\leqq\tau_1\}\ ;\ \pi_1<\tau_2\}.$$

Now, since $\zeta>\tau_2$ w.p.1 and $B_{12}\leqq 0$ we have by the comparison theorem for $Y(0)=(\Delta_1, z)$, $z\leqq\Delta_2$,

(3.38)
$$Y_1(t)\leqq Y_1^{(4)}(t), \qquad t\leqq\tau_2,$$

where $Y_1^{(4)}$ is the solution of

(3.39)
$$dY_1^{(4)}(t)=\sigma_1 dW_3(t)+\{B_1-B_{11}\exp Y_1^{(4)}(t)-B_{12}e^{\Delta_2}\}\,dt, \qquad Y_1^{(4)}(0)=\Delta_1.$$

Consequently, if

(3.40)
$$T_6=\inf\{t\geqq 0:\ Y_1^{(4)}(t)=\Delta_1-1\}\wedge\tau_2$$

then $T_6\geqq\pi_1$, and on $\{\tau_2\leqq\pi_1\}$

$$|Y_1(\tau_2)|\leqq 2|\Delta_1-1|+\sup_{t\leq T_6} Y_1^{(4)}(t).$$

Consequently, by virtue of (3.27), the first term in (3.37) is bounded by

$$e^{4\lambda|\Delta_1-1|}E^{\Delta_1}\exp 2\lambda\sup_{t\leq T_6} Y_1^{(4)}(t)\leqq K_{14}<\infty$$

for $0\leqq\lambda\leqq\lambda_0$ for some $\lambda_0>0$ and $K_{14}<\infty$ independent of $\lambda\leqq\lambda_0$ and $z\leqq\Delta_2$. The second term in (3.37) can be handled in the same way by means of (3.28) if we take into account that on $\{\pi_1<\tau_2\}$ $Y(\pi_1)=(\Delta_1-1, z)$ for some $z<\Delta_2$ and use

$$Y_1(t\,;\ \Delta_1-1,\ z)\geqq Y_1^{(2)}(t\,;\ \Delta_1-1), \qquad t\leqq\tau_2,$$

where $Y_1^{(2)}$ is defined as in (2.40).

Thus, (3.37) is at most $2K_{14}$ for $\lambda\leqq\lambda_0$.

Next we estimate the second factor in the right hand side of (3.36). For $\Delta_2-k-1\leqq z<\Delta_2-k$.

(3.41)
$$P^{(\Delta_1,z)}\{\tau_2\leqq\pi_2\}\leqq P^{(\Delta_1,z)}\{\pi_2>(2B_2)^{-1}k\} + P^{(\Delta_1,z)}\{\max_{t\leq\frac{1}{2}B_2^{-1}k} Y_2(t)-Y_2(0)\geqq k\}.$$

By the comparison lemma (and B_{21}, $B_{22}\geqq 0$)

(3.42)
$$Y_2(t)-Y_2(0)\leqq\sigma_2 W_4(t)+B_2 t$$

so that the last term in (3.41) is at most

(3.43) $$P\Big\{\{\max_{t\leq(2B_2)^{-1}k}\sigma_2 W_4(t)\geqq\frac{1}{2}k\Big\}\leqq K_{15}e^{-K_{16}k}.$$

As for the first term in the right hand side of (3.41),

$$\pi_2=\pi_1+(\tau_1\wedge\tau_2)\cdot\theta_{\pi_1}\leqq T_6+T_7$$

where T_6 is defined in (3.40) and

$$T_7=\inf\{t\geqq 0:\ Y_1^{(2)}(t\,;\ \Delta_1-1)=\Delta_1\}.$$

Since, by (3.29), both $P\{T_6\geqq x\}$ and $P\{T_7\geqq x\}$ decrease exponentially fast as $x\to\infty$, we obtain from (3.42) and (3.43)

$$P^{(\Delta_1,z)}\{\tau_2\leqq\pi_2\}\leqq K_{17}e^{-K_{18}k}.$$

We now substitute these estimates into (3.36) and (3.35) to obtain[6)]

(3.44) $$E^{(\Delta_1,y_2)}e^{\lambda|Y_1(\tau_2)|}\leqq\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(2K_{14}K_{17}e^{-K_{18}k})^{1/2}$$
$$\cdot P^{(\Delta_1,y_2)}\{\pi_{2n}<\tau_2,\ \Delta_2-k-1\leqq Y_2(\pi_{2n})<\Delta_2-k\}$$
$$\leqq K_{19}\sum_{k=0}^{\infty}e^{-1/2K_{18}k}$$
$$\cdot E^{(\Delta_1,y_2)}\#\{n:\ \pi_{2n}<\tau_2,\ \Delta_2-k-1\leqq Y_2(\pi_{2n})<\Delta_2-k\}.$$

To complete the proof of (3.4) we show that the last series in (3.44) is bounded uniformly in $y_2\leqq\Delta_2$. A first entry decomposition shows

(3.45) $$E^{(\Delta_1,y_2)}\#\{n:\ \pi_{2n}<\tau_2,\ \Delta_2-k-1\leqq Y_2(\pi_{2n})<\Delta_2-k\}$$
$$\leqq 1+\sup_{\Delta_2-k-1\leq z<\Delta_2-k}E^{(\Delta_1,z)}\#\{n:\ \pi_{2n}<\tau_2,\ \Delta_2-k-1\leqq Y_2(\pi_{2n})<\Delta_2-k\}.$$

Moreover, for some $K_{20}>0$

(3.46) $$E^{Y(\pi_{2n})}\{\pi_2\}\geqq K_{20}$$

whenever $Y_2(\pi_{2n})<\Delta_2-1$, $\pi_{2n}<\tau_2$. Thus for $k\geqq 1$ and $\Delta_2-k-1\leqq z<\Delta_2-k$

(3.47) $$K_{21}(1+k^2)\geqq E^{(\Delta_1,z)}\{\tau_2\}\quad\text{(by (3.3))}$$
$$\geqq\sum_n E^{(\Delta_1,z)}\{E^{Y(\pi_{2n})}\{\pi_2\}\ ;\ \pi_{2n}<\tau_2,\ \Delta_2-k-1\leqq Y_2(\pi_{2n})<\Delta_1-k\}$$
$$\geqq K_{20}E^{(\Delta_1,z)}\#\{n:\ \pi_{2n}<\tau_2,\ \Delta_2-k-1\leqq Y_2(\pi_{2n})\leqq\Delta_2-k\}.$$

(3.45)-(3.47) take care of all terms in the series in (3.44) with $k\geqq 1$. For $k=0$ we use the simple estimate

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6) $\#A$ denotes the number of elements in the set A.

$$P^{(\Delta_1, z)}\{\#\{n: \pi_{2n}<\tau_2,\ \Delta_2-1\leqq Y_2(\pi_{2n})<\Delta_2\}\geqq r\}\leqq(1-p)^{r-1}$$

where

$$p=\inf_{\Delta_2-1\leq u<\Delta_2} P^{(\Delta_1, u)}\{\tau_2<\pi_2\}>0.$$

(Use the maximum principle ([**1**], Lemma 2.3) and [**4**], Theorem 13.16.) □

The *proof of Lemma* 3 is omitted because it can be obtained from that of Lemma 4 by interchanging the roles of $-Y_1(t)$ and $Y_2(t)$.

PROOF OF LEMMA 4. Let $Y_2^{(2)}(t;y_2)$ be given by (2.40) for $i=2$ and set

$$T_8=\inf\{t\geqq 0:\ Y_2^{(2)}(t;y_2)=\Delta_2\}.$$

As in (2.41),

$$Y_2(t;y_1,y_2)\leqq Y_2^{(2)}(t;y_2), \qquad t<\zeta,$$

and if $y_2\geqq\Delta_2$,

$$\tau_2\leqq T_8, \qquad \text{unless } \zeta\leqq\tau_2\wedge T_8.$$

Exactly as in the proof of (2.42) we now have

$$P^y\{\zeta\leqq T_8\}=0, \qquad y_2\geqq\Delta_2.$$

(We can ignore (2.43) since $Y_2(t)\geqq\Delta_2$ for $t<\zeta\wedge\tau_2$). Therefore, if $y_2\geqq\Delta_2$

$$E^y\{\tau_2^3\}\leqq E^{y_2}\{T_8^3\}=3\int_0^\infty x^2P^{y_2}\{T_8\geqq x\}\,dx \tag{3.48}$$

$$\leqq K_{13}^3|y_2|^3+3\int_{K_{13}|y_2|}^\infty x^2e^{-\lambda x}E^{y_2}e^{\lambda T_8}dx$$

$$\leqq K_{13}^3|y_2|^3+K_{22}(1+|y_2|^2)e^{-\lambda K_{13}|y_2|}E^{y_2}e^{\lambda T_8}.$$

This implies (3.7) since

$$E^{y_2}e^{\lambda T_8}\leqq K_{12}e^{K_{13}\lambda|y_2|},$$

by virtue of (3.29).

As for (3.8), by integrating (2.1) we have

$$Y_1(t)-Y_1(0)=\sigma_1W_3(t)-\frac{B_{11}}{B_{21}}\sigma_2W_4(t) \tag{3.49}$$

$$+\left(B_1-\frac{B_{11}}{B_{21}}B_2\right)t-\left(B_{12}-\frac{B_{11}}{B_{21}}B_{22}\right)\int_0^t e^{Y_2(s)}ds$$

$$+\frac{B_{11}}{B_{21}}(Y_2(t)-Y_2(0)).$$

Consequently,

$$E^y|Y_1(\tau_2)-Y_1(0)|^3\leqq K_{23}\Big\{E^y|W_3(\tau_2)|^3+E^y|W_4(\tau_2)|^3$$

$$+E^y\tau_2^3+E^y|Y_2(\tau_2)|^3+|y_2|^3+E^y\left|\int_0^{\tau_2}e^{Y_2(s)}ds\right|^3\Big\}.$$

Now use the following facts:

$$E^y|W_i(\tau_2)|^3 \leqq K_{24} E^y \tau_2^{3/2} \leqq K_{25}\{E^{y_2} T_8^3\}^{1/2},$$

(see Theorem 2.1 of [3])

$$Y_2(\tau_2) = \Delta_2$$

$$E^y\left|\int_0^{\tau_2} e^{Y_2(s)} ds\right|^3 \leqq E^y\left(\int_0^{T_8} \exp Y_2^{(2)}(s\,;\,y_2) ds\right)^3$$

$$\leqq K_{26} E^y\{|W_4(T_8)|^3 + |Y_2^{(2)}(T_8)|^3 + |y_2|^3 + T_8^3\},$$

together with (3.48) to obtain (3.8).

To prove (3.9) we first show that

$$E^{y_2} T_8 \leqq K_{27} < \infty \qquad \text{for all } y_2 \geqq \Delta_2. \tag{3.50}$$

(3.50) is merely a matter of calculation. The scale function $s_0(\cdot)$ and speed measure $m_0(x)dx$ for $Y_2^{(2)}$ are given by (see [11], p. 13, [2] Ch. 16)

$$s_0(x) = \int_0^x e^{-B_0(y)} dy, \quad m_0(x) = \frac{2}{\sigma^2} e^{B_0(x)},$$

where

$$B_0(x) = \frac{2}{\sigma^2}\int_0^x \{B_2 - B_{22} e^y\}\, dy$$

$$= \frac{2}{\sigma^2}\{B_2 x - B_{22}(e^x - 1)\}.$$

By l'Hopital's rule

$$s_0(x) \sim \frac{\sigma^2}{2B_{22}} e^{-B_0(x) - x}, \qquad x \to \infty,$$

so that

$$\int^\infty s_0(x) m_0(x) dx < \infty.$$

This implies (3.50) (see [2], Theorem 16.36 and Problem 16.6.7).

Now let also $Y_1^{(2)}(t\,;\,y_1)$ be given by (2.40) with $i=1$. Then for $y_1 \geqq \Delta_1$, $y_2 \geqq \Delta_2$

$$Y_1(\tau_2\,;\,y_1,\,y_2) \geqq Y_1^{(2)}(\tau_2\,;\,y_1) \geqq \inf_{t \leq T_8} Y_1^{(2)}(t\,;\,y_1).$$

Consequently,

$$P^{(y_1, y_2)}\{Y_1(\tau_2) < -\Gamma\} \leqq P^{(y_1, y_2)}\left\{T_8 \geqq \frac{2}{\varepsilon} K_{27}\right\} \tag{3.51}$$

$$+ P\{\inf_{t \leq 2K_{27}\varepsilon^{-1}} Y_1^{(2)}(t\,;\,y_1) < -\Gamma\}$$

$$\leqq \frac{\varepsilon}{2} + P\{\inf_{t\leqq 2K_{27}\varepsilon^{-1}} Y_1^{(2)}(t\,;\,y_1) < -\Gamma\} \qquad \text{(by (3.50))}$$

$$\leqq \frac{\varepsilon}{2} + P\{\inf_{t\leqq 2K_{27}\varepsilon^{-1}} Y_1^{(2)}(t\,;\,\Delta_1) < -\Gamma\}$$

(by the comparison lemma). The last member of (3.51) can be made smaller than ε by taking Γ large, and (3.9) is proven.

PROOF OF LEMMA 5. First we observe that on $y_1 \geqq \Delta_1$,

$$b_2(y_1,\ y_2) = B_2 - B_{21}e^{y_1} - B_{22}e^{y_2}$$
$$\leqq B_2 - B_{21}e^{\Delta_1} = -4\sigma_2^2\,.$$

Consequently for $y_1 \geqq \Delta_1$, $y_2 \leqq \Delta_2$

(3.52)
$$Y_2(t\,;\,y_1,\ y_2) \leqq y_2 + \sigma_2 W_4(t) - 4\sigma_2^2 t$$
$$\leqq \Delta_2 + \sigma_2 W_4(t) - 4\sigma_2^2 t\,, \qquad t \in [0,\ \tau_1] \cap [0,\ \zeta)\,.$$

Substitution of (3.52) into (3.49) yields

(3.53)
$$Y_1(t\,;\,y_1,\ y_2) \leqq y_1 + \sigma_1 W_3(t) + \Big(B_1 - \frac{B_{11}}{B_{21}} B_2 - 4\frac{B_{11}}{B_{21}}\sigma_2^2\Big)t$$
$$+ \Big(\frac{B_{11}B_{22}}{B_{21}} - B_{12}\Big)e^{\Delta_2}\int_0^t \exp\{\sigma_2 W_4(s) - 4\sigma_2^2 s\}\,ds$$
$$\leqq y_1 + \sigma_1 W_3(t) - B_3 t$$
$$+ B_4\int_0^t \exp\{\sigma_2 W_4(s) - 4\sigma_2^2 s\}\,ds, \quad t \in [0,\ \tau_1] \cap [0,\ \zeta)\,,$$

where

$$B_3 = 4\frac{B_{11}}{B_{21}}\sigma_2^2 + \frac{B_{11}B_2}{B_{21}} - B_1 > 0 \qquad \text{(see (3.1)), and}$$

$$B_4 = \Big(\frac{B_{11}B_{22}}{B_{21}} - B_{12}\Big)e^{\Delta_2} \geqq 0\,.$$

As with (2.44), one can easily get

(3.54)
$$P^y\{\zeta < \tau_1\} = 0\,, \qquad y_1 \geqq \Delta_1,\ y_2 \leqq \Delta_2\,.$$

Hence from (3.53) it follows that for $y_1 \geqq \Delta_1$, $y_2 \leqq \Delta_2$

(3.55)
$$P^y\{\tau_1 > t\} = P^y\{Y_1(t\,;\,y_1,\ y_2) \geqq \Delta_1 \text{ and } \tau_1 > t\}$$
$$\leqq P\{y_1 + \sigma_1 W_3(t) - B_3 t + B_4\int_0^t \exp\{\sigma_2 W_4(s) - 4\sigma_2^2 s\}\,ds \geqq \Delta_1\}$$

$$\leqq P\left\{\sigma_1 W_3(t) \geqq \frac{1}{2} B_3 t + \Delta_1 - y_1\right\}$$

$$+ P\left\{\int_0^\infty \exp\{\sigma_2 W_4(s) - 4\sigma_2^2 s\} ds \geqq \frac{1}{2B_4} B_3 t\right\}.$$

For $t \leqq 4B_3^{-1}(y_1 - \Delta_1)$ we use the trivial estimate $P^y\{\tau_1 > t\} \leqq 1$, whereas for $t > 4B_3^{-1}(y_1 - \Delta_1)$ (3.55) yields

$$P^y\{\tau_1 > t\} \leqq P\left\{\sigma_1 W_3(t) \geqq \frac{1}{4} B_3 t\right\}$$

$$+\left(\frac{2B_4}{B_3 t}\right)^4 E\left\{\int_0^\infty \exp\{4\sigma_2 W_4(s) - 12\sigma_2^2 s\} ds\right\}\left\{\int_0^\infty e^{-4/3\sigma_2^2 s} ds\right\}^3$$

$$\leqq K_{28}\left\{\exp\left(-\frac{B_3^2 t}{32\sigma_1^2}\right) + \frac{1}{t^4}\right\},$$

because

$$E \exp\{4\sigma_2 W_4(s) - 12\sigma_2^2 s\} = \exp(8\sigma_2^2 s - 12\sigma_2^2 s).$$

(3.10) is an immediate consequence of these estimates.

(3.11) follows now from

$$Y_2(\tau_1; y_1, y_2) \leqq \Delta_2 + \sup_{t \geqq 0}\{\sigma_2 W_4(t) - 4\sigma_2^2 t\} \tag{3.56}$$

and

$$Y_2(\tau_1; y_1, y_2) - y_2 \geqq \sigma_2 W_4(\tau_1) - B_{21}\int_0^{\tau_1} \exp\{Y_1(s)\} ds \tag{3.57}$$

$$- B_{22} e^{\Delta_2}\int_0^\infty \exp\{\sigma_2 W_4(s) - 4\sigma_2^2 s)\} ds,$$

and

$$\int_0^{\tau_1} \exp\{Y_1(s)\} ds = B_{11}^{-1}\left\{y_1 - Y_1(\tau_1) + \sigma_1 W_3(\tau_1) + B_1\tau_1 - B_{12}\int_0^{\tau_1} \exp Y_2(s) ds\right\}$$

$$\leqq B_{11}^{-1}\{y_1 - \Delta_1 + \sigma_1 |W_3(\tau_1)| + B_1\tau_1$$

$$+ |B_{12}| e^{\Delta_2}\int_0^\infty \exp\{\sigma_2 W_4(s) - 4\sigma_2^2 s\} ds.$$

(Compare the proof of (3.8) and use ([13], Ch. 1.5) $P\{\sup_{t \geqq 0}\{\sigma_2 W_4(t) - 4\sigma_2^2 t\} \geqq x\} \leqq e^{-8x}$.)

Finally (3.12) is immediate from (3.56).

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