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A negative answer to a conjecture of conformal transformations of Riemannian manifolds

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1. Introduction.

Let (M, g), or simply M, be an n-dimensional differentiable manifold with Riemannian metric g. We denote by $C_0(M, g)$ the largest connected group of conformal transformations of (M, g), and by $I_0(M, g)$ the largest connected group of isometries of (M, g).

Riemannian manifolds with constant scalar curvature admitting an infinitesimal non-isometric conformal transformation have been extensively studied by various authors, and the following conjecture has been well-known.

CONJECTURE. *Let (M, g) be an n-dimensional compact Riemannian manifold. If*

(i) $n>2$

(ii) *the scalar curvature of (M, g) is constant*

(iii) $C_0(M, g)\neq I_0(M, g)$,

then (M, g) is isometric to a Euclidean n-sphere S^n.

This conjecture has been proved in various forms under some stronger assumptions. Typical results may be quoted as follows.

THEOREM A (Yano and Nagano [8]). *The conjecture is true if, instead of* (ii),

$(\mathrm{ii})_A$ *(M, g) is Einstein.*

THEOREM B (Nagano [6]). *The conjecture is true if, instead of* (ii),

$(\mathrm{ii})_B$ *the Ricci tensor of (M, g) is parallel.*

THEOREM C (Goldberg and Kobayashi [2], [3]). *The conjecture is true if, instead of* (i) *and* (ii),

$(\mathrm{i})_C$ $n>3$

$(\mathrm{ii})_C$ *$I_0(M, g)$ is transitive on M.*

THEOREM D (Lichnerowicz [5]). *The conjecture is true if instead of* (ii),

$(\mathrm{ii})_D$ *the scalar curvature and the length of the Ricci tensor of (M, g) are constant.*

THEOREM E (Hsiung [4]). *The conjecture is true if, instead of* (ii),

$(\mathrm{ii})_E$ *the scalar curvature and the length of curvature tensor of (M, g) are con-*

stant.

THEOREM F (Obata [7]). *The conjecture is true if, instead of* (iii),
$(\mathrm{iii})_F$ $C_0(M, g) \neq I_0(M, e^{2\phi}g)$ *for any smooth function* ϕ *on* M.

The purpose of this paper is to show that the conjecture itself is not true. A counter example will be given by a warped product of a circle and an $(n-1)$-dimensional manifold. Our main theorem can be stated as follows.

THEOREM. *The conjecture is not true. More precisely, let* F *be an* $(n-1)$-*dimensional compact Riemannian manifold with positive constant scalar curvature. Then there exists a positive function* f *on a circle* S^1 *such that the warped product* $S^1 \times_f F$ *satisfies all assumptions of the conjecture.*

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2. Warped products.

In [1], R. L. Bishop and B. O'Neill studied some properties of warped products. Let (B, h) and (F, g) be Riemannian manifolds and f a positive C^∞-function on B. Consider the product manifold $B \times F$ with projections $\pi : B \times F \to B$ and $\varpi : B \times F \to F$. The warped product $B \times_f F$ is the manifold $B \times F$ with Riemannian metric $\tilde{g}$ defined by

$$\tilde{g}(X, Y) = h(\pi_* X, \pi_* Y) + (f(\pi(x)))^2 g(\varpi_* X, \varpi_* Y) \quad \text{for } X, Y \in T_x(B \times F).$$

We say that $X \in T_x(B \times F)$ is horizontal (resp. vertical) if $\varpi_* X = 0$ (resp. $\pi_* X = 0$). We identify $T_x(B \times F)$ with $T_{\pi(x)}(B) + T_{\varpi(x)}(F)$. Note that, for $p \in F$, $\varpi^{-1}(p)$ is totally geodesic in $B \times_f F$ and $\pi | \varpi^{-1}(p) : \varpi^{-1}(p) \to B$ is an isometry. We denote by $\tilde{\nabla}$, ∇ and D the covariant differentiations on $(B \times_f F, \tilde{g})$, (F, g) and (B, h), respectively. We shall review some basic properties of warped products.

LEMMA 2.1 ([1]). *Let* X, Y (*resp.* V, W) *be vector fields on* B (*resp.* F). *Then*

(1) $\tilde{\nabla}_X V = \tilde{\nabla}_V X = (Xf/f) V$,
(2) $\mathfrak{H}(\tilde{\nabla}_V W) = -f \cdot g(V, W) \operatorname{grad} f = -(1/f) \tilde{g}(V, W) \operatorname{grad} f$,
(3) $\mathfrak{V}(\tilde{\nabla}_V W) = \nabla_V W$,

where $\mathfrak{H}$ (*resp.* $\mathfrak{V}$) *denotes the horizontal* (*resp. vertical*) *component.*

We denote by S, R and $\tilde{R}$ the curvature tensor of B, F and $B \times_f F$, respectively.

LEMMA 2.2 ([1]). *Let* X, Y, Z (*resp.* U, V, W) *be vector fields on* B (*resp.* F). *Then*

(1) $\tilde{R}_{UV} W = R_{UV} W - (\|\operatorname{grad} f\|/f)^2 [\tilde{g}(U, W) V - \tilde{g}(V, W) U]$
(2) $\tilde{R}_{XV} Y = -(1/f)((D^2 f)(X, Y)) V = -(1/f) \tilde{g}(D_X \operatorname{grad} f, Y) V$
(3) $\tilde{R}_{XY} U = R_{VW} X = 0$

(4) $\tilde{R}_{XV}W=\tilde{R}_{XW}V=(1/f)\tilde{g}(V, W)\cdot D_X \operatorname{grad} f$

(5) $\tilde{R}_{XY}Z=S_{XY}Z,$

where $D^2 f$ is the Hessian of f.

LEMMA 2.3 ([**1**]). *$B\times_f F$ is complete if and only if B and F are complete.*

LEMMA 2.4. *If $\dim B=1$, then $X=f(d/dt)$ is an infinitesimal conformal transformation of $(B\times_f F, \tilde{g})$ such that $L_X\tilde{g}=2f'\tilde{g}$, where d/dt is a unit vector field on (B, h), L_X is the Lie differentiation in the direction of X and $f'=\nabla_{(d/dt)}f$.*

PROOF. Let V and W be vector fields on F. Then, by Lemma 2.1, we have

$$\begin{aligned}(L_X\tilde{g})(V, W)&=X\cdot\tilde{g}(V, W)-\tilde{g}([X, V], W)-\tilde{g}(V, [X, W])\\ &=f((d/dt)(f^2 g(V, W))=2f^2 f' g(V, W)=2f'\tilde{g}(V, W),\end{aligned}$$

$$(L_X\tilde{g})(d/dt, V)=X\cdot\tilde{g}(d/dt, V)-\tilde{g}([X, d/dt], V)-\tilde{g}(d/dt, [X, V])=0,$$

$$\begin{aligned}(L_X\tilde{g})(d/dt, d/dt)&=X\cdot\tilde{g}(d/dt, d/dt)-2\tilde{g}([X, d/dt], d/dt)\\ &=2f'\tilde{g}(d/dt, d/dt),\end{aligned}$$

which prove that $L_X\tilde{g}=2f'\tilde{g}$. (Q. E. D.)

3. Scalar curvature of $B\times_f F$ with $\dim B=1$.

Let $\boldsymbol{R}$ be the real line with the standard Riemannian metric and (F, g) an $(n-1)$-dimensional Riemannian manifold. Let f be a positive C^∞-function on $\boldsymbol{R}$ and consider the warped product $(\boldsymbol{R}\times_f F, \tilde{g})$ as in § 2. Let ρ and $\tilde{\rho}$ be the scalar curvature of F and $\boldsymbol{R}\times_f F$, respectively. Then we have the following.

LEMMA 3.1.

$$\tilde{\rho}=-2(n-1)f''/f-(n-1)(n-2)(f'/f)^2+(1/f)^2\rho.$$

PROOF. Let $d/dt, e_1, \cdots, e_{n-1}$ be a local field of orthonormal frames of $\boldsymbol{R}\times_f F$, where d/dt denotes a unit vector field on $\boldsymbol{R}$. Then we have

$$\tilde{\rho}=2\sum_{a=1}^{n-1}\tilde{g}(\tilde{R}_{e_a d/dt}e_a, d/dt)+\sum_{a,b=1}^{n-1}\tilde{g}(\tilde{R}_{e_a e_b}e_a, e_b).$$

On the other hand, Lemma 2.2 implies

$$\begin{aligned}&\tilde{g}(\tilde{R}_{e_a d/dt}e_a, d/dt)=-(f''/f)\\ &\tilde{g}(\tilde{R}_{e_a e_b}e_a, e_b)=\tilde{g}(R_{e_a e_b}e_a, e_b)-(f'/f)^2\\ &\qquad=(1/f)^2 g(R_{fe_a fe_b}fe_a, fe_b)-(f'/f)^2.\end{aligned}$$

Since $fe_1, \cdots, fe_{n-1}$ are orthonormal with respect to g, we obtain

$$\tilde{\rho}=-2(n-1)(f''/f)-(n-1)(n-2)(f'/f)^2+(1/f)^2\rho. \qquad \text{(Q. E. D.)}$$

This, combined with Lemma 2.3 and Lemma 2.4, implies the following.

PROPOSITION 3.2. *Let F be an $(n-1)$-dimensional complete Riemannian manifold with constant scalar curvature ρ and let $\tilde{\rho}$ be a constant. If the differential equation in Lemma 3.1 admits a non-constant positive solution f, then $\boldsymbol{R}\times_f F$ is an n-dimensional complete Riemannian manifold with constant scalar curvature $\tilde{\rho}$ admitting an infinitesimal non-isometric conformal transformation.*

We show that the differential equation in Lemma 3.1 admits a positive periodic solution if ρ and $\tilde{\rho}$ are positive constants, that is we have the following.

LEMMA 3.3. *If ρ and $\tilde{\rho}$ are positive constants, then the differential equation*

$$2(n-1)ff''+(n-1)(n-2)(f')^2+\tilde{\rho}f^2-\rho=0$$

admits a positive periodic solution.

PROOF. If we put $x=f(t)$ and $y=f'(t)$, then the differential equation can be written as

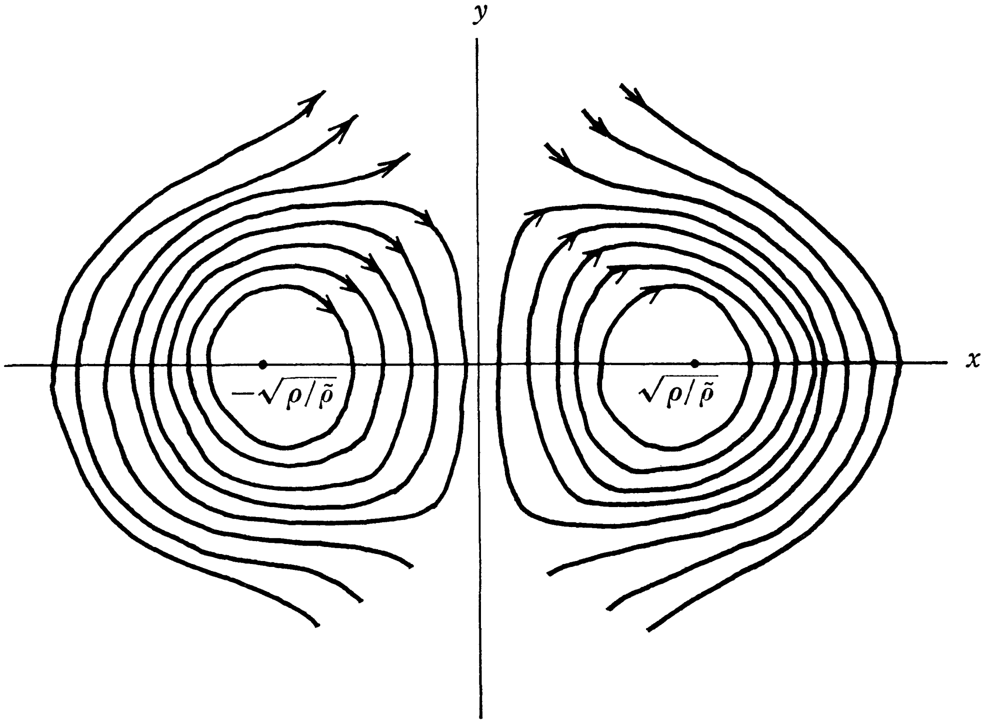
$$(*)\qquad \begin{cases} x'=y \\ y'=-((n-2)/2)(y^2/x)+(\rho/2(n-1))(1/x)-(\tilde{\rho}/2(n-1))x\,. \end{cases}$$

Since (*) is invariant under $(t, x, y)\to(-t, x, -y)$, if $(x(t), y(t))$ is a solution, so is $(x(-t), -y(-t))$. If $(x(t), y(t))$ is a solution of (*) with initial condition $(x(0), y(0))=(a, 0)$, then the solution $(x(-t), -y(-t))$ also satisfies the same initial condition, where $a\neq 0$ is an arbitrary real number. Therefore, by the uniqueness of solution, we have $(x(t), y(t))=(x(-t), -y(-t))$. This implies that an orbit passing through $(a, 0)$ is symmetric (in the reverse sense) with respect to x-axis. On the other hand, if we put $\xi=x-\sqrt{\rho/\tilde{\rho}}$ and $\eta=y$, then (*) implies

$$\begin{pmatrix}\xi' \\ \eta'\end{pmatrix}=\begin{pmatrix}0 & 1 \\ -\tilde{\rho}/(n-1) & 0\end{pmatrix}\begin{pmatrix}\xi \\ \eta\end{pmatrix}+(\text{higher order terms})\,.$$

Since eigenvalues of the matrix representing the linear term are pure imaginary, an orbit $(x(t), y(t))$ with initial condition $(x(0), y(0))=(a, 0)$ for a sufficiently close to $\sqrt{\rho/\tilde{\rho}}$ intersects the x-axis again at $(x(t_0), 0)$ for some $t_0>0$. These imply that an orbit $(x(t), y(t))$ with initial condition $(x(0), y(0))=(a, 0)$ for a sufficiently close to $\sqrt{\rho/\tilde{\rho}}$ is a closed curve. Moreover note that no orbit can intersect the y-axis. Therefore (*) admits positive periodic solutions.

Orbits are illustrated as follows (Note that orbits are symmetric (in the reverse sense) with respect to the y-axis as well, since (*) is invariant under $(t, x, y)\to(-t, -x, y)$);



Since a periodic function on R can be considered as a function on a circle S^1, Proposition 3.2 and Lemma 3.3 yield the following.

THEOREM 3.4. *Let F be an $(n-1)$-dimensional compact Riemannian manifold with positive constant scalar curvature ρ and let $\tilde{\rho}$ be a positive constant. If f is a positive non-constant periodic solution of the differential equation in Lemma 3.3, then $S^1\times_f F$ is an n-dimensional compact Riemannian manifold with positive constant scalar curvature $\tilde{\rho}$ admitting an infinitesimal non-isometric conformal transformation.*

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