

Some theorems on projective hyperbolicity

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Let M be a manifold with a torsionfree affine connection Γ . For such a Γ , S. Kobayashi has recently introduced in [Ko 4] a pseudo-distance p which depends only on the projective structure of Γ , or what is the same thing, on the normal projective connection induced by Γ (cf. [Ko 2], Proposition 7.2 or [K-Na]). Call Γ , or simply M when Γ is understood, *(complete) projective-hyperbolic* if and only if p is a (complete) metric, i. e. $p(x, y)=0 \Rightarrow x=y$ for all $x, y \in M$. Then Kobayashi and T. Sasaki have proved the following theorems:

(A) ([Ko 4]). *If M is a (complete) Riemannian manifold whose Ricci curvature is bounded above by a negative constant, then M is (complete) projective-hyperbolic.*

(B) ([K-S]). *If M is a manifold with a complete torsionfree affine connection whose Ricci tensor is positive semi-definite, then p is identically zero.*

These results are parallel to what is known or what is expected to hold for the Kobayashi metric on complex manifolds (for (A) see [K 1], Theorem 4.11 on p. 61; for (B) see [G-W 1], Conjecture 1a on p. 79). By contrast, we shall prove the following theorems.

THEOREM 1. *Let M be a manifold with a torsionfree affine connection whose Ricci tensor Ric is negative semi-definite. Suppose for each maximal geodesic $\gamma: J \rightarrow M$ where J is an open interval in \mathbf{R} , $\text{Ric}(\dot{\gamma}, \dot{\gamma})$ is never identically zero. Then M is projective-hyperbolic.*

THEOREM 2. *Let M be a compact Riemannian manifold with quasi-negative Ricci curvature (i. e. everywhere nonpositive Ricci curvature which is in addition negative in all directions at a point). Then the group of projective transformations is finite.*

Theorem 2 extends the theorems of Couty [C] and Kobayashi [Ko 4] who proved under the assumption of negative Ricci curvature that this group is discrete and finite, respectively. The analogous fact of Theorem 1 for the Kobayashi metric on complex manifolds is completely false. For instance, there are complete Hermitian metrics on C of negative curvature, yet the Kobayashi

metric of C is identically zero. Theorem 1 has a supplement which guarantees the completeness of p ; for this statement we need the following counterpart of Royden's theorem in the complex case (see [R]).

THEOREM 3. p is the integrated form of an infinitesimal metric P . More precisely, there exists an upper semi-continuous map $P: TM \rightarrow [0, \infty)$ where TM is the tangent bundle, such that

$$p(x, y) = \inf_{\gamma} \int_{\gamma} P,$$

where the infimum is taken over all piecewise C^∞ curves γ joining x to y .

We shall see that this theorem leads to a simple proof of the theorem of Kobayashi-Sasaki ((B) above). Now the supplement to Theorem 1 may be stated as follows.

THEOREM 4. (i) Let M be a Riemannian manifold and let ρ denote the distance from a fixed point $0 \in M$. Suppose for some constant $A > 0$,

$$\text{Ric}(x) \leq -A/(1 + \rho(x)^2)$$

for every $x \in M$, where $\text{Ric}(x)$ denotes the Ricci curvature of every direction at x . Then there exists a positive constant α such that for all tangent vectors $X \in M_x$ and for all x ,

$$|X|_P \geq \frac{\alpha}{(1 + \rho(x)^2)^{1/2}} |X|$$

where $|X|_P$ (resp. $|X|$) denotes the norm of X with respect to P (resp. the Riemannian metric). In particular if the Riemannian metric is complete, then M is complete projective-hyperbolic. (ii) If M is an n -dimensional Riemannian manifold such that $\text{Ric}(x) \leq -A^2$ for some constant $A > 0$, then $A(n-1)^{-1/2} |X| \leq |X|_P$ for all X . If the Riemannian metric is complete and if on the other hand $\text{Ric}(x) \geq -B^2$ for some positive constant B , then $|X|_P \leq B(n-1)^{-1/2} |X|$ for all X .

The analogue of (i) in the complex case is known ([G-W 2]), Theorem E) and the first part of (ii) is implicit in Kobayashi's theorem (A) above. The complex analogue of the second part of (ii) is a conjecture ([G-W 2]), remark after Theorem G). Note that (ii) implies that for an n -dimensional Riemannian manifold of constant Ricci curvature $-A^2$, P is just the Riemannian norm multiplied by $A(n-1)^{-1/2}$.¹⁾ The final theorem is the projective counterpart of Brody's theorem in the complex case ([B]); we refer to § 1 below for the definition of

1) Note added January 3, 1980: the manuscript of this paper has been in circulation since May 1978. In the intervening period, the foregoing result has been proven independently by S. Kobayashi in his paper, Projective invariant metrics for Einstein spaces, Nagoya Math. J., 73 (1979), 171-174.

a projective map from R to M .

THEOREM 5. *A compact manifold M with a torsionfree affine connection is projective-hyperbolic if and only if there is no (nonconstant) projective map from R to M .*

It may not be out of place to make a passing comment on the significance of this projective pseudo-distance p . The classical theory of projective connections is an entirely *local* affair and is totally deficient in global considerations. To illustrate, let R^n be Euclidean space with its canonical flat metric and let M be the n -dimensional simply-connected space form of curvature -1 . Then the classical theory tells us that M and R^n are locally projectively equivalent (vanishing of the Weyl projective tensor for both), but *offers no tool* to decide whether the equivalence could be made global. However, since M is projective-hyperbolic (Kobayashi's theorem (A) above) and $p \equiv 0$ on R^n , this pseudo-distance p gives the first global projective invariant that distinguishes R^n from M . (See the Proposition in §3 for a more general assertion.) Clearly a more globally-oriented study of projective connections deserves the attention of geometers. In this regard, we note that the results of this paper make possible the routine transplant of many theorems in complex hyperbolic manifolds ([Ko 1] and [Ko 3]) to the projective setting. However, in the absence of a clear picture as to what theorems of this type are of real geometric interest, we shall refrain from doing the transplant here. It seems certain that, since the projective structure is much more rigid than the complex structure, their studies must ultimately diverge even in the special case of hyperbolic manifolds.

It is not the intention of this paper to discuss projective connections. It will suffice to cite the following references: Chapter 4 of [C-G], Chapter IV of [Ko 2], [K-Na] and [Y]. The last item gives a detailed account of the relationship between the approaches of E. Cartan and the Princeton school, but it is regrettably inaccessible.

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§ 1.

This section briefly recalls the basic definitions with many details left to [Ko 4], [K-S] and pp. 647-661 of [H], as well as the references there.

Let M be a manifold with a (fixed) torsionless affine connection and $\gamma: J \rightarrow M$ be a geodesic, where J is an interval in R . Assume that γ is one-one; then each $t_0 \in J$ can be identified with $\gamma(t_0)$ so that the parameter t of γ becomes identified with the map from $\gamma(J)$ to J such that $\gamma(t) \rightarrow t$. With this understood, t is called the *affine parameter* of γ or of $\gamma(J)$. Now let $f: J_1 \rightarrow M$ be another

parametrization of $\gamma(J)$; by definition, $f(J_1) \subset \gamma(J)$ and $\dot{f}(u)$, the tangent vector to f at $f(u)$, is nowhere zero for each $u \in J_1$. Also assume f is one-one; then we can identify each $u \in J_1$ with $f(u) \in \gamma(J)$ as before. We may now regard u as a function of t ; equivalently, define $u: J \rightarrow J_1$ by $u(t) = f^{-1}(\gamma(t))$. Similarly, t may be regarded as a function of u : $t(u) \equiv \gamma^{-1}(f(u))$. These conventions will be used without comments in the following. The reason we spend time to cover this elementary material is that unless these identifications are firmly understood, the proofs of the theorems (particularly Theorems 3 and 5) will be hopelessly confusing.

Let J_1 be an open interval in \mathbf{R} and let $f: J_1 \rightarrow M$ be a C^∞ curve in M (with $\dot{f}(u) \neq 0$ for every $u \in J_1$ always understood). f is a *projective map* if and only if (a) $f(J_1)$ is a subset of (the image of) a geodesic $\gamma: J \rightarrow M$, and (b) if t is the affine parameter of γ , then the parameter u of f satisfies:

$$(1) \quad \{u, t\} = \frac{2}{(n-1)} \text{Ric}(\dot{\gamma}(t), \dot{\gamma}(t)) \quad (\equiv 2Q(t)),$$

where $n = \dim M$, Ric is the Ricci tensor of M , and $\{u, t\}$ is the *Schwarzian derivative* of u with respect to t defined by:

$$(2) \quad \{u, t\} \equiv \frac{u'''(t)}{u'(t)} - \frac{3}{2} \left(\frac{u''(t)}{u'(t)} \right)^2.$$

If f is projective, then u is called a *projective parameter* of γ . A projective parameter is unique up to a fractional linear transformation, i. e., if v is another such, then $v = (au + b)/(cu + d)$, where a, b, c, d are real numbers and $ad - bc \neq 0$. Note that it was not assumed that either f or γ is one-one in this definition. However, since f and γ are both locally one-one, equation (1) makes sense in the neighborhood of each $t \in J$, and standard analytic continuation arguments then give meaning to (1) globally on J .

LEMMA 1. Suppose $\gamma: J \rightarrow M$ is a geodesic with affine parameter t and $u: J \rightarrow \mathbf{R}$ is any C^∞ function satisfying (1). Then u is a strictly monotone function of t and the mapping $f: u(J) \rightarrow M$ defined by $f(u(t)) = \gamma(t)$ is projective. ("Any solution of (1) is a projective parameter of γ ".)

PROOF. With the various identifications above understood, it suffices to show that u' is never zero. Recall: if $u(t)$ is a given solution of (1), then there exist linearly independent solutions $y_1(t), y_2(t)$ of the Jacobi equation:

$$(3) \quad y''(t) + Q(t)y(t) = 0,$$

such that

$$(4) \quad u(t) = y_1(t)/y_2(t).$$

Hence $u'(t) = W(y_1, y_2; t)/y_2(t)^2$, where $W(y_1, y_2) \equiv y_2 y_1' - y_2' y_1$ is the Wronskian of (3) and hence is equal to a nonzero constant. The lemma follows. Q.E.D.

In this paper, the symbol I will be reserved exclusively for the open interval $(-1, 1)$. Define on I the metric:

$$(5) \quad dI^2 = \frac{du^2}{(1-u^2)^2}.$$

This is the restriction to $(-1, 1)$ of the complete Riemannian metric on the unit disc of \mathbf{R}^2 with constant curvature -4 (Poincaré metric). Furthermore, dI^2 is homogeneous under the fractional linear maps $u \rightarrow (u+a)/(au+1)$ where $|a| < 1$. If $\rho(u)$ denotes the distance of $u \in I$ from 0 with respect to dI^2 , then

$$(6) \quad \rho(u) = \frac{1}{2} \log \frac{1+|u|}{1-|u|}.$$

Define a pseudo-distance p^* on M (relative to the given torsionfree affine connection) by:

$$(7) \quad p^*(x, y) = \inf \{ \rho(u) : \text{there exists a projective map } f: I \rightarrow M \text{ such that } f(0) = x \text{ and } f(u) = y \}.$$

The projective pseudo-distance p of Kobayashi is defined by:

$$(8) \quad p(x, y) = \inf \sum_{i=1}^l p^*(x_i, x_{i+1}),$$

where the infimum is taken over all finite sequences $\{x = x_1, x_2, \dots, x_l, x_{l+1} = y\}$, with l arbitrary. The motivation for this definition comes from the complex case (see [Ko 1] and [Ko 3]) as well as from Lemma 11 and 13 in §5. p is a projective invariant of the affine connection and enjoys the usual functorial properties; for the latter, we refer to [Ko 1], Chapter IV, or [Ko 3], §2, as well as Proposition 3.5 of [Ko 4].

§ 2.

This section defines the infinitesimal projective metric P along the lines of [R] and proves Theorem 1.

Let M be a manifold with a torsionless affine connection and let TM be its tangent bundle; thus $TM \equiv \{(x, X) : x \in M, X \in M_x\}$. In addition, denote by $|V|_b$ the norm of any tangent vector V of I relative to the metric (5), and by $|V|$ the ordinary Euclidean norm of V . Now define $P: TM \rightarrow [0, \infty)$ by:

$$(9) \quad P(x, X) = \inf \{ |V|_b : V \text{ is tangent to } I \text{ and there exists a projective map } f: I \rightarrow M \text{ such that } df(V) = X \}.$$

Since the metric (5) is homogeneous with respect to all fractional linear transformations of I onto itself, it suffices to restrict V in the above definition to be a tangent vector at the origin 0 of I ; in that event, $|V| = |V|_b$ in view of (5). Thus we may also define $P(x, X)$ by:

$$(10) \quad P(x, X) = \inf\{|V| : V \text{ is tangent to } I \text{ at } 0 \text{ and there exists a projective map } f: I \rightarrow M \text{ such that } df(V) = X\}.$$

P is called the *infinitesimal projective metric*. To proceed, let us assume for the moment the next lemma.

LEMMA 2. $P: TM \rightarrow [0, \infty)$ is upper semi-continuous.

We now define a new pseudo-distance $p_1: M \times M \rightarrow [0, \infty)$ by:

$$(11) \quad p_1(x, y) = \inf_{\gamma} \int_{\gamma} P,$$

where the infimum is taken over all piecewise C^∞ curves joining x to y . Lemma 2 insures that the integral in (11) makes sense. Moreover, definition (9) of P makes it clear that if $f: I \rightarrow M$ is a projective map, then $p_1(f(a), f(b)) \leq \rho(a, b)$ for every $a, b \in I$, where ρ denotes the distance between a and b with respect to dI^2 . By Proposition 3.5 of [Ko 4] (maximality of p among all such projective pseudo-distances), we obtain:

LEMMA 3. $p_1 \leq p$.

In the next section, we will prove the reverse inequality (and thereby Theorem 3). But first, we prove Lemma 2 and Theorem 1.

PROOF OF LEMMA 2. Let $P(x_0, X_0) = \alpha$. Given $\varepsilon > 0$, we must produce a neighborhood U of (x_0, X_0) in TM such that for all $(x, X) \in U$,

$$(12) \quad P(x, X) < \alpha + \varepsilon.$$

We will consistently use definition (10) of P (and not (9)) so that the vectors tangent to I , V_1 , V_2 etc., that come up below will always be understood to be tangent vectors at the origin 0 of I . Now let $f_0: I \rightarrow M$ be a projective map so that $df_0(V_0) = X_0$ and $|V_0| < \alpha + (\varepsilon/4)$. Choose $a \in (0, 1)$ such that $|V_0/a| < |V_0| + (\varepsilon/4)$. Then define $f_1: [-1, 1] \rightarrow M$ by $f_1(u) = f_0(au)$. Observe that f_1 is projective, and $(\text{image } f_1) = f_0([-a, a])$. Moreover, writing $V_1 \equiv V_0/a$, we have:

$$(13) \quad df_1(V_1) = X_0 \quad \text{and} \quad |V_1| < \alpha + (\varepsilon/2).$$

Now let S be an open neighborhood of the compact set $\Gamma \equiv f_1([-1, 1])$ such that the closure of S is compact. By the definition of a projective map, there exists a geodesic $\gamma: [-c, d] \rightarrow M$ such that $\dot{\gamma}(0) = X_0$ and $\gamma([-c, d]) = \Gamma$. Furthermore, since f_1 is defined in a neighborhood of $[-1, 1]$, γ may also be

assumed to be defined in a neighborhood of $[-c, d]$. In the following, we consider only the case where γ is an imbedding; the general case (of γ being an immersion only) can be dealt with by suitable modifications along standard lines. So suppose (x, X) is sufficiently close to (x_0, X_0) ; then by Theorems 7.1 and 7.2 on p. 22 and p. 25 respectively of [C-L] (cf. also the remark in the middle of p. 23 in [C-L]), a geodesic starting at x and tangent to X will be defined on all of $[-c, d]$ and will be uniformly close to γ . More precisely, there is a neighborhood U of (x_0, X_0) in TM and a C^∞ map $\Phi: U \times [-c, d] \rightarrow S$ such that for each $(x, X) \in U$, the map $\gamma_{(x, X)}: [-c, d] \rightarrow S$ defined by $\gamma_{(x, X)}(t) \equiv \Phi(x, X, t)$ is a geodesic of M satisfying $\dot{\gamma}_{(x, X)}(0) = X$. Let t be the affine parameter of each $\gamma_{(x, X)}$; then $-c \leq t \leq d$ and t has a C^∞ extension to an open interval containing $[-c, d]$. On γ , let u be the projective parameter corresponding to f_1 above; u too has a C^∞ extension to an open interval containing $[-1, 1]$. Then u satisfies equation (1) with initial conditions $u(0) = 0$ and

$$(14) \quad u'(0) = |V_1|/|T_0|,$$

where V_1 is the vector in (13) and T_0 is tangent to $[-c, d]$ at 0 such that $d\gamma(T_0) = X_0$.

To return to the geodesics $\gamma_{(x, X)}: [-c, d] \rightarrow S$, consider the solution $u_{(x, X)}$ of

$$(15) \quad \{u_{(x, X)}, t\} = \frac{2}{(n-1)} \text{Ric}(\dot{\gamma}_{(x, X)}(t), \dot{\gamma}_{(x, X)}(t)),$$

with specified initial conditions: $u_{(x, X)}(0) = 0$, $u'_{(x, X)}(0) = u'(0)$ and $u''_{(x, X)}(0) = u''(0)$. Now (15) is a C^∞ perturbation of (1); by Theorems 7.4 and 7.5 on pp. 29-30 of [C-L], there is a neighborhood of (x_0, X_0) , which may be assumed to be U , such that for all $(x, X) \in U$, the solution $u_{(x, X)}$ of (15) is defined on the whole interval $[-c, d]$ and such that the sup norm satisfies $\|u - u_{(x, X)}\| < \delta$ for any pre-assigned positive δ . (All the care in the preceding argument about u being defined on an open interval containing $[-1, 1]$, etc., is to insure at this point that the above sup norm estimate for $u - u_{(x, X)}$ is valid on $[-c, d]$, and not just on a compact subset of $(-c, d)$.) In particular, since $(\text{image } u) = [-1, 1]$, we have that for all $(x, X) \in U$,

$$(16) \quad (-1 + \delta, 1 - \delta) \subset (\text{image } u_{(x, X)}) \subset (-1 - \delta, 1 + \delta).$$

An appropriate choice of δ will be made presently.

Now let $f_{(x, X)}$ be the inverse mapping from $(\text{image } u_{(x, X)})$ to $(\text{image } \gamma_{(x, X)})$. (See the discussion at the beginning of §1 about these matters.) Then $f_{(x, X)}(0) = x$. Let $V(x, X)$ and $T(x, X)$ be tangent vectors at the origins of $(\text{image } u_{(x, X)})$ and $[-c, d]$ respectively, such that $df_{(x, X)}(V(x, X)) = X$ and $d\gamma_{(x, X)}(T(x, X)) = X$. Then as in (14), we have:

$$(17) \quad u'_{(x,X)}(0) = |V(x, X)| / |T(x, X)| = u'(0).$$

Recall now from (14) that $d\gamma(T_0) = X_0$. If (x, X) is close to (x_0, X_0) , then $\gamma_{(x,X)}$ is C^∞ close to γ , $\Rightarrow d\gamma_{(x,X)}(0)$ is close to $d\gamma(0)$, $\Rightarrow T(x, X)$ is close to T_0 in norm. Thus a comparison of (14) with (17) shows that, when U is sufficiently small,

$$(18) \quad ||V_1| - |V(x, X)|| < \varepsilon/4 \quad \text{for all } (x, X) \in U.$$

Next consider the projective map $f_{(x,X)} : (-1+\delta, 1-\delta) \rightarrow S$ (we have here made use of (16)), where $f_{(x,X)}(0) = x$ and $df_{(x,X)}(V(x, X)) = X$. We choose δ at this point to be so small that

$$(19) \quad |(1-\delta)^{-1}V(x, X)| < |V(x, X)| + \varepsilon/4.$$

In view of (18), this choice of δ can be made uniformly in (x, X) . Now define $g_{(x,X)} : (-1, 1) \rightarrow M$ by $g_{(x,X)}(t) = f_{(x,X)}((1-\delta)t)$. Then $g_{(x,X)}$ is projective, $g_{(x,X)}(0) = x$ and $dg_{(x,X)}((1-\delta)^{-1}V(x, X)) = X$. We conclude from definition (10) of P that for all $(x, X) \in U$:

$$\begin{aligned} P(x, X) &\leq |(1-\delta)^{-1}V(x, X)| \\ &< |V(x, X)| + (\varepsilon/4) && \text{(by (19))} \\ &< |V_1| + (\varepsilon/2) && \text{(by (18))} \\ &< \alpha + \varepsilon && \text{(by (13)).} \end{aligned}$$

This proves (12) and hence the lemma. Q. E. D.

PROOF OF THEOREM 1. We first prove the theorem assuming that the Ricci tensor is everywhere negative definite; the essential idea of the proof is already contained in this special case. Let G be a Riemannian metric and let $\|X\|$ denote the G -norm of a tangent vector X . In view of Lemma 3, it suffices to show that for any $x_0 \in M$, there exists a neighborhood U of x_0 and a positive constant α such that

$$(20) \quad P(x, X) \geq \alpha \|X\| \quad \text{for all } X \in M_x, x \in U.$$

Let B be a closed ball around x_0 of radius 2ε (relative to G) which is compact. Choose U to be closed ball of radius ε around x_0 . Let $\gamma : (-c, d) \rightarrow M$ be a geodesic of the given torsionfree affine connection such that $\gamma(0) \in U$ and $\|\dot{\gamma}(0)\| = 1$. On this γ , we shall produce a particular projective parameter which can be estimated. Using the notations of equations (1)-(4), let $y_1(t)$ and $y_2(t)$ be solutions of (3) satisfying the initial conditions: $y_1(0) = 0$, $y_1'(0) = 1$; $y_2(0) = 1$, $y_2'(0) = 0$. Since $Q(t) < 0$, $y_2'' \geq 0$ everywhere so that $y_2 \geq 1$. Define $u(t) = y_1(t)/y_2(t)$. Then $u(t)$ is a projective parameter on γ and the arguments following (4) show that $u'(t) = 1/y_2(t)^2$. Since $u'(0) = 0$,

$$(21) \quad u(t) = \int_0^t 1/y_2^2 \quad \text{for all } t \in (-c, d).$$

It remains to estimate y_2 on $(-c, d)$. By a standard compactness argument, there exist constants η and ξ such that for all geodesics γ with $\gamma(0) \in U$ and $\|\dot{\gamma}(0)\|=1$, we have $\gamma([- \eta, \eta]) \subset B$ and $\frac{1}{(n-1)} \text{Ric}(\dot{\gamma}(t), \dot{\gamma}(t)) \leq -\xi$ for all $t \in [- \eta, \eta]$. The latter uses the negative definiteness of Ric . Let $\varphi: \mathbf{R} \rightarrow [0, \infty)$ be a C^∞ function with support in $[- \eta, \eta]$, equal to ξ in $[- \eta/2, \eta/2]$, and $\leq \xi$ everywhere. Then $-Q(t) \geq \varphi(t)$ for all t , where $Q(t)$ is as in (1). We compare $y_2(t)$, which is the solution of (3), with the solution $z(t)$ of

$$(22) \quad y''(t) - \varphi(t)y(t) = 0,$$

such that $z(t)$ satisfies the initial condition $z(0)=1, z'(0)=0$. Then $-Q(t) \geq \varphi(t)$ implies $y_2(t) \geq z(t)$ wherever $y_2(t)$ is defined. Moreover, since $z(t) \geq 1$ (by (22) and $\varphi \geq 0$), integrating (22) leads to $z'(t) = \int_0^t \varphi z \geq \int_0^t \varphi \geq \frac{1}{2} \xi \eta$ for all $t \geq \eta$. Using the same argument for $t \leq -\eta$, we obtain:

$$(23) \quad \begin{aligned} z'(t) &\geq \frac{1}{2} \xi \eta \quad \text{for } t \geq \eta \\ &\leq -\frac{1}{2} \xi \eta \quad \text{for } t \leq -\eta. \end{aligned}$$

In particular, $z(t) \geq \max\left\{1, \frac{1}{2} \xi \eta (|t| - \eta)\right\}$ for all $t \in \mathbf{R}$, so that $\int_0^\infty 1/z^2 \equiv C(\eta, \xi) < \infty$, where $C(\eta, \xi)$ is a positive constant depending only on ξ and η . Summarizing, we have:

$$(24) \quad u(0)=0, u'(0)=1 \text{ and } |u(t)| < C(\eta, \xi) < \infty.$$

To proceed with the proof of Theorem 1, we need the following lemma.

LEMMA 4. Let $\gamma: (-c, d) \rightarrow M$ be a geodesic in a manifold M with a torsionfree affine connection. Suppose there exists a projective parameter u on γ satisfying (24), then $P(\gamma(0), \dot{\gamma}(0)) \geq 1/C(\eta, \xi)$.

PROOF. For brevity, denote $C(\eta, \xi)$ simply by C_0 . According to definition (10) of P , it suffices to prove that if $f: I \rightarrow M$ is a projective map such that $f(I) \subset (\text{image } \gamma)$, $f(0) = \gamma(0)$, and $df(V) = \dot{\gamma}(0)$, then $|V| \geq 1/C_0$.

Suppose f as above and suppose $|V| = 1/k$. We will show $k \leq C_0$. Let $g: (-k, k) \rightarrow M$ be defined by $g(v) = f(v/k)$ and let v be the projective parameter on γ corresponding to g . Then $v(0) = 0, v'(0) = 1$ and $v(t) \uparrow k, v(t) \downarrow (-k)$ as $t \uparrow d$ and $t \downarrow -c$ respectively. Assume $k > C_0$ and we shall deduce a contradiction. Since projective parameters are unique up to fractional linear transforma-

tions, $v=(a_1u+a_2)/(a_3u+a_4)$ for constants $\{a_i\}$ such that $a_1a_4-a_2a_3\neq 0$. Since $u(0)=v(0)=0$, $a_2=0$; since $u'(0)=v'(0)=1$, $a_1=a_4$. Thus

$$(25) \quad v=u/(\alpha u+1) \quad \text{for some } \alpha \in \mathbf{R}.$$

If $k > C_0$, then there exists $t_1 \in (0, d)$ such that $v(t_1) > C_0$. Since $u(t)$ is strictly monotone on $(-c, d)$ (see Lemma 1) and $u'(0)=1$, $u(t)$ is positive for $t \in (0, d)$ and negative for $t \in (-c, 0)$. In particular, $0 < u(t_1) < C_0$ by (24). Evaluating (25) at t_1 then leads to $\alpha < 0$. However, since also $v(t) \downarrow (-k)$ as $t \downarrow -c$, there is a $t_2 \in (-c, 0)$ such that $v(t_2) < -C_0$. Also $-C_0 < u(t_2) < 0$, by (24). Then evaluating (25) at t_2 leads to $\alpha > 0$. Contradiction. Q. E. D.

Lemma 4 shows that for all $x \in U$ and for all $X \in M_x$ such that $\|X\|=1$, $P(x, X) \geq 1/C(\eta, \xi)$. Since $P(x, \beta X) = |\beta| P(x, X)$, this shows that (20) is valid with $\alpha = 1/C(\eta, \xi)$. Theorem 1 has thus been proved if the negative definiteness of Ric is assumed.

The proof of Theorem 1 under the general assumption of the theorem is merely a technical variation on the preceding argument. Briefly it goes as follows. Notation as before, we wish to prove that given $x_0 \in M$, there exists a neighborhood U of x_0 on which (20) is valid. To this end, let $\pi: SM \rightarrow M$ be the unit sphere bundle of M (relative to the Riemannian metric G above). Take a coordinate neighborhood W of x_0 such that $\pi^{-1}(W)$ is diffeomorphic to $W \times S^{n-1}$ (S^{n-1} = unit sphere in \mathbf{R}^n); identify $\pi^{-1}(W)$ with $W \times S^{n-1}$. Take any unit vector (x_0, X_1) at x_0 . We claim that there is a neighborhood W_1 of x_0 in W , a neighborhood N_1 of X_1 in S^{n-1} and positive numbers t_1, ε_1 which possess the following property: for each $(y, Y) \in W_1 \times N_1$, let $\gamma: (-a, b) \rightarrow M$ be the maximal geodesic (relative to the given affine connection of M) such that $\dot{\gamma}(0) = Y$; then $\text{Ric}(\dot{\gamma}(t), \dot{\gamma}(t)) < 0$ on either $[t_1, t_1 + \varepsilon_1]$, or on $[-t_1 - \varepsilon_1, -t_1]$, or on both. Indeed, if $\gamma_1: (-a_1, b_1) \rightarrow M$ is the maximal geodesic such that $\dot{\gamma}_1(0) = X_1$, then $\text{Ric}(\dot{\gamma}_1(t^*), \dot{\gamma}_1(t^*)) < 0$ for some $t^* \in (-a_1, b_1)$ by hypothesis. There is then a compact neighborhood K of $(\gamma_1(t^*), \dot{\gamma}_1(t^*))$ in TM on which $\text{Ric} < 0$. By the continuous dependence of solutions of ordinary differential equations on initial conditions, there is a neighborhood $W_1 \times N_1^*$ of (x_0, X_1) in TM such that for each $(y, Y) \in W_1 \times N_1^*$, the maximal geodesic $\gamma: (-a, b) \rightarrow M$ with $\dot{\gamma}(0) = Y$ has the property that $(\gamma(t'), \dot{\gamma}(t')) \in K$ for some t' close to t^* . Now define $W_1 \times N_1 \equiv SM \cap (W_1 \times N_1^*)$. The required positive numbers t_1, ε_1 can then be obtained by a straightforward argument.

By the compactness of S^{n-1} , there exist a finite number of such neighborhoods $\{W_i \times N_i: i=1, \dots, k\}$ such that $\bigcup_{i=1}^k N_i = S^{n-1}$. Let $U = \bigcap_i W_i$. Then there exist positive numbers $t_0, \varepsilon_0, \delta_0$ such that for every unit vector $(x, X) \in U \times S^{n-1}$ (i. e. $\|X\|=1$), if $\gamma: (-a, b) \rightarrow M$ is the maximal geodesic satisfying $\dot{\gamma}(0) = X$, then $\text{Ric}(\dot{\gamma}, \dot{\gamma}) > -\delta_0$ on some $[t^*, t^* + \varepsilon_0]$ or $[-t^* - \varepsilon_0, -t^*]$, where $t^* \leq t_0$.

Let us say $\text{Ric}(\dot{\gamma}, \dot{\gamma}) < -\delta_0$ on $[t^*, t^* + \varepsilon_0]$ for a fixed $\dot{\gamma}$. Construct a C^∞ function $\varphi: \mathbf{R} \rightarrow [0, \infty)$ with support in $[t_0, t_0 + \varepsilon_0]$ (caution: here, it is t_0 , not t^*), such that $\varphi \leq \delta_0/(n-1)$ everywhere and $\varphi = \delta_0/(n-1)$ on $[t_0 + \frac{1}{3}\varepsilon_0, t_0 + \frac{2}{3}\varepsilon_0]$. The solution $z(t)$ of $y'' - \varphi y = 0$ which satisfies $z(0) = 1$, $z'(0) = 0$ then also satisfies: $z(t) \geq 1$ everywhere, $z(t) \equiv 1$ on $(-\infty, t_0]$, and $z(t) \geq \frac{\delta_0 \varepsilon_0}{3(n-1)}(t - t_0 - \varepsilon_0)$ on $[t_0 + \varepsilon_0, \infty)$ (cf. (22) and (23) above). In particular,

$$\int_0^\infty 1/z^2 = C(t_0, \varepsilon_0, \delta_0) < \infty$$

where $C(t_0, \varepsilon_0, \delta_0)$ is a positive constant depending only on t_0, ε_0 and δ_0 . Let $y_1(t), y_2(t)$ be solutions of (3) such that $y_1(0) = 0, y_1'(0) = 1; y_2(0) = 1, y_2'(0) = 0$. Let $u_0(t) \equiv y_1(t)/y_2(t)$ be the projective parameter on γ so that (21) is valid with u replaced by u_0 . It is an elementary argument to show that $y_2(t) \geq z(t)$ for all $t \in (-a, b)$ so that $\int_0^\infty 1/y_2^2 \leq C(t_0, \varepsilon_0, \delta_0)$. Summarizing, we have:

$$u_0(0) = 0, u_0'(0) = 1, -\infty < u_0(t) \leq C(t_0, \varepsilon_0, \delta_0).$$

Now define a new projective parameter u on γ by:

$$u(t) = \frac{2Cu(t)}{-u(t) + 2C},$$

where $C \equiv C(t_0, \varepsilon_0, \delta_0)$. Then

$$u(0) = 0, u'(0) = 1, |u(t)| \leq 2C.$$

Applying Lemma 4, we see that $P(\gamma(0), \dot{\gamma}(0)) \geq 1/(2C)$. It follows that (20) is valid with $\alpha = \frac{1}{2}C(t_0, \varepsilon_0, \delta_0)$ and with the choice of U above. Q. E. D.

It remains to give a geometric condition that would guarantee that $\text{Ric}(\dot{\gamma}, \dot{\gamma})$ is never identically zero as stipulated by Theorem 1. Among many complicated alternatives, we take the simplest one. Let *convex neighborhoods* be understood in the sense of J. H. C. Whitehead ([K-No], p. 149). The following is a direct consequence of Theorem 1.

PROPOSITION. *Let M be a manifold with a torsionfree affine connection. Let K be a disjoint union of compact sets such that each is contained in a convex neighborhood. If the Ricci tensor is negative semidefinite on M and negative definite on $M - K$, then M is projective-hyperbolic.*

A *Cartan-Hadamard manifold* is by definition a complete simply-connected Riemannian manifold whose sectional curvature is nonpositive. On the basis of Lemmas 2 and 4, it is also straightforward to prove the following.

PROPOSITION. *Every bounded domain in a Cartan-Hadamard manifold, with the induced metric, is projective-hyperbolic.*

§ 3.

This section proves Theorem 3. Following the proof will be found some immediate corollaries of this theorem as well as a simple proof of (B) at the beginning of this paper (Theorem of [K-S]).

PROOF OF THEOREM 3. By Lemma 3, it suffices to prove $p \leq p_1$. The line of attack is that of Royden [R] (pp. 131-132); the details are of course different. Let $\gamma: [0, 1] \rightarrow M$ be a C^∞ curve ($\dot{\gamma}(u) \neq 0$ being understood for each u), and let a positive number ε as well as an $s \in [0, 1]$ be given. Note that by the upper semi-continuity of P (Lemma 2) and the Lebesgue bounded convergence theorem, there exists a positive continuous function $h: [0, 1] \rightarrow (0, \infty)$ such that:

$$(26) \quad h(t) > P(\gamma(t), \dot{\gamma}(t)) \quad \text{for all } t \in [0, 1],$$

$$(27) \quad \int_0^1 h < \int_0^1 P(\gamma(t), \dot{\gamma}(t)) dt + \varepsilon.$$

Fix this h . We claim: for ε and s as given, there exists an open interval $I_s \equiv (s - t_0, s + t_0)$ for some $t_0 > 0$ such that $p(\gamma(s), \gamma(t)) \leq (1 + \varepsilon)h(s)|s - t|$ for all $t \in I_s$.

To prove this claim, let W be a convex neighborhood of $x \equiv \gamma(s)$ in M and let $\{w^i\}$ denote geodesic coordinates normal at x and such that $w^i(x) = 0$ for $i = 1, \dots, n$. We regard W as part of Euclidean space \mathbf{R}^n by declaring $\{w^i\}$ to be the ordinary coordinate functions on \mathbf{R}^n ; after re-scaling, one may also assume $\dot{\gamma}(s)$ to be a unit vector at the origin, i.e. $|\dot{\gamma}(x)| = 1$. Choose $t_0 > 0$ such that $\gamma(I_s) \subset W$, where $I_s \equiv (s - t_0, s + t_0)$ as above. There will be other restrictions on t_0 . For each $t \in I_s$, $\gamma(t) \in W$ and, by the convexity of W , there is a unique geodesic of the given connection $\xi_t: [0, v_t] \rightarrow W$ such that $\xi_t(0) = x$, $\xi_t(v_t) = \gamma(t)$, and $|\dot{\xi}_t(0)| = 1$. Note that ξ_t is a radial line segment with unit tangent vectors in the Euclidean coordinate system $\{w^i\}$ and that

$$v_t = |\gamma(t)| = |\dot{\gamma}(s)(t - s) + O(|t - s|^2)|.$$

Recall that $|\dot{\gamma}(s)| = 1$. Thus we may assume t_0 is so small that

$$(28) \quad v_t \leq (1 + \varepsilon)^{1/3} |t - s| \quad \text{for all } t \in I_s.$$

Now choose a neighborhood N of $(\gamma(s), \dot{\gamma}(s))$ in TM so that for all $(y, Y) \in N$, $P(y, Y) < h(s)$. This is guaranteed by (26) and Lemma 2. Now restrict t_0 to be so small that for the geodesic ξ_t above, $\dot{\xi}_t(0) \in N$ whenever $t \in I_s$. This is possible because: $|\dot{\xi}_t(0)| = |\dot{\gamma}(s)| = 1$ and since each ξ_t is the radial line segment

in the coordinate system $\{w^i\}$ joining $0 (=x)$ to $\gamma(t)$, $\dot{\xi}_t(0) \rightarrow \dot{\gamma}(s)$ as $t \rightarrow s$. In particular,

$$P(x, \dot{\xi}_t(0)) < h(s)$$

for all $t \in I_s$.

Now extend ξ_t to a maximal geodesic $\xi_t^*: J \rightarrow M$. By definition (10) of P , there exists a projective map $f: I \rightarrow M$ such that $f(I) \subset \xi_t^*(J)$, $f(0) = x$ and $df(V) = \dot{\xi}_t(0)$, where $|V| \leq P(x, \dot{\xi}_t(0)) + \delta$, and δ is any pre-assigned number. It follows that $|V| \leq h(s) + \delta$. Since $h > 0$, we may choose δ so small that

$$(29) \quad |V| < (1 + \varepsilon)^{1/3} h(s).$$

Let $u_t \in I$ be the number satisfying $f(u_t) = \xi_t(v_t) = \gamma(t)$, where v_t and t are as above. By the definition of p (cf. (6)-(8)):

$$p(\gamma(s), \gamma(t)) \leq \frac{1}{2} \log \frac{1 + u_t}{1 - u_t} = u_t + O(|u_t|^3).$$

Let u be the projective parameter corresponding to f and consider u as a function $u(v)$ of the affine parameter v of ξ_t^* . Then $u(v)$ satisfies: $u(0) = 0$, $u'(0) = |V|$ and $u(v_t) = u_t$. Hence,

$$u_t = u(v_t) = |V| v_t + O(|v_t|^2),$$

so that

$$p(\gamma(s), \gamma(t)) \leq |V| v_t + O(|v_t|^2).$$

Recall that v_t is small if t_0 is small (cf. (28)). Our final restriction on t_0 is that it be so small that $h(s)v_t + O(|v_t|^2) \leq (1 + \varepsilon)^{1/3} v_t h(s)$. This together with (29) now lead to:

$$\begin{aligned} p(\gamma(s), \gamma(t)) &\leq (1 + \varepsilon)^{2/3} v_t h(s) \\ &\leq (1 + \varepsilon) h(s) |t - s|, \end{aligned}$$

by virtue of (28). This proves the claim.

The proof of Theorem 3 can now be concluded following the idea of [R] pp. 131-2. Indeed, by (27) and by the continuity of h , there exists an $\eta > 0$ such that if $0 < t_0 < t_1 < \dots < t_k = 1$ is any subdivision of $[0, 1]$ with $t_{i+1} - t_i < \eta$ for all i , then

$$(30) \quad \sum_{i=1}^k h(s_i)(t_i - t_{i-1}) < \int_I P + \varepsilon$$

where $s_i \in (t_{i-1}, t_i)$ for each i . Now in the above claim, there is no loss of generality in assuming that each I_s has length less than η . By the compactness of $[0, 1]$, there exist $(m+1)$ points $0 = s_0 < s_1 < \dots < s_m = 1$ whose corresponding intervals $\{I_1, \dots, I_m\}$ as guaranteed by the claim cover $[0, 1]$, and are

irredundant in the sense that $I_i \not\subset I_j$ for any $i \neq j$. It follows that $I_{i-1} \cap I_i \neq \emptyset$ for all i . Now pick $t_i \in I_{i-1} \cap I_i$ for each i so that $s_{i-1} < t_i < s_i$; note that $0 < t_1 < \dots < t_m < 1$ and $t_i - t_{i-1} < \eta$. By the *claim*, we have:

$$\begin{aligned} p(\gamma(s_i), \gamma(s_{i-1})) &\leq p(\gamma(s_i), \gamma(t_i)) + p(\gamma(t_i), \gamma(s_{i-1})) \\ &\leq (1 + \varepsilon) \{h(s_i)(s_i - t_i) + h(s_{i-1})(t_i - s_{i-1})\}. \end{aligned}$$

Account being taken of (3), it follows that

$$\begin{aligned} p(\gamma(0), \gamma(1)) &\leq \sum_i p(\gamma(s_i), \gamma(s_{i-1})) \\ &\leq (1 + \varepsilon) \sum_i h(s_i)(t_{i+1} - t_i) \\ &< (1 + \varepsilon) \left\{ \int_{\gamma} P + \varepsilon \right\}. \end{aligned}$$

Since ε is arbitrary, $p(\gamma(0), \gamma(1)) \leq \int_{\gamma} P$ for every C^∞ curve γ . If γ were a piecewise C^∞ curve instead, then the same argument applied to each C^∞ segment separately would yield the same inequality. By (11), this implies $p \leq p_1$.

Q. E. D.

The rest of this paper consists of applications of Theorem 3. An immediate one is the following proof of (B) at the beginning of this paper, namely, the theorem of Kobayashi-Sasaki ([K-S]). First a lemma:

LEMMA 5. Let $y_1(t)$ and $y_2(t)$ be solutions of $y'' + Qy = 0$ on \mathbf{R} which satisfy $y_1(0) = 0$, $y_1'(0) = 1$; $y_2(0) = 1$, $y_2'(0) = 0$. If $Q(t) \geq 0$ for all $t \in \mathbf{R}$, then there exists an interval $(-a, b) \subset \mathbf{R}$ such that the function $u(t) \equiv y_1(t)/y_2(t)$ maps $(-a, b)$ onto \mathbf{R} .

PROOF. It follows from the proof of Lemma 1 that $u(t)$ is strictly increasing. Consider $u(t)$ on $[0, \infty)$. If $Q(t) \equiv 0$ on $[0, \infty)$, u maps $[0, b)$ onto $[0, \infty)$, where $b = \infty$. If $Q(t_0) > 0$ for some $t_0 > 0$, then an elementary argument shows that $y_2(t)$ has a first zero at some $b > 0$. Sturm's separation theorem implies that $y_1(t) > 0$ on $(0, b)$. Thus in this case, u also maps $[0, b)$ onto $[0, \infty)$. Similarly, u maps some $(-a, 0]$ onto $(-\infty, 0]$.

Q. E. D.

PROOF OF (B). From the known relationship between equations (1) and (3), Lemma 5 implies that given any maximal geodesic $\gamma: \mathbf{R} \rightarrow M$ in the manifold M , there is a projective map $f: \mathbf{R} \rightarrow M$ such that $f(0) = \gamma(0)$. By (10), this means $P(\gamma(0), \dot{\gamma}(0)) = 0$. Since γ is arbitrary, $P \equiv 0$ on TM and hence $p \equiv 0$ on M (Theorem 3).

Q. E. D.

Given a manifold M with a torsionless affine connection, M (or more precisely, the affine connection) is *projective-hyperbolic* at $x \in M$ if and only if there exists a neighborhood U of x and a positive constant such that $P(y, Y) \geq \alpha |Y|$

for all $y \in U$, and all $Y \in M_y$, where $|Y|$ denotes the norm of Y relative to some Riemannian metric on M . With this defined, Theorem 2 of [R] on p. 133 (and essentially also its proof) is now valid for the projective pseudo-distance p . This includes the projective analogue of the theorem of Eisenman-Kiernan ([E], [Ki]) to the effect that hyperbolicity is equivalent to tightness (in the sense of [W-1]). It also includes the following:

LEMMA 6. *A manifold M with a torsionfree affine connection is projective-hyperbolic (i. e., p is a metric) if and only if M is projective-hyperbolic at each $x \in M$.*

The proof of Theorem 1 above implies the following fact:

LEMMA 7. *Let M be a manifold with a torsionless affine connection such that its Ricci tensor is negative semi-definite. If Ric is negative definite at some x , then M is projective-hyperbolic at x .*

From the proof of Theorem 1 and from (B) ([K-S]), we can deduce immediately:

PROPOSITION. *Let Γ_1 and Γ_2 be two torsionfree affine connections on a manifold M such that Γ_1 is complete and has positive semi-definite Ricci tensor while Γ_2 has negative semi-definite tensor which is not identically zero. Then Γ_1 and Γ_2 are not projectively equivalent.*

QUESTION. If M is a manifold with a complete torsionfree affine connection which is not projective-hyperbolic, does there exist an $(x, X) \in TM$ such that $P(x, X) = 0$? (Theorem 5 answers this affirmatively if M is in addition compact. In general, there is some heuristic evidence to show that the answer may be negative.)

§ 4.

This section is devoted to the proof of Theorems 2 and 5, both being concerned with compact manifolds.

PROOF OF THEOREM 2. According to an observation in [W-2], the hypothesis of the theorem implies that the group G of projective transformations of M is discrete; this is a mild improvement of a theorem of Couty ([C], p. 209) to the effect that the same conclusion holds if everywhere negative Ricci curvature is assumed. It thus suffices to prove that G is compact.

Let H be the points of M at which M is projective-hyperbolic (see definition at the end of the last section). By Lemma 7 and by the assumption of quasi-negative Ricci curvature, H is a nonempty open set. In view of Theorem 3, the restriction of the projective pseudo-distance p of M to H is a metric, i. e. for all $x, y \in H$, $p(x, y) = 0 \Rightarrow x = y$. An argument of [R] (specifically, (iv) \Rightarrow (v) in Theorem 2) furthermore proves that the restriction of p to H induces the usual topology on H . Let $x_0 \in H$ and let CH denote the complement of H in M .

If $p(x_0, CH) = 2\varepsilon$ for some positive ε , then define $H_0 = \{y \in H : p(y, CH) \geq \varepsilon\}$; H_0 is a closed subset of M with interior. We claim that the number of components of H_0 is finite.

If not, then there exists a sequence $\{x_i\}$ of points in M such that no two of them belong to the same component of H_0 . We may assume that x_i converges to some $x \in M$. Let γ be a piecewise C^∞ curve joining x_i to x_j ($i \neq j$) and let π_i be the first point on γ leaving the component of x_i and π_j be the last point on γ entering the component of x_j . Then the length of γ with respect to P between x_i and π_i is at least ε by the definition of H_0 ; similarly the length of γ from x_j to π_j is at least ε . Thus $\int_\gamma P \geq 2\varepsilon$ and hence $p(x_i, x_j) \geq 2\varepsilon$ if $i \neq j$ (Theorem 3). Now $P: TM \rightarrow [0, \infty)$ is upper semi-continuous and M is compact. Hence for some positive constant A , $P(y, Y) \leq A|Y|$, where $|Y|$ denotes the Riemannian norm of Y . Theorem 3 again tells us that $p(x, y) \leq Ad(x, y)$ for all $x, y \in M$, where d denotes the Riemannian distance function. For x as above, define $U = \{y \in M : d(y, x) \leq \varepsilon/(2A)\}$. Then for all $y, y' \in U$, $p(y, y') \leq Ad(y, y') \leq Ad(y, x) + Ad(y', x) \leq \varepsilon$. However, for all sufficiently large i, j , $x_i, x_j \in U$ while $p(x_i, x_j) \geq 2\varepsilon$. This contradiction proves the claim.

With respect to p , H_0 is then a compact metric space with a finite number of components. Its group of isometry $I(H_0)$ is therefore compact with respect to the compact-open topology (theorem of van Dantzig-van der Waerden; cf. Corollary 4.10 on p. 50 of [K-No]). Now if $f \in G$ (the group of projective transformations of M), then f preserves P and p , and hence $f(H_0) \subset H_0$. The restriction $f|_{H_0}$ is therefore an element of $I(H_0)$ and the map $\tau: G \rightarrow I(H_0)$ defined by $\tau(f) = f|_{H_0}$ is a continuous homomorphism. Let y be an interior point of H_0 ; then $\tau(f)$ determines the behavior of f at x up to any number of derivatives. Since the second order jet of a projective transformation at one point determines completely the transformation everywhere, $\tau(f)$ determines f and hence τ is injective. We are now in a position to prove that G is a compact group.

If G is noncompact, let $\{g_i\}$ be an infinite sequence of distinct elements of G . By deleting a subsequence if necessary, we may assume that $\tau(g_i)$ converges to some element in the compact group $I(H_0)$. In particular, the sequence $\{\tau(h_{ij})\}$ where $h_{ij} = g_i g_j^{-1}$ converges to the identity e of $I(H_0)$ in the sense that given a neighborhood N of e , there is an i_0 such that $i, j \geq i_0 \Rightarrow \tau(h_{ij}) \in N$. We claim that this implies h_{ij} converges in G to the identity e_0 of G , i.e., e_0 is the identity map $M \rightarrow M$. Assuming this for the moment, we see that (since G has the discrete topology) for some i_1 , $i, j \geq i_1 \Rightarrow h_{ij} = e_0$. Thus $g_i = g_{i_1}$ for all $i \geq i_1$. This contradicts the fact that the sequence $\{g_i\}$ is distinct. Hence G is compact.

To prove the preceding claim and hence the theorem, it suffices to show:

LEMMA 8. *Let M be a manifold with a torsionfree affine connection and let $\{f_i\}$ be a sequence of projective transformations. Suppose on a nonempty open set S of M , $\{f_i\}$ converges (in the compact-open topology) to the identity map of S . Then $\{f_i\}$ converges everywhere to the identity map of M .*

PROOF. In this proof only, we let P be the projective structure of M defined by the given affine connection; thus P is a principal subbundle of the second order frame bundle $P^2(M)$ ([Ko 2], p. 142). For each diffeomorphism $h: M \rightarrow M$, let j^2h denote the naturally induced map of $P^2(M) \rightarrow P^2(M)$ by h ([Ko 2], p. 139). Then $\{j^2f_i\}$ is a sequence of diffeomorphisms of P onto itself preserving the absolute parallelism of P defined by the normal projective connection on P ([Ko 2], p. 143); in the language of [Ko 2], each j^2f_i is an automorphism of the $\{1\}$ -structure of P defined by the connection forms. Moreover, the hypothesis of the lemma implies that j^2f_i converges to the identity map on $P|S$ (=the subset of P over S). It is straightforward to show that j^2f_i in fact converges to the identity map of P on all of P (cf. Lemma 1 on p. 15 of [Ko 2]). In particular, f_i converges to the identity map on M .

Q. E. D.

We remark that the proof of Theorem 2 also shows that the following is valid.

PROPOSITION. *Let M be a compact manifold with a torsionfree affine connection whose Ricci tensor is negative semidefinite and in addition negative definite at a point. Then the group of projective transformations (and in particular the group of affine transformations) is a compact Lie group.*

In the proof of Theorem 5 below, the key observation is Lemma 9. This lemma has had a rather curious history which deserves some comments. It first appeared in Landau's 1929 paper [L], § 4 on pp. 618–619, in the context of holomorphic functions on the unit disc. Apparently without being aware of Landau's work, Brody in his thesis [B] proved the same lemma for holomorphic mappings from the unit disc with the same technique and exploited it with great effectiveness in proving (among other things) the complex counterpart of Theorem 5 of the present paper. Independently and concurrently, Zalcman [Z] had arrived at this lemma for meromorphic functions on the unit disc again with the same proof as Landau, and used it to characterize normal families of meromorphic functions on the disc; with a trivial change of terminology, Zalcman's arguments would have proved Brody's theorem. However, the credit of fully realizing the scope of this method is indisputably Brody's. Zalcman was aware of Landau's work; see [Z] for further detail.

PROOF OF THEOREM 5. If there is (nonconstant) projective map $f: \mathbf{R} \rightarrow M$, then $P(f(t), \dot{f}(t)) \equiv 0$ for all t . Conversely, we shall prove that not being projective-hyperbolic implies the existence of such an f . Put some Riemannian

metric on T^*M and denote the norm of X in this metric by $|X|$. By the compactness of T^*M and the upper semi-continuity of P , there is a positive constant A such that :

$$(31) \quad P(x, X) \leq A|X| \quad \text{for all tangent vectors } X.$$

Suppose M is not projective-hyperbolic. By Lemma 6, there exists a sequence $\{(x_n, X_n)\}$ in TM such that $P(x_n, X_n) \rightarrow 0$. By (31) and (10), this implies the existence of a sequence of projective maps $f_n: I \rightarrow M$ such that

$$(32) \quad |\dot{f}_n(0)| > n \quad \text{for all } n.$$

We now need the following renormalization lemma.

LEMMA 9. *Let $h: I \rightarrow M$ be a projective map. If for some positive constant c , $|\dot{h}(0)| > c$, then there exists a projective map $k: I \rightarrow M$ such that: (i) $|\dot{k}(0)| = c$, (ii) $|\dot{k}(u)|(1-u^2) \leq c$ for all $u \in I$, and (iii) $k(I) \subset h(I)$.*

PROOF OF LEMMA 9. We follow §4 of [L] closely. Let $s < 1$. Replacing $h(u)$ by $h(su)$ for s sufficiently close to 1 if necessary, we may assume that h is defined on $[-1, 1]$ and that $|\dot{h}(0)| > c$ is still valid. Now for each $t \in [0, 1]$, define $h_t: I \rightarrow M$ by $h_t(u) = h(tu)$. Furthermore, define :

$$M_t = \max_{u \in I} |\dot{h}_t(u)|(1-u^2).$$

Since also $M_t = t \max |\dot{h}(tu)|(1-u^2)$, M_t is a continuous function of t . Moreover, $M_0 = 0$ and $M_1 = \max |\dot{h}| > c$. Hence $M_s = c$ for some $s \in (0, 1)$, and hence for some $u_0 \in I$,

$$c = M_s = |\dot{h}_s(u_0)|(1-u_0^2).$$

Let $\varphi: I \rightarrow I$ be defined by $\varphi(u) = (u + u_0)/(1 + uu_0)$ and let $k = h_s \circ \varphi$. Then $k: I \rightarrow M$, k is projective, and $(1-u^2)|\dot{k}(u)| = |\dot{h}_s(\varphi(u))||\varphi'(u)|(1-u^2) = |\dot{h}_s(\varphi(u))|(1-\varphi(u)^2) \leq M_s = c$. Also $|\dot{k}(0)| = |\dot{h}_s(u_0)|(1-u_0^2) = M_s = c$. Q. E. D.

To continue from (32), define for each n a map $f_n^*: (-n, n) \rightarrow M$ by $f_n^*(v) = f_n(v/n)$. Then $|f_n^*(0)| > 1$. By Lemma 9, there exists a sequence g_n such that $g_n: (-n, n) \rightarrow M$, g_n is projective, $|\dot{g}_n(0)| = 1$ and $|\dot{g}_n(v)| \left(\frac{n^2 - v^2}{n^2} \right) \leq 1$ for all $v \in (-n, n)$. Following the scheme in [B], we shall show that on each closed interval $[-n, n]$ ($n \geq 1$), a subsequence of $\{g_n\}$ converges uniformly to a projective map. Repeating the argument successively for $[-1, 1]$, $[-2, 2]$, ... and passing to a subsequence at each step, we obtain a sequence of projective maps which converges uniformly on compact subsets to a projective map from \mathbf{R} to M . It suffices to do this in detail for $[-1, 1]$. To this end, we begin with the larger interval $[-2, 2]$.

On $[-2, 2]$, $|g_n(0)| = 1$ for each n and, considering only those $n \geq 3$ from now on, we get :

$$(33) \quad |\dot{g}_n(v)| \leq 2 \quad \text{for all } n, \text{ for all } v \in [-2, 2].$$

Thus $\{g_n\}$ is an equi-continuous family on $[-2, 2]$ and therefore a subsequence $\{g_i\}$ converges uniformly to some $g: [-2, 2] \rightarrow M$. We will show that g is a projective map on $[-1, 1]$ (and hence C^∞ there).

By the definition of a projective map, there exists for each i a geodesic $\xi_i: \mathbf{R} \rightarrow M$ such that $g_i([-2, 2]) \subset \xi_i(\mathbf{R})$ (ξ_i is defined on \mathbf{R} because the connection is complete). Normalize ξ_i by: if $x_i \equiv g_i(0)$, then $\xi_i(0) = x_i$ and $|\dot{\xi}_i(0)| = 1$. Let $x \equiv g(0)$. Then $x_i \rightarrow x$ and (again passing to subsequence if necessary) $\dot{\xi}_i(0)$ converges to some $X \in M_x$ such that $|X| = 1$. Let $\xi: \mathbf{R} \rightarrow M$ be the maximal geodesic such that $\dot{\xi}(0) = X$. By the C^∞ dependence of solutions of ordinary differential equations on initial data, $\xi_i \rightarrow \xi$ (C^∞) uniformly on compact sets (the symbol (C^∞) is to signify that the convergence is for all derivatives). For each i , define the intervals $[-a_i, b_i] \subset \mathbf{R}$ by: a_i, b_i are the smallest numbers such that $g_i([-2, 2]) = \xi_i([-a_i, b_i])$. Since $\xi_i \rightarrow \xi$ uniformly on compact sets, $g([-2, 2]) \subset \xi(\mathbf{R})$. Thus we can similarly define $[a, b]$ such that $g([-2, 2]) = \xi([-a, b])$. It follows that $a_i \rightarrow a$ and $b_i \rightarrow b$. Given an $\varepsilon > 0$, there exists an i_0 such that

$$(34) \quad [-a + \varepsilon, b - \varepsilon] \subset [-a_i, b_i] \quad \text{for all } i \geq i_0.$$

Let v_i be the projective parameter on ξ_i corresponding to g_i . Thus $v_i: [-a_i, b_i] \rightarrow [-2, 2]$, $v_i(0) = 0$, $|v'_i(0)| = 1$ and for each $t \in [-a + \varepsilon, b - \varepsilon]$,

$$|v'_i(t)| = |\dot{\xi}_i(t)| / |\dot{g}_i(v_i(t))|$$

(cf. (14)). Note that the set $K \equiv \{(\xi_i(t), \dot{\xi}_i(t)) : \text{all } i \text{ and all } t \in [-a + \varepsilon, b - \varepsilon]\}$ is a relatively compact subset of TM because $\xi_i \rightarrow \xi$ (C^∞) uniformly on $[-a + \varepsilon, b - \varepsilon]$. Thus there is a $B > 0$ such that $|\dot{\xi}_i(t)| \geq B$ for all $i \geq i_0$ and $t \in [-a + \varepsilon, b - \varepsilon]$. In view of (33), we have:

$$(35) \quad |v'_i(t)| \geq B/2 \quad \text{for all } i \geq i_0, t \in [-a + \varepsilon, b - \varepsilon].$$

As in the proof of Lemma 1, each $v_i(t)$ can be expressed as a quotient $\alpha_i(t)/\beta_i(t)$, where $\alpha_i(t)$ and $\beta_i(t)$ are solutions of:

$$(36) \quad y''(t) + \frac{1}{(n-1)} \text{Ric}(\dot{\xi}_i(t), \dot{\xi}_i(t))y(t) = 0.$$

Normalize β_i by setting $\beta_i(0) = 1$. Then $v_i(0) = 0$ implies $\alpha_i(0) = 0$. A simple calculation (cf. [H], p. 648) leads to $\alpha'_i(0) = 1$ and $\beta'_i(0) = -v''_i(0)$. We wish to show that α_i and β_i are separately convergent. The case of β_i is more difficult so we do it first. By the argument after equation (4), $v'_i(t) = 1/\beta_i(t)^2$. By (35), we get:

$$(37) \quad |\beta_i(t)| \leq \sqrt{2}/\sqrt{B} \quad \text{for all } i \geq i_0, t \in [-a+\varepsilon, b-\varepsilon].$$

We estimate β_i'' on $[-a+\varepsilon, b-\varepsilon]$ as follows. Let C be the upper bound of the function $X \rightarrow |\text{Ric}(X, X)|$ on the relatively compact set K above. Since each β_i satisfies (36), we obtain:

$$(38) \quad |\beta_i''(t)| \leq \frac{\sqrt{2}C}{(n-1)\sqrt{B}} \quad \text{for all } i \geq i_0, t \in [-a+\varepsilon, b-\varepsilon].$$

Now let $\|h\|$ denote the sup norm of any function on $[-a+\varepsilon, b-\varepsilon]$. Denote the length of $[-a+\varepsilon, b-\varepsilon]$ by l . Then an elementary argument using the Taylor expansion gives:

$$\|\beta_i'\| \leq \frac{1}{l} \left\{ 2\|\beta_i\| + \frac{1}{2}l^2\|\beta_i''\| \right\}.$$

Together with (37) and (38), this implies that $\{\beta_i\}$ is a sequence with uniformly bounded first derivatives on $[-a+\varepsilon, b-\varepsilon]$; in particular, $\{\beta_i\}$ is an equi-continuous family. Since $\beta_i(0)=1$, a subsequence of $\{\beta_i\}$ (to be still denoted by $\{\beta_i\}$) is uniformly convergent on $[-a+2\varepsilon, b-2\varepsilon]$ to a continuous function β .

Since $\beta_i'' + \frac{1}{(n-1)}\text{Ric}(\dot{\xi}_i, \dot{\xi}_i)\beta_i = 0$, taking limit as $i \rightarrow \infty$ in the space of distributions yields:

$$(39) \quad \beta'' + \frac{1}{(n-1)}\text{Ric}(\dot{\xi}, \dot{\xi})\beta = 0$$

in the sense of distributions. Since $\text{Ric}(\dot{\xi}, \dot{\xi})$ is a C^∞ function of t , the regularity theorem of ordinary differential equations implies β is C^∞ . Note that $\beta(0)=1$.

We claim: β is nowhere zero on $[-a+3\varepsilon, b-3\varepsilon]$. Suppose β vanishes at $x_0 \in [-a+3\varepsilon, b-3\varepsilon]$. Since β is not the zero solution of (39), $\beta'(x_0) \neq 0$. Thus β changes sign near x_0 . Since $\beta_i \rightarrow \beta$ uniformly on the larger interval $[-a+2\varepsilon, b-2\varepsilon]$, each β_i also changes sign near x_0 for i large. Thus for such i , β_i vanishes in $(-a+2\varepsilon, b-2\varepsilon)$, which implies that $v_i (= \alpha_i/\beta_i)$ is not C^∞ in $(-a+2\varepsilon, b-2\varepsilon)$ because α_i is nonzero at the points where β_i vanishes. This is a contradiction and the claim is proved. Thus β is a positive C^∞ function on $[-a+3\varepsilon, b-3\varepsilon]$.

Finally, define $\alpha : [-a+2\varepsilon, b-2\varepsilon] \rightarrow \mathbf{R}$ as the solution of $y'' + \frac{1}{(n-1)}\text{Ric}(\dot{\xi}, \dot{\xi})y = 0$ with initial condition $\alpha(0)=0$, $\alpha'(0)=1$. Recall that each α_i satisfies (36) with the same initial condition $\alpha_i(0)=0$, $\alpha_i'(0)=1$ for all $i \geq i_0$. Since $\text{Ric}(\dot{\xi}_i, \dot{\xi}_i) \rightarrow \text{Ric}(\dot{\xi}, \dot{\xi})$ (C^∞) uniformly on $[-a+2\varepsilon, b+2\varepsilon]$, it follows from the basic theory of ordinary differential equations that $\alpha_i \rightarrow \alpha$ (C^∞) uniformly on the smaller interval $[-a+3\varepsilon, b-3\varepsilon]$. Now define $v = \alpha/\beta$. Then $v : [-a+3\varepsilon, b-3\varepsilon] \rightarrow [-2, 2]$, v

is a projective parameter on ξ (Lemma 1) and $v_i \rightarrow v$ (C^∞) uniformly on $[-a+3\varepsilon, b-3\varepsilon]$. Since for all $i \geq i_0$, $v_i([-a_i, b_i]) = [-2, 2]$, (34) implies that for ε sufficiently small, $v_i([-a+3\varepsilon, b-3\varepsilon]) \supset [-3/2, 3/2]$. In particular, the projective map g^* which corresponds to v is defined on $[-1, 1]$, and each g_i of the subsequence of (33) converges uniformly to g^* on $[-1, 1]$. By uniqueness of limit, g^* equals the g previously defined. Thus we have proved that the uniform limit of $\{g_i\}$ is a projective map on $[-1, 1]$. Q. E. D.

REMARK. The preceding proof in fact also proves the following theorem: if $P(x, X) = 0$ for some $(x, X) \in TM$ and if the hypotheses of Theorem 5 are in force, then there is a projective map $f: \mathbf{R} \rightarrow M$ such that $f(\mathbf{R})$ lies in the unique maximal geodesic through x and tangent to X .

§ 5.

We finally prove Theorem 4. This requires a sequence of lemmas. It may be remarked that a proof of part (ii) will be given without using any kind of Schwarz lemma.

LEMMA 10. Let $\varphi: \mathbf{R} \rightarrow [0, \infty)$ be a given function which is C^∞ , even (i. e. $\varphi(t) = \varphi(-t)$ for all t) and not identically zero, and let u be the natural parameter on I . Then there exists a unique C^∞ function $s: I \rightarrow \mathbf{R}$ such that: (i) $s(I) = \mathbf{R}$, (ii) $s' > 0$, (iii) s is odd, i. e. $s(-u) = -s(u)$ and (iv) $\{s, u\} = \varphi(s) (ds/du)^2$.

PROOF. Let s denote the natural parameter of \mathbf{R} . Let y_1, y_2 be the solutions of $y''(s) - \frac{1}{2}\varphi(s)y(s) = 0$ satisfying $y_1(0) = 0, y_1'(0) = 1; y_2(0) = 1, y_2'(0) = 0$. Define $u_0(s) = y_1(s)/y_2(s)$. Then $u_0(0) = 0$ and $u_0'(s) = 1/y_2(s)^2$ (cf. (3), (4), (22)). Since $\varphi \geq 0, y_2 \geq 1$. Hence u_0 is a C^∞ function on \mathbf{R} such that $u_0' > 0$. The same argument which led to (24) also shows that $y_2(s) > \varepsilon s$ on $[a, \infty)$, where a, ε are some positive constants. Thus for all $s \geq 0; u_0(s) = \int_0^s u_0' < \int_0^\infty u_0' = \int_0^\infty 1/y_2^2 \equiv A < \infty$ for some $A > 0$. By the evenness of $\varphi, u_0(s) > -A$ for all $s \leq 0$. Note also that $\{u_0, s\} = -\varphi$.

Now define $u = u_0/A$. Thus $u: \mathbf{R} \rightarrow I$ and u has the following properties:

$$(40) \quad u(\mathbf{R}) = I, u' > 0, u(0) = 0, \{u, s\} = -\varphi \text{ and } u(s) = -u(-s).$$

Let $s: I \rightarrow \mathbf{R}$ denote the inverse function of u . Then this s satisfies all the assertions of the lemma in view of

$$(41) \quad \{s, u\} = -\{u, s\} (ds/du)^2.$$

It remains to prove the uniqueness of s . We shall prove the equivalent statement that any $u: \mathbf{R} \rightarrow I$ satisfying (40) is unique. Since $\{u, s\} = -\varphi$, we may write $u = y_1/y_2$ where y_1 and y_2 are solutions of $y'' - \frac{1}{2}\varphi y = 0$. Normalize

y_2 by setting $y_2(0)=1$. Then (cf. (36) and the argument following it): $y_1(0)=0$, and $y_2'(0)=-u''(0)=0$ because u is odd. This shows that $y_2(0)$ and $y_2'(0)$ are determined by (40) and hence so is y_2 itself. Suppose we can show $y_1'(0)$ is determined by y_2 , then coupled with $y_1(0)=0$, this would show y_1 is determined by (40) and hence u itself would be determined by (40). To compute $y_1'(0)$, we follow the argument after (4) to conclude that $u'(s)=y_1'(0)/y_2(s)^2$. Hence for $s \geq 0$, $u(s)=\int_0^s [y_1'(0)/y_2(t)^2] dt$. Since $u(s) \uparrow 1$ as $s \uparrow 1$, $y_1'(0)=\left(\int_0^1 1/y_2^2\right)^{-1}$. Q. E. D.

The following is a direct consequence of Lemma 10 and a simple computation; it brings out the special property of the metric dI^2 in (5) and (6).

LEMMA 11. *The function which is the odd extension to I of the function $s_0 = \frac{1}{2} \log \frac{1+u}{1-u}$ on $[0, 1)$ is the unique solution in Lemma 10 corresponding to $\varphi \equiv 2$. Furthermore, $dI^2 = [s_0'(u)du]^2$.*

The next lemma is the projective counterpart of Proposition 5.7 in [G-W 2].

LEMMA 12 (Schwarz Lemma I). *Let $s: I \rightarrow \mathbf{R}$ be the unique solution in Lemma 10 corresponding to $\varphi(s) = \frac{A}{1+s^2}$, where A is a positive constant. Let $t: I \rightarrow \mathbf{R}$ satisfy $t(0)=0$, $t'(u)>0$, and $\{t, u\} \geq \frac{A}{1+t^2} \left(\frac{dt}{du}\right)^2$. Then $t' \leq s'$.*

PROOF. Define $h=t'/s'$; we must show $h \leq 1$. Using the standard trick originating with Ahlfors (cf. [Ko 1], p. 4 or [G-W 2], proof of Proposition 5.7), we may assume that h assumes an interior maximum at $u_0 \in I$. Hence $(\log h)'(u_0)=0$ and $(\log h)''(u_0) \leq 0$. It follows that at u_0 :

$$0 = (\log h)' = t''/t' - s''/s',$$

$$\begin{aligned} 0 &\geq (\log h)'' = \{t, u\} - \{s, u\} + \frac{1}{2}(t''/t')^2 - \frac{1}{2}(s''/s')^2 \\ &= \{t, u\} - \{s, u\} \\ &= \frac{A(t')^2}{1+t^2} - \frac{A(s')^2}{1+s^2}, \end{aligned}$$

so that $(t')^2/(1+t^2) \leq (s')^2/(1+s^2)$ at u_0 . Suppose $h(u_0)=a>1$. Then at u_0 : $a^2/(1+t^2) \leq 1/(1+s^2)$, $\Rightarrow 1+a^2s^2 < 1+t^2$, $\Rightarrow a^2s(u_0)^2 < t(u_0)^2$. But since $t' \leq as'$ on I and $f(0)=s(0)=0$, $t^2 \leq a^2s^2$ on I . This is a contradiction. So $a \leq 1$ and $t' \leq s'$. Q. E. D.

LEMMA 13 (Schwarz Lemma II). *Let $t: I \rightarrow \mathbf{R}$ satisfy $t(0)=0$, $t'(u)>0$ and $\{u, t\} \leq -A^2$. Then $t'(u) \leq \frac{\sqrt{2}}{A(1-u^2)}$ for all $u \in I$.*

PROOF. This is essentially proved in (4.1) of [Ko 4]. In the present framework, the hypothesis implies $\{t, u\} \geq A^2(dt/du)^2$ (cf. (41)). Also, the odd func-

tion $s_1(u)$ which equals $\frac{1}{\sqrt{2}A} \log \frac{1+u}{1-u}$ on $[1, 0)$ is the solution in Lemma 10 corresponding to $\varphi \equiv A^2$. Then exactly the same proof as in Lemma 12 applies here. Q. E. D.

We are now in a position to prove Theorem 4. First prove part (i). Suppose ρ denotes distance from a fixed $0 \in M$ and by hypothesis:

$$(41) \quad \text{Ric}(x) \leq \frac{-A}{1+\rho(x)^2} \quad \text{for all } x \in M.$$

Let $x \in M$ and let $X \in M_x$, $|X|=1$. Let $\gamma: J \rightarrow M$ be the geodesic such that $\dot{\gamma}(0)=X$. If $f: I \rightarrow M$ is a projective map such that $df(V)=X$, where V is a tangent vector at the origin 0 of I , then according to (10) it suffices to prove that for some positive constant α , α independent of (x, X) :

$$|V| \geq \frac{\alpha}{(1+\rho(x)^2)^{1/2}}.$$

Let u be the projective parameter corresponding to f and let t denote the affine parameter of γ . Then the function $t(u)$ satisfies: $t(0)=0$, $t' > 0$ and $|V| = u'(0) = 1/t'(0)$. Thus it reduces to proving:

$$(42) \quad t'(0) \leq \beta(1+\rho(x)^2)^{1/2}$$

for some positive constant β which is independent of (x, X) . Now $\{u, t\} = \frac{2}{(n-1)} \text{Ric}(\dot{\gamma}(t), \dot{\gamma}(t))$. Define a constant $\lambda \equiv (1+2\rho(x)^2)$ and let $t_0(u) \equiv \sqrt{2/\lambda} t(u)$. We claim:

$$(43) \quad \{t_0, u\} \geq \frac{A/(n-1)}{1+t_0^2} \left(\frac{dt_0}{du} \right)^2.$$

The idea of the proof of (43) is essentially that used in proving Theorem E of [G-W 2], and it goes as follows. Since the triangle inequality implies $\rho(\gamma(t))^2 \leq 2[\rho(x)^2 + d(\gamma(0), \gamma(t))^2] \leq 2\rho(x)^2 + 2t^2$, we have from (41) that

$$\text{Ric}(\dot{\gamma}(t), \dot{\gamma}(t)) \leq \frac{-A}{\lambda + 2t^2} \leq \frac{-A/\lambda}{1+t_0^2}.$$

Thus $\{t_0, u\} = \{t, u\} = -\{u, t\} (dt/du)^2$ (by (41))

$$\begin{aligned} &= \frac{2}{(n-1)} \text{Ric}(\dot{\gamma}(t), \dot{\gamma}(t)) \left(\frac{dt}{dt_0} \right)^2 \left(\frac{dt_0}{du} \right)^2 \\ &\geq \frac{A/(n-1)}{1+t_0^2} \left(\frac{dt_0}{du} \right)^2, \end{aligned}$$

which is (43).

Now according to Lemma 12, $t'_0(u) \leq s'(u)$, where $s(u)$ is the unique solution in Lemma 10 corresponding to $\varphi(s) = \frac{A/(n-1)}{(1+s^2)}$. This function $s(u)$ is of course independent of (x, X) . Thus $t'(0) \leq \frac{1}{\sqrt{2}}(1+2\rho(x)^2)^{1/2}s'(0) \leq s'(0)(1+\rho(x)^2)^{1/2}$. This is (42) with $\beta = s'(0)$.

To prove part (ii) of Theorem 4, first assume $\text{Ric}(x) \leq -A^2$ for all $x \in M$. Standard arguments using the Schwarz Lemma II (Lemma 13) prove immediately that $P(x, X) \geq A/\sqrt{n-1}$ for all unit vector X . It may be of some interest to prove this inequality directly without recourse to the Schwarz lemma. Thus let $X \in M_x$, $|X|=1$, and let $\gamma: (-a, b) \rightarrow M$ be the maximal geodesic such that $\dot{\gamma}(0) = X$. If y_1, y_2 are solutions of $y'' + \frac{1}{(n-1)}\text{Ric}(\dot{\gamma}, \dot{\gamma})y = 0$ satisfying $y_1(0) = 0$, $y'_1(0) = 1$, $y_2(0) = 1$, $y'_2(0) = 0$, then the function $u(t) \equiv y_1(t)/y_2(t)$ on $(-a, b)$ is a projective parameter of γ such that $u(0) = 0$, $u'(0) = 1$ and $u'(t) = 1/y_2(t)^2$ for all $t \in (-a, b)$. Compare y_2 with the solution $z(t)$ of $z'' - \frac{1}{(n-1)}A^2z = 0$ which satisfies $z(0) = 1$, $z'(0) = 0$; then the inequality $\text{Ric}(x) \leq -A^2$ implies that $y_2(t) \geq z(t)$ for all $t \geq 0$. Since $z(t) = \cosh\left(\frac{A}{\sqrt{n-1}}t\right)$, we see that for all $t \geq 0$,

$$\begin{aligned} u(t) &= \int_0^t 1/y_2^2 \leq \int_0^t 1/z^2 \\ &\leq \int_0^\infty \text{sech}^2\left(\frac{A}{\sqrt{n-1}}s\right) ds = \frac{\sqrt{n-1}}{A}. \end{aligned}$$

A similar argument for $t \leq 0$ shows that $u(t) \geq -\sqrt{n-1}/A$ for $t \leq 0$. Thus $|u(t)| \leq \sqrt{n-1}/A$. By Lemma 4, $P(x, X) = P(\gamma(0), \dot{\gamma}(0)) \geq A/\sqrt{n-1}$, as desired.

Finally, assume M is complete and $\text{Ric}(x) \geq -B^2$ for all $x \in M$, and we must show $P(x, X) \leq B/\sqrt{n-1}$ for every unit vector $X \in M_x$, for every $x \in M$. Let $\gamma: \mathbf{R} \rightarrow M$ be the geodesic such that $\gamma(0) = X$ and let $f: I \rightarrow M$ be a projective map such that $df(V) = X$ and $f(I) \subset \gamma(\mathbf{R})$, where V is tangent to I at 0. Then $|V| = u'(0)$, where u is the projective parameter on γ corresponding to f . We may therefore reformulate (10) as $P(x, X) = \inf |u'(0)|$, where the infimum is taken over all projective parameters u on γ such that $u(0) = 0$ and $u(\mathbf{R}) \supset I$. It suffices to produce *one* such u which satisfies $|u'(0)| \leq B\sqrt{n-1}$. First define a projective parameter $u_0 = y_1/y_2$, where y_1 and y_2 are solutions of $y'' + \frac{1}{(n-1)}\text{Ric}(\dot{\gamma}, \dot{\gamma})y = 0$ which satisfy $y_1(0) = 0$, $y'_1(0) = 1$, $y_2(0) = 1$, $y'_2(0) = 0$. Then $u_0(0) = 0$ and $u'_0 = 1/y_2^2$, so that as $t \uparrow \infty$, $u_0(t) \uparrow \int_0^\infty 1/y_2^2$. Comparing y_2 with the solution ζ of $\zeta'' - \frac{1}{(n-1)}B^2\zeta = 0$ which satisfies $\zeta(0) = 1$, $\zeta'(0) = 0$ and using the

inequality $\text{Ric}(x) \geq -B^2$, we obtain as in the last paragraph: $u_0(t) \uparrow \int_0^\infty 1/y_2^2 \geq \int_0^\infty \text{sech}^2 \frac{B}{\sqrt{n-1}} t dt = \sqrt{n-1}/B$, as $t \uparrow \infty$. Similarly, as $t \downarrow -\infty$, $u_0(t) \downarrow -\int_0^\infty 1/y_2^2 \geq -\sqrt{n-1}/B$. Define now the projective parameter u on γ by $u(t) = (B/\sqrt{n-1})u_0(t)$. Then $u(\mathbf{R}) \supset I$ and $u'(0) = B/\sqrt{n-1}$ because $u'_0(0) = 1$.

Q. E. D.

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