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On conformal diffeomorphisms between complete product Riemannian manifolds

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Introduction.

Several authors have been concerned with a problem:

Does there globally exist a non-homothetic conformal diffeomorphism between complete product Riemannian manifolds of dimension $n \geqq 3$?

Since there are conformally flat product Riemannian manifolds in the local, cf. M. Kurita [1], it is assured that a conformal diffeomorphism locally exists between two of such manifolds. On the other hand, N. Tanaka [4], T. Nagano [2], Y. Tashiro and K. Miyashita [8] showed the non-existence of such global diffeomorphism between complete Riemannian manifolds with parallel Ricci tensor, and the non-existence of infinitesimal conformal transformation generating a global 1-parameter group in a product Riemannian manifold was shown by S. Tachibana [3] in the case of compact manifold and by Y. Tashiro and K. Miyashita [7] in the case of complete manifold.

Let M and M^* be product Riemannian manifolds of dimension $n \geqq 3$, and denote the structures by (M, g, F) and (M^*, g^*, G) respectively. Under a diffeomorphism f of M to M^*, the image of a quantity on M^* to M by the induced map f^* of f will be denoted by the same letter as the original one. For example, we write g^* for f^*g^* and G for f^*G on M. If $FG=GF$ at a point $\mathrm{P} \in M$, then we say that the structures F and G are *commutative* at P with one another under f.

A purpose of the present paper is to establish the following

THEOREM. *There is no global conformal diffeomorphism between complete product Riemannian manifolds* M *and* M^* *such that the product structures* F *and* G *are not commutative under it in a dense subset of* M.

Another purpose is to give an affirmative example of a global conformal diffeomorphism making the product structures commutative.

By virtue of the well known de Rham decomposition theorem, the productness of manifolds in the theorem can be replaced by *reducibility* of manifolds by considering the universal covering spaces of them.

After preliminaries on product structures and conformal diffeomorphisms in

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Paragraph 1, we shall obtain lemmas concerning the commutativity of structures and the scalar field ρ associated with a conformal diffeomorphism in Paragraph 2. In the three paragraphs following, we shall derive various equations on the field ρ. Paragraph 6 is devoted to a proof of the theorem. In the seventh and last paragraph, we shall give an example of a global conformal diffeomorphism between Riemannian manifolds constructed on a 3-dimensional torus.

1. Product Riemannian structures and conformal diffeomorphism.

Throughout this paper, we assume that manifolds are connected, the dimension n is $\geqq 3$ and the differentiability is of class C^∞. For indicating components of tensors, Greek indices run on the range 1 to n.

Let M and M^* be n-dimensional product Riemannian manifolds $M=M_1\times M_2$ and $M^*=M_1^*\times M_2^*$ with structure (M, g, F) and (M^*, g^*, G), where g and g^* are the metric tensors and F and G the product structures of M and M^* respectively. The product structures F and G are by definition (1, 1)-tensor fields $(F_\lambda{}^\kappa)$ and $(G_\lambda{}^\kappa)$ different from the unit tensor $I=(\delta_\lambda^\kappa)$ and satisfying $F^2=I$ and $G^2=I$. The dimensions n_1 and n_2 $(n_1+n_2=n)$ of the parts M_1 and M_2 may be different from those of M_1^* and M_2^*. The covariant differentiations in M and M^* will be denoted by ∇ and ∇^* respectively.

Conditions for the structures (M, g, F) and (M^*, g^*, G) to be almost product Riemannian are

$$(1.1)\qquad g_{\nu\mu}F_\lambda{}^\nu F_\kappa{}^\mu=g_{\lambda\kappa}\,,\qquad g^*_{\nu\mu}G_\lambda{}^\nu G_\kappa{}^\mu=g^*_{\lambda\kappa}\,,$$

and integrability conditions for them to be product Riemannian are

$$(1.2)\qquad \nabla_\mu F_\lambda{}^\kappa=0\,,\qquad \nabla^*_\mu G_\lambda{}^\kappa=0\,.$$

The covariant tensors $F_{\mu\lambda}$ and $G^*_{\mu\lambda}$ defined by $F_{\mu\lambda}=F_\mu{}^\kappa g_{\lambda\kappa}$ and $G^*_{\mu\lambda}=G_\mu{}^\kappa g^*_{\lambda\kappa}$ are symmetric and the conditions (1.2) are equivalent to

$$(1.3)\qquad \nabla_\mu F_{\lambda\kappa}=0\,,\qquad \nabla^*_\mu G^*_{\lambda\kappa}=0\,.$$

If there is given a diffeomorphism f of M to M^* and the product structures F and G are commutative under f, then we have the equation $FGF=G$ or

$$F_\nu{}^\mu G_\mu{}^\lambda F_\lambda{}^\kappa=G_\nu{}^\kappa\,,$$

which is equivalent to the purity of the tensor G with respect to the product structure F.

A conformal diffeomorphism f of M to M^* is characterized by the change

$$(1.4)\qquad g^*_{\mu\lambda}=\frac{1}{\rho^2}g_{\mu\lambda}$$

of the metric tensors, where ρ is a positive valued scalar field and said to be *associated* with the conformal diffeomorphism f. The definition adopted here of associated scalar field is the reciprocal of usual one, but it is of more convenience. We shall put $\rho_\lambda=\nabla_\lambda\rho$ and denote by Y the gradient vector field (ρ^κ) of ρ. Then the Christoffel symbol is transformed by the formula

$$\left\{{\kappa \atop \mu\lambda}\right\}^*=\left\{{\kappa \atop \mu\lambda}\right\}-\frac{1}{\rho}(\delta^\kappa_\mu\rho_\lambda+\delta^\kappa_\lambda\rho_\mu-g_{\mu\lambda}\rho^\kappa). \tag{1.5}$$

If the associated scalar field ρ satisfies the equation

$$\nabla_\mu\rho_\lambda=\phi g_{\mu\lambda}, \tag{1.6}$$

ϕ being a scalar field, then f carries Riemannian circles in M to those in M^* and is called a *concircular* diffeomorphism of M to M^*. More generally a scalar field ρ satisfying the equation of the form (1.6) will be called a *concircular scalar field*, and a *special concircular* one if the equation (1.6) is reduced to a special form

$$\nabla_\mu\rho_\lambda=(k\rho+b)g_{\mu\lambda},$$

k and b being constants. As to properties of concircular scalar fields, refer to [**5**] or [**6**].

Under a conformal diffeomorphism f of M to M^*, we have the equations

$$G_\mu{}^\lambda G_\lambda{}^\kappa=\delta^\kappa_\mu, \qquad g_{\nu\mu}G_\lambda{}^\nu G_\kappa{}^\mu=g_{\lambda\kappa}, \tag{1.7}$$

which mean that the induced tensor G constitutes an almost product Riemannian structure together with the metric g on M but not necessarily integrable. The covariant tensor $G_{\mu\lambda}$ defined by $G_{\mu\lambda}=G_\mu{}^\kappa g_{\lambda\kappa}$ is symmetric. Substituting $G^*_{\mu\lambda}=G_{\mu\lambda}/\rho^2$ into the second equation of (1.3) and using the transformation formula (1.5), we can obtain the differential equation

$$\nabla_\mu G_{\lambda\kappa}=-\frac{1}{\rho}(G_{\mu\lambda}\rho_\kappa+G_{\mu\kappa}\rho_\lambda-g_{\mu\lambda}G_{\kappa\omega}\rho^\omega-g_{\mu\kappa}G_{\lambda\omega}\rho^\omega). \tag{1.8}$$

Applying Ricci's formula to this equation and by straightforward computations, we have the equation

$$\begin{aligned}
&\rho(K_{\nu\mu\lambda}{}^\omega G_{\omega\kappa}+K_{\nu\mu\kappa}{}^\omega G_{\lambda\omega})\\
&\quad=G_{\mu\lambda}\nabla_\nu\rho_\kappa+G_{\mu\kappa}\nabla_\nu\rho_\lambda-G_{\nu\lambda}\nabla_\mu\rho_\kappa-G_{\nu\kappa}\nabla_\mu\rho_\lambda\\
&\qquad-g_{\mu\lambda}\left[(\nabla_\nu\rho^\omega)G_{\kappa\omega}-\frac{\Phi}{\rho}G_{\nu\kappa}\right]-g_{\mu\kappa}\left[(\nabla_\nu\rho^\omega)G_{\lambda\omega}-\frac{\Phi}{\rho}G_{\nu\lambda}\right]\\
&\qquad+g_{\nu\lambda}\left[(\nabla_\mu\rho^\omega)G_{\kappa\omega}-\frac{\Phi}{\rho}G_{\mu\kappa}\right]+g_{\nu\kappa}\left[(\nabla_\mu\rho^\omega)G_{\lambda\omega}-\frac{\Phi}{\rho}G_{\mu\lambda}\right],
\end{aligned} \tag{1.9}$$

where $K_{\nu\mu\lambda}{}^{\kappa}$ is the curvature tensor of M and Φ the square of the length of the gradient vector field $Y=(\rho^{\kappa})$:

$$\Phi=|Y|^2=\rho_{\kappa}\rho^{\kappa}. \tag{1.10}$$

The equation (1.8) is equivalent to the integrability condition of G in M^*, the equation (1.9) to the purity of the curvature tensor $K^*_{\nu\mu\lambda\kappa}$ of M^* with respect to the product structure G, and these equations play important roles in our discussions.

2. Separate coordinate system and lemmas.

In the following, Latin indices run on the ranges

$$h,\ i,\ j,\ k=1,\ 2,\ \cdots,\ n_1;$$

$$p,\ q,\ r,\ s=n_1+1,\ \cdots,\ n,$$

respectively. Summation convention is also adopted to repeated Latin indices over their own range unless otherwise is stated.

In the product Riemannian manifold $M=M_1\times M_2$, there is a local coordinate system (x^h, x^p), called a *separate coordinate system,* such that the metric form of M is expressed as

$$ds^2=g_{ji}(x^h)dx^jdx^i+g_{rq}(x^p)dx^rdx^q,$$

where (x^h) and (x^p) are local coordinate systems in the parts M_1 and M_2 and g_{ji} and g_{rq} are components of the metric tensors g_1 and g_2 of M_1 and M_2 respectively. The product structure F has components

$$\begin{pmatrix} \delta_i^h & 0 \\ 0 & -\delta_q^p \end{pmatrix}$$

with respect to such a coordinate system, to within the signature. The Christoffel symbol $\left\{ {\kappa \atop \mu\lambda} \right\}$ and the curvature tensor $K_{\kappa\mu\lambda}{}^{\kappa}$ of M have pure components only. The covariant differentiations ∇_i along M_1 and ∇_q along M_2 are commutative with one another.

We shall show the following lemma in the local:

LEMMA 1. *A conformal diffeomorphism f of a product Riemannian manifold (M, g, F) to (M^*, g^*, G) is a homothety if and only if the induced tensor G constitutes a product Riemannian structure together with g on M, that is,*

$$\nabla_{\mu}G_{\lambda\kappa}=0. \tag{2.1}$$

Then the structures F and G are commutative under f.

PROOF. If f is a homothety, then ρ is a constant and we have the equation (2.1) from (1.8). Conversely, if the equation (2.1) is satisfied, then the equation (1.8) contracted with ρ^{κ} gives

$$\Phi G_{\mu\lambda}-g_{\mu\lambda}G_{\kappa\omega}\rho^{\kappa}\rho^{\omega}=\rho_{\mu}G_{\lambda\omega}\rho^{\omega}-\rho_{\lambda}G_{\mu\omega}\rho^{\omega}.$$

Since the left hand side is symmetric in the indices λ and μ and the right hand side anti-symmetric, both of the sides are equal to 0 and we can see that $\rho_{\lambda}=0$ and ρ is a constant by account of $G\neq\pm I$.

Then we decompose M into the product of a number of irreducible parts. Taking account of the equation (2.1) on each part and the property $G^2=I$, we can see that the product structure G is a diagonal matrix having ± 1 as diagonal components with respect to a suitable separate coordinate system in M. Hence we have $FG=GF$. Q.E.D.

We shall denote by Y_1 and Y_2 the parts (ρ^h) and (ρ^p) of the gradient vector field $Y=(\rho^{\kappa})$ belonging to M_1 and M_2 respectively. The associated scalar field ρ may be a function depending on one or both of the parts. If ρ is independent of points of M_2, then $Y_2=(\rho^{\kappa})$ vanishes identically.

As many equations referred to the parts M_1 and M_2 will appear as pairs, we shall sometimes derive one of a pair and state the other without repeating similar arguments. When we refer an equation to a separate coordinate system and restrict indices to the parts, for example, $\kappa=\lambda=i$, $\mu=j$, $\nu=p$ in the equation (1.9), we indicate $(\kappa, \lambda, \mu, \nu)=(i, i, j, p)$.

LEMMA 2. *If the product structures F and G are commutative under a non-homothetic conformal diffeomorphism f, then the associated scalar field ρ is a function on either of the parts M_1 or M_2 only.*

PROOF. Since the structure G is pure with respect to the structure F, the hybrid components G_{pi} of G in a separate coordinate system in M all vanish. Putting the indices $(\kappa, \lambda, \mu, \nu)=(i, i, j, p)$ in (1.9), we have

$$G_{ji}\nabla_{p}\rho_{i}=g_{ji}(\nabla_{p}\rho^{h})G_{ih} \qquad \text{(not summed in } i).$$

If there would be a component $\nabla_{p}\rho_{i}\neq 0$ for fixed indices i and p, then we may put

$$G_{ji}=\gamma_{i}g_{ji} \qquad \text{(not summed in } i)$$

for any index j. If in addition $\nabla_{q}\rho_{h}\neq 0$ for fixed h and q, then we may put

$$G_{jh}=\gamma_{h}g_{jh} \qquad \text{(not summed in } h)$$

for any j, and the comparison of this expression with the above yields $\gamma_i=\gamma_h$ provided $g_{ih}\neq 0$. If $g_{ih}=0$, then $G_{ih}=0$, we put $(\kappa, \lambda, \mu, \nu)=(h, i, i, q)$ in (1.9) and obtain the relations

$$G_{ii}\nabla_q\rho_h=g_{ii}(\nabla_q\rho^l)G_{lh} \qquad \text{(not summed in } i)$$

or

$$\gamma_i g_{ii}\nabla_q\rho_h=\gamma_h g_{ii}\nabla_q\rho_h \qquad \text{(not summed in } h \text{ and } i),$$

from which we again see $\gamma_i=\gamma_h$. If $\nabla_q\rho_h=0$ for fixed h and any q, we put $(\kappa, \lambda, \mu, \nu)=(h, h, h, q)$ in (1.9) and have

$$(\nabla_q\rho^l)G_{lh}=0\,.$$

Moreover, putting $(\kappa, \lambda, \mu, \nu)=(h, i, j, p)$ in (1.9), we have

$$G_{jh}\nabla_p\rho_i=\gamma_i g_{jh}(\nabla_p\rho_i) \qquad \text{(not summed in } i)$$

and see $\gamma_i=\gamma_h$. Thus we may put

$$G_{ji}=\gamma_1 g_{ji}\,, \qquad G_{qp}=\gamma_2 g_{qp}$$

or

$$G_j{}^h=\gamma_1\delta_j^h\,, \qquad G_q{}^p=\gamma_2\delta_q^p\,.$$

By means of the property $G^2=I$, we see that $\gamma_1=\gamma_2=\pm1$ or $\gamma_1=-\gamma_2=\pm1$, and consequently the structure G is identical with $\pm I$ or $\pm F$, and f would be homothetic by means of Lemma 1. This is a contradiction. Therefore ρ should satisfy $\nabla_q\rho_i=0$ for all indices i and q and be decomposable in M.

Putting $(\kappa, \lambda, \mu)=(i, j, q)$ and (p, q, j) in (1.8), we have

$$\nabla_q G_{ji}=0\,, \qquad \nabla_j G_{qp}=0\,,$$

which means that, in a separate coordinate system, the components G_{ji} belonging to M_1 are independent of (x^p) and G_{qp} belonging to M_2 independent of (x^h). Putting $(\kappa, \lambda, \mu)=(h, i, j)$ in (1.8), we have

$$\rho\nabla_j G_{ih}=-(G_{ji}\rho_h+G_{jh}\rho_i-g_{ji}G_{hk}\rho^k-g_{jh}G_{ik}\rho^k)\,.$$

Since the right hand side is independent of (x^p), so is ρ if $\nabla_j G_{ih}\neq0$. If in addition $\nabla_r G_{qp}\neq0$, then we see that ρ is independent of (x^h) and consequently ρ is a constant. This contradicts the non-homothety of f. Therefore ρ should depend on one part, say M_1, only and we have $\nabla_j G_{ih}\neq0$ and $\nabla_r G_{qp}=0$. Q.E.D.

A converse to Lemma 2 is the following

LEMMA 3. *If the associated scalar field ρ depends on one part, say M_1, only but not a constant, then the structure G is commutative with F under f, or the field ρ is a special concircular one satisfying the equation*

$$\nabla_j\rho_i=c^2\rho g_{ji} \tag{2.2}$$

on M_1, where c is a positive constant.

PROOF. By the assumption, $\rho_p=0$ and $\nabla_p\rho_\lambda=0$. We suppose that G is not commutative with F, and a hybrid component G_{qi} does not vanish. Putting

$(\kappa, \lambda, \mu, \nu)=(i, i, j, q)$ in (1.9), we have

$$G_{qi}\left(\nabla_j \rho_i - \frac{\Phi}{\rho} g_{ji}\right)=0 \qquad \text{(not summed in } i\text{)}$$

and hence

$$\nabla_j \rho_i = \frac{\Phi}{\rho} g_{ji} \tag{2.3}$$

for a fixed i and any j. Putting again $(\kappa, \lambda, \mu, \nu)=(h, i, j, q)$, we can see the equation (2.3) is valid for any pair of indices i and j. By contracting the equation (2.3) with $2\rho^i$ and integrating, we have

$$\Phi = \rho_i \rho^i = c^2 \rho^2, \tag{2.4}$$

because $\rho^2>0$, $\Phi \geqq 0$ and $\Phi(\mathrm{P})>0$ at some point P. Substituting (2.4) into (2.3), we obtain the equation (2.2). Q. E. D.

We put the subsets

$$N_1=\{\mathrm{P} \mid Y_1(\mathrm{P})=0\},$$

$$N_2=\{\mathrm{P} \mid Y_2(\mathrm{P})=0\},$$

$$U=\{\mathrm{P} \mid Y_1(\mathrm{P}) \neq 0,\ Y_2(\mathrm{P}) \neq 0\},$$

$$V=\{\mathrm{P} \mid FG \neq GF \text{ at P}\}$$

in M. The subsets N_1 and N_2 are closed, U and V are open, and the inclusions $U=M-N_1 \cup N_2 \subset V \subset M-N_1 \cap N_2$ are obvious by means of Lemmas 1 and 2.

3. Equations in the subset U.

We suppose that the open subset U is not empty. As we first deal with equations in a connected component of U, we suppose that U itself is connected. The structures F and G are not commutative under f in U by virtue of Lemma 2. Hence G is not pure in U and the covariant tensor $G_{\mu\lambda}$ has non-vanishing hybrid components G_{pi} with respect to a separate coordinate system in U. We denote by G' the block of hybrid components G_{pi} of the matrix $(G_{\mu\lambda})$.

Putting $(\kappa, \lambda, \mu, \nu)=(i, i, p, q)$ and (p, p, i, j) in (1.9), we have the relations

$$\begin{cases} G_{pi} \nabla_q \rho_i = G_{qi} \nabla_p \rho_i & \text{(not summed in } i\text{)}, \\ G_{ip} \nabla_j \rho_p = G_{jp} \nabla_i \rho_p & \text{(not summed in } p\text{)}. \end{cases} \tag{3.1}$$

If a component G_{pi} for fixed indices i and p does not vanish, then it follows from the equation (3.1) that we may put

$$(3.2)\qquad \begin{cases} \nabla_q\rho_i=\phi_{pi}G_{qi} & \text{(not summed in } i), \\ \nabla_j\rho_p=\phi_{pi}G_{pi} & \text{(not summed in } p) \end{cases}$$

for any q and any j. Moreover, if there are non-zero components G_{qi} in the i-th row or G_{pj} in the p-th column of G', then the proportional factor ϕ_{pi} is common with the q-th column or the j-th row. By rearrangement of rows and columns of the matrix G' by this property, G' is divided into blocks such as

$$G'=\begin{pmatrix} G_1' & 0 & \cdots & 0 \\ 0 & G_2' & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

and the equations (3.2) are rewritten in the form

$$\nabla_q\rho_i=\phi_tG_{qi}\,, \qquad \nabla_p\rho_j=\phi_tG_{pj}$$

for the indices i and p belonging to the t-th block except the last null block. Consider, for example, the first and second blocks, and suppose that components G_{pi} in G_1' and G_{qj} in G_2' are not equal to zero. Then, putting $(\kappa, \lambda, \mu, \nu)=(j, i, p, q)$ in (1.9), we have

$$G_{pi}\nabla_q\rho_j-G_{qj}\nabla_p\rho_i=(\phi_2-\phi_1)G_{pi}G_{qj}=0$$

and hence $\phi_1=\phi_2$. If G_{qj} is a component in the last null block, then we have similarly

$$G_{pi}\nabla_q\rho_j=0 \qquad \text{or} \qquad \nabla_q\rho_j=0\,.$$

Therefore we may put

$$(3.3)\qquad \nabla_p\rho_i=\phi G_{pi}$$

with a proportional factor ϕ for all rows and columns of the block G'.

Putting $(\kappa, \lambda, \mu, \nu)=(i, i, j, q)$ in (1.9), we have

$$G_{ji}\nabla_q\rho_i-G_{qi}\nabla_j\rho_i=g_{ji}\left[(\nabla_q\rho^\omega)G_{\omega i}-\frac{\Phi}{\rho}G_{qi}\right]$$

(not summed in i),

or, taking account of (3.3),

$$(3.4)\qquad G_{qi}\left(\phi G_{ji}-\nabla_j\rho_i+\frac{\Phi}{\rho}g_{ji}\right)=g_{ji}(\nabla_q\rho_\omega)G^\omega{}_i$$

(not summed in i).

If the i-th row of G' contains a non-zero component G_{qi}, then we may put

(3.5) $$\nabla_j \rho_i = \phi_1 g_{ji} + \psi G_{ji}\,,$$

where ϕ_1 is a function satisfying the relation

(3.6) $$G_{qi}\left(-\phi_1 + \frac{\Phi}{\rho}\right) = (\nabla_q \rho^\omega) G_{\omega i}\,.$$

If a row, say the h-th, in the block G' has all zero components, then $\nabla_q \rho_h = 0$. Putting $(\kappa, \lambda, \mu, \nu) = (h, i, j, q)$ in (1.9), we have

$$G_{jh} \nabla_q \rho_i - G_{qi} \nabla_j \rho_h - g_{ji} (\nabla_q \rho^k) G_{kh} - g_{jh}\left[(\nabla_q \rho^\omega) G_{\omega i} - \frac{\Phi}{\rho} G_{qi}\right] = 0\,.$$

Substituting (3.3) and (3.6) into this equation, we can see the expression (3.5) also valid for the h-th row. Applying similar arguments to $\nabla_q \rho_p$ and gathering the equations (3.3) and (3.5), we have the system of differential equations

(3.7) $$\begin{cases} \nabla_j \rho_i = \phi_1 g_{ji} + \psi G_{ji}\,, \\ \nabla_q \rho_i = \qquad\quad \psi G_{qi}\,, \\ \nabla_q \rho_p = \phi_2 g_{qp} + \psi G_{qp} \end{cases}$$

on the associated scalar field ρ. We shall indicate the quotation of an equation of a system by a number following comma, e.g., (3.7, 1) for the first of the equations (3.7).

Substituting the equations (3.7) into (3.4), we have

$$\begin{aligned} G_{qi}\left(-\phi_1 + \frac{\Phi}{\rho}\right) g_{ji} &= g_{ji}[\psi G_{qh} G^h{}_i + (\phi_2 g_{qp} + \psi G_{qp}) G^p{}_i] \\ &= g_{ji}[\psi G_{q\omega} G^\omega{}_i + \phi_2 g_{qp} G^p{}_i] \\ &= \phi_2 g_{ji} G_{qi} \qquad \text{(not summed in } i) \end{aligned}$$

because of $G_{q\omega} G^\omega{}_i = g_{qi} = 0$, and consequently the factors ϕ_1 and ϕ_2 satisfy the relation

(3.8) $$\phi_1 + \phi_2 = \frac{\Phi}{\rho} = \frac{1}{\rho} \rho_\omega \rho^\omega\,.$$

By means of (3.7, 2) and (1.8), the covariant derivative $\nabla_q \nabla_j \rho_i$ is equal to

(3.9) $$\begin{aligned} \nabla_j \nabla_q \rho_i &= (\nabla_j \psi) G_{qi} + \psi \nabla_j G_{qi} \\ &= (\nabla_j \psi) G_{qi} - \frac{\psi}{\rho}(G_{ji} \rho_q + G_{jq} \rho_i - g_{ji} G_{q\omega} \rho^\omega)\,. \end{aligned}$$

Since $\nabla_q \nabla_j \rho_i$ is symmetric in i and j, we have

$$(\rho \nabla_j \psi + \psi \rho_j) G_{qi} = (\rho \nabla_i \psi + \psi \rho_i) G_{qj}$$

and this is rewritten as

$$(3.10)\qquad \begin{cases} \tau_j G_{qi} = \tau_i G_{qj}\,, \\ \tau_q G_{pi} = \tau_p G_{qi}\,, \end{cases}$$

where we have put

$$(3.11)\qquad \tau_\lambda = \nabla_\lambda \rho\phi\,.$$

On the other hand, deriving the equation (3.7, 1) in x^q and using (1.8), we obtain the expression

$$\nabla_q \nabla_j \rho_i = (\nabla_q \phi_1) g_{ji} + (\nabla_q \phi) G_{ji} - \frac{\phi}{\rho}(G_{qj}\rho_i + G_{qi}\rho_j)\,.$$

Equating this to (3.9), we have the relations

$$(3.12)\qquad \begin{cases} \tau_j G_{qi} - \tau_q G_{ji} = (\rho \nabla_q \phi_1 - \phi G_{q\omega}\rho^\omega) g_{ji}\,, \\ \tau_q G_{jp} - \tau_j G_{qp} = (\rho \nabla_j \phi_2 - \phi G_{j\omega}\rho^\omega) g_{qp}\,. \end{cases}$$

4. The vanishing of the vector field τ.

The gradient vector field (τ^κ) in U will be denoted by τ and the parts (τ^h) and (τ^p) belonging to M_1 and M_2 by τ_1 and τ_2 respectively. The aim of this paragraph is to show the following

LEMMA 4. *The gradient vector field τ identically vanishes in the subset U.*

PROOF. We suppose that the field τ would not vanish in an open subset U' in U. First consider the case where $\tau_2 = 0$ identically but $\tau_1 \neq 0$ in U'. Then, by means of (3.10, 1), we may put

$$G_{qi} = \mu_q \tau_i\,,$$

μ_q being proportional factors. From the equation (3.12, 1) follows the equation

$$\mu_q \tau_j \tau_i = (\rho \nabla_q \phi_1 - \phi G_{q\omega}\rho^\omega) g_{ji}\,.$$

Taking account of the rank of the metric tensor g_1, we have $\mu_q = 0$ and $G_{qi} = 0$ unless $n_1 = 1$. This contradicts our assumption. If $n_1 = 1$, then $n_2 \geqq 2$ and the metric tensor g is of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & g_{qp} \end{pmatrix}.$$

From (3.12, 2) substituted with $\tau_q = 0$, we may put $G_{qp} = \gamma_2 g_{qp}$. The product structure G is then given by

$$G = \begin{pmatrix} G_1{}^1 & G_q{}^1 \\ G_1{}^p & \gamma_2 \delta_q^p \end{pmatrix}.$$

Substituting these components into (1.7, 1), we can see that $G_q{}^1=0$, contradicting the non-purity of G in U. Thus it does not rise that either the part τ_1 or τ_2 of τ vanishes.

Consequently, by means of the equation (3.10), we may put

$$G_{qi}=\lambda\tau_q\tau_i\,, \tag{4.1}$$

λ being a non-zero proportional factor. Contracting the equations (3.12, 1) with τ^q and (3.12, 2) with τ^j, we may put

$$\begin{cases} G_{ji}=\gamma_1 g_{ji}+\lambda\tau_j\tau_i\,, \\ G_{qp}=\gamma_2 g_{qp}+\lambda\tau_q\tau_p\,, \end{cases} \tag{4.2}$$

γ_1 and γ_2 being proportional factors. Then the components of the product structure G are given by

$$\begin{pmatrix} G_i{}^h & G_q{}^h \\ G_i{}^p & G_q{}^p \end{pmatrix}=\begin{pmatrix} \gamma_1\delta_i^h+\lambda\tau_i\tau^h & \lambda\tau_q\tau^h \\ \lambda\tau_i\tau^p & \gamma_2\delta_q^p+\lambda\tau_q\tau^p \end{pmatrix}. \tag{4.3}$$

Substituting these components into the equation (1.7, 1), we have

$$\begin{cases} \gamma_1^2\delta_j^h+(2\gamma_1\lambda+\lambda^2|\tau|^2)\tau_j\tau^h=\delta_j^h\,, \\ (\gamma_1+\gamma_2+\lambda|\tau|^2)\tau_j\tau^p=0\,, \\ \gamma_2^2\delta_q^p+(2\gamma_2\lambda+\lambda^2|\tau|^2)\tau_q\tau^p=\delta_q^p\,, \end{cases}$$

where $|\tau|$ is the length of the gradient vector τ, i.e., $|\tau|^2=\tau_\omega\tau^\omega$. From these equations we see

$$\gamma_1=\gamma_2=\pm1\,,\qquad \lambda=\mp2/|\tau|^2$$

with double signs in same order.

If we take the unit vector field $v=(v^\kappa)$ in the direction of τ, the components $G_{\mu\lambda}$ are equal to

$$\begin{cases} G_{ji}=g_{ji}-2v_jv_i\,, \\ G_{qi}=\qquad -2v_qv_i\,, \\ G_{qp}=g_{qp}-2v_qv_p\,, \end{cases} \tag{4.4}$$

to within signature, or we obtain the tensor equation

$$G_{\mu\lambda}=g_{\mu\lambda}-2v_\mu v_\lambda\,. \tag{4.5}$$

Therefore the product structure G is of the form

$$G_\lambda{}^\kappa=\delta_\lambda^\kappa-2v_\lambda v^\kappa\,. \tag{4.6}$$

A vector field $\xi=(\xi^\kappa)$ defined by

$$\xi^\kappa=\rho v^\kappa$$

is unit with respect to the metric g^* of M^*. Denoting the covariant components of ξ with respect to the metrics g and g^* by ξ_λ and ξ^*_λ respectively, we have the relations

$$\xi_\lambda=\rho v_\lambda\,, \qquad \xi^*_\lambda=g^*_{\lambda\kappa}\xi^\kappa=\frac{1}{\rho}v_\lambda\,,$$

and the product structure G is written in the form

$$G_\lambda{}^\kappa=\delta^\kappa_\lambda-2\xi^*_\lambda\xi^\kappa\,.$$

The integrability condition (1.2, 2) of G in M^* is equivalent to

$$\nabla^*_\mu\xi^\kappa=0\,.$$

Substituting the transformation formula (1.5) into this equation, we obtain the equation

$$\nabla_\mu\xi^\kappa=\frac{1}{\rho}(\delta^\kappa_\mu\rho_\lambda\xi^\lambda+\rho_\mu\xi^\kappa-\xi_\mu\rho^\kappa)$$

or

$$\nabla_\mu\xi_\lambda=\rho_\mu v_\lambda-v_\mu\rho_\lambda+v^\omega\rho_\omega g_{\mu\lambda}\,. \tag{4.7}$$

It follows from this equation that

$$\nabla_\mu\xi_\lambda+\nabla_\lambda\xi_\mu=2\sigma g_{\mu\lambda}\,, \tag{4.8}$$

where we have put

$$\sigma=v^\omega\rho_\omega\,. \tag{4.9}$$

Hence the vector field ξ is an infinitesimal conformal transformation of the metric g in U'. It follows also from (4.7) that

$$\nabla_\mu v_\lambda=\frac{1}{\rho}(\sigma g_{\mu\lambda}-v_\mu\rho_\lambda)\,. \tag{4.10}$$

Substituting (4.4) into (3.7), we have the equations

$$\begin{cases} \nabla_j\rho_i=(\phi_1+\psi)g_{ji}-2\psi v_j v_i\,,\\ \nabla_q\rho_i=\qquad\qquad -2\psi v_q v_i\,,\\ \nabla_q\rho_p=(\phi_2+\psi)g_{qp}-2\psi v_q v_p\,. \end{cases} \tag{4.11}$$

Contraction of these equations with $2\rho^\lambda$ yields

$$\begin{cases} \nabla_j\Phi=2(\phi_1+\psi)\rho_j-4\sigma\psi v_j\,,\\ \nabla_q\Phi=2(\phi_2+\psi)\rho_q-4\sigma\psi v_q\,. \end{cases} \tag{4.12}$$

By use of (3.8), (4.10) and (4.11), the derivatives $\sigma_\lambda=\nabla_\lambda\sigma$ of σ given by (4.9) are equal to

$$\text{(4.13)}\qquad \begin{cases} \sigma_j=\dfrac{\sigma}{\rho}\rho_j-(\phi_2+\psi)v_j\,, \\ \sigma_q=\dfrac{\sigma}{\rho}\rho_q-(\phi_1+\psi)v_q\,, \end{cases}$$

and they are rewritten as

$$\text{(4.14)}\qquad \begin{cases} \nabla_j\dfrac{\sigma}{\rho}=-\dfrac{1}{\rho}(\phi_2+\psi)v_j\,, \\ \nabla_q\dfrac{\sigma}{\rho}=-\dfrac{1}{\rho}(\phi_1+\psi)v_q\,. \end{cases}$$

Covariantly differentiating the equations (4.11, 1) in x^q and (4.11, 2) in x^j and putting $\psi_\lambda=\nabla_\lambda\psi$, we have

$$\nabla_q\nabla_j\rho_i=\nabla_q(\phi_1+\psi)g_{ji}-2\psi_q v_j v_i+\frac{2\psi}{\rho}v_q(\rho_j v_i+v_j\rho_i)\,,$$

$$\nabla_j\nabla_q\rho_i=-2\psi_j v_q v_i+\frac{2\psi}{\rho}v_j\rho_q v_i-\frac{2\psi}{\rho}v_q(\sigma g_{ji}-v_j\rho_i)$$

by means of (4.10). Since these expressions are equal to one another, we see

$$\left[\nabla_q(\phi_1+\psi)+\frac{2\psi\sigma}{\rho}v_q\right]g_{ji}=\frac{2}{\rho}(\rho\psi_q v_j+\psi\rho_q v_j-\rho\psi_j v_q-\psi\rho_j v_q)v_i$$

$$=\frac{2}{\rho}(\tau_q v_j-\tau_j v_q)v_i=0$$

because of the parallelism of $\tau=(\tau^\kappa)$ to v. By putting the expression in the brackets equal to zero, we have

$$\text{(4.15)}\qquad \begin{cases} \nabla_j(\phi_2+\psi)=-\dfrac{2\psi\sigma}{\rho}v_j\,, \\ \nabla_q(\phi_1+\psi)=-\dfrac{2\psi\sigma}{\rho}v_q\,. \end{cases}$$

Covariantly differentiating (4.13) and using themselves, (4.11), (4.15) and (4.10), we obtain the equations

$$\text{(4.16)}\qquad \begin{cases} \nabla_j\sigma_i=\dfrac{\sigma}{\rho}(\phi_1-\phi_2)g_{ji}\,, \\ \nabla_q\sigma_p=\dfrac{\sigma}{\rho}(\phi_2-\phi_1)g_{qp}\,. \end{cases}$$

Putting

$$\Psi=\frac{\sigma}{\rho}(\phi_1-\phi_2) \qquad \text{and} \qquad \Psi_\lambda=\nabla_\lambda\Psi$$

and applying Ricci's formula to the equation (4.16, 1), we have the equation

$$-K_{kji}{}^h\sigma_h=\Psi_k g_{ji}-\Psi_j g_{ki}\,.$$

If $\sigma_i\neq 0$ and $n_1\geqq 2$, then, contracting this equation with σ^i, we see that Ψ_j is proportional to σ_j, i. e.,

$$\Psi_j=\alpha\sigma_j\,, \tag{4.17}$$

α being a factor, and obtain

$$-K_{kji}{}^h\sigma_h=\alpha(\sigma_k g_{ji}-\sigma_j g_{ki})\,. \tag{4.18}$$

On the other hand, applying Ricci's formula to the equation (4.10) for $(\kappa, \lambda)=(i, j)$, we have

$$-K_{kji}{}^h v_h=\frac{1}{\rho}(\phi_1-\phi_2)(v_k g_{ji}-v_j g_{ki})\,. \tag{4.19}$$

The equation (4.18) contracted with v^i has opposite signature to the equation (4.19) contracted with σ^i, and we see that

$$\alpha=\frac{1}{\rho}(\phi_1-\phi_2) \qquad \text{and} \qquad \Psi=\alpha\sigma\,, \tag{4.20}$$

whether v_j is proportional to σ_j or not. Comparing the derivative of (4.20, 2) with (4.17), we see that $\nabla_j\alpha=0$ and the proportional factor α is independent of (x^h) in U'. If in addition $n_2\geqq 2$, then α is also independent of (x^p) in U', and hence a constant in U'. If $n_2=1$, then let P and Q be two arbitrary points in U' on the same part M_2. The equation (4.18) is valid at the points P and Q, and the components g_{ji} and $K_{kji}{}^h$ are independent of P and Q. By contraction of the equation at P with $\sigma^i(\mathrm{Q})$ and the equation at Q with $\sigma^i(\mathrm{P})$, we see that $\alpha(\mathrm{P})=\alpha(\mathrm{Q})$ whether $\sigma^h(\mathrm{P})$ is proportional to $\sigma^h(\mathrm{Q})$ or not, that is, α is independent of points of M_2. Thus the factor α is a constant, say h, and the equations (4.16) turn to

$$\begin{cases} \nabla_j\sigma_i= h\sigma g_{ji}\,, \\ \nabla_q\sigma_p=-h\sigma g_{qp} \end{cases} \tag{4.21}$$

in U'. If σ is independent of points of M_1, then we have $h=0$.

From (4.20) follows the relation

$$\phi_1-\phi_2=h\rho\,. \tag{4.22}$$

Comparing this with (3.8), we see that

$$\left\{\begin{aligned} \phi_1 &= \frac{1}{2\rho}(\Phi + h\rho^2), \\ \phi_2 &= \frac{1}{2\rho}(\Phi - h\rho^2). \end{aligned}\right. \tag{4.23}$$

Differentiating the second equation in x^i and using (3.8), (4.12) and (4.21), we have

$$\begin{aligned} \nabla_i\phi_2 &= \frac{1}{2\rho}\left[2(\phi_1+\psi)\rho_i - 4\sigma\psi v_i - \frac{\Phi}{\rho}\rho_i - h\rho\rho_i\right] \\ &= \frac{1}{2\rho}[2\psi\rho_i + (\phi_1-\phi_2)\rho_i - 4\sigma\psi v_i - h\rho\rho_i] \\ &= \frac{1}{\rho}(\psi\rho_i - 2\sigma\psi v_i) \end{aligned}$$

that is

$$\left\{\begin{aligned} \nabla_i\phi_2 &= \frac{1}{\rho}(\psi\rho_i - 2\sigma\psi v_i), \\ \nabla_p\phi_1 &= \frac{1}{\rho}(\psi\rho_p - 2\sigma\psi v_p). \end{aligned}\right. \tag{4.24}$$

It follows from the equations (4.15) and (4.24) that

$$\left\{\begin{aligned} \rho\nabla_i\psi + \psi\rho_i &= \nabla_i(\rho\psi) = 0, \\ \rho\nabla_p\psi + \psi\rho_p &= \nabla_p(\rho\psi) = 0, \end{aligned}\right.$$

or

$$\tau_\lambda = \nabla_\lambda(\rho\psi) = 0.$$

Thus the vector field τ identically vanishes in U'. This contradicts the assumption $\tau \neq 0$ in U'. Therefore the vector field τ identically vanishes in U. Q.E.D.

5. Further equation in the subset U.

By means of Lemma 4, we may put $\rho\psi = C$ in U, or

$$\psi = \frac{C}{\rho}, \tag{5.1}$$

C being a constant. Then the equations (3.7) turn to

$$\left\{\begin{aligned} \nabla_j\rho_i &= \phi_1 g_{ji} + \frac{C}{\rho}G_{ji}, \\ \nabla_q\rho_i &= \frac{C}{\rho}G_{qi}, \\ \nabla_q\rho_p &= \phi_2 g_{qp} + \frac{C}{\rho}G_{qp}. \end{aligned}\right. \tag{5.2}$$

It follows from (3.12) with $\tau=0$ that we have

$$\begin{cases} \nabla_j\phi_2=\dfrac{C}{\rho^2}G_{j\omega}\rho^{\omega}, \\ \nabla_q\phi_1=\dfrac{C}{\rho^2}G_{q\omega}\rho^{\omega}. \end{cases} \tag{5.3}$$

Applying Ricci's formula to the equations (5.2, 1) and (5.2, 3) and substituting (1.8), we have the equation

$$-K_{kji}{}^{h}\rho_h=\left(\nabla_k\phi_1-\frac{C}{\rho^2}G_{k\omega}\rho^{\omega}\right)g_{ji}-\left(\nabla_j\phi_1-\frac{C}{\rho^2}G_{j\omega}\rho^{\omega}\right)g_{ki}.$$

Contracting this equation with ρ^i, we may put

$$\begin{cases} \nabla_j\phi_1=\dfrac{C}{\rho^2}G_{j\omega}\rho^{\omega}+\alpha_1\rho_j, \\ \nabla_q\phi_2=\dfrac{C}{\rho^2}G_{q\omega}\rho^{\omega}+\alpha_2\rho_q, \end{cases} \tag{5.4}$$

α_1 and α_2 being proportional factors. Differentiating the relation (3.8) and substituting (5.3) and (5.4), we have the relation

$$\alpha_1=-\alpha_2=\frac{1}{\rho}(\phi_1-\phi_2).$$

The differences of the equations (5.3) and (5.4) make all together the tensor equation

$$\nabla_\mu(\phi_1-\phi_2)=\frac{1}{\rho}(\phi_1-\phi_2)\rho_\mu,$$

hence we may put

$$\phi_1-\phi_2=k\rho, \tag{5.5}$$

k being a constant. Comparing this relation with (3.8), we obtain again

$$\begin{cases} \phi_1=\dfrac{1}{2\rho}(\Phi+k\rho^2), \\ \phi_2=\dfrac{1}{2\rho}(\Phi-k\rho^2), \end{cases} \tag{5.6}$$

and the equations (5.2) turn to

$$\begin{cases} \nabla_j\rho_i=\dfrac{1}{2\rho}(\Phi+k\rho^2)g_{ji}+\dfrac{C}{\rho}G_{ji}, \\ \nabla_q\rho_i=\dfrac{C}{\rho}G_{qi}, \\ \nabla_q\rho_p=\dfrac{1}{2\rho}(\Phi-k\rho^2)g_{qp}+\dfrac{C}{\rho}G_{qp}. \end{cases} \tag{5.7}$$

Now the derivatives of $\Phi=\rho_\omega\rho^\omega$ are equal to

$$(5.8)\qquad \begin{cases} \nabla_i\Phi=\dfrac{1}{\rho}[(\Phi+k\rho^2)\rho_i+2CG_{i\omega}\rho^\omega], \\ \nabla_p\Phi=\dfrac{1}{\rho}[(\Phi-k\rho^2)\rho_p+2CG_{p\omega}\rho^\omega]. \end{cases}$$

Covariantly differentiating the first equation in x^q and using the second, (1.8) and (5.2), we have

$$(5.9)\qquad \begin{aligned} \nabla_q\nabla_i\Phi=\frac{1}{\rho}[\{(\Phi-k\rho^2)\rho_q+2C\rho G_{q\omega}\rho^\omega-\Phi\rho_q+k\rho^2\rho_q\}\rho_i \\ +(\Phi+k\rho^2)CG_{qi}-2C\rho_q G_{i\omega}\rho^\omega \\ -2C(G_{qi}\rho_\kappa+G_{q\kappa}\rho_i-g_{q\kappa}G_{i\omega}\rho^\omega)\rho^\kappa \\ +CG_{i\omega}\{(\Phi-k\rho^2)\delta_q^\omega+2CG_q{}^\omega\}]=0. \end{aligned}$$

Therefore the squared length Φ is decomposable in U, that is, it is the sum

$$(5.10)\qquad \Phi=\rho_\omega\rho^\omega=\Phi_1+\Phi_2$$

of functions Φ_1 of (x^h) and Φ_2 of (x^p). By similar computations, we have also the equations

$$(5.11)\qquad \begin{cases} \nabla_j\nabla_i\Phi=2k(\rho_j\rho_i+CG_{ji})+\dfrac{2}{\rho^2}\left[\dfrac{1}{4}(\Phi+k\rho^2)^2+CG_{\lambda\kappa}\rho^\lambda\rho^\kappa+C^2\right]g_{ji}, \\ \nabla_q\nabla_p\Phi=-2k(\rho_q\rho_p+CG_{qp})+\dfrac{2}{\rho^2}\left[\dfrac{1}{4}(\Phi-k\rho^2)^2+CG_{\lambda\kappa}\rho^\lambda\rho^\kappa+C^2\right]g_{qp}. \end{cases}$$

Moreover, by use of (5.7), we have

$$(5.12)\qquad \begin{cases} \nabla_j\nabla_i\rho^2=(\Phi+k\rho^2)g_{ji}+2CG_{ji}+2\rho_j\rho_i, \\ \nabla_q\nabla_i\rho^2=\qquad\qquad\qquad 2CG_{qi}+2\rho_q\rho_i, \\ \nabla_q\nabla_p\rho^2=(\Phi-k\rho^2)g_{qp}+2CG_{qp}+2\rho_q\rho_p, \end{cases}$$

and therefore

$$(5.13)\qquad \begin{cases} \nabla_j\nabla_i(\Phi-k\rho^2)=\Omega g_{ji}, \\ \nabla_q\nabla_p(\Phi+k\rho^2)=\Omega g_{qp}, \end{cases}$$

where we have put

$$(5.14)\qquad \Omega=\frac{1}{2\rho^2}(\Phi^2-k^2\rho^4+4CG_{\lambda\kappa}\rho^\lambda\rho^\kappa+4C^2).$$

By use of (1.8), (5.8) and (5.10), the derivative of the equation (5.12, 1) in x^p is equal to

$$\nabla_p\nabla_j\nabla_i\rho^2=\frac{1}{\rho}[(\Phi+k\rho^2)\rho_p+2CG_{p\omega}\rho^\omega]g_{ji}$$
$$=(\nabla_p\Phi+2k\rho\rho_p)g_{ji}$$
$$=\nabla_p(\Phi+k\rho^2)g_{ji},$$

thus we have

$$\left\{\begin{aligned}&\nabla_p\nabla_j\nabla_i\rho^2=\nabla_p(\Phi_2+k\rho^2)g_{ji},\\&\nabla_i\nabla_q\nabla_p\rho^2=\nabla_i(\Phi_1-k\rho^2)g_{qp}.\end{aligned}\right.\tag{5.15}$$

On the other hand, by taking account of the decomposability (5.9) of Φ, the derivatives of (5.13) are given by

$$\left\{\begin{aligned}&-k\nabla_p\nabla_j\nabla_i\rho^2=(\nabla_p\Omega)g_{ji},\\&\quad k\nabla_i\nabla_q\nabla_p\rho^2=(\nabla_i\Omega)g_{qp}.\end{aligned}\right.\tag{5.16}$$

Comparing the equations (5.15) with (5.16), we have

$$\left\{\begin{aligned}&\nabla_i\Omega=\quad k\nabla_i(\Phi_1-k\rho^2),\\&\nabla_p\Omega=-k\nabla_p(\Phi_2+k\rho^2),\end{aligned}\right.$$

and consequently the function Ω is equal to

$$\Omega=k(\Phi_1-\Phi_2-k\rho^2)+b,\tag{5.17}$$

b being a constant. Then the equations (5.13) turn to

$$\left\{\begin{aligned}&\nabla_j\nabla_i(\Phi_1-k\rho^2)=[k(\Phi_1-\Phi_2-k\rho^2)+b]g_{ji},\\&\nabla_q\nabla_p(\Phi_2+k\rho^2)=[k(\Phi_1-\Phi_2-k\rho^2)+b]g_{pq}.\end{aligned}\right.\tag{5.18}$$

Covariantly differentiating the equation (5.12, 1) and using (1.8), (5.8) and (5.10), we have the equation

$$\nabla_k\nabla_j\nabla_i\rho^2=(\nabla_k\Phi_1+k\nabla_k\rho^2)g_{ji}$$
$$-\frac{2C}{\rho}(G_{kj}\rho_i+G_{ki}\rho_j-g_{kj}G_{i\omega}\rho^\omega-g_{ki}G_{j\omega}\rho^\omega)$$
$$+\frac{1}{\rho}[(\Phi+k\rho^2)g_{kj}+2CG_{kj}]\rho_i$$
$$+\frac{1}{\rho}[(\Phi+k\rho^2)g_{ki}+2CG_{ki}]\rho_j$$
$$=(\nabla_k\Phi_1+k\nabla_k\rho^2)g_{ji}+g_{kj}\nabla_i\Phi+g_{ki}\nabla_j\Phi$$
$$=(\nabla_k\Phi_1+k\nabla_k\rho^2)g_{ji}+g_{kj}\nabla_i\Phi_1+g_{ki}\nabla_j\Phi_1.$$

Adding k times of this expression to the covariant derivative of the equation (5.18, 1), we obtain the equations

$$(5.19)\qquad \begin{cases} \nabla_k\nabla_j\nabla_i\Phi_1= \ k(2g_{ji}\nabla_k\Phi_1+g_{kj}\nabla_i\Phi_1+g_{ki}\nabla_j\Phi_1), \\ \nabla_r\nabla_q\nabla_p\Phi_2=-k(2g_{qp}\nabla_r\Phi_2+g_{rq}\nabla_p\Phi_2+g_{rp}\nabla_q\Phi_2) \end{cases}$$

in U. The functions Φ_1 and Φ_2 in the equations (5.15) and (5.19) can be replaced with the function $\Phi=|Y|^2$ and these equations are extended on the closure $\bar{U}$ of a connected component U because ρ is differentiable in M from the outset.

6. Proof of the theorem.

The following lemma is a combination of Lemmas 2 and 3 in [8] and will be used.

LEMMA 5. *Let (M, g) and (M^*, g^*) be complete Riemannian manifolds and f a diffeomorphism of M onto M^*. If the length of a differentiable curve Γ in M is bounded, then so is the length of the image $f(\Gamma)$ in M^*.*

The assumption that the open subset V is dense in M implies that the complement $M-V$ is a border set. As is noticed at the end of Paragraph 2, $M-N_1\cap N_2\supset V$ and hence the closed subset $N_1\cap N_2$ is a border set too.

Case (1). First we consider the case where U is empty. Then $M=N_1\cup N_2$, $(M-N_1)\cap(M-N_2)=\emptyset$ and no point of one of $M-N_1$ and $M-N_2$ is on the boundary of the other. The intersection $(M-N_1)\cap V$ is dense in $M-N_1$ and so is $(M-N_2)\cap V$ in $M-N_2$. By means of Lemma 3, we have an equation of type (2.2) in first in each connected component of $M-N_1$ and $M-N_2$ and secondly in the closure of the component. The coefficient c^2 might be different from one of components to another.

Suppose that $M-N_1$ is not empty and let W be its connected component. Denote the part M_1 through a point $\mathrm{P}\in W$ by $M_1(\mathrm{P})$ and put $M_1(W)=\bigcup_{\mathrm{P}\in W} M_1(\mathrm{P})$. Since the stationary point of a concircular scalar field ρ satisfying (2.2) is at most one, the boundary of the set $M_1(\mathrm{P})\cap(M-N_1)$ should consist of at most one point. Hence the equation (2.2) is valid in the whole part $M_1(\mathrm{P})$ through any point $\mathrm{P}\in W$, and so over the open subset $M_1(W)$.

Since the manifold M is complete, so are the parts M_1 and M_2. Any geodesic in a complete manifold is extendable to the infinity. Let Γ be a geodesic curve lying in $M_1(\mathrm{P})$ through $\mathrm{P}\in W$ and s the arc length of Γ. The ordinary derivatives in s will be denoted by prime. The equation (2.2) is reduced to the ordinary differential equation

$$\rho''(s)=c^2\rho$$

along Γ and the general solution is given in the form

$$\rho(s)=Ae^{cs}+Be^{-cs} \tag{6.1}$$

A and B being arbitrary constants.

The integral curves, called *ρ-curves*, of the gradient vector field Y of a concircular scalar field ρ are defined by differential equations

$$\frac{dx^\kappa}{dt}=\rho^\kappa$$

with respect to the canonical parameter t, or

$$\frac{dx^\kappa}{ds}=\frac{1}{|Y|}\rho^\kappa$$

with respect to the arc length s. The ρ-curves are geodesic. In the present case, the ρ-curve passing through P lies in the part $M_1(\mathrm{P})$. It follows from the equation (2.4) that we have the ordinary differential equation

$$\rho'(s)=\rho_i\frac{dx^i}{ds}=\frac{1}{|Y|}\rho_i\rho^i=c\rho \tag{6.2}$$

along a ρ-curve. The general solution is given by

$$\rho(s)=Ae^{cs},$$

A being a positive constant.

Let Γ^* be the image $f(\Gamma)$ of a ρ-curve Γ by the conformal diffeomorphism, and s^* the arc length of Γ^* such as $s^*=0$ corresponding to $s=0$. Then they are related by the ordinary differential equation

$$\frac{ds^*}{ds}=\frac{1}{\rho}=\frac{1}{A}e^{-cs}$$

or, by integration, by the equation

$$s^*=\frac{1}{Ac}(1-e^{-cs})<\frac{1}{Ac}.$$

Therefore the length of the image Γ^* is bounded as s tends to the infinity along the ρ-curve Γ. This contradicts the completeness of M^* by virtue of Lemma 5. Thus there is no global conformal diffeomorphism in this case.

We notice that $M=N_2$, i.e., there is no point such that $Y_2\neq 0$ in this case. If there would be a point Q such that $Y_2(\mathrm{Q})=0$, then there would be an open subset W' in $M-N_2$ like W in $M-N_1$ and the vector field Y_2 would not identically vanish in $M_2(W')$, and hence U would not be empty, in contradiction to the assumption.

Case (2). Next we consider the case where the subset U is not empty and the constant k in (5.5) is equal to zero in every component of U. Now the equations (5.7), (5.8) and (5.12) turn to the tensor equations

(6.3) $$\nabla_\mu \rho_\lambda = \frac{1}{2\rho}(\Phi g_{\mu\lambda} + 2CG_{\mu\lambda}),$$

(6.4) $$\nabla_\lambda \Phi = \frac{1}{\rho}(\Phi \rho_\lambda + 2CG_{\lambda\kappa}\rho^\kappa)$$

and

(6.5) $$\nabla_\mu \nabla_\lambda \rho^2 = \Phi g_{\mu\lambda} + 2CG_{\mu\lambda} + \rho_\mu \rho_\lambda$$

in $\bar{U}$, respectively. Covariant differentiating (6.5), substituting (1.8) and (6.3) and taking account of (6.4), we obtain the equation

(6.6) $$\nabla_\nu \nabla_\mu \nabla_\lambda \rho^2 = g_{\nu\mu} \nabla_\lambda \Phi + g_{\nu\lambda} \nabla_\mu \Phi + g_{\mu\lambda} \nabla_\nu \Phi .$$

It follows also from (5.17) that $\Omega = b$ and the equations (5.9) and (5.11) turn to

(6.7) $$\nabla_q \nabla_i \Phi = 0$$

and

(6.8) $$\begin{cases} \nabla_j \nabla_i \Phi = \nabla_j \nabla_i \Phi_1 = b g_{ji}, \\ \nabla_q \nabla_p \Phi = \nabla_q \nabla_p \Phi_2 = b g_{qp}, \end{cases}$$

respectively, and altogether to the tensor equation

(6.9) $$\nabla_\mu \nabla_\lambda \Phi = b g_{\mu\lambda}$$

in $\bar{U}$. It is noticed that the constant b might be different from a component of U to another.

If the closed subset N_2 contains inner points and contacts with the closure of a component of U at a point O, then $\Phi = \rho_i \rho^i$ is dependent of (x^h) only in N_2 and we have the equation (2.2) and consequently

$$\nabla_j \nabla_i \Phi = 2(c^4 \rho^2 g_{ji} + c^2 \rho_j \rho_i)$$

in N_2. Comparing this with the equations (6.8) at O, we see first $b=0$ and then we have the equation

$$c^2(c^2 \rho^2 g_{ji} + \rho_j \rho_i) = 0$$

and, by contraction with g^{ji},

$$c^2(n_1 c^2 \rho^2 + \Phi) = 0$$

at O. Since the expression in the parentheses is positive, the coefficient c should be equal to zero. This contradicts the positiveness of c in Lemma 3. Therefore the subset N_2 is a border set in M and similarly so is the subset N_1 in this case, even if they exist.

If the closures of two components of U contact with one another at a point O of $N_1 \cup N_2$, then we compare the equations of type (6.9) in the closures at O and see that the constant b is common with all components of U. Therefore the equations from (6.3) to (6.9) are valid on the whole manifold M.

If the length $|Y|$ of Y is constant, say $|Y|=B$, then we have

$$\nabla_\lambda \Phi = 2\rho^\mu \nabla_\mu \rho_\lambda = 0 ,$$

which means that the integral curves of the gradient vector field $Y=(\rho^\kappa)$ are geodesic. Let Γ be one of the integral curves. Along Γ we have

$$\rho'(s)=B$$

by a similar way to (6.2), and the solution is given in the form

$$\rho = Bs + C , \tag{6.10}$$

C being a constant. Then ρ has negative values on a half infinite interval of s. This does not occur.

If Φ is not constant, then the gradient vector field, denoted by $Z=(\Phi^\kappa)$, of Φ is parallel or concurrent according as b is equal to zero or not. The integral curves of the field Z are geodesic. Let Γ be one of the integral curves. The equations (6.6) and (6.9) are reduced to

$$(\rho^2(s))''' = 3\Phi'(s) \tag{6.11}$$

and

$$\Phi''(s) = b \tag{6.12}$$

along Γ.

Provided $b=0$, the solution of (6.12) is given in the form

$$\Phi = |Y|^2 = as + c \tag{6.13}$$

a and c being constant and $a>0$. Then the squared length Φ of Y has negative values in a half infinite interval of s. This is a contradiction.

Provided $b \neq 0$, the squared length Φ has one stationary point, say O, and any integral curve Γ of Z issues from the point O. We choose the arc length s of Γ such as $s=0$ at O. Then we have

$$\Phi'(s) = bs$$

and the general solution of the equation (6.11) is given in the form

$$\rho^2 = \frac{1}{8} bs^4 + As^2 + Bs + C , \tag{6.14}$$

A, B and C being arbitrary constants. Since ρ is positive for any value of s, the constant b should be positive and we put $b=16a^2$, $a>0$. Take a value s_0 so large that the inequality

$$\rho > as^2$$

holds for $s>s_0$. The arc length s^* of the image $\Gamma^*=f(\Gamma)$ is related to s by the differential equation

$$\frac{ds^*}{ds}=\frac{1}{\rho}<\frac{1}{as^2}.$$

Integrating this equation and denoting by s_0^* the value of s corresponding to s_0, we obtain the inequality

$$s^*-s_0^*<\frac{1}{a}\left(\frac{1}{s_0}-\frac{1}{s}\right)<\frac{1}{as_0}.$$

Hence the length of the curve Γ^* is bounded as s tends to the infinity. This leads to a contradiction to Lemma 5.

Case (3). We finally consider the case where the subset U is not empty and the constant k appearing in (5.5) is not equal to zero at least in one connected component, say U_0, of U. Let Γ be a geodesic curve lying in the part $M_1(\mathrm{P})$ passing through a point $\mathrm{P}\in U_0$. Along the arc of Γ contained in $\bar{U}_0$, the equations (5.18, 1) and (5.19, 1) are reduced to the ordinary differential equations

$$(\Phi_1(s)-k\rho^2(s))''=k(\Phi_1-k\rho^2)+b-k\Phi_2(\mathrm{P}) \tag{6.15}$$

and

$$\Phi_1'''(s)=4k\Phi_1'(s) \tag{6.16}$$

respectively. We put $k=c^2$ or $k=-c^2$ according as $k>0$ or $k<0$. Then the general solution of (6.16) is written in the form

$$\Phi_1=\begin{cases} Ae^{2cs}+Be^{-2cs}+C & (k=c^2), \\ A\cos 2cs+B\sin 2cs+C & (k=-c^2), \end{cases} \tag{6.17}$$

and consequently, by means of the equations (6.15), ρ^2 is given in the form

$$\rho^2=\begin{cases} \frac{1}{c^2}(Ae^{2cs}+Be^{-2cs})+A_1e^{cs}+B_1e^{-cs}+C_1, \\ \frac{1}{c^2}(A\cos 2cs+B\sin 2cs)+A_1\cos cs+B_1\sin cs+C_1 \end{cases} \tag{6.18}$$

respectively, where the coefficients A, B, C and so on are arbitrary constants.

Without loss of generality, we may suppose $k>0$ in the component U_0 and let Γ be the geodesic curve tangent to $Y_1(\mathrm{P})$ at the point P. Crossing the boundary of U_0 at a point O, the geodesic curve Γ enters another connected component of U or the subset $N_1 \cup N_2$.

If Γ enters another component of U, then ρ^2 is given by one of the expressions (6.18) on the arc of Γ contained in the component. Since ρ^2 is differentiable on the geodesic curve Γ, we compare the derivatives of ρ^2 in s at O in the two components, and see that the constant k is common with the components and ρ^2 has the expression (6.18, 1) along the whole curve Γ.

If Γ enters the closed subset N_2, then ρ^2 has the expression (6.1) on the arc of Γ contained in N_2 and this is a special one of (6.18, 1). If Γ enters the closed subset N_1, then ρ^2 is constant on the arc and this does not happen by comparison with the expression (6.18, 1). Therefore we have the expression (6.18, 1) of ρ^2 on the whole geodesic curve Γ.

At least one of the coefficients A, B, A_1 and B_1 of the expression (6.18, 1) is different from zero. If, for example, $A \neq 0$, then A should be positive and we put $A=a^2$, $a>0$. Take a value s_0 so large that the inequality

$$\rho > \frac{a}{2c} e^{cs}$$

holds for $s>s_0$. Let Γ^* be the image $f(\Gamma)$, s^* the arc length of Γ^* and s_0^* the value corresponding to s_0. Then s^* is related to s by the differential equation

$$\frac{ds^*}{ds} = \frac{1}{\rho} < \frac{2c}{a} e^{-cs},$$

and, by integration, we have the inequality

$$s^* - s_0^* < \frac{2}{a} e^{-cs_0}.$$

Hence the length of the curve Γ^* is bounded as s tends to the infinity. By virtue of Lemma 5, this is a contradiction. Thus we have completed the proof of the theorem.

7. An example of conformal diffeomorphism.

Let S be a unit circle and T^3 a 3-dimensional torus, the product $S \times S \times S$ of three copies of S. Denote by x, y, z the arc lengths modulo 2π of the copies. Take a positive valued function $\rho(y)$ of y with period 2π, for example, $\rho = \sin y + 2$. Consider two Riemannian manifolds M and M^* on the same underlying manifold T^3 having the metrics

$$ds^2 = \{\rho(y)\}^2 dx^2 + dy^2 + dz^2,$$

$$ds^{*2}=dx^2+\frac{1}{\{\rho(y)\}^2}(dy^2+dz^2)$$

respectively. These are conformally related with $\rho(y)$ as associated scalar field. The first manifold M is the product $M_1\times M_2$ of a 2-dimensional manifold M_1 with metric $\rho^2dx^2+dy^2$ on T^2 and a circle $M_2=S$. The second M^* is the product $M_1^*\times M_2^*$ of a circle $M_1^*=S$ and a 2-dimensional manifold M_2^* with metric $(dy^2+dz^2)/\rho^2$ on T^2. These manifolds M and M^* are compact and consequently complete. The product structures F and G are given by

$$F=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix},\quad G=\begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

respectively, and are commutative with one another. This is a desired example.

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