

Strong Picard principle

By Mitsuru NAKAI

Consider the punctured unit disk $\Omega: 0 < |z| < 1$. We view Ω as the interior of the bordered Riemann surface $\bar{\Omega}: 0 < |z| \leq 1$ with the relative boundary $\partial\Omega: |z|=1$ and the ideal boundary $z=0$. By a density $P(z)$ on Ω we mean a non-negative locally Hölder continuous function $P(z)$ on $\bar{\Omega}$. We say that the *strong Picard principle* is valid for a density P on Ω at $z=0$ if

$$(1) \quad u(z) = \mathcal{O}(-\log|z|) \quad (z \rightarrow 0)$$

for every nonnegative solution u of the equation $Lu \equiv (\Delta - P)u = 0$ on Ω . The *purpose* of this paper is to characterize completely those densities P for which the strong Picard principle is valid as follows:

THEOREM. *The strong Picard principle is valid for a density $P(z)$ on Ω at $z=0$ if and only if the condition*

$$(2) \quad -\int_{\Omega-E} P(z) \log|z| dx dy < +\infty \quad (z=x+iy)$$

is satisfied for a closed subset E of Ω thin at $z=0$.

The proof will be given in nos. 3-4 after some preparations in nos. 1-2. An open question related to a generalization of the above theorem to a Riemann surface is stated in no. 5.

1. We denote by \mathcal{P} the family of nonnegative solutions of the equation $Lu = (\Delta - P)u = 0$ on Ω with vanishing boundary values on $\partial\Omega$. The family \mathcal{P} forms a halfmodule and its dimension is referred to as the *elliptic dimension* of P (or L) at $z=0$, $\dim P$ in notation. More precisely $\dim P$ is the cardinal number of the set β of extreme points of the convex set

$$\left\{ u \in \mathcal{P}; -\int_0^{2\pi} \left[\frac{\partial}{\partial r} u(re^{i\theta}) \right]_{r=1} d\theta = 1 \right\}.$$

By the Martin-Choquet theorem there exists a unique measure μ_u on β for each u in \mathcal{P} such that $u = \int_{\beta} v d\mu_u(v)$. It is easy to see that $\dim P \geq 1$. After Bouligand we say that the (ordinary) *Picard principle* is valid for P on Ω at

This research was partially supported by Grant-in-Aid for Scientific Research (No. 454027), Ministry of Education.

$z=0$ if $\dim P=1$. In view of the above integral representation the condition is equivalent to that \mathcal{P} is generated by a single element $u: \mathcal{P}=\mathbf{R}^+u$ where \mathbf{R}^+ is the set of nonnegative members in the field \mathbf{R} of real numbers. As for the relation between the strong and ordinary Picard principle, which are equivalent to each other for the zero density known as the *principle of positive singularity* cf. e. g. [2] for harmonic functions, we have the following:

PROPOSITION. *If the strong Picard principle is valid for a density P , then the ordinary Picard principle is valid for P . The converse of this is not necessarily true.*

PROOF. Let Ω_n be the annulus $1/n < |z| < 1$ ($n=2, 3, \dots$) so that the border of Ω_n consists of $\partial\Omega: |z|=1$ and $\Gamma_n: |z|=1/n$. Suppose that the strong Picard principle is valid for P . Take an arbitrary u in \mathcal{P} corresponding to P . Then, by (1), there exists a nonnegative constant c with $-c \log|z| \geq u(z)$ on Ω . Let h_n be the harmonic function on Ω_n with boundary values u on $\Gamma_n \cup \partial\Omega$. Observe that $-c \log|z| \geq h_{n+1} \geq h_n \geq u$ on Ω_n ($n=2, 3, \dots$) and therefore $\lim_{n \rightarrow \infty} h_n(z) = -b(u) \log|z|$ on Ω with a nonnegative constant $b(u)$. In terms of the harmonic Green's function $g_n(z, \zeta)$ on Ω_n the functions u and h_n are related as

$$h_n(z) = u(z) + (2\pi)^{-1} \int_{\Omega_n} g_n(z, \zeta) P(\zeta) u(\zeta) d\xi d\eta \quad (\zeta = \xi + i\eta).$$

The integrand of the above increases as n does and accordingly the Lebesgue-Fatou theorem yields the identity

$$(3) \quad b(u)g(z, 0) = u(z) + (2\pi)^{-1} \int_{\Omega} g(z, \zeta) P(\zeta) u(\zeta) d\xi d\eta$$

where $g(z, \zeta) = \log(|1 - \bar{\zeta}z| / |z - \zeta|)$ is the harmonic Green's function on Ω and hence on the unit disk $|z| < 1$. Since $u \rightarrow b(u)$ is positively linear and $b(u) = 0$ if and only if $u = 0$, the fact that $\dim P \geq 1$ implies the existence of a u_0 in \mathcal{P} with $b(u_0) = 1$. From the identities (3) for u and u_0 , it follows that

$$v(z) = -(2\pi)^{-1} \int_{\Omega} g(z, \zeta) P(\zeta) v(\zeta) d\xi d\eta$$

for $v = u - b(u)u_0$ and therefore

$$|v(z)| \leq (2\pi)^{-1} \int_{\Omega} g(z, \zeta) P(\zeta) |v(\zeta)| d\xi d\eta.$$

Since $|v(z)|$ is nonnegative and subharmonic on Ω and dominated by the harmonic Green potential on Ω of the measure $(2\pi)^{-1} P(\zeta) |v(\zeta)|$, we may conclude that $|v(\zeta)| = 0$ on Ω , i. e. $u = b(u)u_0$. Therefore \mathcal{P} is generated by a single element u_0 and thus the ordinary Picard principle is valid for P .

To show that the converse is not necessarily true, we consider the density

$P_2(z)=4/|z|^2$ on Ω . Observe that $u_2(z)=|z|^{-2}$ is a positive solution of $\Delta u=P_2u$ on Ω , and hence the strong Picard principle is not valid for P_2 . That the ordinary Picard principle is valid for P_2 is known (cf. [7]), and in fact many proofs can be considered. We present here a proof, probably the simplest but not too direct, based on a criterion of the Picard principle valid for rotation free densities P , i.e. $P(z)=P(|z|)$, established in [5] (see also Godefroid [4]). Let $Q(t)=e^{-2t}P(e^{-t})$ ($0 \leq t < +\infty$) and $a(t)$ be the Riccati component of Q , i.e. the unique nonnegative solution of $-a' + a^2 = Q$ on $[0, +\infty)$. Then the Picard principle is valid for P if and only if $\int_0^\infty (1+a(t))^{-1} dt = +\infty$. For our present P_2 , $Q_2(t) = e^{-2t}P_2(e^{-t}) = 4$ and its Riccati component $a_2 = 2$. Therefore $\int_0^\infty (1+a_2(t))^{-1} dt = \int_0^\infty 3^{-1} dt = +\infty$ and the Picard principle is valid for P_2 .

2. Before proceeding to the proof of the theorem we pause here to state a remark. The prototype of the theorem was, in essence, shown by BreLOT [1] for rotation free densities for which exceptional thin sets need not come in. It is, however, inevitable to consider thin sets for the case of general densities as we are going to consider. A closed set E in Ω is thin at $z=0$, by definition, if the closure of E considered in the complex plane does not contain $z=0$ or else there exists a superharmonic function s on a certain disk $|z| < \rho < 1$ such that

$$s(0) < \liminf_{z \in E, z \rightarrow 0} s(z).$$

We denote by \mathcal{E} the family of closed subsets E of Ω thin at $z=0$. It is easy to see that $E_1 \cup E_2$ and $E_1 \cap E_2$ belong to \mathcal{E} along with E_1 and E_2 . For convenience we list below BreLOT's results on properties of sets in \mathcal{E} which we will make use of in our proof and their proofs can be found in BreLOT [2, 3].

(a) Let $\{s\}$ be the family of nonnegative superharmonic functions s on $\Omega - E$ with $E \in \mathcal{E}$ and satisfying $\liminf_{z \rightarrow \theta} s(z) \geq 1$ for every θ in \mathbf{R} and h be the lower envelope of the family $\{s\}$ which is harmonic in $\Omega - E$. Then $\limsup_{z \rightarrow 0} h(z) > 0$.

(b) For any $E \in \mathcal{E}$ there exists a decreasing zero sequence $\{a_n\}$ of positive numbers $a_n < 1$ such that the circles $\{|z|=a_n\} \cap E = \emptyset$ ($n=1, 2, \dots$).

(c) Let u be a bounded subharmonic function on Ω and $\limsup_{z \rightarrow 0} u(z) = \eta$. Then $\{z \in \Omega; u(z) \leq \sigma\}$ belongs to \mathcal{E} for any $\sigma < \eta$.

(d) Let μ be a Borel measure on $|z| < 1$ such that $\mu(0) = 0$ and $g_\mu(z) = \int_\Omega g(z, \zeta) d\mu(\zeta) \neq +\infty$. Then $\{z \in \Omega; g_\mu(z)/g(z, 0) \geq \varepsilon\}$ belongs to \mathcal{E} for every $\varepsilon > 0$.

3. PROOF OF THE NECESSITY OF (2). We assume the validity of the *strong* Picard principle for P at $z=0$. By the proposition in no. 1 the corresponding \mathcal{P} is generated by a single element u_0 with $b(u_0)=1$:

$$(4) \quad g(z, 0)=u_0(z)+g_\mu(z)$$

where $d\mu(\zeta)=(2\pi)^{-1}P(\zeta)u_0(\zeta)d\xi d\eta$. By (d) in no. 2 the set $\{z \in \Omega; g_\mu(z)/g(z, 0) \geq \varepsilon\}$ belongs to \mathcal{E} for every $\varepsilon > 0$. Therefore, in view of (4), the set

$$E_\varepsilon = \{z \in \Omega; u_0(z)/g(z, 0) \leq 1 - \varepsilon\}$$

is closed in Ω and *thin* at $z=0$ for every $\varepsilon > 0$. From $g_\mu(z) < +\infty$ for every z in Ω it follows that

$$\int_{\Omega} P(\zeta)u_0(\zeta) d\xi d\eta < +\infty.$$

For definiteness let $E = E_{1/2}$. Since $u_0(\zeta) > -2^{-1} \log|\zeta|$ on $\Omega - E$, the above implies (2) for P .

4. PROOF OF THE SUFFICIENCY OF (2). We assume that P satisfies (2). We first show that the *ordinary* Picard principle is valid for P . This was already shown in [6] but we give here a relatively simpler proof based on the properties of sets in \mathcal{E} mentioned in no. 2.

There exists a unique bounded solution e of $Lu=0$ on Ω with boundary values 1 on $\partial\Omega$, which is referred to as the *P-unit*. It may be of some independent interest to observe that (2) is equivalent to

$$(5) \quad a = \limsup_{z \rightarrow 0} e(z) > 0,$$

although we only need the implication from (2) to (5). In fact, we have

$$1 = e(z) + (2\pi)^{-1} \int_{\Omega} g(z, \zeta) P(\zeta) e(\zeta) d\xi d\eta.$$

If we assume (5), then, by (c) in no. 2, the set $E = \{z \in \Omega; e(z) \leq a/2\}$ belongs to \mathcal{E} and the above identity implies (2). Less trivial is the converse. We assume the validity of (2). By (b) in no. 2 there exists a decreasing zero sequence $\{a_n\}$ in $(0, 1)$ such that $\{|z| = a_n\} \cap E = \emptyset$ ($n=1, 2, \dots$). We denote by Ω_n the punctured disk $0 < |z| < a_n$ and set $\Omega'_n = \Omega_n - E$. Let h_n be as in (a) in no. 2 constructed for Ω'_n instead of $\Omega - E$. The integral equation

$$h_n(z) = e_n(z) + (2\pi)^{-1} \int_{\Omega'_n} g'_n(z, \zeta) P(\zeta) e_n(\zeta) d\xi d\eta$$

has the unique solution e_n which is the minimal positive solution of $Lu=0$ on

Ω'_n with boundary values 1 on $\partial\Omega_n: |z|=a_n$, where g'_n is the harmonic Green's function on Ω'_n . This follows from the fact that

$$\int_{\Omega'_n} g'_n(z, \zeta) P(\zeta) d\xi d\eta \leq \int_{\Omega-E} g(z, \zeta) P(\zeta) d\xi d\eta < +\infty$$

which is the consequence of (2). Observe that h_n and e_n are subharmonic on Ω_n by putting $h_n=e_n=0$ on $E \cap \Omega_n$ and that $0 < h_n < h_{n+1} < 1$ and $0 < e_n < e_{n+1} < 1$ on Ω_{n+1} , and that

$$h_n(z) \leq e_n(z) + (2\pi)^{-1} \int_{\Omega'_n} g(z, \zeta) P(\zeta) d\xi d\eta.$$

Using the notation $f^*(r) = (2\pi)^{-1} \int_0^{2\pi} f(re^{i\theta}) d\theta$ for functions f on Ω , and integrating of both sides of the above with respect to $d\theta$ over $[0, 2\pi)$, we get

$$h_n^*(r) \leq e_n^*(r) + (2\pi)^{-1} \int_{\Omega'_n} \min(\log r^{-1}, \log |\zeta|^{-1}) P(\zeta) d\xi d\eta.$$

Since $A_n = \limsup_{z \rightarrow 0} h_n(z) = \lim_{r \rightarrow 0} h_n^*(r)$ and $B_n = \limsup_{z \rightarrow 0} e_n(z) = \lim_{r \rightarrow 0} e_n^*(r)$, on letting $r \rightarrow 0$ in the above, we obtain

$$A_n \leq B_n + C_n, \quad C_n = (2\pi)^{-1} \int_{\Omega_n-E} P(\zeta) \log |\zeta|^{-1} d\xi d\eta.$$

Here $A_n > 0$ by (a) in no. 2. Clearly the sequence $\{A_n\}$ is increasing and the sequence $\{C_n\}$ is decreasing and convergent to zero. Therefore there exists an n such that $B_n \geq A_n - C_n > 0$. Since $e(z) \geq \alpha e_n(z)$ on Ω_n with $\alpha = \min_{\partial\Omega_n} e$, $a = \limsup_{z \rightarrow 0} e(z) \geq \alpha \limsup_{z \rightarrow 0} e_n(z) = \alpha B_n > 0$, i. e. (2) implies (5).

Consider the associated operator $\hat{L}v \equiv \Delta v + 2\nabla \log e \cdot \nabla v$ to $L = \Delta - P$. We say that the (weak) Riemann theorem is valid for \hat{L} if every bounded solution v of the equation $\hat{L}v = 0$ on Ω has a limit at $z = 0$. We recall the following duality theorem (cf. e. g. [7]): The Picard principle is valid for L (i. e. for P) if and only if the Riemann theorem is valid for \hat{L} . We thus have to show that v has a limit at $z = 0$ for every bounded solution v of $\hat{L}v = 0$ on Ω . By adding a suitable constant we may assume that $v > 0$ on Ω . Observe that $u = ev$ is a bounded solution of $Lu = 0$ on Ω . Let $b = \limsup_{z \rightarrow 0} u(z)$. Since u and e are bounded and subharmonic on Ω , (c) in no. 2 implies that the set

$$E_\varepsilon = \{z \in \Omega; e(z) \leq a - \varepsilon\} \cup \{z \in \Omega; u(z) \leq b - \varepsilon\}$$

belongs to \mathcal{E} for any ε in $(0, a)$. Let $\{a_n\}$ be the sequence in (b) in no. 2 for the set E_ε and $\Omega_n = \{z \in \Omega; 0 < |z| < a_n\}$. Clearly

$$a + \varepsilon \geq e(z) \geq a - \varepsilon, \quad b + \varepsilon \geq u(z) > b - \varepsilon$$

on $\partial\Omega_n$ for sufficiently large n and a fortiori

$$(b+\varepsilon)/(a-\varepsilon) \geq v(z) \geq (b-\varepsilon)/(a+\varepsilon)$$

on $\partial\Omega_n$. Since the maximum and minimum of v on $\bar{\Omega}_n: 0 < |z| \leq a_n$ are attained on $\partial\Omega_n: |z|=a_n$, the above inequality for v holds on $\bar{\Omega}_n$. Therefore $\lim_{z \rightarrow 0} v(z) = b/a$ exists and the ordinary Picard principle is valid for P .

We denote by $G(z, \zeta)$ the Green's function on Ω for the operator L . Let $\{z_n\}$ be the zero sequence in Ω such that $e(z_n) \rightarrow a$ ($n \rightarrow +\infty$). Consider the function $u_n(\zeta) = G(z_n, \zeta)/e(z_n)$ which is a positive solution of $Lu=0$ on $\Omega - \{z_n\}$ with vanishing boundary values on $\partial\Omega$. Since

$$e(z_n) = -(2\pi)^{-1} \int_0^{2\pi} \left[\frac{\partial}{\partial r} G(z_n, re^{i\theta}) \right]_{r=1} d\theta,$$

we see that

$$-(2\pi)^{-1} \int_0^{2\pi} \left[\frac{\partial}{\partial r} u_n(re^{i\theta}) \right]_{r=1} d\theta = 1.$$

In view of the Harnack principle we may assume that $\{u_n\}$ converges to a positive solution u_0 on Ω by choosing a suitable subsequence of $\{u_n\}$ if necessary. Since the Picard principle is valid for P and u_0 belongs to \mathcal{P} , \mathcal{P} is generated by u_0 , i. e. $\mathcal{P} = \mathbf{R}^+ u_0$. By the Green formula

$$\int_{\Omega} G(z_n, \zeta) P(\zeta) d\xi d\eta = 2\pi(1 - e(z_n))$$

and accordingly

$$\int_{\Omega} u_n(\zeta) P(\zeta) d\xi d\eta = 2\pi e(z_n)^{-1} (1 - e(z_n)).$$

On taking the inferior limits of both sides of the above, we conclude, by the Fatou lemma, that

$$\int_{\Omega} u_0(\zeta) P(\zeta) d\xi d\eta \leq 2\pi a^{-1} (1 - a) < +\infty.$$

Hence we can define a function h on Ω by

$$h(z) = u_0(z) + (2\pi)^{-1} \int_{\Omega} g(z, \zeta) u_0(\zeta) P(\zeta) d\xi d\eta.$$

It is easy to see that $h(z)$ is a positive harmonic function on Ω with vanishing boundary values on $\partial\Omega$ and therefore, by the classical Picard principle, $h(z) = cg(z, 0)$ with c in \mathbf{R}^+ . In particular $u_0(z) \leq cg(z, 0)$ on Ω and (1) is valid for u_0 . Any positive solution u of $Lu=0$ on Ω satisfies $u(z) = \mathcal{O}(u_0(z))$ ($z \rightarrow 0$), we finally conclude the validity of the *strong* Picard principle for P .

5. Let Ω be a *parabolic end* in the sense of Heins. A *density* P on Ω is a 2-form $P(z)dxdy$ on $\bar{\Omega}$ such that $P(z)$ is a nonnegative locally Hölder continuous function of each local parameter $z=x+iy$. Let $\mathcal{P}=\mathcal{P}_P$ be the family of nonnegative solutions u of the equation $(Lu(z))dxdy=0$ with $L=\Delta-P$ on Ω with vanishing boundary values on $\partial\Omega$. The dimension of the half module \mathcal{P} is referred to as the *elliptic dimension*, $\dim P$ in notation, of P (or L) at the ideal boundary δ of Ω . In particular $\dim 0$ is referred to as the *harmonic dimension* of (the ideal boundary of) Ω . Let $g(z, \zeta)$ be the harmonic Green's function on Ω . Assume Ω is of harmonic dimension one, then

$$g(z, \delta)=\lim_{\zeta \rightarrow \delta} g(z, \zeta)$$

exists and we call it an *Evans harmonic function* on Ω . We say that the (ordinary) *Picard principle* is valid for P (or L) at δ if $\dim P=1$ and that the *strong Picard principle* is valid for P (or L) at Ω if $u(z)=\mathcal{O}(g(z, \delta))$ ($z \rightarrow \delta$) for every nonnegative solution u of $Lu=0$ on Ω . By a verbatim application of the proof for the proposition in no. 1, we see that the strong Picard principle implies the ordinary Picard principle. The condition corresponding to (2) is

$$(6) \quad \int_{\Omega-E} P(z)g(z, \delta)dxdy < +\infty,$$

where E is a closed subset of Ω "thin" at δ . The *P-unit* e is the unique bounded solution of $Lu=0$ on Ω with boundary values 1 on $\partial\Omega$. The condition corresponding to (5) is

$$(7) \quad \limsup_{z \rightarrow \delta} e(z) > 0.$$

Although we are unable to demonstrate at present, it sounds plausible that (6) and (7) are equivalent and that (6) is necessary and sufficient for the validity of the strong Picard principle for P . These are certainly true if the properties (a)-(d) in no. 2 hold for the present end Ω , most of which are very far from being able to settle at present.

Still assuming that the harmonic dimension of Ω is one, let us consider the following condition

$$(8) \quad \int_{\Omega} P(z)g(z, \delta)dxdy < +\infty$$

much stronger than (6). In view of the identity

$$1=e(z)+(2\pi)^{-1}\int_{\Omega} g(z, \zeta)P(\zeta)e(\zeta) d\xi d\eta$$

it is easy to see that (8) is equivalent to the following condition

$$(9) \quad \liminf_{z \rightarrow \delta} e(z) > 0,$$

which is also much stronger than (7). Under the condition (9), Ozawa [9] proved that \mathcal{P}_P and \mathcal{P}_0 are isomorphic, and therefore $\dim P=1$. In view of this, by a similar reasoning as in the last part of no. 4, we conclude that (8) *implies the validity of the strong Picard principle for P*. This is the motivation why we venture to state the above conjecture.

References

- [1] M. BreLOT, Sur le principe des singularites positives et la notion de source pour l'équation $\Delta u(M) = c(M)u(M)$ ($c \geq 0$), Ann. Univ. Lyon Sci. Math. Astro., 11 (1948), 9-19.
- [2] M. BreLOT, Éléments de la théorie classique du potentiel, Les cours de Sorbonne, 1959.
- [3] M. BreLOT, On topologies and boundaries in potential theory, Lecture Notes in Math., 175, Springer, 1971.
- [4] M. Godefroid, Sur un article de Kawamura et Nakai a propos du principe de Picard, Bull. Sci. Math., 102 (1978), 295-303.
- [5] M. Kawamura and M. Nakai, A test of Picard principle for rotation free densities, II, J. Math. Soc. Japan, 28 (1976), 323-342.
- [6] M. Nakai, A test for Picard principle, Nagoya Math. J., 56 (1974), 363-370.
- [7] M. Nakai, A test of Picard Principle for rotation free densities, J. Math. Soc. Japan, 27 (1975), 412-431.
- [8] M. Nakai, Picard principle and Riemann theorem, Tôhoku Math. J., 28 (1976), 277-292.
- [9] M. Ozawa, Some classes of positive solutions of $\Delta u = Pu$ on Riemann surfaces I, II, Kôdai Math. Sem. Rep., 6 (1954), 121-126, 7 (1955), 15-20.

Mitsuru NAKAI

Department of Mathematics
Nagoya Institute of Technology
Gokiso, Showa, Nagoya 466
Japan

Added in proof. After the present paper had been accepted for publication, the following paper, closely related to ours methodologically, appeared: A. Boukricha, Das Picard-Prinzip und verwandte Fragen bei Störung von harmonischen Räumen, Math. Ann., 239 (1979), 247-270.