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Minimal length of Liouville chain for solutions of an algebraic differential equation

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§ 0. Introduction.

The author [5] proved that the order of a liouvillian element in Liouville's sense is at most 3 if it satisfies an algebraic differential equation of the first order. Here, we shall generalize his theorem as follows: The order of a liouvillian element in Liouville's sense is at most $3n$ if it satisfies an algebraic differential equation of order n.

Let k be an ordinary differential field of characteristic 0, and Ω be a universal extension of k. We assume that the field of constants k_0 of k is algebraically closed. A finite chain of extending differential subfields $L_0 \subset L_1 \subset \cdots \subset L_n$ in Ω is called a *Liouville chain* over k if the following three conditions are satisfied:

(i) L_0 is an algebraic extension of k of finite degree:

(ii) The field of constants of L_n is k_0:

(iii) For each i $(1 \leqq i \leqq n)$ there exists a finite system of elements $w_1, \cdots, w_r$ of L_i which satisfies the following two conditions; either $w'_j \in L_{i-1}$ or w'_j/w_j is the derivative of an element of L_{i-1} for each j $(1 \leqq j \leqq r)$; L_i is an algebraic extension of $L_{i-1}(w_1, \cdots, w_r)$ of finite degree.

A subfield L of Ω is called a *liouvillian extension* of k if there exists a Liouville chain over k which ends with L. Let z be an element of Ω. Then, z is called a *liouvillian element* over k if there exists a Liouville chain over k such that its end contains z. In particular, if $k = k_0(x)$ with $x' = 1$, then a liouvillian element over k is called an *elementary transcendental function* of x over k_0 (cf. Watson [9, p. 111]). The following definition is due to Liouville [3]: A liouvillian element z over k is said to be of *order* m if m is the minimum of those n such that the end of a Liouville chain $L_0 \subset \cdots \subset L_n$ over k contains z.

THEOREM. *The order of a liouvillian element over k satisfying an algebraic differential equation over k of order n is at most $3n$.*

It follows from the following:

LEMMA. *Let k^* be a finitely generated differential extension field of k in Ω*

whose field of constants is k_0, and L^* be a differential extension field of k^* in Ω. Suppose that L^* is contained in a liouvillian extension K^* of k^*. Then, there exists such an extending chain $L_0 \subset \cdots \subset L_n$ of differential subfields of Ω that satisfies the following conditions: L_0 is an algebraic extension of k^* of finite degree; $n = \text{tr.}\deg_{k^*} L^*$; $L^* \subset L_n$; for each i $(0 < i \leqq n)$ there exist an element y_i of L_i and elements α_i, β_i of L_{i-1} such that $y_i' = \alpha_i y_i + \beta_i$ and L_i is an algebraic extension of $L_{i-1}(y_i)$ of finite degree.

In § 1 we shall show that Theorem follows from Lemma. In § 3 we shall prove Lemma. In the last § 4 an example of liouvillian element in Theorem whose order attains $3n$ will be given. In § 2 we shall show the following:

PROPOSITION. *Let A be a differential extension field of k in Ω. Suppose that two elements t_1, t_2 of Ω are algebraically independent over A and satisfy $t_i' = a_i t_i + b_i$ $(i=1, 2)$; here we assume that each of a_1, b_1 is algebraic over A and each of a_2, b_2 is algebraic over $A(t_1)$. Let B be a differential extension field of A in Ω. Suppose that t_1 is transcendental over B, t_2 is algebraic over $B(t_1)$ and the field of constants of $B(t_1)$ is k_0. Then, there exist an element t of B_1 and elements a, b of A_1 such that t is transcendental over A and $t' = at + b$, where A_1 and B_1 are the algebraic closures of A and B respectively.*

REMARK 1. If we replace "$w_j' \in L_{i-1}$" in the definition of a liouvillian element by "$w_j' = a'/a$, $a \in L_{i-1}$", then we have an "elementary" liouvillian element. For such an element let us modify the definition of "order" by the above replacement. Then the order of an elementary liouvillian element satisfying an algebraic differential equation of order n is at most $2n$. This theorem is due to Singer [**8**] (cf. Rosenlicht and Singer [**7**, Theorem 1]). In the special case where $n=1$ and $k = \boldsymbol{C}(x)$ with $x'=1$ it is due to Mordukhai-Boltovskoi [**4**] (cf. Ritt [**6**, p. 86]).

REMARK 2. Our definition of "liouvillian extension" is slightly stronger than the ordinary one. If we replace "$w_j'/w_j = a'$, $a \in L_{i-1}$" by "$w_j'/w_j \in L_{i-1}$", then we have the ordinary definition. The difference is not essential (cf. § 1). If we modify our definition of "order" by this replacement, then $3n$ in our result is replaced by $2n$.

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§ 1. Kolchin's existence theorem.

Kolchin [**2**] obtained the following theorem: Let Λ be a proper prime differential ideal in the differential polynomial algebra $k\{u_1, \cdots, u_n\}$ over k, and let J be an element of $k\{u_1, \cdots, u_n\}$ which is not in Λ. Then Λ has a solution $(\eta_1, \cdots, \eta_n)$ such that $J(\eta_1, \cdots, \eta_n) \neq 0$ and the field of constants of $k\langle \eta_1, \cdots, \eta_n \rangle$

is k_0.

By this theorem we shall show that our Theorem follows from Lemma: Suppose that y is a liouvillian element over k satisfying an algebraic differential equation over k of order m. Let L denote $k\langle y\rangle$. Then, L is contained in a liouvillian extension of k. The transcendence degree n of L over k is at most m. If we set $k=k^*$ and $L=L^*$, then the assumption of Lemma is satisfied. Hence there exists such an extending chain $L_0\subset\cdots\subset L_n$ of differential subfields in Ω as stated in Lemma. There exist elements $v_{01},\cdots,v_{0n},w_{01},\cdots,w_{0n},z_{01},\cdots,z_{0n}$ of Ω such that

$$v'_{0i}=\alpha_i\,;\quad w'_{0i}=v'_{0i}w_{0i},\quad w_{0i}\neq 0\,;\quad w_{0i}z'_{0i}=\beta_i$$

for each $i\,(1\leqq i\leqq n)$. We set $K=L_n$ for simplicity. Let Σ be a prime differential ideal in the differential polynomial algebra

$$K\{V_1,\cdots,V_n,W_1,\cdots,W_n,Z_1,\cdots,Z_n\}$$

over K whose generic point is $(v_{01},\cdots,v_{0n},w_{01},\cdots,w_{0n},z_{01},\cdots,z_{0n})$, and T be $\prod W_i\ (1\leqq i\leqq n)$. Then, $T\notin\Sigma$. The field of constants of K is k_0, since K is an algebraic extension of L. By the existence theorem of Kolchin there exists a zero $(v_1,\cdots,v_n,w_1,\cdots,w_n,z_1,\cdots,z_n)$ of Σ in Ω such that $T(w_1,\cdots,w_n)\neq 0$ and the field of constants of

$$K\langle v_1,\cdots,v_n,w_1,\cdots,w_n,z_1,\cdots,z_n\rangle$$

is k_0. We have

$$(y_i/w_i)'-z'_i=0\,,\qquad 1\leqq i\leqq n$$

by $y'_i=\alpha_iy_i+\beta_i$. Then, there exists a constant c_i such that $y_i=w_i(z_i+c_i)$ for each $i\ (1\leqq i\leqq n)$. We have $c_i\in k_0$. Since K is an algebraic extension of $k(y_1,\cdots,y_n)$ of finite degree, there exists an element t of K such that $K=k(y_1,\cdots,y_n,t)$. We define a chain $M_0\subset M_1\subset\cdots\subset M_{3n}$ by

$$M_0=k(\alpha_1,\beta_1)\,,$$
$$M_{3i-2}=M_{3i-3}(v_i)\qquad(1\leqq i\leqq n)\,,$$
$$M_{3i-1}=M_{3i-2}(w_i)\qquad(1\leqq i\leqq n)\,,$$
$$M_{3i}=M_{3i-1}(z_i,\alpha_{i+1},\beta_{i+1})\qquad(1\leqq i<n)\,,$$
$$M_{3n}=M_{3n-1}(z_n,t)\,.$$

This is a Liouville chain over k and M_{3n} contains y.

§ 2. Proof of Proposition.

We shall prove that there exist an element t_3 of $B_1(t_1, a_2, b_2)$ and elements a_3, b_3 of $A_1(t_1, a_2, b_2)$ such that t_3 is transcendental over $A_1(t_1)$ and $t_3'=a_3t_3+b_3$. Let C and D denote $A_1(t_1, a_2, b_2)$ and $B_1(t_1, a_2, b_2)$ respectively, and G be the minimal polynomial of t_2 over D:

$$G(T)=T^g+v_1T^{g-1}+\cdots+v_g, \qquad v_i\in D \ (1\leqq i\leqq g).$$

Then, differentiating $G(t_2)=0$ we have

$$t_2'\{gt_2^{g-1}+(g-1)v_1t_2^{g-2}+\cdots+v_{g-1}\}+v_1't_2^{g-1}+\cdots+v_g'=0.$$

Hence

$$v_1'=v_1a_2-gb_2$$

by $G(t_2)=0$ and $t_2'=a_2t_2+b_2$. Thus

$$(gt_2+v_1)'=a_2(gt_2+v_1).$$

Suppose that v_1 is algebraic over C. Then gt_2+v_1 is transcendental over C and algebraic over D. Since k_0 is algebraically closed, the field of constants of $D(t_2)$ is k_0. Hence there exists a positive integer q such that $(gt_2+v_1)^q\in D$. As t_3 we can take $(gt_2+v_1)^q$: $t_3'=qa_2t_3$. If v_1 is transcendental over C, then we can take v_1 as t_3. Thus the existence of t_3, a_3 and b_3 is proved. We consider C and D as one-dimensional algebraic function fields over A_1 and B_1 respectively. There exists a prime divisor P of C such that $\nu_P(t_1)<0$, where ν_P is the normalized valuation belonging to P. Let τ be prime element in P such that $\tau\in C$. Then $\nu_P(\tau')>0$ by $t_1'=a_1t_1+b_1$: For, $t_1=\sigma^{-e}$ with some prime element σ in P: We have $-e\sigma'=a_1\sigma+b_1\sigma^{e+1}$ and $\nu_P(\sigma')>0$. There exists uniquely a prime divisor Q of D such that the restriction of ν_Q^* to C is ν_P, where ν_Q^* is the normalized valuation belonging to Q. In this Q, τ is a prime element. The completion C_P of C with respect to P is a differential extension of C, and the completion D_Q of D with respect to Q is a differential extension of D; the differentiation is continuous in each completion (cf. Chevalley [**1**, p. 114]). The latter D_Q is a differential extension of the former C_P. In D_Q we have

$$\tau'=\sum f_i\tau^{i+d}, \qquad f_0\neq 0,\ f_i\in A_1,\ d>0,$$

$$a_3=\sum a_i^*\tau^{i+s}, \qquad a_0^*\neq 0,\ a_i^*\in A_1,$$

$$b_3=\sum b_i^*\tau^{i+r}, \qquad b_0^*\neq 0,\ b_i^*\in A_1,$$

$$t_3=\sum \gamma_i\tau^{i+p}, \qquad \gamma_0\neq 0,\ \gamma_i\in B_1,\ 0\leqq i<\infty;$$

here we assume that $f_i=a_i^*=b_i^*=0$ if $i<0$. We shall prove that $\gamma_j\notin A_1$ for some j. To the contrary suppose that each of γ_i is in A_1. Since t_1 and t_3 are algebraically dependent over B_1, we have

$$\sum e_{ij} t_1^i t_3^j = 0 \qquad (1 \leqq i,\ j \leqq \lambda),\ e_{ij} \in B_1\,,$$

where some e_{ij} is not 0. Let $\{\omega_1, \cdots, \omega_\mu\}$ be a basis of the linear space spanned by all e_{ij} over A_1. Then for each i, j $(1 \leqq i,\ j \leqq \lambda)$

$$e_{ij} = \sum \delta_{ijh} \omega_h \qquad (1 \leqq h \leqq \mu),\ \delta_{ijh} \in A_1\,.$$

We have

$$0 = \sum \{\sum \delta_{ijh} t_1^i t_3^j\} \omega_h \qquad (1 \leqq h \leqq \mu\,;\ 1 \leqq i,\ j \leqq \lambda)\,.$$

By our assumption $t_1,\ t_3 \in A_1((\tau))$. Hence we have

(1) $$\sum \delta_{ijh} t_1^i t_3^j \in A_1((\tau)) \qquad (1 \leqq i,\ j \leqq \lambda)$$

for each h $(1 \leqq h \leqq \mu)$. Since $\omega_1, \cdots, \omega_\mu$ are linearly independent over A_1, each of (1) is 0. Since t_1, t_3 are algebraically independent over A_1, we have

$$\delta_{ijh} = 0\,, \qquad 1 \leqq i,\ j \leqq \lambda\,;\ 1 \leqq h \leqq \mu\,.$$

Hence, each of e_{ij} is 0. This is a contradiction. Thus we may suppose that $\gamma_j \notin A_1$ and $\gamma_i \in A_1$ $(0 \leqq i < j)$ for some j $(j \geqq 0)$. Differentiating the expression of t_3 in D_Q we have

(2) $$\sum \gamma_i' \tau^{i+p} + \{\sum (i+p) \gamma_i \tau^{i+p-1}\} \{\sum f_i \tau^{i+d}\}$$
$$= \{\sum a_i^* \tau^{i+s}\} \{\sum \gamma_i \tau^{i+p}\} + \sum b_i^* \tau^{i+r} \qquad (0 \leqq i < \infty)$$

by $t_3' = a_3 t_3 + b_3$. We shall see that $s \geqq 0$. To the contrary suppose that $s < 0$. Then comparing the coefficients of τ^{s+p+j}, we have

$$\gamma_{s+j}' + \sum (i+p) \gamma_i f_m \qquad (i+m+d-1-s = j\,;\ i \geqq 0)$$
$$= b_{s+p+j-r}^* + \sum a_i^* \gamma_m \qquad (i+m = j\,;\ m \geqq 0)\,;$$

here we assume that $\gamma_{s+j} = 0$ if $s+j < 0$. Since $a_0^* \neq 0$, we have $\gamma_j \in A_1$. This is a contradiction. Hence $s \geqq 0$. Comparing the coefficients of τ^{p+j} in (2), we have

$$\gamma_j' + \sum (i+p) \gamma_i f_m \qquad (i+m+d-1 = j\,;\ i \geqq 0)$$
$$= b_{p+j-r}^* + \sum a_i^* \gamma_m \qquad (i+m+s = j\,;\ m \geqq 0)\,.$$

Hence, $\gamma_j' = a \gamma_j + b$ with $a = a_{-s}^* - (j+p) f_{1-d}$ and

$$b = b_{p+j-r}^* + \sum \{a_{j-i-s}^* + (i+p) f_{j-i-d+1}\} \gamma_i :$$

Here i runs through $0, \cdots, j-1$, because $d > 0$ and $s \geqq 0$. We have $b \in A_1$. Since γ_j is transcendental over A_1, we can take γ_j as t.

§ 3. Proof of Lemma.

Let Λ be the set of all pairs (k^*, L^*) satisfying the assumption of Lemma. For each pair (k^*, L^*) of Λ there exists such an extending chain $N_0 \subset N_1 \subset \cdots \subset N_f$ in Ω that satisfies the following condition:

(iv) N_0 is an algebraic extension of k^* of finite degree; L^* is contained in N_f; $\mathrm{tr.deg}_{k^*} N_f = f$; the field of constants of N_f is k_0; for each i $(1 \leqq i \leqq f)$ there exist an element t_i of N_i and elements a_i, b_i of N_{i-1} such that $t_i' = a_i t_i + b_i$ and N_i is an algebraic extension of $N_{i-1}(t_i)$ of finite degree. For example we can make it from a Liouville chain over k^* which ends with K^*. For a pair (k^*, L^*) of Λ let us define $f(k^*, L^*)$ as the minimum of those f such that the condition (iv) is satisfied. Then

$$f(k^*, L^*) \geqq \mathrm{tr.deg}_{k^*} L^* .$$

Our Lemma asserts that the equality holds. To the contrary suppose that the subset Γ of all pairs (k^*, L^*) of Λ for which the equality does not hold is not empty. Let Γ_e be the set of all pairs (k^*, L^*) of Γ such that $\mathrm{tr.deg}_{k^*} L^* = e$. Let n be the minimum of those e such that Γ_e is not empty, and m be the minimum of $f(k^*, L^*)$ where (k^*, L^*) runs over all elements of Γ_n. Then, $m > n \geqq 1$. We assume that $(k^*, L^*) \in \Gamma_n$, $f(k^*, L^*) = m$ and $N_0 \subset \cdots \subset N_m$ satisfies (iv) for (k^*, L^*) with $f = m$. Consider $(N_1, N_1(L^*))$. It belongs to Λ (cf. § 1). We have $\mathrm{tr.deg}_{N_1} N_1(L^*) \leqq n$ and $N_1 \subset \cdots \subset N_m$ satisfies (iv) for $(N_1, N_1(L^*))$. Hence, $(N_1, N_1(L^*)) \notin \Gamma$ because of the minimality of m and n. Let e be the transcendence degree of $N_1(L^*)$ over N_1. Then, there exists an extending chain $H_0 \subset \cdots \subset H_e$ satisfying the condition (iv) for $(N_1, N_1(L^*))$. The chain $N_0 \subset H_0 \subset \cdots \subset H_e$ satisfies the condition (iv) for (k^*, L^*). Hence, $e = n$ and $m = n+1$. Thus, t_1 is transcendental over L^* and t_2 is algebraic over $L^*(t_1)$. Two elements t_1 and t_2 are algebraically independent over k^* by the assumption (iv). By Proposition there exist an element t of the algebraic closure of L^* and elements a, b of the algebraic closure of k^* such that $t' = at + b$ and t is transcendental over k^*. The transcendence degree of $L^*(a, b, t)$ over $k^*(a, b, t)$ is $n-1$. Hence, $(k^*(a, b, t), L^*(a, b, t)) \in \Lambda - \Gamma$. There exists a chain $H_0^* \subset H_1^* \subset \cdots \subset H_{n-1}^*$ satisfying the condition (iv) for $(k^*(a, b, t), L^*(a, b, t))$. The chain $k^*(a, b) \subset H_0^* \subset H_1^* \subset \cdots \subset H_{n-1}^*$ satisfies (iv) for (k^*, L^*). Thus $f(k^*, L^*)$ is n. This is a contradiction.

§ 4. An example.

We assume that $k = k_0(x)$ with $x' = 1$. In the differential polynomial algebra $k\{u_1, \cdots, u_n\}$ over k we define F_i $(1 \leqq i \leqq n)$ by

$$F_1 = u_1' - u_1/(\alpha x) - 1/(\alpha x + 1), \qquad \alpha \in k_0,\ \alpha \neq 0,$$

$$F_i = (u_{i-1}+1)(u_{i-1}u_i' - u_i) - u_{i-1}, \qquad (2 \leqq i \leqq n).$$

There exists a solution $(y_1, \cdots, y_n)$ of $F_1 = F_2 = \cdots = F_n = 0$ in Ω. Suppose that α is not a rational number. Then, the element y_n is proved to be a liouvillian one over k. It satisfies an algebraic differential equation over k of order n. It can be shown that the order of y_n over k is $3n$.

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