Holomorphic continuation of solutions of partial differential equations across the multiple characteristic surface

By Yoshimichi TSUNO

§ 1. Introduction.

Let $P(z, \partial_z)$ be a linear partial differential operator of order m with holomorphic coefficients defined near a point p in C^n and Ω be an open set such that the point p belongs to the C^2-boundary $\partial\Omega$. In this paper we shall study the following problem:

(Q) Let u be a holomorphic solution of $Pu=f$ in Ω, where f is holomorphic near p. What conditions on Ω and P garantees that u can be continued across p?

The results already obtained to this problem are in the cases where $\partial\Omega$ is tangent to the holomorphic hypersurface S which is non-characteristic with respect to P (Bony-Schapira [2], Zerner [10]) or simple characteristic (Pallu de La Barriere [5], Persson [6], Tsuno [7]) or some other special cases ([5], [8], [9]). In [5], the case where the tangential operator of P on $\partial\Omega$ has the regular characteristic variety is studied. In [6] Persson also studied the general case using the so called "cones of analytic continuation". The purpose of this paper is to extend these results to the case where $P(z, \partial_z)$ is highly degenerated at p. Since the problem (Q) is invariant under the holomorphic change of variables, it is desirable to describe the results free from the choice of the local coordinates. But the treatment of the normal direction of $\partial\Omega$ at p and the tangential directions of $\partial\Omega$ at p is different, so we introduce the weighted coordinates in the next section. The weighted coordinates are systematically used to determine the type of $\partial\Omega$ at p by T. Bloom and I. Graham [1]. The weighted coordinate system used in this article is the simplest one such that the complex normal coordinate z_1 of $\partial\Omega$ at p is assigned the weight 2, while the complex tangential coordinates $z_2, z_3, \cdots, z_n$ are each assigned the weight 1. In the second section, some invariances are shown under the equivalent change of the weighted coordinates. Then in the third section, we state the basic theorem under some fixed local coordinates. The

method used here is the analogy of that of Zerner [10]. Zerner's proof is based on the quantitative estimate of the existence-domain of solutions of the Cauchy-Kowalewsky theorem. We use the Goursat problem instead of the Cauchy problem. All the necessary estimates to obtain the basic theorem are already known (see Hörmander [3] Theorem 5.1.1). As a corollary of this basic theorem, another proof of the simple characteristic case ([7], Corollary 1) is also obtained. In the last section, § 4, we study the geometric conditions on $P(z, \partial_z)$ and $\partial\Omega$ to insure the existence of the local coordinates in the third section. All the conditions in this section are invariant under some equivalent change of variables. Interesting is the fact that these conditions are formally similar to the Levi-condition of the constant multiple characteristic surface S and the bicharacteristic space of the localization of P in the normal direction at p which is due to Hörmander [4]. The results in this paper in the two dimensional case is already announced in [11]. The author wishes to thank Professor J. Persson who points out to him the mistakes of the original manuscript.

§ 2. Weighted coordinates.

Since the problem (Q) is local, we always assume that p is the origin of the local coordinates. Let $(z_1, z_2, \cdots, z_n)$ be a local coordinate system. Then we say that $(z_1, \cdots, z_n)$ is the weighted coordinate system with the weights $(2, 1, \cdots, 1)$ if the coordinate function z_1 has the weight 2, while z_j $(j=2, \cdots, n)$ has the weight 1. A monomial in z has the weight l if the sum of the weights of z_j which occur is l. For a holomorphic function $f(z)$, we say that $f(z)$ has the weight l if, among the monomials in the Taylor series expansion of $f(z)$, there is one of weight l but none of lower weight. For convenience, the weight of $f=0$ is assigned $+\infty$. It is easy to see that if f_1 and f_2 are holomorphic functions,

(2.1)   weight $(f_1+f_2) \geqq$ min (weight (f_1), weight (f_2))

(2.2)   weight $(f_1 f_2)=$ weight $(f_1)+$ weight (f_2) .

Next we assign corresponding negative weights to differential operators. We begin by assigning weight -2 to $\partial/\partial z_1$ and weight -1 to $\partial/\partial z_j$ $(j=2, \cdots, n)$. The weight of differential monomial $(\partial/\partial z)^\alpha = (\partial/\partial z_1)^{\alpha_1} \cdots (\partial/\partial z_n)^{\alpha_n}$, where $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index, is determined by $-\alpha_1 - |\alpha| = -(2\alpha_1 + \alpha_2 + \cdots + \alpha_n)$. Similarly the weight of $a(z)(\partial/\partial z)^\alpha$ is defined by weight $(a(z))+$ weight $(\partial/\partial z)^\alpha$. Lastly the weight of a linear partial differential operator $P(z, \partial_z) = \sum_{|\alpha| \leq m} a_\alpha(z)(\partial/\partial z)^\alpha$ is determined by min weight $(a_\alpha(z)(\partial/\partial z)^\alpha)$. These definitions

are due to T. Bloom and I. Graham [1]. Corresponding to the properties (2.1) and (2.2), the following relations are easily derived.

(2.3) $\text{weight } (P_1 + P_2) \geqq \min (\text{weight } (P_1), \text{weight } (P_2))$

(2.4) $\text{weight } (P_1 \circ P_2) \geqq \text{weight } (P_1) + \text{weight } (P_2)$

(2.5) $\text{weight } (Pf) \geqq \text{weight } (P) + \text{weight } (f)$

for non-zero differential operators P, P_1, P_2 and a holomorphic function f.

DEFINITION 2.1 (T. Bloom and I. Graham [1]). Let $(z_1, \cdots, z_n)$ and $(w_1, \cdots, w_n)$ be local coordinates with the same origin. We say that these coordinate systems are equivalent as the weighted coordinates if the coordinate function w_j has the same weight as z_j as a holomorphic function of z and the converse is also true.

REMARK 2.1. In this paper the weight of the coordinate system is always $(2, 1, \cdots, 1)$. Therefore $(z_1, \cdots, z_n)$ and $(w_1, \cdots, w_n)$ are equivalent if and only if

$$\frac{\partial(w_1, \cdots, w_n)}{\partial(z_1, \cdots, z_n)}(0) = \begin{vmatrix} c & 0 & \cdots\cdots & 0 \\ a_2 & c_{22} & \cdots\cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & & \vdots \\ a_n & c_{n2} & \cdots\cdots & c_{nn} \end{vmatrix} \neq 0 .$$

REMARK 2.2. The weights of functions or differential operators are invariant under the equivalent weighted coordinate system.

By this remark the following proposition is easy to prove.

PROPOSITION 2.1. *Let* $P(z, \partial_z)$ *be a linear differential operator of the weight* l *and denote by* $Q(z, \partial_z)$ *the sum of the terms in* P *with the weight* l. *Then* $Q(z, \partial_z)$ *is invariant modulo differential operators of the weight larger than* l *under the equivalent change of the weighted coordinate systems.*

PROOF. If we decompose $P = Q + Q'$ such that the weight of Q' is larger than l, then under the equivalent change of variables, Q' is transformed to the operator of the weight $>l$. While the weight of P is invariant. This shows that Q is invariant modulo differential operators of the weight larger than l.

DEFINITION 2.2. The sum of the terms in $P(z, \partial_z)$ with the lowest weight is said to be the weighted principal part of $P(z, \partial_z)$.

Lastly we introduce some notion concerning the relation between a complex hypersurface S and a real C^2-hypersurface M. Let T_p be the holomorphic tangent spase of S at p. If A' is a complex linear subspace of T_p, and M is tangent to S at p, then we make the following definition.

DEFINITION 2.3. The real hypersurface M is said to be tangent to S at p of the second order holomorphically in A' if for some defining function ρ of

M, $(XY\rho)(p)=0$ for all holomorphic vector fields X and Y such that X and Y are tangent to S and $X(p)$ or $Y(p)$ is in Λ'.

REMARK 2.3. This definition is independent of the choice of the defining function ρ of M. Indeed if ρ' is another defining function of M then there exists a positive C^1 function g such that $\rho'=g\rho$ and $\rho(D^2g)$, which is well defined where $\rho\neq0$, is uniquely extended as 0 when $\rho=0$, where D^2 denotes any differentiation of order 2.

REMARK 2.4. It is well known that for a suitable choice of holomorphic coordinates with the origin p, M has the defining function ρ such that

$$\rho(z,\ \bar{z})=z_1+\bar{z}_1+\sum_{j,\ k\geq2}c_{j\bar{k}}z_j\bar{z}_k+o(|z|^2).$$

In this case M is tangent to $S=\{z_1=0\}$ at 0 of the second order holomorphically in any subspace Λ'. The restrictive case is happened when the surface S is previously given.

§3. The basic theorem.

We begin this section by recalling the following theorem which is in Hörmander [3].

THEOREM 3.1 ([3] Theorem 5.1.1). *Consider a differential equation*

$$(3.1) \qquad \left(\frac{\partial}{\partial z}\right)^\beta u=\sum_{|\alpha|\leq|\beta|}a_\alpha(z)\left(\frac{\partial}{\partial z}\right)^\alpha u+f,$$

where the coefficients $a_\alpha(z)$ are holomorphic on the closed polydisc $\Omega=\{z|\ |z_j|\leq r_j,$ $j=1,\cdots,n\}$. Pose the boundary conditions

$$(3.2) \qquad \left(\frac{\partial}{\partial z_j}\right)^k(u-\phi)=0 \qquad \text{when } z_j=0 \quad \text{if } 0\leq k<\beta_j;\ j=1,\cdots,n.$$

Set $\displaystyle A=\max_{z\in\Omega}\sum_{|\alpha|\leq|\beta|}\frac{r^\beta}{r^\alpha}|a_\alpha(z)|$ *where $r=(r_1,\cdots,r_n)$ and assume that $A(2e)^{|\beta|}<1$. Then the boundary problem (3.1), (3.2) has one and only one holomorphic solution u in*

$$\Omega'=\left\{z;\ z\in\Omega,\ \prod_1^n\left(1-\frac{|z_j|}{r_j}\right)^m>A(2e)^{|\beta|}\right\}$$

for arbitrary functions f and ϕ which are holomorphic on Ω.

REMARK 3.1. It is well known that if the functions ϕ_j^k, $0\leq k\leq\alpha_j$, are holomorphic in the plane $z_j=0$ and satisfy the compatibility conditions

$$\left(\frac{\partial}{\partial z_i}\right)^l\phi_j^k=\left(\frac{\partial}{\partial z_j}\right)^k\phi_i^l \qquad \text{on } z_i=z_j=0,$$

then there exists a function ϕ holomorphic on $\{z;\ |z_j|\leqq r_j\}$, satisfying the boundary conditions

$$\left(\frac{\partial}{\partial z_j}\right)^k\phi\,|_{z_j=0}=\phi_j^k.$$

We now study the problem (Q) in the introduction. The differential operator studied in this section is the following one:

$$(3.3)\qquad P(z,\partial_z)=\left(\frac{\partial}{\partial z_1}\right)^{m-l}\left(\frac{\partial}{\partial z_2}\right)^l+\sum_\alpha a_\alpha(z)\left(\frac{\partial}{\partial z}\right)^\alpha\qquad (1\leqq l<m)$$

where $a_\alpha(z)$ are holomorphic in some neighborhood U of 0 and the summation is taken over the multi-indices α such that $|\alpha|\leqq m$ and $\alpha\neq(m-l,l,0,\cdots,0)$. The domain Ω is given by

$$(3.4)\qquad \Omega=\{z\in U\mid \rho(z,\bar z)<0\}$$

where ρ is a real-valued C^2 function such that

$$(\Omega.1)\qquad \rho(0)=0,\quad \frac{\partial\rho}{\partial z_1}(0)=1,\quad \frac{\partial\rho}{\partial z_j}(0)=0\qquad j=2,\cdots,n.$$

The local coordinates $(z_1,\cdots,z_n)$ are always fixed in this section and regarded as the weighted coordinate system with the weights $(2,1,\cdots,1)$. Then we make the following conditions on the operator $P(z,\partial_z)$.

(P.1) Every weight of $a_\alpha(z)(\partial/\partial z)^\alpha$ in $P(z,\partial_z)$ is larger than or equal to $l-2m$ =the weight of $(\partial/\partial z_1)^{m-l}(\partial/\partial z_2)^l$ and also every weight of $(a_\alpha(z)-a_\alpha(0))$ $\times(\partial/\partial z)^\alpha$ is larger than $l-2m$.

(P.2) There exists a positive integer μ $(2\leqq\mu\leqq n)$ such that if the weight of $a_\alpha(z)(\partial/\partial z)^\alpha$ is equal to $l-2m$, then $\alpha_{\mu+1}=\cdots=\alpha_n=0$.

(P.3) There are no terms except $(\partial/\partial z_1)^{m-l}(\partial/\partial z_2)^l$ of the weight $l-2m$ which are generated only by $\partial/\partial z_1$ and $\partial/\partial z_2$.

We remark here that if $\mu=2$ in (P.2), then these conditions mean that there is no term of the weight $l-2m$ in P except $(\partial/\partial z_1)^{m-l}(\partial/\partial z_2)^l$.

REMARK 3.2. If P is simple characteristic at $(0,N)$ with $N=(1,0,\cdots,0)$, then it is always possible to choose the local coordinates so that P is in the form (3.3) with $l=1$, and $\alpha_1<m-1$ in the sum of the second terms. In this case, all conditions (P.1), (P.2) with $\mu=2$ and (P.3) are automatically satisfied.

REMARK 3.3. It is easy to see that these conditions (P.1), (P.2) and (P.3) are only restrictive on the terms of order larger than or equal to $m-l/2$.

Concerning the boundary function $\rho(z,\bar z)$ of $\partial\Omega$, the following conditions are made in addition to $(\Omega.1)$.

(Ω.2) $\dfrac{\partial^2 \rho}{\partial x_2^2}(0) < 0$

where x_2 is the real part of z_2.

(Ω.3) $\dfrac{\partial^2 \rho}{\partial z_i \partial z_j}(0) = 0$ if $3 \leq i$ or $j \leq \mu$, $2 \leq i$, $j \leq n$

(Ω.4) $\dfrac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}(0) = 0$ if $3 \leq i$ or $j \leq \mu$, $2 \leq i$, $j \leq n$

where μ is the number taken in the condition (P.2). If $\mu = 2$, then conditions (Ω.3) and (Ω.4) become empty. Under these preparations we can state the basic theorem.

THEOREM 3.2. *Let $P(z, \partial_z)$ be a differential operator of the form (3.3) which satisfies the conditions (P.1), (P.2) and (P.3), and Ω be an open set given by (3.4) with the conditions (Ω.1), (Ω.2), (Ω.3) and (Ω.4). If $u(z)$ is a holomorphic solution of $Pu = f$ in Ω where f is holomorphic near 0, then $u(z)$ can be holomorphically prolonged across $\partial\Omega$ at 0.*

REMARK 3.4. If P is simple characteristic at $(0, N)$, then by the remark 3.2, (P.1), (P.2) and (P.3) always hold and the conditions (Ω.3), (Ω.4) are empty because $\mu = 2$. Moreover in this case z_2-axis is considered as the bicharacteristic curve. Thus this theorem is exactly reduced to the result of [7, Corollary 1].

For the rest of this section we devote ourselves to prove this theorem.

LEMMA 3.1. *Let $\rho(z, \bar{z})$ be a real valued C^2 function satisfying the conditions (Ω.1), (Ω.2), (Ω.3) and (Ω.4). Then there exist positive constants M and α such that for any small positive number ε*

(3.5) $\rho \leq \rho' = 2x_1 - \alpha x_2^2 + \dfrac{M}{\varepsilon}|z_1|^2 + \varepsilon(|z_3|^2 + \cdots + |z_\mu|^2)$

$\qquad\qquad\qquad + M(y_2^2 + |z_{\mu+1}|^2 + \cdots + |z_n|^2)$

if z is sufficiently near 0, where $z_j = x_j + \sqrt{-1}\,y_j$.

PROOF. We expand ρ in the Taylor series up to the order 2. The first order part is equal to $z_1 + \bar{z}_1 = 2x_1$. For the terms of the second order we divide into the next four groups by (Ω.3) and (Ω.4): (i) terms containing only z_2 and $\bar{z}_2$, (ii) terms containing z_1 or $\bar{z}_1$, (iii) terms consisted by the products of z_2 or $\bar{z}_2$ and $z_{\mu+1}, \cdots, z_n$ or $\bar{z}_{\mu+1}, \cdots, \bar{z}_n$, (iv) quadratic terms of $z_{\mu+1}, \cdots, z_n$ and these complex adjoints. For the sums of the terms in each groups (i), (ii), (iii) and (iv), we estimate these by $-\alpha' x_2^2 + \beta y_2^2$ with $\alpha' > 0$, $\beta > 0$ because of (Ω.2), $(A/\varepsilon')|z_1|^2 + \varepsilon'(|z_2|^2 + \cdots + |z_n|^2)$ for some constant A and any $\varepsilon' > 0$, $\lambda|z_2|^2 + (B/\lambda)(|z_{\mu+1}|^2 + \cdots + |z_n|^2)$ for some B and any $\lambda > 0$, and $C(|z_{\mu+1}|^2 + \cdots + |z_n|^2)$, respectively. If we take $\lambda + \varepsilon' < \alpha'$ then (3.5) is easily derived.

LEMMA 3.2. *For any small positive numbers δ and λ ($\lambda < \varepsilon/2M$), the polydisc*

$$\{z \mid z_1 = -\lambda, \ |z_2 - \delta| \leqq r_2, \ |z_j| \leqq r_j, \ j = 3, \cdots, n\}$$

on the surface $z_1 = -\lambda$ is contained in Ω provided that

(3.6)
$$\begin{cases} r_2^2 = \dfrac{\lambda}{M} + \dfrac{\alpha}{\alpha + M}\delta^2 \\[2mm] r_3^2 = \cdots = r_\mu^2 = \dfrac{\lambda}{2n\varepsilon} \\[2mm] r_{\mu+1}^2 = \cdots = r_n^2 = \dfrac{\lambda}{2nM} \end{cases}$$

PROOF. On the surface $z_1 = -\lambda$,

$$\rho' = -2\lambda - \alpha x_2^2 + \frac{M\lambda^2}{\varepsilon} + \varepsilon(|z_3|^2 + \cdots + |z_\mu|^2)$$

$$+ M(y_2^2 + |z_{\mu+1}|^2 + \cdots + |z_n|^2).$$

If $|z_j|^2 \leqq r_j^2$ ($j \geqq 3$) and $\lambda < \varepsilon/2M$, then

$$\rho' < -2\lambda - \alpha x_2^2 + \frac{M\lambda^2}{\varepsilon} + My_2^2 + \frac{\lambda}{2}$$

$$< -\lambda - \alpha x_2^2 + My_2^2.$$

Then it is easy to see that if $(x_2 - \delta)^2 + y_2^2 \leqq \dfrac{\lambda}{M} + \dfrac{\alpha}{\alpha + M}\delta^2$, then $\rho' < 0$. This proves the lemma.

LEMMA 3.3. *For any small positive number δ and λ $\left(\delta \leqq 4\sqrt{\dfrac{\varepsilon}{\alpha M}}, \ \lambda < \dfrac{\varepsilon}{M}\right)$, the polydisc*

$$\{z \mid |z_1 + \lambda| \leqq r_1, \ z_2 = \delta, \ |z_j| \leqq r_j, \ j = 3, \cdots, n\}$$

on the surface $z_2 = \delta$ is contained in Ω provided that

(3.7)
$$\begin{cases} r_1 = \lambda + \dfrac{1}{8}\alpha\delta^2 \\[2mm] r_3^2 = \cdots = r_\mu^2 = \dfrac{\alpha}{2n\varepsilon}\delta^2 \\[2mm] r_{\mu+1}^2 = \cdots = r_n^2 = \dfrac{\alpha}{2nM}\delta^2. \end{cases}$$

PROOF. On the surface $z_2 = \delta$

$$\rho' = 2x_1 - \alpha\delta^2 + \frac{M}{\varepsilon}(x_1^2 + y_1^2) + \varepsilon(|z_3|^2 + \cdots + |z_\mu|^2)$$

$$+ M(|z_{\mu+1}|^2 + \cdots + |z_n|^2).$$

If $|z_j|^2 \leq r_j^2$ $(j \geq 3)$, then

$$\rho' < \frac{M}{\varepsilon} \left\{ \left(x_1 + \frac{\varepsilon}{M} \right)^2 + y_1^2 - \left(\frac{1}{2}\alpha\delta^2 + \frac{\varepsilon}{M} \right)\frac{\varepsilon}{M} \right\}.$$

Here if we take $\delta \leq 4(\varepsilon/\alpha M)^{1/2}$ then

$$\left\{ \left(\frac{1}{2}\alpha\delta^2 + \frac{\varepsilon}{M} \right)\left(\frac{\varepsilon}{M} \right) \right\}^{1/2} \geq \frac{\varepsilon}{M} + \frac{1}{8}\alpha\delta^2.$$

Thus $\rho' < 0$ for z_1 satisfying the inequality $|z_1 + \lambda| \leq \lambda + (1/8)\alpha\delta^2$ if $\lambda < \varepsilon/M$. This proves the lemma.

LEMMA 3.4. *There exist constants k $(k>1)$ and γ $(\gamma > \alpha)$ which depend only on α and M, such that*

$$\Delta_1 = \{z \mid z_1 = -\gamma\delta^2, \ |z_2 - \delta| \leq r_2, \ |z_j| \leq r_j \quad j = 3, \cdots, n\}$$

and

$$\Delta_2 = \{z \mid |z_1 + \gamma\delta^2| \leq r_1, \ z_2 = \delta, \ |z_j| \leq r_j \quad j = 3, \cdots, n\}$$

are contained in Ω and the inequalities

(3.8) $r_1 \geq \gamma\delta^2 k, \qquad r_2 \geq k\delta$

hold for sufficiently small δ $(\delta > 0)$, where

(3.9)
$$\begin{cases} r_1 = \left(\gamma + \dfrac{\alpha}{8} \right)\delta^2 \\[2mm] r_2^2 = \left(\dfrac{\gamma}{M} + \dfrac{\alpha}{\alpha+M} \right)\delta^2 \\[2mm] r_3^2 = \cdots = r_\mu^2 = \dfrac{\alpha}{2n\varepsilon}\delta^2 \\[2mm] r_{\mu+1}^2 = \cdots = r_n^2 = \dfrac{\alpha}{2nM}\delta^2. \end{cases}$$

PROOF. Take k $(1 < k < 9/8)$ and γ $(\gamma > \alpha)$ so that the next inequalities hold:

(3.10) $\dfrac{\alpha}{8(k-1)} \geq \gamma \geq M\left(k^2 - \dfrac{\alpha}{\alpha+M} \right).$

And set $\lambda = \gamma\delta^2$ in the lemmas 3.2 and 3.3. Then Δ_1 and Δ_2 are contained in Ω for sufficiently small δ. The inequalities (3.8) are easily derived from (3.10). This completes the proof.

PROOF OF THE THEOREM 3.2. Let $P(z, \partial_z)$ be a differential operator considered in the theorem 3.2 and $\Omega(r(\delta))$ be the polydisc $\{z \mid |z_j| \leq r_j\}$ where r_j is given by (3.9). Set $A = \max\limits_{z \in \Omega(r)} \sum\limits_\alpha \dfrac{r_1^{m-l}r_2^l}{r^\alpha} |a_\alpha(z)|$. Then if the weight of

$a_\alpha(z)(\partial/\partial z)^\alpha$ is larger than $l-2m$, then

$$\frac{r_1^{m-l}r_2^l}{r^\alpha}|a_\alpha(z)|=o(1) \qquad \text{as } \delta\to 0.$$

As for the terms of the weight $l-2m$,

$$\frac{r_1^{m-l}r_2^l}{r^\alpha}|a_\alpha(z)|\leq\frac{r_1^{m-l}r_2^l}{r^\alpha}|a_\alpha(z)-a_\alpha(0)|+\frac{r_1^{m-l}r_2^l}{r^\alpha}|a_\alpha(0)|$$

$$=O(\delta)+\frac{r_1^{m-l}r_2^l}{r^\alpha}|a_\alpha(0)|.$$

But in the second terms of these estimates, $\alpha_j\neq 0$ for some j $(3\leq j\leq\mu)$ by the conditions (P.2) and (P.3). Thus if we take ε small enough, then these terms are also arbitrary small since $r_j^{-1}=\text{const. }\varepsilon^{1/2}$. Therefore the constant A becomes arbitrary small if we take ε and δ small enough. Now we fix ε and δ_0 so that $A(2e)^m<(1-k^{-1})^{2m}$ for all $\delta\leq\delta_0$ where k is the number in the lemma 3.4. Let $u(z)$ be any solution of $Pu=f$ in Ω. We consider $u(z)$ to be the solution of the Goursat problem with the boundary conditions given on $\varDelta_1$ and $\varDelta_2$ for sufficiently small δ so that $\varDelta_1$ and $\varDelta_2$ are contained in $\Omega(r(\delta_0))$. Applying the theorem 3.1, $u(z)$ is holomorphic in

$$\varDelta=\left\{z\left|\left(1-\left|\frac{z_1+\gamma\delta^2}{r_1}\right|\right)^m\left(1-\left|\frac{z_2-\delta}{r_2}\right|\right)^m\prod_3^n\left(1-\left|\frac{z_j}{r_j}\right|\right)^m>A(2e)^m\right.\right\}.$$

Then by (3.8), the origin is contained in $\varDelta$, which proves the theorem.

REMARK 3.5. Persson [6] constructed the solution $u(z)$ of $Pu=0$ which had singularities on the set $\text{Re }z_1>0$ where P is in the similar form as (3.3).

§4. Choice of the local coordinates in the basic theorem.

In this section we seek the geometric conditions on $P(z, \partial_z)$ and Ω which insure the existence of a local coordinate system under which $P(z, \partial_z)$ and Ω satisfy all the conditions in the basic theorem. Let $(z_1, \cdots, z_n)$ be a local coordinates such that the surface $z_1=0$ is tangent to $\partial\Omega$ at $z=0$. We consider this coordinates at the weighted coordinate system with the weights $(2, 1, \cdots, 1)$. Let $P(z, \partial_z)$ be a linear differential operator of order m with holomorphic coefficients which is characteristic at 0 in the cotangential direction $N=(1, 0, \cdots, 0)$ and l $(1\leq l<m)$ be the multiplicity of P at $(0, N)$. That is

(4.1) $\qquad P_m(0, N+t\zeta)=p(\zeta)t^l+\text{higher order terms of } t$

where P_m is the principal part of P and $p(\zeta)$ is a non-zero polynomial of ζ. (4.1) means that in $P_m(0, \partial/\partial z)$ there is none of the terms of order larger than

$m-l$ with respect to $\partial/\partial z_1$ and the sum of the coefficients of $(\partial/\partial z_1)^{m-l}$ is equal to $p(\partial/\partial z_2, \cdots, \partial/\partial z_n)$. Since p is the homogeneous polynomial of order l, the weight of $p(\partial/\partial z_2, \cdots, \partial/\partial z_n)(\partial/\partial z_1)^{m-l}$ is $l-2m$. We now assume that

(P.I)   the weight of $P(z, \partial_z)$ is equal to $l-2m$.

DEFINITION 4.1.   A holomorphic function $\phi(z)$ with $\mathrm{grad}_z\phi(0)=N$ is said to be a weighted characteristic function of $P(z, \partial_z)$ if it satisfies the following condition

(4.2) $e^{-t\phi(z)}Q(z, \partial_z)e^{t\phi(z)}\equiv 0$ (mod weight $l-2m+1$)

where $Q(z, \partial_z)$ is the weighted principal part of P and the complex parameter t is assigned the weight -2.

Since $Q(z, \partial_z)$ is invariantly defined modulo operators of the weight larger than or equal to $l-2m+1$ (Proposition 2.1), the above definition is also invariant under the equivarent weighted coordinate change. To find such a weighted characteristic function $\phi(z)$, it is sufficient that ϕ is in the form

$$\phi(z)=z_1+\sum_{i, j\geq 2} a_{ij}z_iz_j.$$

Assume that

(P.II)   there exists a weighted characteristic function $\phi(z)$.

By the suitable equivalent change of the weighted coordinates, we can assume that $\phi(z)=z_1$. Then we have the next proposition.

PROPOSITION 4.1.   *If $\phi(z)=z_1$, then (4.2) is equivalent to that there is none of the differential monomials in $Q(z, \partial_z)$ which are generated only by $\partial/\partial z_1$.*

PROOF.   It is easy and omitted.

We now fix some weighted characteristic function $\phi(z)$ and consider the local coordinates $(z_1, z_2, \cdots, z_n)$ as $\phi(z)=z_1$ and each z_j $(j\geq 2)$ has the weight 1. This means that the coordinate transformation considered from now on is always in the following form:

(4.3) $\begin{cases} w_1=z_1 \\ w_j=a_jz_1+\sum\limits_{k=2}^{n}c_{jk}z_k+f_j(z) \qquad j=2, \cdots, n \end{cases}$

where $\det(c_{jk})\neq 0$ and $f_j(z)=O(|z|^2)$ (cf. Remark 2.1). Under some of these coordinate system if $P(z, \partial_z)$ is written as

$$P(z, \partial_z)=\sum_{k=0}^{m} p_k(z, \partial/\partial z')(\partial/\partial z_1)^k$$

where $\partial/\partial z'=(\partial/\partial z_2, \cdots, \partial/\partial z_n)$, then the weight of p_k is larger than or equal to $l-2m+2k$. We set $Q_k(z, \partial_{z'})$ the sum of the terms in p_k of the weight $l-2m+2k$.

PROPOSITION 4.2. *Each $Q_k(z, \partial_{z'})$ is invariant modulo differential operators of the weight larger than $l-2m+2k$ under the coordinate change of the form (4.3).*

PROOF. By (4.3),

$$\frac{\partial}{\partial z_1} = \frac{\partial}{\partial w_1} + \sum_{j=2}^{n} a_j \frac{\partial}{\partial w_j} + \text{terms of the weight} \geq 0$$

$$\frac{\partial}{\partial z_j} = \sum_{k=2}^{n} c_{kj} \frac{\partial}{\partial w_k} + \text{terms of the weight} \geq 0.$$

Therefore by recalling the relation (2.4)

$$(4.4) \quad \begin{cases} \left(\frac{\partial}{\partial z_1}\right)^p = \left(\frac{\partial}{\partial w_1}\right)^p + \text{terms of the weight} > -2p \\ \left(\frac{\partial}{\partial z_j}\right)^q = \left(\sum_{k=2}^{n} c_{kj} \frac{\partial}{\partial w_k}\right)^q + \text{terms of the weight} > -q. \end{cases}$$

This proves that each sum of the coefficients of $(\partial/\partial z_1)^k$ of the lowest weight $l-2m+2k$, $Q_k(z, \partial_{z'})$, is invariant modulo operators of the weight larger than $l-2m+2k$ under these transformation.

PROPOSITION 4.3. *The polynomial $Q_k(0, \xi')$ is invariant under the transformation (4.3) if ξ' is transformed as the cotangent vector at 0.*

PROOF. By the same way of the preceding proof this is easily derived from (4.4) and the details are omitted.

Relating to the polynomial $Q_k(0, \xi')$, we introduce some complex linear subspaces in the holomorphic tangent space T_0 and the cotangent space T_0^* of the surface $S = \{z_1 = 0\}$ at 0. Define the complex subspaces $\Lambda(Q_k)$ and $\Lambda(Q)$ as follows:

$$(4.5) \quad \begin{cases} \Lambda(Q_k) = \{\eta' \in T_0^* \mid Q_k(0, \xi' + t\eta') = Q_k(0, \xi') \quad \text{for any } t, \xi'\} \\ \Lambda(Q) = \bigcap_k \Lambda(Q_k). \end{cases}$$

Then $\Lambda'(Q)$ is settled by

$$\Lambda'(Q) = \{v \in T_0 \mid \langle v, \eta \rangle = 0 \quad \text{for any } \eta \in \Lambda(Q)\}$$

where $\langle , \rangle$ denotes the contraction between cotangent vectors and tangent vectors. By the proposition 4.3, these subspaces are invariant for the coordinate transformation of the type (4.3). We remark here that these subspaces Λ and Λ' are originally introduced by Hörmander [4] to analyse the singular support of distribution solutions of $P(D)u=0$.

DEFINITION 4.1. $\Lambda'(Q)$ is called the weighted bicharacteristic space of P at $(0, N)$ with respect to the weighted characteristic function $\phi(z)$.

REMARK 4.1. If P is simple characteristic at $(0, N)$ then $\Lambda'(Q)$ is generated

by the vector $(P_m^{(1)}(0, N), \cdots, P_m^{(n)}(0, N))$ which is the bicharacteristic direction of P at $(0, N)$.

We now make the third assumption for P as follows:

(P.III)   weight $(Q_k(z, \partial_{z'}) - Q_k(0, \partial_{z'})) \geq l - 2m + 2k + 1$.

By the proposition 4.2, this assumption is invariant for the change of variables of the form (4.3).

REMARK 4.2. Since Q_k is the coefficient of $(\partial/\partial z_1)^k$, its order of differentiation is not larger than $m - k$. Moreover $Q_k(0, \xi')$ is a homogeneous polynomial in ξ' of degree $2m - 2k - l$, because the weight of $Q_k(0, \partial_{z'})(\partial/\partial z_1)^k$ is equal to $l - 2m$. Thus the inequalities $0 \leq 2m - 2k - l \leq m - k$ implies that $m - l \leq k \leq m - l/2$. If $k = m - l/2$ (l even), $Q_k(0, \xi')$ becomes constant. Then by the proposition 4.1 $Q_k(0, \xi') = 0$. Consequently $Q_k(0, \xi')$ is not necessarily zero only if $m - l \leq k < m - l/2$.

REMARK 4.3. $Q_{m-l}(0, \xi')$ is equal to the localization of $P_m(z, \xi)$ at $(0, N)$, which is due to Hörmander. Indeed the relation

$$P_m(0, N + t\xi) = Q_{m-l}(0, \xi')t^l + \text{higher order terms of } t$$

is easy to prove.

The last assumption for P is the following one:

(P.IV)   there exists a non-zero cotangent vector ξ_0' at 0 such that

(i) $Q_{m-l}(0, \xi_0') \neq 0$

(ii) $Q_k(0, \xi_0') = 0$ $(m - l < k < m - l/2)$.

We remark that ξ_0' is not contained in $\Lambda(Q)$ because by (i) we have the relation

$$Q_{m-l}(0, \xi_0') \neq 0 = Q_{m-l}(0, \xi_0' + (-1)\xi_0')$$

which shows that $\xi_0' \notin \Lambda(Q_{m-l})$.

Now we proceed to examine the conditions on Ω under the assumptions (P.I), (P.II), (P.III) and (P.IV).

Let ξ_0' be the covector in (P.IV) and set

(4.6) $\begin{cases} \Lambda'(\xi_0') = \{v \in T_0 \mid \langle v, \xi_0' \rangle = 0\} \\ \Lambda_1' = \Lambda'(\xi_0') \cap \Lambda'(Q). \end{cases}$

Since $\xi_0' \notin \Lambda(Q)$, $\Lambda'(\xi_0')$ is not contained in $\Lambda'(Q)$ and its dimension is equal to $n - 2$. If the dimension of $\Lambda'(Q)$ is $\mu - 1$, where $\mu \geq 2$ because Q is not identically zero, we assume that

(Ω.I) $\dim \Lambda_1' = \mu - 2$.

In addition to this assumption, we suppose that

(Ω.II)   the Levi form of $\partial\Omega$ is degenerate on Λ_1',

(Ω.III) $\partial\Omega$ is tangent to $S=\{z_1=0\}$ at 0 of the second order holomorphically in Λ_1' (cf. Definition 2.3).

If Ω is given by $\{\rho(z)<0\}$ and the complex Hessian form of ρ at 0 is denoted by

$$H_{\rho(0)}(t, s) = \sum_{i,j} \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}(0) dz_i(t) d\bar{z}_j(\bar{s})$$

for two holomorphic tangent vectors t and s in T_0, then (Ω.II) means that the linear form on T_0, $H_{\rho(0)}(\cdot, s)$, vanishes for all s in Λ_1'.

REMARK 4.4.   These conditions are invariant under the holomorphic change of variables. Moreover they are also invariant from the choice of the defining function $\rho(z)$ (see Remark 2.3).

The last assumption for $\partial\Omega$ is the following one:

(Ω.IV)   there exist a holomorphic curve $\zeta(t)$ in $S=\{z_1=0\}$ and a non-zero complex number t_0 such that

$$\zeta(0)=0, \quad d\zeta(0)\in\Lambda'(Q), \quad \langle d\zeta(0), \xi_0'\rangle\neq 0$$

and

$$\frac{d^2}{d\tau^2}\rho(\zeta(t_0\tau))|_{\tau=0}<0$$

for a real parameter τ.

Now we construct the local coordinates $(z_1, \cdots, z_n)$ so that the operator $P(z, \partial_z)$ is reduced to the form (3.3) and all assumptions in the basic theorem are fulfilled.

First we fix some weighted characteristic function $\phi(z)$ and set $\phi(z)=z_1$.

Secondly we fix the tangential coordinates $(z_2, \cdots, z_n)$ so that the next conditions are satisfied:

(4.7) $\begin{cases} \text{(i)} \quad \partial/\partial z_2 = d\zeta(0) \\ \text{(ii)} \quad \Lambda_1' \text{ is generated by } \partial/\partial z_3, \cdots, \partial/\partial z_\mu . \end{cases}$

This is indeed possible because $d\zeta(0)$ is not contained in Λ_1'. Under these coordinates $\Lambda'(Q)$ is generated by the vectors $\partial/\partial z_2, \partial/\partial z_3, \cdots, \partial/\partial z_\mu$ because the codimension of Λ_1' in $\Lambda'(Q)$ is equal to 1.

Now we check all the conditions in the basic theorem.

PROPOSITION 4.4.   *Under these coordinates we have that* $Q_{m-l}(0, dz_2)\neq 0$, $Q_k(0, dz_2)=0$ $(m-l<k<m-l/2)$.

PROOF. If we write the covector ξ_0' as

$$\xi_0' = c_2 dz_2 + \cdots + c_n dz_n$$

then by (4.6)

$$\langle \xi_0', \partial/\partial z_j \rangle = 0 \qquad j = 3, \cdots, \mu.$$

Thus $\xi_0' = c_2 dz_2 + \eta'$ where $\eta' = c_{\mu+1} dz_{\mu+1} + \cdots + c_n dz_n$ which is in $A(Q)$. Therefore by (4.5)

$$Q_k(0, \xi_0') = Q_k(0, \xi_0' - \eta') = Q_k(0, c_2 dz_2)$$

which proves the proposition.

PROPOSITION 4.5. *Under these coordinates, $P(z, \partial_z)$ can be written as in the form (3.3).*

POOOF. This is easy from the preceding proposition because Q_{m-l} is the coefficient of $(\partial/\partial z_1)^{m-l}$.

PROPOSITION 4.6. (P.1) *is equivalent to* (P.I) *and* (P.III).

PROOF. Trivial.

PROPOSITION 4.7. (P.2) *follows from* (P.III) *and* (4.7).

PROOF. If the weight of $a_\alpha(z)(\partial/\partial z)^\alpha$ in $P(z, \partial_z)$ is equal to $l-2m$, then by (P.III), $a_\alpha(z)$ does not vanish at 0. Therefore $a_\alpha(0)\xi^\alpha$ is the term in $Q(0, \xi)$ which is by (4.7) generated only by $\xi_2, \cdots, \xi_\mu$.

PROPOSITION 4.8. (P.3) *follows from* (P.III) *and* (P.IV).

PROOF. Let $a(z)(\partial/\partial z_1)^p(\partial/\partial z_2)^q$ be the term in $P(z, \partial_z)$ with the weight $l-2m$. Then by (P.III) $a(0)$ is not zero. Thus the weight of $a(z)$ is equal to zero. Therefore we have the equality $l-2m = -2p-q = -p-(p+q)$, which shows that $p \geq m-l$ (because $p+q \leq m$). If $p > m-l$, $a(0)\xi_2^q$ is the term in $Q_p(0, \xi')$ which contradicts the proposition 4.4.

Since the assumption $(\Omega.1)$ is easily followed from the choice of the weighted characteristic function, we examine the rest of the conditions with respect to $\partial\Omega$.

PROPOSITION 4.9. $(\Omega.2)$, $(\Omega.3)$ and $(\Omega.4)$ are derived from $(\Omega.\mathrm{II})$, $(\Omega.\mathrm{III})$ and $(\Omega.\mathrm{IV})$.

PROOF. It is easy and omitted.

Summing up these propositions we have the next theorem.

THEOREM 4.1. *Let $P(z, \partial_z)$ be a differential operator of order m with holomorphic coefficients in a neighborhood of p and Ω be an open set with C^2-boundary $\partial\Omega \ni p$. We suppose that $P(z, \partial_z)$ and Ω satisfy the conditions* (P.I)$\sim$(P.IV) *and* $(\Omega.\mathrm{I})\sim(\Omega.\mathrm{IV})$. *Under these assumptions if $u(z)$ is a holomorphic solution of $Pu = f$ in Ω where f is holomorphic near p, then $u(z)$ can be holomorphically prolonged across $\partial\Omega$ at p.*

References

[1] T. Bloom and I. Graham, On 'Type' Conditions for Generic Real Submanifolds of C^n, Invent. Math., **40** (1977), 212-243.

[2] J. M. Bony and P. Schapira, Existence et prolongement des solutions holomorphes des équations aux dérivées partielles, Invent. Math., **17** (1972), 95-105.

[3] L. Hörmander, Linear Partial Differential Operators, 2nd revised printing, SpringerVerlag, 1964.

[4] L. Hörmander, On the singularities of solutions of partial differential equations, Comm. Pure Appl. Math., **23** (1970), 329-358.

[5] P. Pallu de La Barrière, Existence et prolongement des solutions holomophes des équations aux dérivées partielles, J. Math. Pures Appl., **55** (1976), 21-46.

[6] J. Persson, Local analytic continuation of holomorphic solutions of partial differential Equations, Ann. Mat. Pura Appl., **112** (1977), 193-204.

[7] Y. Tsuno, On the prolongation of local holomorphic solutions of partial differential equations, J. Math. Soc. Japan, **26** (1974), 523-548.

[8] Y. Tsuno, On the prolongation of local holomorphic solutions of partial differential equations, II, prolongation across the pluri-harmonic hypersurface, J. Math. Soc. Japan, **28** (1976), 304-306.

[9] Y. Tsuno, On the prolongation of local holomorphic solutions of partial differential equations, III, equations of the Fuchsian type, J. Math. Soc. Japan, **28** (1976), 611-616.

[10] M. Zerner, Domaines d'holomorphie des fonctions vérifiant une équation aux dérivées partielles, C. R. Acad. Sci. Paris, **272** (1971), 1646-1648.

[11] Y. Tsuno, Holomorphic continuation of solutions of degenerate partial differential equations in two variables, Proc. Japan Acad., **54** (1978), 226-229.

Yoshimichi TSUNO

Department of Mathematics
Okayama University
Tsushima, Okayama 700
Japan