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Cartan subgroups of a Lie group

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§1. Introduction.

By an analytic group and by an analytic subgroup of a Lie group, we mean a connected Lie group and a connected Lie subgroup, respectively. An analytic subgroup and the corresponding Lie algebra will be denoted by the same capital script and capital Roman letter, respectively. For example, if $\mathcal{L}$ is an analytic group and $\mathcal{G}$ an analytic subgroup of $\mathcal{L}$, then L will denote the Lie algebra of $\mathcal{L}$ and G the subalgebra of L corresponding to $\mathcal{G}$.

Let $\mathcal{G}$ be an analytic group, and H a Cartan subalgebra of G. We shall call $\mathcal{H}$ a *Cartan subgroup* of $\mathcal{G}$. Any Cartan subgroup of $\mathcal{G}$ is closed in $\mathcal{G}$. For the closure of a subset $\mathcal{M}$ of a topological space, we adopt the notation $\overline{\mathcal{M}}$. In this paper, we shall prove the following theorems:

THEOREM 5. *Let $\mathcal{L}$ be an analytic group, and $\mathcal{G}$ an analytic subgroup of $\mathcal{L}$. Then there exists a Cartan subgroup $\mathcal{H}$ of $\mathcal{G}$ such that $\bar{\mathcal{G}}=\overline{\mathcal{H}}\mathcal{G}$. This implies, in particular, if $\mathcal{G}$ is non-closed, then so is $\mathcal{H}$.*

THEOREM 6. *Let $\mathcal{G}$ be an analytic subgroup of $GL(n, \boldsymbol{R})$. For any Cartan subgroup $\mathcal{H}$ of $\mathcal{G}$ we have $\bar{\mathcal{G}}=\overline{\mathcal{H}}\mathcal{G}$.*

REMARK. In Theorem 5 and 6, we can replace $\bar{\mathcal{G}}=\overline{\mathcal{H}}\mathcal{G}$ by $\bar{\mathcal{G}}=\overline{Z(\mathcal{H})}\mathcal{G}$ where $Z(\mathcal{H})$ is the center of $\mathcal{H}$. Indeed, since $\mathcal{H}$ is nilpotent, we have that $\overline{\mathcal{H}}=\overline{Z(\mathcal{H})}\mathcal{H}$.

Let L be a Lie algebra. A Cartan subalgebra of L is, by definition, the eigenspace corresponding to 0 of any regular inner derivation of L. Also we know that a subalgebra H of L is Cartan if and only if H is nilpotent and coincides with its normalizer.

In [2], Gantmacher proved that for complex semisimple Lie algebras we can define Cartan subalgebras using inner automorphisms rather than derivations. We shall generalize the result of Gantmacher, and get a new definition of Cartan subalgebras. Let $\mathcal{L}$ be an analytic group. For $a\in\mathcal{L}$, we denote by $\mathrm{Ad}\,a$ the (inner) automorphism of L induced by the inner automorphism $x\mapsto axa^{-1}$ of the group $\mathcal{L}$.

THEOREM 1. *Let $\mathcal{L}$ be an analytic group, and let x be a regular element of $\mathcal{L}$, i.e., the multiplicity of the eigenvalue one of* $\mathrm{Ad}\,x$ *is the smallest. Let H be the eigenspace corresponding to the eigenvalue one of* $\mathrm{Ad}\,x$.

(i) *H is a Cartan subalgebra, and any Cartan subalgebra of L can be obtained in this manner.*

(ii) *If in particular* $\mathcal{L}$ *is a complex analytic group, then* $x\in\mathcal{H}$.

As a consequence of Theorem 1, we get

THEOREM 2. *Let* $\mathcal{L}$ *be an analytic group, and* $\mathcal{G}$ *a dense analytic subgroup of* $\mathcal{L}$.

(i) *For any Cartan subalgebra H of L, the intersection* $H\cap G$ *is a Cartan subalgebra of G, and* $\mathcal{L}=\mathcal{H}\mathcal{G}$.

(ii) *Let* H_G *be a Cartan subalgebra of G. Then there exists a Cartan subalgebra H of L such that*

$$H_G=H\cap G\,.$$

In this paper, we call two subalgebras of a Lie algebra (or two subgroups of a group) being *conjugated* if they are conjugated to each other with respect to inner automorphisms.

Let $\mathcal{L}$ be an analytic group. All maximal toral subgroups of $\mathcal{L}$ are conjugated to each other. The following results give relations between Cartan subgroups and toral subgroups.

THEOREM 3. *Let* $\mathcal{L}$ *be an analytic group.*

(i) *For any maximal toral subgroup* $\mathcal{T}$ *of* $\mathcal{L}$, *there exists a Cartan subgroup* $\mathcal{H}$ *with* $\mathcal{H}\supset\mathcal{T}$.

(ii) *Any Cartan subgroup of* $\mathcal{L}$ *contains some maximal toral subgroup* $\mathcal{T}_{\mathcal{R}}$ *of the radical* $\mathcal{R}$ *of* $\mathcal{L}$.

In order to prove Theorem 6, we use a slight extension of a theorem in Goto [**5**]:

THEOREM 4. *Let* $\mathcal{G}$ *be an analytic group, and let f be a continuous one-one homomorphism from* $\mathcal{G}$ *into* $GL(n, \boldsymbol{R})$. *Then we can find a closed subgroup* $\mathcal{V}\cong\boldsymbol{R}^k$ *in the radical of* $\mathcal{G}$, *and a closed connected normal subgroup* $\mathcal{N}$, *of* $\mathcal{G}$, *such that* $\mathcal{G}$ *is a semi-direct product*

$$\mathcal{G}=\mathcal{V}\mathcal{N}\,,\qquad \mathcal{V}\cap\mathcal{N}=\{1\}\,,$$

$\overline{f(\mathcal{V})}$ *is a toral group,* $f(\mathcal{N})$ *is closed, and* $\overline{f(\mathcal{G})}$ *is an almost semi-direct product of* $\overline{f(\mathcal{V})}$ *and* $f(\mathcal{N})$:

$$\overline{f(\mathcal{G})}=\overline{f(\mathcal{V})}f(\mathcal{N})\,,\qquad \overline{f(\mathcal{V})}\cap f(\mathcal{N})\ \ \textit{is finite.}$$

In this case, $\overline{f(\mathcal{G})}$ *is diffeomorphic with the direct product space* $\overline{f(\mathcal{V})}\times\mathcal{N}$.

Theorem 6 cannot be extended to the general case. A counter-example will be given in the end of the paper.

For the sake of completeness, some of the known results are reproved in this paper.

§ 2. Cartan subalgebras.

Let W be a vector space of dimension n over $\boldsymbol{R}$ (or $\boldsymbol{C}$), and let σ be a linear transformation of W. For $\alpha \in \boldsymbol{R}$ (or $\boldsymbol{C}$), we adopt the notation

$$W(\sigma\,;\,\alpha)=\{w\in W\,;\,(\sigma-\alpha)^n w=0\}\,.$$

Then $W=W(\sigma\,;\,\alpha)+(\sigma-\alpha)^n W$ (direct sum of vector spaces.)

We first recall the classical definition of Cartan subalgebras. Let L be a Lie algebra of dimension n (over $\boldsymbol{R}$ or $\boldsymbol{C}$). For any X in L, the space $L_0=L(\operatorname{ad} X\,;\,0)$ forms a subalgebra of L, and denoting $L_*=(\operatorname{ad} X)^n L$ we have

$$L=L_0+L_*\,,\qquad L_0\cap L_*=\{0\}\,,$$

$$[X,\,L_*]=[L_0,\,L_*]=L_*\,.$$

For a subalgebra S of L, let $n(S)$ denote the normalizer of S: $n(S)=\{X\in L\,;\,[X,\,S]\subset S\}$. Then we have $n(L_0)=L_0$. Indeed, if $Y\in L_*$ normalizes L_0, i.e. $[Y,\,L_0]\subset L_0$, then $[Y,\,L_0]\subset L_*\cap L_0=\{0\}$ and $[Y,\,X]=0$, whence $Y\in L_0$ and $Y=0$.

PROPOSITION 1. *Let X_0 be in L and*

$$L_0=L(\operatorname{ad} X_0\,;\,0)\,,\qquad L_*=(\operatorname{ad} X_0)^n L\,.$$

Then the map

$$\varphi:\ L_0\times L_*\ni(X,\,Y)\mapsto\exp(\operatorname{ad} Y)\cdot(X_0+X)\in L$$

is a diffeomorphism from a suitable neighborhood of $(0,\,0)$ *in* $L_0\times L_*$ *onto a neighborhood of X_0 in L.*

PROOF. For a real parameter t

$$\exp(\operatorname{ad}(tY))\cdot(X_0+tX)=X_0+t(X-(\operatorname{ad} X_0)Y)+0(t^2)\,,$$

where $0(t^2)$ is a power series of t starting with a term of t^2. Since $X+Y\mapsto X-(\operatorname{ad} X_0)Y$ is a non-singular linear transformation of L, the map φ is a local diffeomorphism at $(0,\,0)$. Q. E. D.

For X in L we put

$$\lambda(X)=\dim L(\operatorname{ad} X\,;\,0)\,,$$

and call the minimum value of $\lambda(X)$ the *rank* of L: $\operatorname{rank} L=l$. An element X of L is said to be *regular* if $\lambda(X)=l$. For a regular X, the subalgebra $L(\operatorname{ad} X\,;\,0)$ is called *Cartan*. A Cartan subalgebra is nilpotent. Conversely, if a subalgebra H of L is nilpotent and $n(H)=H$, then H is known to be a Cartan subalgebra.

For X in L let $c(t,\,X)$ denote the characteristic polynomial of $\operatorname{ad} X$:

$$c(t, X) = \text{the determinant of } (t - \text{ad } X)$$

$$= t^n + c_{n-1}(X)t^{n-1} + \cdots + c_l(X)t^l ,$$

where $c_i(X)$ are polynomial functions of X and $c_l \neq 0$. Then X is regular if and only if $c_l(X) \neq 0$. Let L_{reg} denote the set of all regular elements of L. Since $c_l = 0$ defines an algebraic set in L, the set L_{reg} is open and dense in L, and if in particular L is a Lie algebra over $\boldsymbol{C}$ then we have that L_{reg} is connected, because the topological codimension of $L - L_{\text{reg}}$ is at least two.

PROPOSITION 2 (C. Chevalley). *All Cartan subalgebras of a complex Lie algebra L are conjugated to each other.*

PROOF. Let X_0 be a regular element in L. By Proposition 1, any element in a suitable neighborhood of X_0 can be written in a form

$$X = \exp(\text{ad } Y)X_1 , \qquad X_1 \in L(\text{ad } X_0 ;\ 0) ,$$

where we can suppose that X is regular, and so is X_1.

Since $L(\text{ad } X_0 ;\ 0)$ is a nilpotent Lie algebra, we have that $L(\text{ad } X_1 ;\ 0) \supset L(\text{ad } X_0 ;\ 0)$. Because X_1 is regular, this implies that $L(\text{ad } X_1 ;\ 0) = L(\text{ad } X_0 ;\ 0)$. Hence

$$L(\text{ad } X;\ 0) = \exp(\text{ad } Y)L(\text{ad } X_1 ;\ 0) = \exp(\text{ad } Y)L(\text{ad } X_0 ;\ 0) .$$

Thus there exists a neighborhood U of X_0 in L_{reg} such that $L(\text{ad } X;\ 0)$ is conjugated with $L(\text{ad } X_0 ;\ 0)$ for all $X \in U$. Since L_{reg} is open and connected, we have the proposition. Q. E. D.

§ 3. A definition of Cartan subalgebras.

In order to establish Theorem 1, we follow a method due to Gantmacher [2] and Matsushima [8].

Let $\mathcal{L}$ be an analytic group of dimension n. In a similar way as in § 2, we define

$$\mu(x) = \dim L(\text{Ad } x ;\ 1) \qquad \text{for} \quad x \in \mathcal{L} ;$$

call x (or $\text{Ad } x$) *regular* if $\mu(x)$ attains the minimum at x ; denote by $\mathcal{L}_{\text{reg}}$ the set of all regular elements in $\mathcal{L}$; then $\mathcal{L} - \mathcal{L}_{\text{reg}}$ is an analytic subset, and $\mathcal{L}_{\text{reg}}$ is open and dense in $\mathcal{L}$; if in particular $\mathcal{L}$ is a complex analytic group then $\mathcal{L}_{\text{reg}}$ is connected.

PROPOSITION 3. *Let x be in $\mathcal{L}$. We put*

$$L^1 = L(\text{Ad } x ;\ 1) , \qquad L^* = (\text{Ad } x - 1)^n L ,$$

and get $L = L^1 + L^*$, $L^1 \cap L^* = \{0\}$,

$$[L^1, L^1]\subset L^1, \qquad [L^1, L^*]\subset L^*, \qquad (\mathrm{Ad}\, x-1)L^*=L^*.$$

Furthermore, the map

$$\psi:\ L^1\times L^*\ni(X, Y)\mapsto \exp Y\cdot\exp X\cdot x\cdot\exp(-Y)$$

is a diffeomorphism from a suitable neighborhood of $(0, 0)$ *in* $L^1\times L^*$ *onto a neighborhood of* x *in* $\mathcal{L}$.

PROOF. About the first part see e. g. Goto-Grosshans [6]. Next we write

$$\exp tY\cdot\exp tX\cdot x\cdot\exp(-tY)=\exp f(t)\cdot x \qquad \text{for} \quad t\in\boldsymbol{R},$$

and get

$$f(t)=(Y+X-\mathrm{Ad}\, x\cdot Y)t+0(t^2).$$

On the other hand, the map $X+Y\mapsto X+(1-\mathrm{Ad}\, x)Y$ is a non-singular linear transformation. Q. E. D.

Next, we choose a neighborhood U of 0 in L so small that for every eigenvalue γ of $\mathrm{ad}\, X$ ($X\in U$), the absolute value of γ is less than 2π. Then we have

$$L(\mathrm{ad}\, X;\ 0)=L(\mathrm{Ad}\,(\exp X);\ 1)$$

and $\lambda(X)=\mu(\exp X)$. Hence the minimum of $\mu(x)$, $x\in\mathcal{L}$, coincides with rank L.

PROPOSITION 4. *Let* $\mathcal{L}$ *be an analytic group and let* $\mathcal{L}_{\mathrm{reg}}$ *denote the set of all regular elements in* $\mathcal{L}$. *For* $x\in\mathcal{L}_{\mathrm{reg}}$, *let* $\mathcal{L}(x)$ *denote the analytic subgroup of* $\mathcal{L}$ *corresponding to the Lie algebra* $L(\mathrm{Ad}\, x;\ 1)$. *Then for any* $x\in\mathcal{L}_{\mathrm{reg}}$, *there is a neighborhood* $\mathcal{V}(x)$ *of* x *such that* $\mathcal{V}(x)\subset\mathcal{L}_{\mathrm{reg}}$ *and for any* $y\in\mathcal{V}(x)$, $\mathcal{L}(y)$ *is conjugated with* $\mathcal{L}(x)$. *Furthermore the set* $\{x\in\mathcal{L}_{\mathrm{reg}};\ \mathcal{L}(x)\ni x\}$ *is open and closed in* $\mathcal{L}_{\mathrm{reg}}$.

PROOF. Let x be in $\mathcal{L}_{\mathrm{reg}}$. We put

$$L^1=L(\mathrm{Ad}\, x;\ 1), \qquad L^*=(\mathrm{Ad}\, x-1)^n L.$$

By Proposition 3, there exists a neighborhood $\mathcal{V}$ of x, $\mathcal{V}\subset\mathcal{L}_{\mathrm{reg}}$, such that any y in $\mathcal{V}$ is of the form

$$y=ax'a^{-1} \quad (x'\in\mathcal{L}(x)x,\ a\in\exp L^*),$$

where we can suppose that $(\mathrm{Ad}\, x'-1)^n: L^*\mapsto L$ is one-one. Since $(\mathrm{Ad}\, x')L^1=L^1$, we have that $L(\mathrm{Ad}\, x';\ 1)=L^1$. Hence

$$L(\mathrm{Ad}\, y;\ 1)=(\mathrm{Ad}\, a)L(\mathrm{Ad}\, x';\ 1)=(\mathrm{Ad}\, a)L^1,$$

and

$$\mathcal{L}(y)=a\mathcal{L}(x)a^{-1}.$$

On the other hand, $\mathcal{L}(x)\ni x$ if and only if $\mathcal{L}(x)\ni x'$. Hence $y=ax'a^{-1}\in\mathcal{L}(y)=a\mathcal{L}(x)a^{-1}$ if and only if $x\in\mathcal{L}(x)$. Q. E. D.

THEOREM 1. *Let $\mathcal{L}$ be an analytic group, and let x be a regular element of $\mathcal{L}$. Then*

(i) $L(\mathrm{Ad}\, x\,;\,1)$ *is a Cartan subalgebra, and conversely any Cartan subalgebra can be obtained in this way.*

(ii) *If in particular $\mathcal{L}$ is a complex analytic group, then* $x\in\exp L(\mathrm{Ad}\, x\,;\,1)$.

PROOF. First suppose that L is complex. For a regular element X in L, sufficiently close to 0, we have that $L(\mathrm{Ad}(\exp X)\,;\,1)=L(\mathrm{ad}\, X\,;\,0)$ is a Cartan subalgebra and $\mathcal{L}(\exp X)\ni\exp X$. Because $\mathcal{L}_{\mathrm{reg}}$ is connected, we have $\mathcal{L}(x)\ni x$ for all $x\in\mathcal{L}_{\mathrm{reg}}$, and all $\mathcal{L}(x)$ are conjugated, by Proposition 4. Since a Cartan subalgebra is nilpotent, and the exponential map of a nilpotent Lie group is surjective, we have

$$L(x)=\exp L(\mathrm{Ad}\, x\,;\,1)\,.$$

Next, we shall consider the general case. Let L^C denote the complexification of L. Then $\mathrm{Ad}\, x$ is regular and $L^C(\mathrm{Ad}\, x\,;\,1)$ is a Cartan subalgebra of L^C. Hence $L(\mathrm{Ad}\, x\,;\,1)=L^C(\mathrm{Ad}\, x\,;\,1)\cap L$ is a Cartan subalgebra of L. Also for any Cartan subalgebra H of L, we can find a regular element $X\in H$, arbitrary close to 0. Then $H=L(\mathrm{Ad}(\exp X)\,;\,1)$. Q. E. D.

COROLLARY. *Let $\mathcal{L}$ be a complex analytic group. For any x in $\mathcal{L}$, there exist X and Y in L with $x=\exp X\cdot\exp Y$.*

REMARK. The algebraic group version of the results in this section has been done in C. Chevalley, Theorie des Groupes de Lie III, Paris, Hermann, 1955.

§ 4. Cartan subgroups of a dense analytic group.

Let $\mathcal{L}$ be an analytic group, and let $\mathcal{G}$ be a dense analytic subgroup of $\mathcal{L}$. Then $[G, G]=[L, L]$, see e. g. Goto [4], and in particular

$$(\mathrm{ad}\, X)L\subset G \qquad \text{for} \quad X\in L\,,$$

whence we have $(\mathrm{Ad}\, x-1)L\subset G$ for all $x\in\mathcal{L}$. We define $\mu(x)=\dim L(\mathrm{Ad}\, x\,;\,1)$ and $\mu'(x)=\dim G(\mathrm{Ad}\, x\,;\,1)$ for $x\in\mathcal{L}$, and we get

$$\mu'(x)=\mu(x)-m \quad (m=\dim L-\dim G)\,.$$

Let x be a regular element of $\mathcal{L}$. By Proposition 4, there exists a neighborhood $\mathcal{V}$ of x such that for any $y\in\mathcal{V}$,

$$L(\mathrm{Ad}\, y\,;\,1)=(\mathrm{Ad}\, a)L(\mathrm{Ad}\, x\,;\,1) \qquad \text{for some} \quad a\in\mathcal{L}\,.$$

Since $\mathcal{G}$ is dense in $\mathcal{L}$, we can pick $y \in \mathcal{V} \cap \mathcal{G}$. Obviously, y is a regular element of $\mathcal{G}$. Hence $L(\mathrm{Ad}\, y\,;\, 1)$ is a Cartan subalgebra of L, $G(\mathrm{Ad}\, y\,;\, 1)$ is a Cartan subalgebra of G, and we have

$$L(\mathrm{Ad}\, y\,;\, 1) \cap G = G(\mathrm{Ad}\, y\,;\, 1)\,.$$

Hence

$$\begin{aligned} G(\mathrm{Ad}\, x\,;\, 1) &= (\mathrm{Ad}\, a^{-1}) L(\mathrm{Ad}\, y\,;\, 1) \cap G \\ &= (\mathrm{Ad}\, a^{-1})(L(\mathrm{Ad}\, y\,;\, 1) \cap G) \\ &= (\mathrm{Ad}\, a^{-1}) G(\mathrm{Ad}\, y\,;\, 1) \end{aligned}$$

is a Cartan subalgebra of G.

Conversely suppose that y is a regular element of $\mathcal{G}$. Since rank G $=$rank $L-m$, we have $\mu(y)=$rank L and y is regular in $\mathcal{L}$. Hence $L(\mathrm{Ad}\, y\,;\, 1)$ is a Cartan subalgebra of L.

Next, for any Cartan subalgebra H of L we have $L=H+[L, L]=H+[G, G]$ $=H+G$, whence $\mathcal{L}=\mathcal{H}\mathcal{G}$.

Thus we have the following theorem.

THEOREM 2. *Let $\mathcal{L}$ be an analytic group, and $\mathcal{G}$ a dense analytic subgroup of $\mathcal{L}$.*

(i) *For any Cartan subalgebra H of L, the intersection $H \cap G$ is a Cartan subalgebra of G, and $\mathcal{L}=\mathcal{H}\mathcal{G}$.*

(ii) *Let H_G be a Cartan subalgebra of G. Then there exists a Cartan subalgebra H of L such that $H_G=H \cap G$.*

§ 5. Maximal toral subgroups.

Let $\mathcal{L}$ be an analytic group. All maximal compact subgroups of $\mathcal{L}$ are connected and conjugated to each other. Let $\mathcal{K}$ be a compact analytic group. All maximal toral subgroups of $\mathcal{K}$ are conjugated to each other. Hence all maximal toral subgroups of $\mathcal{L}$ are conjugated to each other.

THEOREM 3 (i). *Let $\mathcal{L}$ be an analytic group, and $\mathcal{T}$ a maximal toral subgroup of $\mathcal{L}$. Then there exists a Cartan subalgebra H of L with $H \supset T$.*

PROOF. We set

$$L_0 = \{X \in L\,;\, [X, Y]=0 \quad \text{for all} \quad Y \in T\}\,.$$

We pick $Y_0 \in T$ such that $\exp \boldsymbol{R} Y_0$ is a dense one-parameter subgroup of $\mathcal{T}$ Then $L_0=L(\mathrm{ad}\, Y_0\,;\, 0)$. Let H be a Cartan subalgebra of L_0. Since T is central in L_0 we have that $H \supset T$. We shall prove that H is a Cartan subalgebra of L. Since H is nilpotent, it suffices to show that H coincides with the normalizer $n(H)$.

We put $L_*=(\mathrm{ad}\,Y_0)^{\dim L}L$, and

$$L=L_0+L_*, \quad L_0\cap L_*=\{0\}, \quad [Y_0, L_*]=[L_0, L_*]=L_*.$$

Let $X=Y+Z$ ($Y\in L_0$, $Z\in L_*$) be in $n(H)$. Then $[X, H]\subset[Y, H]+[Z, H]\subset H \subset L_0$, where $[Y, H]\subset L_0$ and $[Z, H]\subset L_*$. Hence we have $[Z, H]=0$ and $[Y, H]\subset H$. Therefore $Y\in H$ and $Z\in L_0\cap L_*=\{0\}$. Q. E. D.

§ 6. Cartan subalgebras of an algebraic Lie algebra.

Let L be a subalgebra of the Lie algebra $gl(n, \boldsymbol{R})$. If there is an algebraic group in $GL(n, \boldsymbol{R})$, whose Lie algebra is L, we call L *algebraic*. Let L be an algebraic Lie algebra, R the radical of L, and let N be the nil-radical of L, i. e. N is the set of all nilpotent matrices in R. Then we can find a maximal semisimple subalgebra S and an abelian algebraic subalgebra A composed of semisimple matrices such that

$$L=S+A+N, \quad \text{(direct sum of vector spaces)}$$

$$R=A+N, \quad [S, A]=\{0\}.$$

$S+A$ is called a *reductive part* of L. Let C be a reductive (=completely reducible) subalgebra of L. Then there exists an inner automorphism σ of L such that $\sigma C\subset S+A$. If in particular L is a nilpotent Lie algebra, then A is uniquely determined and $L=A+N$ is a direct sum of ideals. On these results see e. g. Borel [1].

Although the following proposition is known by Iwahori-Satake [7], we shall give a proof here, for the sake of completeness.

PROPOSITION 5. *Let L be an algebraic Lie algebra, and R the radical of L. Then any Cartan subalgebra of L contains a reductive part of R.*

PROOF. Let X_0 be a regular element of L. We put

$$L_0=L(\mathrm{ad}\,X_0\,;\,0), \quad L_*=(\mathrm{ad}\,X_0)^{\dim L}L$$

and

$$L=L_0+L_*, \quad L_0\cap L_*=\{0\}, \quad [X_0, L_*]=[L_0, L_*]=L_*.$$

Since R is invariant under $\mathrm{ad}\,X_0$,

$$R=R_0+R_*, \quad R_0\cap R_*=\{0\}, \quad [X_0, R_*]=R_*,$$

where $R_0=R\cap L_0$ and $R_*=R\cap L_*$. Both L_0 and R are algebraic, and so is R_0.

Let N be the nil-radical of L. We put $P=\boldsymbol{R}X_0+R$. Then P is a solvable Lie algebra and the commutator subalgebra $[P, P]$ is composed of nilpotent matrices. Hence $R_*\subset[P, P]\subset N$.

Let $R_0=A+N_0$ be the decomposition of the nilpotent algebraic Lie algebra R_0 into the reductive part A and the nil-radical N_0. Then $N_0\subset N$ and

$$R=A+N_0+R_*=A+N.$$

Hence A is a reductive part of R, and $A\subset L_0$. Q.E.D.

§7. Maximal toral subgroups of the radical.

THEOREM 3 (ii). *Let $\mathcal{L}$ be an analytic group, $\mathcal{R}$ the radical, and $\mathcal{H}$ a Cartan subgroup, of $\mathcal{L}$. Then $\mathcal{H}$ contains some maximal toral subgroup of $\mathcal{R}$.*

PROOF. (a) First suppose that $\mathcal{L}$ is an analytic subgroup of $GL(n, \boldsymbol{R})$.

For an analytic subgroup $\mathcal{S}$ of $GL(n, \boldsymbol{R})$, let $\mathcal{A}(\mathcal{S})$ denote the identity component of the smallest algebraic group containing $\mathcal{S}$. The Lie algebra of $\mathcal{A}(\mathcal{S})$ will be denoted by $A(S)\subset gl(n, \boldsymbol{R})$.

By Goto [3], $A(H)$ is a Cartan subalgebra of $A(L)$,

$$A(L)=A(H)+[L, L], \qquad H=L\cap A(H),$$

and $A(R)$ is the radical of $A(L)$. By Proposition 5, $A(H)$ contains a reductive part A of $A(R)$. Let $\mathcal{T}$ be a maximal compact subgroup of $\mathcal{R}$. Then $\mathcal{T}$ is a toral group, and there exists $a\in\mathcal{A}(\mathcal{R})$ such that $aTa^{-1}\subset\mathcal{A}$. Hence $H=L\cap A(H)\supset(\mathrm{Ad}\, a)T$, i.e. $\mathcal{H}\supset aTa^{-1}$. Since $\mathcal{R}$ is a normal subgroup of $\mathcal{A}(\mathcal{R})$, the group aTa^{-1} is maximal toral in $\mathcal{R}$.

(b) General case.

Let L be a Lie algebra, and H a Cartan subalgebra of L. Let $\mathrm{ad}\, L =\{\mathrm{ad}\, X ;\ X\in L\}$ and $H_1=\{\mathrm{ad}\, Y ;\ Y\in H\}$. It is straightforward to see that H_1 is a Cartan subalgebra of $\mathrm{ad}\, L$.

Let $\mathcal{T}$ be a maximal toral subgroup of $\mathcal{R}$. Let φ denote the adjoint representation of $\mathcal{L}$:

$$\mathcal{L}\ni x\mapsto\varphi(x)=\mathrm{Ad}\, x\in\mathrm{Ad}\,(L).$$

Then $\varphi(\mathcal{T})$ being a toral subgroup of the radical of $\mathrm{Ad}\,(L)$, there exists a maximal toral subgroup $\mathcal{T}_1$ of the radical of $\mathrm{Ad}\,(L)$ such that $\varphi(\mathcal{T})\subset\mathcal{T}_1$.

Let $\mathcal{H}$ be a Cartan subgroup of $\mathcal{L}$. Then $\varphi(H)$ is a Cartan subgroup of $\mathrm{Ad}\,(L)$ and by (a) there exists a maximal toral subgroup, say $\varphi(a)\mathcal{T}_1\varphi(a^{-1}) =\varphi(a\mathcal{T}_1a^{-1})$, of the radical of $\mathrm{Ad}\,(L)$ with $\varphi(a\mathcal{T}_1a^{-1})\subset\varphi(\mathcal{H})$, where $a\in\mathcal{R}$. Hence H contains the Lie algebra $(\mathrm{Ad}\, a)T$ of the group $a\mathcal{T}a^{-1}$. Q.E.D.

§8. A remark on linear Lie groups.

In [5], the author proved the following theorem,

THEOREM 4^-. *Let $\mathcal{G}$ be an analytic group, and let f be a continuous one-one homomorphism from $\mathcal{G}$ into $GL(n, \boldsymbol{R})$. Then we can find a closed subgroup*

$\mathcal{V}\cong\boldsymbol{R}^k$, and a closed connected normal subgroup $\mathcal{N}$, of $\mathcal{G}$, such that $\mathcal{G}$ is a semi-direct product

$$\mathcal{G}=\mathcal{V}\mathcal{N}, \qquad \mathcal{V}\cap\mathcal{N}=\{1\},$$

$\overline{f(\mathcal{V})}$ is a toral group, $f(\mathcal{N})$ is closed, and $\overline{f(\mathcal{G})}$ is an almost semi-direct product of $\overline{f(\mathcal{V})}$ and $f(\mathcal{N})$:

$$f(\mathcal{G})=f(\mathcal{V})f(\mathcal{N}), \qquad f(\mathcal{V})\cap f(\mathcal{N}) \quad \text{is finite.}$$

In this case, $\overline{f(\mathcal{G})}$ is diffeomorphic with the direct product space $\overline{f(\mathcal{V})}\times\mathcal{N}$.

Here we shall add some remark to the theorem.

THEOREM 4. *In Theorem 4^-, we can suppose that $\mathcal{V}$ is in the radical of $\mathcal{G}$.*

PROOF. Let $\mathcal{G}'$ and $\mathcal{R}$ denote the commutator subgroup and the radical of $\mathcal{G}$, respectively. Then $f(\mathcal{G}')$ is closed in $GL(n, \boldsymbol{R})$ and coincides with the commutator subgroup of $\overline{f(\mathcal{G})}$; $f(\mathcal{G})$ is closed if and only if $f(\mathcal{R})$ is closed, see Goto [4]. Let $\mathcal{R}_1$ be the radical of $\mathcal{G}'$. Then $R_1=G'\cap R$. Since $f(\mathcal{R}_1)$ is closed, we can pick a maximal analytic subgroup $\mathcal{M}$ of $\mathcal{R}$, such that $\mathcal{M}\supset\mathcal{R}_1$ and $f(\mathcal{M})$ is closed. By the proof in [5], there is a closed subgroup $\mathcal{V}\cong\boldsymbol{R}^k$ of $\mathcal{R}$ such that $\mathcal{R}=\mathcal{V}\mathcal{M}$, $\mathcal{V}\cap\mathcal{M}=\{1\}$. Let $\mathcal{S}$ be a maximal semi-simple analytic subgroup of $\mathcal{G}$. Then $\mathcal{N}=\mathcal{M}\mathcal{S}$ is a maximal analytic subgroup of $\mathcal{G}$ with respect to the properties $\mathcal{N}\supset\mathcal{G}'$ and $f(\mathcal{N})$ is closed. Then $\mathcal{G}/\mathcal{N}\cong\boldsymbol{R}^k$ by the proof in [5] again. On the other hand,

$$\mathcal{G}/\mathcal{N}=\mathcal{V}\mathcal{N}/\mathcal{N}\cong\mathcal{V}/\mathcal{V}\cap\mathcal{N}$$

and $\mathcal{V}\cap\mathcal{N}$ is discrete. Hence $\mathcal{V}\cap\mathcal{N}=\{1\}$. Q. E. D.

§9. Proof of Theorem 5.

THEOREM 5. *Let $\mathcal{L}$ be an analytic group, and $\mathcal{G}$ an analytic subgroup of $\mathcal{L}$. Then there exists a Cartan subgroup $\mathcal{H}$ of $\mathcal{G}$ such that $\overline{\mathcal{G}}=\overline{\mathcal{H}}\mathcal{G}$. This implies, in particular, that if $\mathcal{G}$ is not closed, there is a Cartan subgroup of $\mathcal{G}$ which is not closed in $\mathcal{L}$.*

PROOF. Suppose that $\mathcal{G}$ is not closed in $\mathcal{L}$. By Malcev, there exists a one-dimensional analytic subgroup $\mathcal{X}$ of $\mathcal{G}$ such that $\overline{\mathcal{G}}=\overline{\mathcal{X}}\mathcal{G}$, see Goto [4]. Since $\overline{\mathcal{X}}$ is a toral group, there exists a Cartan subgroup $\mathcal{H}_1$ of $\mathcal{L}$ with $\mathcal{H}_1\supset\overline{\mathcal{X}}$, by Theorem 3(i). By Theorem 2, $H=H_1\cap G$ is a Cartan subalgebra of G, and $H\supset X$. Hence $\overline{\mathcal{H}}\mathcal{G}\supset\overline{\mathcal{X}}\mathcal{G}=\overline{\mathcal{G}}$. Q. E. D.

COROLLARY. *Let $\mathcal{L}$ be an analytic group, and $\mathcal{G}$ a solvable analytic subgroup of $\mathcal{L}$. If $\mathcal{G}$ is non-closed, then all the Cartan subgroups of $\mathcal{G}$ are non-closed in $\mathcal{L}$.*

PROOF. By Iwahori-Satake [7], all the Cartan subgroups of $\mathcal{G}$ are conjugated to each other. Q. E. D.

§ 10. Proof of Theorem 6.

THEOREM 6. *Let $\mathcal{G}$ be a non-closed analytic subgroup of $GL(n, \boldsymbol{R})$, and $\mathcal{H}$ a Cartan subgroup of $\mathcal{G}$. Then $\mathcal{H}$ is not closed in $GL(n, \boldsymbol{R})$ and $\overline{\mathcal{G}}=\overline{\mathcal{H}}\mathcal{G}$.*

PROOF. By Theorem 4, there exists a vector subgroup $\mathcal{V}$ in the radical of $\mathcal{G}$ such that $\overline{\mathcal{G}}=\overline{\mathcal{V}}\mathcal{G}$. The toral group $\overline{\mathcal{V}}$ is in the radical of $\mathcal{L}=\overline{\mathcal{G}}$. For the Cartan subalgebra H of G, we can find a Cartan subalgebra H_1 of L such that $H=H_1\cap G$, by Theorem 2. Then by Theorem 3 (ii), $\mathcal{H}_1$ contains a maximal toral subgroup $\mathcal{T}$ of the radical of $\mathcal{L}$. Hence we can find $a\in\mathcal{L}$ with $a\overline{\mathcal{V}}a^{-1}\subset\mathcal{T}$, and $a\mathcal{V}a^{-1}\subset\mathcal{H}$. Therefore

$$\overline{\mathcal{H}}\mathcal{G}\supset a\overline{\mathcal{V}}a^{-1}\mathcal{G}=\overline{\mathcal{V}}\mathcal{G}=\overline{\mathcal{G}}\,. \qquad \text{Q. E. D.}$$

§ 11. An example.

Let $\mathcal{G}$ be the universal covering group of $SL(2, \boldsymbol{R})$. The center $\mathcal{Z}$ of $\mathcal{G}$ is an infinite cyclic group. Let z be a generator of $\mathcal{Z}$. Let $\mathcal{T}$ denote the toral group of dimension two, and t an element of $\mathcal{T}$ generating a dense subgroup. In the direct product group $\mathcal{G}\times\mathcal{T}$, let $\mathcal{D}$ denote the discrete central subgroup generated by (z, t) and we put $\mathcal{L}=(\mathcal{G}\times\mathcal{T})/\mathcal{D}$. Then we have a continuous one-one homomorphism $\iota: \mathcal{G}\to\mathcal{L}$ such that $\overline{\iota(\mathcal{G})}=\mathcal{L}$.

The Lie algebra of $\mathcal{G}$ is identified with

$$sl(2, \boldsymbol{R})=\{X\in gl(2, \boldsymbol{R})\,;\ \text{trace}\, X=0\}\,,$$

and $H_1=\boldsymbol{R}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ and $H_2=\boldsymbol{R}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ are Cartan subalgebras of $sl(2, \boldsymbol{R})$.

The Cartan subgroup $\iota(\mathcal{H}_1)$ is non-closed and $\iota(\mathcal{H}_2)$ is closed in $\mathcal{L}$.

Appendix. Added June 20, 1979.

After this paper was submitted to the Journal, the author found out that it can be improved by introducing the notion of "standard Cartan subgroups", of which he will explain briefly here.

Since then the author again found another notion "*gm*-tori of an analytic group", which is more convenient than standard Cartan subgroups for our purposes, and using this as one of the principal tools established the main results in

Immersions of Lie groups, J. Math. Soc. Japan, forthcoming.

The proofs of the following results can be obtained easily by studying the methods in the above paper.

PROPOSITION 1. *Let $\mathfrak{G}$ be an analytic group.*

(i) *There exists a Cartan subgroup $\mathfrak{H}$ containing a given maximal torus of $\mathfrak{G}$.*

(ii) *If, in particular, $\mathfrak{G}$ is faithfully representable, then the choice of $\mathfrak{H}$ containing some maximal torus is unique up to inner automorphism.*

DEFINITION. Let G be a Lie algebra of finite dimension over $\boldsymbol{R}$. A Cartan subalgebra H of G is said to be *standard* if the analytic subgroup of $\mathrm{Ad}(G)$ corresponding to $H^*=\{\mathrm{ad}\, X;\ X\in H\}$ contains a maximal torus of $\mathrm{Ad}(G)$. The Cartan subgroup corresponding to a standard Cartan subalgebra is called *standard.*

PROPOSITION 2. (i) *The choice of standard Cartan subalgebras is unique up to inner automorphisms.*

(ii) *Let $\mathfrak{G}$ be an analytic group, and $\mathfrak{H}$ a standard Cartan subgroup. Then $\mathfrak{H}$ contains the center and a maximal torus of $\mathfrak{G}$.*

(iii) *If $\mathfrak{G}$ is faithfully representable, then a Cartan subgroup containing a maximal torus is standard.*

THEOREM 5A. *In Theorem* 5, *if $\mathfrak{H}$ is a standard Cartan subgroup, then $\overline{\mathfrak{G}}=\overline{\mathfrak{H}}\mathfrak{G}$.*

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