J. Math. Soc. Japan
Vol. 32, No. 1, 1980

Remarks on a geometric constant of Yau

By David HOFFMAN

(Received April 26, 1978)
(Revised Sept. 28, 1978)

§ 1. Introduction.

Let M be a compact m-dimensional Riemannian manifold with or without boundary. In [2], Yau defines an isoperimetric constant $I(M)$ as follows: $I(M)=\inf\frac{\mathrm{Vol}(\partial M_1\cap\partial M_2)}{\min(\mathrm{Vol}\,M_1,\ \mathrm{Vol}\,M_2)}$, the infimum being taken over all decompositions $M=M_1\cup M_2$ with $\mathrm{Vol}(M_1\cap M_2)=0$. By standard methods, Yau shows that

$$(1)\qquad I(M)=\inf\left\{\int_M|\nabla f|\Big/\inf_{\beta\in R}\int_M|f-\beta|\ \Big|\ f\in C^1(M)\right\}.$$

$I(M)$ is useful for estimating eigenvalues of the Laplacian from below. In this note we wish to investigate and clarify a geometric quantity ω associated to M (and defined below) that arises in trying to estimate $I(M)$. At each $p\in M$, consider a subset $\mathcal{R}$ of T_p^1M with the following property: The set of all points of M reachable by minimal geodesics from p with initial direction in $\mathcal{R}$ has volume equal to or greater than $\mathrm{Vol}M/2$. Then ω_p is equal to the infimum of the $(m-1)$-dimensional areas of all such $\mathcal{R}$ and $\omega=\inf_{p\in M}\omega_p$. Let $\alpha_{m-1}=(\mathrm{m}-1)$-dimensional area of $S^{m-1}\subset \boldsymbol{R}^m$. Clearly $0<\omega/\alpha_{m-1}\leqq 1/2$ and for $M=S^m$, $\omega=\omega_p=\alpha_{m-1}/2$. One estimate we make is contained in the following proposition.

PROPOSITON 1. *Suppose the Ricci curvature of M is equal to or greater than $(m-1)a^2$. Then for all $p\in M$*

$$\omega_p/\alpha_{m-1}\geqq\omega/\alpha_{m-1}\geqq(1/2)V(a,\ d(M))^{-1}\cdot\mathrm{Vol}(M).$$

Here $V(a,\rho)$ is the volume of the solid ball of radius ρ in the space form of constant curvature a^2, (a may be a real positive or a purely imaginary number) and $d(M)$ is the diameter of M. The proof of Proposition 1 follows in § 3.

§ 2. To begin with, for $p\in M$ let (r,θ) denote polar coordinates on T_pM; $\theta\in T_p^1M$, $r\geqq 0$. For each $\theta\in T_p^1M$, let $r(\theta)$ be the distance to the cut locus of

Partially supported by NSF Grant MCS-7606752.

p in the direction θ. Let

$$D_p=\{(r, \theta) \mid \theta\in T_p^1M,\ 0\leqq r\leqq r(\theta)\}.$$

For a set $E\subset T_pM$, define the following associated sets

$$\mathcal{S}_p(E)=\{\theta\in T_p^1M \mid \exists\, \bar{r}\leqq r(\theta),\ \ni(\bar{r}, \theta)\in E\}$$

$$C_p(E)=\{(r, \theta) \mid \theta\in\mathcal{S}_p(E),\ 0\leqq r\leqq r(\theta)\}.$$

Within the set D_p, $C_p(E)$ may be thought of as the cone over $\mathcal{S}_p(E)$ and $\mathcal{S}_p(E)$ itself is the set of directions in which points of E are first visible. Let $\omega_p(E)$ denote the (m−1)-dimensional volume of $\mathcal{S}_p(E)\subset T_p^1M$. Yau defines

$$\omega=\inf\{\omega_p(E) \mid p\in M,\ \mathrm{Vol}(\exp_p(E))\geqq \mathrm{Vol}M/2\}.$$

We note that in fact

$$\omega=\inf\{\omega_p(C_p(E)) \mid p\in M,\ \mathrm{Vol}(\exp_p(C_p(E))\geqq \mathrm{Vol}M/2\}.$$

Therefore

$$\omega=\inf_{p\in M}\omega_p,\ \text{where}\ \omega_p=\inf\{\mathrm{Vol}_{m-1}(\mathcal{R}) \mid \mathcal{R}\subset T_p^1M,\ \mathrm{Vol}(\exp C_p(\mathcal{R}))\geqq \mathrm{Vol}M/2\}.$$

Intuitively ω_p/α_{m-1} is the smallest proportion of the unit sphere of T_pM necessary to "view" at least half of M.

LEMMA 1. *Let h be a C^1 function on M. If* $\mathrm{Vol}(h^{-1}(0))\geqq \mathrm{Vol}M/2$,

$$\int_M |h|\leqq\int_M |\nabla h|\cdot(\omega/\alpha_{m-1})^{-1}d(M). \tag{2}$$

PROOF. If we define $\widetilde{(r, \theta)}:=\exp_p(r, \theta)$, then formulae (6.9) and (6.10) of [2], page 502 combine to give

$$\int_M |h|\leqq\left(\int_M |\nabla h|\right)\cdot\sup_{p\in M}\int_{T_p^1M}\int_0^{r(\theta)}[\omega_{\widetilde{(r,\theta)}}(\exp^{-1}_{\widetilde{(r,\theta)}}(h^{-1}(0))]^{-1}drd\theta.$$

Therefore

$$\int_M |h|\leqq\left(\int_M |\nabla h|\right)(\omega/\alpha_{m-1})^{-1}\int_0^{r(\theta)}dr\leqq\left(\int_M |\nabla h|\right)(\omega/\alpha_{m-1})^{-1}\cdot d(M).\quad \text{q.e.d.}$$

PROPOSITION 2. *If M is compact and $\partial M=\emptyset$, then*

$$\lambda_1=\inf_{\int_M f=0}\left(\int|\nabla f|^2\Big/\int f^2\right)\geqq 1/4(\omega/\alpha_{m-1})^2d(M)^{-2}.$$

PROOF. (Similar to [2], (5.5).) Given $f\in C^1(M)$, there always exists a constant β_0 such that both $\mathrm{Vol}f^{-1}([\beta_0, \infty))$ and $\mathrm{Vol}f^{-1}((-\infty, \beta_0])$ are equal to or greater than $\mathrm{Vol}M/2$. By the previous lemma applied repeatedly to $(f^+)^2$ and $(f^-)^2$, where

$$f^+(x)=\max\{f(x)-\beta_0, 0\}$$

$$f^-(x)=-\min\{f(x)-\beta_0, 0\},$$

we have

$$\int_M |f-\beta_0|^2 \leqq \int_M (f^+)^2+(f^-)^2+2f^+f^- = \int_M (f^+)^2+(f^-)^2$$

$$\leqq (\omega/\alpha_{m-1})^{-1} d(M)\cdot 2\int_M f^+|\nabla f^+|+f^-|\nabla f^-|$$

$$=2(\omega/\alpha_{m-1})^{-1} d(M)\int_M (f^++f^-)|\nabla(f^++f^-)|$$

$$=2(\omega/\alpha_{m-1})^{-1} d(M)\int_M |f-\beta_0|\,|\nabla(f-\beta_0)|$$

$$\leqq 2(\omega/\alpha_{m-1})^{-1} d(M)\Big(\int_M |f-\beta_0|^2\Big)^{1/2}\Big(\int_M |\nabla f|^2\Big)^{1/2}.$$

Hence $\int_M |f-\beta_0|^2 \Big/ \int_M |\nabla f|^2 \leqq 4(\omega/\alpha_{m-1})^{-2} d(M)^2$. But if $\int_M f=0$, then $\int_M |f-\beta_0|^2 \geqq \int_M f^2$. q. e. d.

The calculations of the previous lemma and proposition imply the following.

PROPOSITION 3. (Yau) *Let $f\in C^1(M)$, then*

$$\Big(\int_M |\nabla f|^2\Big)^{1/2} \geqq I(M)\Big/2\Big(\int_M |f-\beta_0|^2\Big)^{1/2},$$

where β_0 is as in the previous proposition.

§3. We now prove Proposition 1. Let $\mathfrak{R}\subset T_p^1(M)$ satisfy $\mathrm{Vol}M/2\leqq \mathrm{Vol}(\exp_p(C_p\mathfrak{R}))$. If dV is the volume element of M, we mav write, in terms of the coordinate system $(r, \theta)\in D_p \to \exp_p(r, \theta)$,

$$dV=\sqrt{g_p(r, \theta)}\, r^{n-1} dr\, d\theta.$$

Then

$$\mathrm{Vol}M/2 \leqq \int_{\mathfrak{R}}\int_0^{r(\theta)} \sqrt{g_p(r, \theta)}\, r^{m-1} dr\, d\theta.$$

Since $\mathrm{Ric}\, M \geqq (m-1)a^2$,

$$\sqrt{g_p(r, \theta)} \leqq \Big[\frac{\sin ar}{ar}\Big]^{m-1}, \tag{3}$$

where we use the complex extension of sin if $a^2<0$, and consider $(\sin ar)/ar=1$ if $a=0$, (see [**1**], pp. 253-257). Consequently,

$$\mathrm{Vol}M/2 \leqq \int_{\mathfrak{R}} \int_0^{d(M)} \left[\frac{\sin ar}{ar}\right]^{m-1} r^{m-1} dr\, d\theta$$

$$\leqq (\mathrm{Vol}_{m-1}(\mathfrak{R})/\alpha_{m-1})\, \alpha_{m-1} \int_0^{d(M)} \left[\frac{\sin ar}{ar}\right]^{m-1} r^{m-1} dr\,.$$

Therfore

$$(4) \qquad \mathrm{Vol}M/2 \leqq (\omega_p/\alpha_{m-1})\alpha_{m-1} \int_0^{d(M)} \left[\frac{\sin ar}{ar}\right]^{m-1} r^{m-1} dr \leqq (\omega_p/\alpha_{m-1}) \cdot V(\alpha,\ d(M))\,.$$

COROLLARY 1. *If M is a compact Riemannian manifold with $\partial M = \emptyset$ on which* $\mathrm{Ric}\,(M) \geqq (\mathrm{m}-1)a^2$, *then*

$$\lambda_1 \geqq \frac{1}{16} (\mathrm{Vol}M/V(a,\ d(M)))^2 d(M)^{-2}\,.$$

If in addition the sectional curvature of M is bounded above by b^2,

$$\lambda_1 \geqq \frac{1}{16} \left(\frac{V(b,\ r)}{V(a,\ d(M))}\right)^2 d(M)^{-2}\,, \qquad \textit{where } r = \textit{maximum injectivity radius of } M.$$

§4. We conclude with a proposition which provides a lower bound for $I(M)$ in terms of $d(M)$ and ω. As a corollary to this proposition, we have a Theorem of Yau giving a lower bound for $I(M)$ on a closed manifold M with Ricci curvature bounded from below.

PROPOSITION 4. *Suppose M is compact without boundary. Then*

$$(5) \qquad I(M) \geqq d(M)^{-1} \cdot \omega/\alpha_{m-1}\,.$$

COROLLARY 2. (Yau). *If M is as in Proposition 4 and also has Ricci curvature bounded below by $(m-1)a^2$, then*

$$(6) \qquad I(M) \geqq \mathrm{Vol}M/2 \left[d(M)\, \alpha_{m-1} \int_0^{d(M)} \left[\frac{\sin ar}{a}\right]^{m-1} dr\right]^{-1}.$$

PROOF. Let $f \in C^1(M)$ and choose β_0 as in the proof of Proposition 2. Then the functions

$$f^+ = \max\{f(x) - \beta_0,\ 0\}$$

and

$$f^- = -\min\{f(x) - \beta_0,\ 0\}$$

both satisfy the hypothesis of Lemma 1, namely

$$\mathrm{Vol}((f^+)^{-1}(0)) \geqq \mathrm{Vol}M/2$$

$$\mathrm{Vol}((f^-)^{-1}(0)) \geqq \mathrm{Vol}M/2\,.$$

Moreover f^+ and f^- are C^1 except on a set of measure zero on which f^+ and

f^- both vanish. The conclusion of Lemma 1 clearly holds for these functions. Now

$$f-\beta_0=f^+-f^-$$

$$|f-\beta_0|=f^++f^-=|f^+|+|f^-|$$

and

$$|\nabla f|=|\nabla f^+-\nabla f^-|=|\nabla f^+|+|\nabla f^-|$$

almost everywhere. By (2) together with the above equalities we have

$$\int_M|f-\beta_0|=\int_M|f^+|+\int_M|f^-|$$

$$\leqq\left(\int_M|\nabla f^+|+\int_M|\nabla f^-|\right)(\omega/\alpha_{m-1})^{-1}d(M)$$

$$=\left(\int|\nabla f|\right)(\omega/\alpha_{m-1})^{-1}d(M).$$

Thus

$$\int_M|\nabla f|\Big/\left(\inf_{\beta\in R}\cdot\int_M|f-\beta|\right)\geqq(\omega/\alpha_{m-1})\,d(M)^{-1}. \tag{7}$$

By using equation (1) we conclude that

$$I(M)\geqq d(M)^{-1}\cdot\omega/\alpha_{m-1}.$$

This proves the proposition. We now prove Corollary 2. If $\mathrm{Ric}(M)\geqq(m-1)a^2$, then estimate (3) holds and we may use the estimate (4) in the proof of Proposition 1 to conclude that

$$\omega/\alpha_{m-1}\geqq\mathrm{Vol}M/2\left[\alpha_{m-1}\int_0^{d(M)}\left[\frac{\sin ar}{a}\right]^{m-1}dr\right]^{-1}. \tag{8}$$

Combined with (5) this proves (6).

References

[1] R. Bishop and R. Crittenden, Geometry of Manifolds, Academic Press, New York, 1964.

[2] Shing-Tung Yau, Isoperimetric constants and the first eigenvalue of a compact Riemannian manifold, Ann. Sci. École Norm. Sup., 4, **8** (1975), 487-507.

David HOFFMAN

Department of Mathematics and Statistics
University of Massachusetts
Amherst, Mass. 01003
U. S. A.