J. Math. Soc. Japan
Vol. 32, No. 1, 1980

On the topology of the Newton boundary II

(generic weighted homogeneous singularity)

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(Received April 10, 1978)
(Revised Oct. 23, 1978)

0. Introduction.

Since E. Brieskorn found exotic spheres as a neighborhood variety of an isolated singularity of a hypersurface defined by a certain type of polynomial, various remarkable results have been obtained by many mathematicians. ([2], [12], [10], [11], [1], [3], [5], [18] etc.) However most of them treat only isolated singularities of hypersurfaces and few results are known about the topology of non-isolated singularities. For example, if an analytic function $f(z_1, \cdots, z_{n+1})$ has an isolated critical point at the origin, the local Milnor fiber is $(n-1)$-connected by Milnor [10] and a fair amount of information can be obtained from the Milnor number ($=n$-th Betti number of the Milnor fiber). But if f has a non-isolated critical point at the origin, the connectivity of the fiber goes down so that we have no general method to determine the respective Betti numbers or even the Euler-Poincaré characteristic of the fiber.

In this paper we study the topology of the singularity of a weighted homogeneous polynomial which has a non-degenerate Newton boundary. Our main theorem is a kind of generalization of the above connectivity theorem of Milnor. This paper is a continuation of [16] and it consists of the following sections.

1. Notations and main results.
2. Fundamental groups and monodromy maps.
3. Lifting principle.
4. A special case of Theorem (1.1).
5. Topology of a globally non-degenerate polynomial.
6. Proof of (ii) of Theorem (5.3).
7. Topology of F.

* The author is partially supported by Sakkokai Foundation.

1. Notations and main results.

We use the following notations throughout this paper.

$\boldsymbol{Z}$: group of integers

$\boldsymbol{N}$: non-negative integers

$\boldsymbol{C}$: complex numbers, $\boldsymbol{R}$: real numbers

$\boldsymbol{C}^*$: $\boldsymbol{C}-\{0\}$

$$z^{\nu}=z_1^{\nu_1}z_2^{\nu_2}\cdots z_{n+1}^{\nu_{n+1}}, \quad |\nu|=\sum_{j=1}^{n+1}\nu_j \quad \text{for } z\in\boldsymbol{C}^{n+1} \text{ where } \nu=(\nu_1,\cdots,\nu_{n+1}).$$

$$\boldsymbol{R}^I=\{x=(x_1,\cdots,x_{n+1})\in\boldsymbol{R}^{n+1};\ x_j=0 \quad \text{for} \quad j\notin I\}$$

$$\boldsymbol{C}^I=\{z=(z_1,\cdots,z_{n+1})\in\boldsymbol{C}^{n+1};\ z_j=0 \quad \text{for} \quad j\notin I\}$$

where I is a subset of $\{1, 2, \cdots, n+1\}$.

m-volume (P): m-dimensional Euclidean volume of a polyhedron P of $\boldsymbol{R}^m$

$\det N$: the determinant of a matrix N

$$\binom{n}{j}=\frac{n!}{j!(n-j)!}$$

$A:=B$ A is B by definition.

Let $f(z_1,\cdots,z_{n+1})=\sum_{\nu\in\boldsymbol{N}^{n+1}}c_\nu z^\nu$ be an analytic function in a neighborhood of the origin. We denote the Newton boundary of f by $\Gamma(f)$. (See [7] or [16] for the definition.) f is called to be *non-degenerate in the sense of the Newton boundary if* $\frac{\partial f_\Delta}{\partial z_1}=\cdots=\frac{\partial f_\Delta}{\partial z_{n+1}}=0$ has no solution in $(\boldsymbol{C}^*)^{n+1}$ for any closed face Δ of $\Gamma(f)$ where $f_\Delta(z)=\sum_{\nu\in\Delta}c_\nu z^\nu$. In this paper, we use "non-degenerate" in the above sense unless otherwise stated.

Now let $f(z)=\sum_{j=1}^{m}c_j z^{\nu^j}$ $(c_j\neq 0,\ j=1,\cdots,m)$ be a non-degenerate weighted homogeneous polynomial with the Newton boundary Δ. Then Δ is a convex polyhedron spanned by the vertices $\nu^1,\cdots,\nu^m$ and $\dim\Delta=\operatorname{rank}\{\nu^1,\cdots,\nu^m\}-1\leqq n$. We consider the canonical fibration $f:(\boldsymbol{C}^*)^{n+1}-f^{-1}(0)\to\boldsymbol{C}^*$ which is the restriction of the Milnor fibration $f:\boldsymbol{C}^{n+1}-f^{-1}(0)\to\boldsymbol{C}^*$. Let $F^*=\{z\in(\boldsymbol{C}^*)^{n+1};\ f(z)=1\}$ and $F=\{z\in\boldsymbol{C}^{n+1};\ f(z)=1\}$ be the respective fibers and let $\rho: F^*\to(\boldsymbol{C}^*)^{n+1}$ be the inclusion map. Then F^* is completely described by the following theorem.

THEOREM (1.1). *Assume that* $\dim\Delta=n$. *Then*

(i) *the Euler-Poincaré characteristic* $\chi(F^*)$ *is* $(-1)^n(n+1)!$ $(n+1)$-*volume* $(\Delta(0))$ *where* $\Delta(0)$ *is the geometric cone of* Δ *and the origin in* $\boldsymbol{R}^{n+1}$

(ii) *ρ is an n-equivalence, i.e. $\rho_{\#}:\pi_j(F^*)\to\pi_j((\boldsymbol{C}^*)^{n+1})$ is bijective for $j<n$ and surjective for $j=n$.*

By Whitehead theorem, we get:

COROLLARY (1.1.1). *$\rho_*: H_j(F^*;\boldsymbol{Z})\to H_j((\boldsymbol{C}^*)^{n+1};\boldsymbol{Z})$ is bijective for $j<n$ and surjective for $j=n$.*

Let $(a_1,\cdots,a_{n+1};d)\in\boldsymbol{N}^{n+1}\times\boldsymbol{N}$ be the weight of f i.e. G.C.M$(a_1,\cdots,a_{n+1})=1$ and $(a,\nu^j)=\sum_{k=1}^{n+1}a_k\nu_k^j=d$ for $j=1,\cdots,m$ where $\nu^j=(\nu_1^j,\cdots,\nu_{n+1}^j)$. The monodromy map $h^*:F^*\to F^*$ of the above fibration is defined by $h^*(z)=\left(z_1\exp\frac{a_1}{d}2\pi\sqrt{-1},\cdots,z_{n+1}\exp\frac{a_{n+1}}{d}2\pi\sqrt{-1}\right)$. Let $\zeta(h^*;t)$ be the zeta function of h^*. (See [**10**] for definition.)

COROLLARY (1.1.2). (i) *$(h^*)_*: H_j(F^*;\boldsymbol{Z})\to H_j(F^*;\boldsymbol{Z})$ is the identity map for $j<n$.*

(ii) *$\zeta(h^*;t)=(1-t^d)^{-\frac{\chi(F^*)}{d}}$.*

COROLLARY (1.2). *Let $f(z)$ be a weighted homogeneous polynomial with a non-degenerate Newton boundary Δ. Assume that $r:=\dim\Delta<n$. Then we can find a non-degenerate weighted homogeneous polynomial $f_N(z_1,\cdots,z_{r+1})$ of $(r+1)$-variables such that F^* is diffeomorphic to $G^*\times(\boldsymbol{C}^*)^{n-r}$ where $G^*=\{(z_1,\cdots,z_{r+1})\in(\boldsymbol{C}^*)^{r+1};f_N(z)=1\}$. In particular* (i) *$\chi(F^*)=0$ and ρ is an r-equivalence and* (ii) *$\zeta(h^*;t)=1$.*

Theorem (1.1) and the above corollaries are proved in §§2~6. The topology of the Milnor fiber F is studied in §7.

2. Fundamental groups and monodromy maps.

In this section we investigate the relation between the fundamental group $\pi_1(F^*)$ and the monodromy map $(h^*)_*: H_1(F^*;\boldsymbol{Z})\to H_1(F^*;\boldsymbol{Z})$. Hereafter the homology groups are assumed to have $\boldsymbol{Z}$ as the coefficient group. Let $f(z)$ be a weighted homogeneous polynomial of type $(a_1,\cdots,a_{n+1};d)$ and we assume:

(2.1.1) $V^*=f^{-1}(0)\cap(\boldsymbol{C}^*)^{n+1}$ is not empty and

(2.1.2) F^* is connected (or F is connected).

Let A^* be the Laurent polynomial ring $\boldsymbol{C}[z_1,z_1^{-1},\cdots,z_{n+1},z_{n+1}^{-1}]$. Then $\{(2.1.1)+(2.1.2)\}$ is equivalent to:

(2.1.3) $f(z)$ is not a monomial and $f=g^m$ for some $g\in A^*$ and $m\in\boldsymbol{Z}$ in A^* implies $m=1$ and $f=g$.

LEMMA (2.2). *The following are equivalent.*

(i) *$\pi_1((\boldsymbol{C}^*)^{n+1}-f^{-1}(0))$ is an abelian group.*

(ii) $\pi_1(F^*)$ *is abelian and* $(h^*)_*: H_1(F^*; Z) \to H_1(F^*; Z)$ *is the identity map.*
Assume that f *is irreducible in* A^*. *Then the above conditions are equivalent to:*
(iii) $\rho_\#: \pi_1(F^*) \to \pi_1((\boldsymbol{C}^*)^{n+1})$ *is bijective where* $\rho: F^* \to (\boldsymbol{C}^*)^{n+1}$ *is the inclusion map.*

PROOF. The proof is almost parallel to the proof of Proposition 5 of [**14**]. From the fibration $f: (\boldsymbol{C}^*)^{n+1} - f^{-1}(0) \to \boldsymbol{C}^*$, we get the exact sequence:

$$0 \longrightarrow \pi_1(F^*) \xrightarrow{k_*} \pi_1((\boldsymbol{C}^*)^{n+1} - f^{-1}(0)) \xrightarrow{f_*} \pi_1(\boldsymbol{C}^*) \longrightarrow 0 \tag{2.2.1}$$

where k is the inclusion map $F^* \subsetneq (\boldsymbol{C}^*)^{n+1} - f^{-1}(0)$. In the following argument we fix respective base points in a suitable manner.
(i)$\Leftrightarrow$(ii): Let σ be the element of $\pi_1((\boldsymbol{C}^*)^{n+1} - f^{-1}(0))$ which is represented by $[\sigma_1 \circ \sigma_2]$ where σ_1 is the orbit path of the base point $*$ by the monodromy map i. e. $\sigma_1(t) = \left(*_1 \exp \frac{a_1}{d} 2\pi\sqrt{-1}, \cdots, *_{n+1} \exp \frac{a_{n+1}}{d} 2\pi\sqrt{-1}\right)$ $(0 \leqq t \leqq 1)$ and σ_2 is a fixed path joining $h^*(*)$ and $*$ in F^*. σ induces a canonical cross-section of $f_\#$ in (2.2.1) so that $\pi_1((\boldsymbol{C}^*)^{n+1} - f^{-1}(0))$ is a semi-direct product of $\pi_1(F^*)$ and $\pi_1(\boldsymbol{C}^*)$ via σ. Thus to see the equivalence (i)$\Leftrightarrow$(ii), we need only the next commutative diagram

$$\begin{tikzcd} \pi_1(F^*) \arrow[r, "\mathrm{ad}(\sigma)"] \arrow[d, "\xi"] \arrow[dr, phantom, "\circlearrowright"] & \pi_1(F^*) \arrow[d, "\xi"] \\ H_1(F^*) \arrow[r, "(h^*)_*"] & H_1(F^*) \end{tikzcd} \tag{2.2.2}$$

where $\mathrm{ad}(\sigma)$ is defined by $\mathrm{ad}(\sigma) \cdot \omega = \sigma^{-1}\omega\sigma$ and ξ is the Hurewicz homomorphism. (iii)$\Rightarrow$(i): For this direction, it is not necessary to assume that f is irreducible in A^*. Using the homomorphism φ which is the composition of $\pi_1((\boldsymbol{C}^*)^{n+1} - f^{-1}(0)) \xrightarrow{\tilde{\rho}_\#} \pi_1(\boldsymbol{C}^*)^{n+1} \xrightarrow{\rho_\#^{-1}} \pi_1(F^*)$ where $\tilde{\rho}$ is the inclusion map $(\boldsymbol{C}^*)^{n+1} - f^{-1}(0) \subsetneq (\boldsymbol{C}^*)^{n+1}$, we get a splitting of the exact sequence (2.2.1) i. e. $\varphi \cdot k_\#$ is the identity map. As φ is a homomorphism into an abelian group $\pi_1(F^*)$, we have that $\varphi(\sigma^{-1}\omega\sigma) = \varphi(\omega)$ for any $\omega \in \pi_1((\boldsymbol{C}^*)^{n+1} - f^{-1}(0))$. In particular, σ commutes with $k_\#\pi_1(F^*)$ and $\pi_1((\boldsymbol{C}^*)^{n+1} - f^{-1}(0))$ is a free abelian group of rank $n+2$. (i)$\Rightarrow$(iii): Assume that f is irreducible in A^*. By the Alexander duality, $\pi_1((\boldsymbol{C}^*)^{n+1} - f^{-1}(0)) \cong H_1((\boldsymbol{C}^*)^{n+1} - f^{-1}(0))$ is freely generated by $e_1, \cdots, e_{n+1}, e_f$ which are defined in the following way (cf. §4 of [**15**]): Take a small loop s_j $(j=1, \cdots, n+1)$ (respectively s_f) winding once around $L_j := \{z_j = 0\}$ at a regular point of $\bigcup_{j=1}^{n+1} L_j \cup \{f=0\}$ (respectively around $\{f=0\}$ at a regular point in $(\boldsymbol{C}^*)^{n+1}$) and take paths l_j and l_f in $(\boldsymbol{C}^*)^{n+1}-f^{-1}(0)$ which connect s_j and s_f with the base point respectively. Put $e_j=[l_j \cdot s_j \cdot l_j^{-1}]$ and $e_f=[l_f \cdot s_j \cdot l_f^{-1}]$. See Figure (2.2.3).

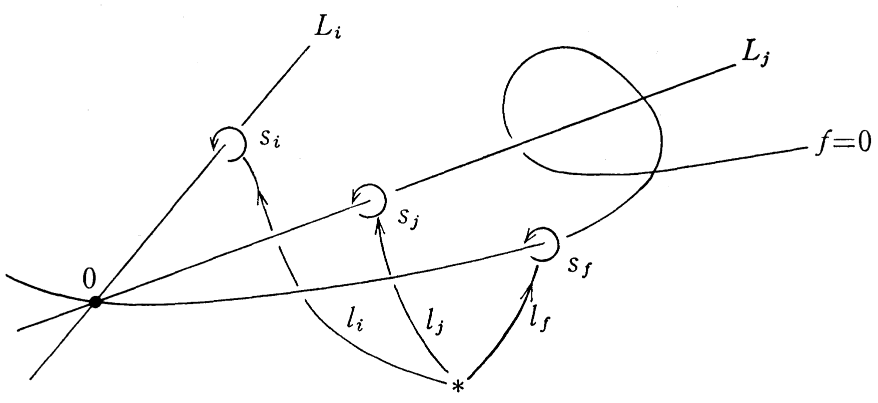


Figure (2.2.3)

We identify $\pi_1(\boldsymbol{C}^*)\cong Z$ and $\pi_1((\boldsymbol{C}^*)^{n+1})\cong \boldsymbol{Z}^{n+1}$ in a canonical way. Taking the orientations of s_j and s_f in a suitable manner, we may assume:

(2.2.4) $$f_{\#}(e_f)=1 \quad \text{and}$$

(2.2.5) $$\tilde{\rho}_{\#}(e_j)=(0, \cdots, \overset{j}{\check{1}}, \cdots, 0) \ (j=1, 2, \cdots, n+1) \quad \text{and} \quad \tilde{\rho}_{\#}(e_f)=0\,.$$

Let $m_j=f_{\#}(e_j)$ and put $e_j'=e_j-m_je_f$ $(j=1, \cdots n+1)$. Clearly $e_1', \cdots, e_{n+1}'$ generate a subgroup N of $\pi_1(F^*)$. However $e_1', \cdots, e_{n+1}'$ and e_f are still generators of $\pi_1((\boldsymbol{C}^*)^{n+1}-f^{-1}(0))$ and (2.2.1) and (2.2.4) implies $N=\pi_1(F^*)$. Now it is an easy task to see that $\pi_1(F^*)\xrightarrow{\rho_{\#}}\pi_1((\boldsymbol{C}^*)^{n+1})$ is bijective. Q. E. D.

REMARK (2.3). (iii) implies the irreducibility of f in A^*. In fact we have shown in the argument (iii)$\Rightarrow$(i) that $\pi_1((\boldsymbol{C}^*)^{n+1}-f^{-1}(0))$ is a free abelian group of rank $n+2$. On the other hand the rank of $H_1((\boldsymbol{C}^*)^{n+1}-f^{-1}(0))=n+1+r$ where r is the number of irreducible components of $\{f=0\}$ in $(\boldsymbol{C}^*)^{n+1}$ by the Alexander duality.

LEMMA (2.4). *Let $f(z)$ be as in Theorem* (1.1). *Then* (2.1.3) *is true for f.*

PROOF. By the assumption $\dim \Delta=n$, f is not a monomial. Assume that $f=g^m$ for some $m>1$ and $g\in A^*$. Then it is not a unit element of A^*. Thus $g^{-1}(0)\cap(\boldsymbol{C}^*)^{n+1}$ is not empty and any point $z\in g^{-1}(0)\cap(\boldsymbol{C}^*)^{n+1}$ is a solution of the equation $\frac{\partial f}{\partial z_1}=\frac{\partial f}{\partial z_2}=\cdots=\frac{\partial f}{\partial z_{n+1}}=0$. This contradicts the non-degeneracy assumption for f. Q. E. D.

REMARK (2.5). Theorem (1.1) and Corollary (1.2) imply that f is irreducible in A^* if f is a non-degenerate weighted homogeneous polynomial such that $\dim \Delta>0$.

3. Lifting principle.

In this section we study a technique to reduce Theorem (1.1) to the case of homogeneous polynomials. Let $f(z)=\sum_{j=1}^{m} c_j z^{\nu^j}$ be as in Theorem (1.1).

DEFINITION (3.1). ν^j is called *an effective vertex of* Δ when the convex polyhedron spanned by ν^i $(i\neq j)$ is strictly smaller than Δ. By the definition, Δ is spanned by effective vertices of Δ.

LEMMA (3.2). *The fibrations* $f: \boldsymbol{C}^{n+1}-f^{-1}(0)\to \boldsymbol{C}^*$ *and* $f:(\boldsymbol{C}^*)^{n+1}-f^{-1}(0)\to \boldsymbol{C}^*$ *are uniquely determined by* Δ *and they do not depend on the particular choice of the coefficients.*

The proof is more or less a general nonsense argument and is given in the Appendix. By virtue of this lemma, we may assume hereafter that $\nu^j, \cdots, \nu^m$ are effective vertices.

Let $N=(b_j^i)=(b^1, \cdots, b^{n+1})$ be a matrix in $GL(n+1, \boldsymbol{Z})$ where b^j are column vectors and define $|\det N|$-fold covering $\varphi_N:(\boldsymbol{C}^*)^{n+1}\to(\boldsymbol{C}^*)^{n+1}$ by $\varphi_N(z)=(z^{b^1}, \cdots, z^{b^{n+1}})$. For the sake of brevity, we assume that $\mu^j:=N\nu^j$ are contained in $\boldsymbol{N}^{n+1}$ for $j=1, \cdots, m$. Here ν^j is considered as a column vector. Putting $f_N(z)=f(\varphi_N(z))$, we can easily see that $f_N(z)=\sum_{j=1}^{m} c_j z^{\mu^j}$ and that f_N is a non-degenerate weighted homogeneous polynomial with the Newton boundary $N\Delta$ which is spanned by effective vertices $\mu^1, \cdots, \mu^m$. Let $F_N^*=f_N^{-1}(1)\cap(\boldsymbol{C}^*)^{n+1}$ and $\rho_N: F_N^*\to(\boldsymbol{C}^*)^{n+1}$ is the inclusion map. Then

LEMMA (3.3). $\rho_\#:\pi_1(F^*)\to\pi_1((\boldsymbol{C}^*)^{n+1})$ *is bijective if and only if* $(\rho_N)_\#: \pi_1(F_N^*)\to\pi_1((\boldsymbol{C}^*)^{n+1})$ *is bijective.*

PROOF. Note that $\rho_\#$ and $(\rho_N)_\#$ are always surjective because we can take elements $e_1, \cdots e_{n+1}, e_f$ of $\pi_1((\boldsymbol{C}^*)^{n+1}-f^{-1}(0))$ (respectively of $\pi_1((\boldsymbol{C}^*)^{n+1}-f_N^{-1}(0))$) satisfying (2.2.4) and (2.2.5). The restriction $\varphi_N|_{F_N^*}: F_N^*\to F^*$ is also a $|\det N|$-fold covering map. Thus the assertion is an immediate consequence of the Five lemma applied in the following:

$$\begin{tikzcd}
0 \arrow[r] & \pi_1(F_N^*) \arrow[r, "(\varphi_N)_\#"] \arrow[d, "(\rho_N)_\#"] \arrow[dr, phantom, "\circlearrowright"] & \pi_1(F^*) \arrow[r] \arrow[d, "\rho_\#"] \arrow[dr, phantom, "\circlearrowright"] & G_1 \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \pi_1((\boldsymbol{C}^*)^{n+1}) \arrow[r, "(\varphi_N)_\#"] & \pi_1((\boldsymbol{C}^*)^{n+1}) \arrow[r] & G_2 \arrow[r] & 0
\end{tikzcd}$$

where G_i $(i=1, 2)$ are finite groups of order $|\det N|$ defined by the respective cokernels. Q. E. D.

COROLLARY (3.3.1). (*Lifting principle*) *Under the same assumption as above, the assertion for* $f(z)$ *in Theorem* (1.1) *is equivalent to the corresponding assertion for* $f_N(z)$.

PROOF. Use the equalities: $\chi(F_N^*)=|\det N|\cdot\chi(F^*)$ and $(n+1)$-volume $N\Delta(0)$ $=|\det N|\cdot(n+1)$-volume $\Delta(0)$. Q. E. D.

Returning to the situation in Theorem (1.1), we take

$$N=\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_{n+1} \end{pmatrix}.$$

Then f_N is a homogeneous polynomial of degree d. Thus from now on we can restrict ourselves to the case that $f(z)$ is a non-degenerate homogeneous polynomial of degree d. Let $V=\{[z]\in \boldsymbol{P}^n\,;f(z)=0\}$ and $L_j=\{[z]\in \boldsymbol{P}^n\,;z_j=0\}$ and $Y^*=P^n-V\cup L_1\cup\cdots\cup L_{n+1}$. The monodromy map h^* on F^* is defined in this case by the coordinate-wise multiplication of $\exp\frac{2\pi}{d}\sqrt{-1}$ and it naturally induces a free $\boldsymbol{Z}/d\boldsymbol{Z}$-action on F^*. Considering the restriction of the Hopf fibration $(\boldsymbol{C}^*)^{n+1}-f^{-1}(0)\xrightarrow{\boldsymbol{C}^*}Y^*$, we can see that the orbit space of F^* by the above $\boldsymbol{Z}/d\boldsymbol{Z}$-action is nothing but Y^*. (Compare Proposition 1, §4 of [**15**].) The following lemma describes the relation between F^* and Y^*.

LEMMA (3.4).

(i) $d\cdot\chi(Y^*)=\chi(F^*)$.

(ii) $\pi_1(Y^*)$ *is a free abelian group of rank* $n+1$ *if and only if* $\pi_1(F^*)$ *is a free abelian group of rank* $n+1$.

(iii) $\pi_i(Y^*)=\pi_i(F^*)$ *for* $i>1$.

PROOF. We prove only (ii) as the other assertions are obvious. Assume that $\pi_1(F^*)$ is a free abelian group of rank $n+1$. Then $\rho_\#:\pi_1(F^*)\to\pi_1((\boldsymbol{C}^*)^{n+1})$ is bijective because $\rho_\#$ is always surjective if $\dim\Delta\geqq 1$. Thus by Lemma (2.2) $\pi_1((\boldsymbol{C}^*)^{n+1}-f^{-1}(0))$ is abelian which implies $\pi_1(Y^*)$ is abelian by the exact sequence:

$$\pi_1(C^*)\longrightarrow\pi_1((\boldsymbol{C}^*)^{n+1}-f^{-1}(0))\longrightarrow\pi_1(Y^*)\longrightarrow 0\,.$$

Then $\pi_1(Y^*)\cong H_1(Y^*)\cong \boldsymbol{Z}^{n+1}$ by the Lefschetz duality. The opposite direction is obvious from the fact that $\pi_1(F^*)$ is a subgroup of $\pi_1(Y^*)$ of index d (cf. Proposition 5, §2 of [**14**].) Q. E. D.

4. A special case of Theorem (1.1).

In this section we study the case that Δ has $(n+1)$ effective vertices in the situation of Theorem (1.1). Let $f(z)=\sum_{j=0}^{n+1} c_j z^{\nu^j}$ and put $M=(\nu_k^j)\in GL(n+1, \boldsymbol{Z})$ and put $N=|\det M|\cdot M^{-1}\in GL(n+1, \boldsymbol{Z})$. Then it is easy to see that $f_N(z)=c_1z_1^p+\cdots+c_{n+1}z_{n+1}^p$ where $p=|\det M|$.

Let

$$V_N=\{[z]\in \boldsymbol{P}^n\,;f_N(z)=0\}\,,\qquad L_j=\{[z]\in \boldsymbol{P}^n\,;z_j=0\}$$

and $Y_N^*=\boldsymbol{P}^n-V_N\cup L_1\cup\cdots\cup L_{n+1}$. It is clear that the hypersurfaces V_N, $L_1,\cdots$, L_{n+1} are in a general position in the sense of [15]. Thus by Theorem 4 of [15], $\pi_1(Y_N^*)$ is an abelian group of rank $n+1$ and $\pi_j(Y_N^*)=0$ for $1<j<n$. As for the Euler characteristic, we get $\chi(Y_N^*)=(-1)^n p^n$ by a slight calculation using $\chi(V_N)=(n+1)-\frac{1}{p}\{1+(-1)^n(p-1)^{n+1}\}$ (cf. [13]). Combining Lemma (3.4) and Corollary (3.3.1), we conclude that Theorem (1.1) is true for the above case.

This is the first step of the proof of Theorem (1.1) which is carried out by the induction on the number of effective vertices in §§ 5-6.

5. Topology of a globally non-degenerate polynomial.

Let $g(u)=\sum_{j=1}^{m} b_j u^{\mu^j}$ be a Laurent polynomial in n variables $u_1,\cdots,u_n$. *The support of* g is the convex polyhedron in $\boldsymbol{R}^n$ which is spanned by μ^j for which $b_j\neq 0$ and we denote it by $S(g)$. We say that g is *globally non-degenerate on* Ξ if the equation $g_\Xi(u)=\frac{\partial g_\Xi}{\partial u_1}(u)=\cdots=\frac{\partial g_\Xi}{\partial u_n}(u)=0$ has no solution in $(\boldsymbol{C}^*)^n$ where $g_\Xi(u)=\sum_{\mu^j\in\Xi} b_j u^{\mu^j}$. When the minimal plane $L(\Xi)$ containing Ξ does not pass through the origin, $g_\Xi(u)$ becomes a weighted homogeneous polynomial and $g_\Xi(u)$ can be written as $\sum_{i=1}^{n} m_i u_i \frac{\partial g_\Xi}{\partial u_i}$ for some $m=(m_1,\cdots,m_n)\in Q^n-\{0\}$. Thus the global non-degeneracy on Ξ is nothing but the non-degeneracy on Ξ in the sense of [7]. However if $L(\Xi)$ contains the origin, the equation $\frac{\partial g_\Xi}{\partial u_1}(u)=\cdots=\frac{\partial g_\Xi}{\partial u_n}(u)=0$ may have solutions in $(\boldsymbol{C}^*)^n$ for any coefficients but it has no solution in $(\boldsymbol{C}^*)^n\cap g_\Xi^{-1}(0)$ for generic coefficients. (See Lemma (5.2) below and Theorem 6.1 of [7].)

Definition (5.1). We say that g is *a globally non-degenerate Laurent polynomial* if g is globally non-degenerate on each face of $S(g)$. Unless otherwise stated, the face of a polyhedron is closed.

Let $f(z)=\sum_{j=1}^{m} c_j z^{\nu^j}$ be a homogeneous polynomial of degree d with the Newton boundary Δ. Define a polynomial $\pi f(u)$ by $\pi f(u_1, \cdots, u_n, 1)$. Then we have

LEMMA (5.2). (i) *$f(z)$ is a non-degenerate homogeneous polynomial if and only if $\pi f(u)$ is a globally non-degenerate polynomial.*
(ii) *$n!$ n-volume $S(\pi f)=(n+1)!$ $(n+1)$-volume $\Delta(0)/d$.*

PROOF. Let $\pi: \boldsymbol{R}^{n+1} \to \boldsymbol{R}^n$ be the projection $(x_1, \cdots, x_{n+1}) \mapsto (x_1, \cdots, x_n)$. $S(\pi f)$ is spanned by $\pi(\nu^j)$ and $\pi: \Delta \to S(\pi f)$ is a piecewise linear homeomorphism. Let Ξ be a closed face of Δ. Then $f_\Xi(z)=\frac{1}{d}\sum_{i=1}^{n+1} z_i \frac{\partial f_\Xi}{\partial z_i}$ and $g_{\pi\Xi}(u)=\frac{1}{d}\sum_{i=1}^{n} u_i \frac{\partial g_{\pi\Xi}}{\partial u_i}+\pi\left(\frac{\partial f_\Xi}{\partial z_{n+1}}\right)(u)$ where $g=\pi f$. Therefore (i) is obvious from the definition. To prove (ii), we may assume by the linearity of the corresponding volumes that Δ is spanned by $(n+1)$ vertices $\nu^1, \cdots, \nu^{n+1}$.
Then

$$
\begin{aligned}
(n+1)!\,(n+1)\text{-volume } \Delta(0) &= \left| \det \begin{pmatrix} \nu_1^1, & \cdots\cdots, & \nu_1^{n+1} \\ \vdots & & \vdots \\ \nu_{n+1}^1, & \cdots, & \nu_{n+1}^{n+1} \end{pmatrix} \right| \\
&= \left| \det \begin{pmatrix} \nu_1^1, & \cdots, & \nu_1^{n+1} \\ \vdots & & \\ \nu_n^1, & \cdots, & \nu_n^{n+1} \\ d, & \cdots, & d \end{pmatrix} \right| \quad \text{because } |\nu^i|=d \\
&= d\cdot|\det(\nu_j^i-\nu_j^{n+1})|,\ 1\leqq i,\ j\leqq n \\
&= d\cdot n!\ n\text{-volume } \pi\Delta. \qquad \text{Q.E.D.}
\end{aligned}
$$

Now using Lemma (3.4) and Corollary (3.3.1) we have no difficulty to show that Theorem (1.1) is equivalent to the following:

THEOREM (5.3). *Let $g(u)$ be a globally non-degenerate polynomial such that* $\dim S(g)=n$. *Then*
(i) $\chi((\boldsymbol{C}^*)^n-g^{-1}(0))=(-1)^n n!$ *n-volume $S(g)$.*
(ii) *If $n\geqq 1$, $\pi_1((\boldsymbol{C}^*)^n-g^{-1}(0))$ is a free abelian group of rank $n+1$ and $\pi_j((\boldsymbol{C}^*)^n-g^{-1}(0))=0$ for $1<j<n$.*

Let $g(u)=\sum_{j=1}^{m} b_j u^{\mu^j}$ be as in Theorem (5.3). Let A be an affine transformation of $\boldsymbol{R}^n$ defined by $Ax=Nx+\lambda$ where $N\in GL(n, \boldsymbol{Z})$ and $\lambda\in\boldsymbol{Z}^n$ and let $g_A(u)=\sum_{j=1}^{m} b_j u^{A\mu^j}$.

LEMMA (5.4). *The assertion for $g(u)$ in Theorem (5.3) is equivalent to the corresponding assertion for $g_A(u)$.*

PROOF. The parallel translation by λ does not change the space in Theorem (5.3) and the volume of the support. Thus we may assume $\lambda=0$. Let $f(z)$ be a non-degenerate homogeneous polynomial such that $\pi f=g$. Let $Y^*=(\boldsymbol{C}^*)^n-g^{-1}(0)$, $F^*=f^{-1}(1)\cap(\boldsymbol{C}^*)^{n+1}$, $Y_N^*=(\boldsymbol{C}^*)^n-g_N^{-1}(0)$ and $F^*_{\tilde{N}}=f_{\tilde{N}}^{-1}(1)\cap(\boldsymbol{C}^*)^{n+1}$ where $\tilde{N}=\begin{bmatrix} N, & 0 \\ 0, & 1\end{bmatrix}\in GL(n+1, \boldsymbol{Z})$. Then $\pi f_{\tilde{N}}=g_N$ and the assertion is immediate from Corollary (3.3.1) and Lemma (3.4). Q. E. D.

DEFINITION (5.5). A Laurent polynomial g is said to be *regular* if for any face Ξ of $S(g)$ of dimension less than n, $L(\Xi)$ does not pass through the origin where $L(\Xi)$ is the plane of dimension $\dim \Xi$ containing Ξ.

LEMMA (5.6). *For any Laurent polynomial $g(u)$, we can find a vector $\lambda\in\boldsymbol{Z}^n$ so that $g_\lambda(u)=g(u)\cdot u^\lambda$ is regular.*

This is obvious from the definition.

LEMMA (5.7). *Assume that $g(u)$ is a regular globally non-degenerate polynomial such that $\dim S(g)=n$. Then $g(u)$ has $n!$ n-volume $S(g)$ critical points in $(\boldsymbol{C}^*)^n$ counting the multiplicity. The corresponding critical values are non-zero. Moreover if we choose nice coefficients, all the critical points become simple.*

PROOF. Let $f(z)$ be a non-degenerate homogeneous polynomial of degree d with the Newton boundary Δ such that $\pi f=g$. Let $\alpha\in(\boldsymbol{C}^*)^n$ be a critical point of g i. e.

$$\frac{\partial g}{\partial u_1}(\alpha)=\cdots=\frac{\partial g}{\partial u_n}(\alpha)=0, \qquad \alpha\in(\boldsymbol{C}^*)^n. \tag{5.7.1}$$

Then by the global non-degeneracy of g, $g(\alpha)\neq 0$. The solutions of (5.7.1) in $(\boldsymbol{C}^*)^n$ is obtained from the solutions of

$$\frac{\partial f}{\partial z_1}(\beta)=\cdots=\frac{\partial f}{\partial z_n}(\beta)=0, \qquad \beta\in(\boldsymbol{C}^*)^{n+1} \tag{5.7.2}$$

by the correspondence $\beta\mapsto\alpha=(\beta_1/\beta_{n+1}, \cdots, \beta_n/\beta_{n+1})$. Let us consider the equation

$$\beta_1\frac{\partial f}{\partial z_1}(\beta)=\cdots=\beta_n\frac{\partial f}{\partial z_n}(\beta)=0 \quad\text{and}\quad \beta_{n+1}\frac{\partial f}{\partial z_{n+1}}(\beta)=t. \tag{5.7.2 ; t}$$

The vector $e_{n+1}=(0, \cdots, 0, 1)\in\boldsymbol{R}^{n+1}$ is regular for Δ in the sense of [**16**] by the regularity of g. By Lemma (4.3) and Lemma (4.3)$'$ (in the Appendix) of [**16**], (5.7.2 ; 1) has $(n+1)!$ $(n+1)$-volume $\Delta(0)$ solutions in $(\boldsymbol{C}^*)^{n+1}$ counting the multiplicity. However by the regularity of e_{n+1}, all the solutions must be contained in $(\boldsymbol{C}^*)^{n+1}$. Let $\mathcal{S}=\{\beta^1, \cdots, \beta^q\}$ be the solutions of (5.7.2 ; 1). As $\mathcal{S}$ is clearly invariant under the monodromy $\boldsymbol{Z}/d\boldsymbol{Z}$-action, let $\beta^1, \cdots, \beta^{q/d}$ be the representatives of the respective orbits. We assert that $\beta^1, \cdots, \beta^{q/d}$ correspond to mutually distinct solutions of (5.7.1) and this correspondence is surjective. By the argument of Lemma (4.3) of [**16**], the solutions of (5.7.2; t) for $t\neq 0$ are $\beta^j\zeta_k(t)$ where $j=1, \cdots, q/d$ and $\zeta_k(t)$ $(k=1, \cdots, d)$ are the d-th roots of t. As $\beta^j\zeta_k(t)$ corresponds to $\alpha^j=(\beta_1^j/\beta_{n+1}^j, \cdots, \beta_n^j/\beta_{n+1}^j)$ and α^j are clearly mutually distinct, the assertion is obvious. Now we assume the coefficients are sufficiently generic so that $\beta^1, \cdots, \beta^q$ are simple and in particular $q=(n+1)!$ $(n+1)$-volume $\Delta(0)$. The simplicity of α^j $(j=1, \cdots, q/d)$ is obtained from the simplicity of β^j as follows.

$$0\neq \det\left\{\partial\left(z_i\frac{\partial f}{\partial z_i}\right)\Big/\partial z_k\right\} \quad (z=\beta^j)$$

$$=\det\begin{pmatrix} \beta_1^j\dfrac{\partial^2 f}{\partial z_1\partial z_1}(\beta^j), & \cdots, & \beta_1^j\dfrac{\partial^2 f}{\partial z_1\partial z_n}(\beta^j) \\ \cdots & \cdots & \cdots \\ \beta_n^j\dfrac{\partial^2 f}{\partial z_n\partial z_1}(\beta^j), & \cdots, & \beta_n^j\dfrac{\partial^2 f}{\partial z_n\partial z_n}(\beta^j)\end{pmatrix}\cdot d\frac{\partial f}{\partial z_{n+1}}(\beta^j)$$

$$=\det\frac{\partial^2 g}{\partial u_i\partial u_k}(\alpha^j)\cdot \text{non-zero complex number.}$$

To get the first equality, we have used the equality $\dfrac{\partial f}{\partial z_i}=\dfrac{1}{d-1}\sum_{k=1}^{n+1} z_k\dfrac{\partial^2 f}{\partial z_i\partial z_k}$. Thus α^j are simple. To show that the number of the solutions of (5.7.1) is independent of the coefficients, we note that the local sum of the multiplicities is constant under a small deformation of the coefficients by virtue of Rouche's principle. Now Lemma (5.7) is an immediate consequence of Lemma (4.3) and Lemma (4.3)′ of [**16**] and Lemma (5.2). Q. E. D.

The following lemma is one of the key steps of the proof of Theorem (5.3).

LEMMA (5.8). *Let $g(u)$ be as in Lemma* (5.7) *and we suppose, by a parallel translation if necessary, that $S(g)$ is included in $\{x\in \boldsymbol{R}^n; x_i\geqq 1$ for $i=1, \cdots, n\}$. Then the family of hypersurfaces $\{g^{-1}(\eta)\}$, $\eta\in \boldsymbol{C}^*$, are controlled by the spheres $S_R=\{u\in \boldsymbol{C}^n; \|u\|^2=|u_1|^2+\cdots+|u_n|^2=R^2\}$ near the infinity in the following sense: For any $0<\varepsilon<1$, there is a positive number $R(\varepsilon)$ so that $g^{-1}(\eta)$ intersects transversely with the spheres S_R for any $\varepsilon<|\eta|<\varepsilon^{-1}$ and $R>R(\varepsilon)$.*

PROOF. Assume the non-existence of $R(\varepsilon)$ as above for some ε. We apply the Curve selection lemma ([**10**]) to the real algebraic variety W and an open set U where $U=\{[z]\in \boldsymbol{P}^n; z_1z_2\cdots z_{n+1}\neq 0,\ \varepsilon^2\cdot|z_{n+1}|^{2d}<|f(z)|^2<\varepsilon^{-2}|z_{n+1}|^{2d}\}$ and $W=\{[z]\in \boldsymbol{P}^n;\ \left(\dfrac{\partial f}{\partial z_1}, \cdots, \dfrac{\partial f}{\partial z_n}\right)$ and $(\bar z_1, \cdots, \bar z_n)$ are linearly dependent over $\boldsymbol{C}\}$. Here $f(z_1, \cdots, z_{n+1})$ is a fixed homogeneous polynomial of degree d such that $\pi f=g$. It is clear that $W\cap U$ correspond bijectively to the set $\{u\in(\boldsymbol{C}^*)^n;$

$\varepsilon<|g(u)|<\varepsilon^{-1}$, $g^{-1}(g(u))$ and $S_{\|u\|}$ are not transverse at $u\}$ by $z\mapsto u=(z_1/z_{n+1}, \cdots z_n/z_{n+1})$. We can find a curve $u(s)$ $(0\leqq s\leqq 1)$ expanded in a Laurent series

$$u(s)=(\alpha_1 s^{a_1}+\cdots, \cdots, \alpha_n s^{a_n}+\cdots)$$

such that (i) $\alpha=(\alpha_1, \cdots, \alpha_n)\in(\boldsymbol{C}^*)^n$, (ii) $a_0:=\text{minimum}\ \{a_1, \cdots, a_n\}$ is a negative integer, (iii) $\varepsilon<|g(u(s))|<\varepsilon^{-1}$ for $s\neq 0$ and

$$\text{(iv)}\qquad \frac{\partial g}{\partial u_i}(u(s))=\lambda(s)\bar{u}_i(s)\quad (i=1, \cdots, n)\qquad \text{for some}\ \ \lambda(s)\in\boldsymbol{C}.$$

(ii) say that $\|u(s)\|\to\infty$ when $s\to 0$. By (iv) $\lambda(s)$ can be automatically expanded in a Laurent series and $\lambda(s)\neq 0$ by Lemma (5.7). For a Laurent series $p(s)=\sum_{j\geq m} p_j s^j$, $p_m\neq 0$, we call m the order of $p(s)$ and denote it by ord $p(s)$. Let $d(a)=\text{minimum}\{(x, a)=\sum_{j=1}^n x_j a_j\,;\, x\in S(g)\}$ and let $\Delta_a=\{x\in S(g)\,;\,(x, a)=d(a)\}$. $d(a)$ is clearly an integer. Let $I=\{1\leqq i\leqq n\,;\, a_i=a_0\}$. Comparing the orders of both side of (iv) we get:

$$\text{(5.8.1)}\qquad \text{ord}\left\{u_i(s)\frac{\partial g}{\partial u_i}(u(s))\right\}=\text{ord}\,\lambda(s)+2a_i\,,\qquad i=1, \cdots, n\,.$$

On the other hand, ord $\left\{u_i(s)\frac{\partial g}{\partial u_i}(u(s))\right\}\geqq d(a)$ and the equality holds if and only if $\partial g_{\Delta_a}/\partial u_i(\alpha)\neq 0$. By the global non-degeneracy and the regularity assumption for $g(u)$, there exists an i_0, $1\leqq i_0\leqq n$, such that $\partial g_{\Delta_a}/\partial u_{i_0}(\alpha)\neq 0$. From these arguments and (5.8.1), we get:

$$\text{(5.8.2)}\qquad d(a)=\text{ord}\,\lambda(s)+2a_0\quad\text{and}$$

$$\text{(5.8.3)}\qquad \alpha_i\partial g_{\Delta_a}/\partial u_i(\alpha)=\begin{cases}\lambda_0|\alpha_i|^2, & i\in I\\ 0 & \text{otherwise}\end{cases}$$

where λ_0 is the leading coefficient of $\lambda(s)$. Assume $d(a)=0$. Then we get $\frac{dg(u(s))}{ds}=\sum_i\frac{\partial g}{\partial u_i}(u(s))\cdot\frac{du_i(s)}{ds}=a_0\lambda_0\sum_{i\in I}|\alpha_i|^2\cdot s^{-1}+$higher terms. Integrating this, we get a contradiction to (iii). Thus $d(a)\neq 0$ but in this case we get $g(u(s))=\lambda_0\frac{a}{d(a)}\sum_{i\in I}|\alpha_i|^2\cdot s^{d(a)}+$higher terms which contradicts (iii) again.

Q. E. D.

Let ε_n $(n=1, 2, \cdots)$ be a monotone decreasing sequence such that $0<\varepsilon_n<1$ and $\lim_{n\to\infty}\varepsilon_n=0$ and let R_n be a monotone increasing sequence so that (ε_n, R_n) satisfies the assertion in Lemma (5.8). We can construct a controlled plane field ξ of the map $g:\boldsymbol{C}^n-g^{-1}(\Sigma)\to\boldsymbol{C}-\Sigma$ in the following sense where Σ is the set of the critical values of g. (i) $\dim_{\boldsymbol{R}}\xi(u)=2$ and $dg_u:\xi(u)\to T_{g(u)}(\boldsymbol{C}-\Sigma)$

is bijective for $u\in C^n-g^{-1}(\Sigma)$ and (ii) $\xi(u)\subset T_u S_{\|u\|}$ for u such that $\varepsilon_n<|g(u)|<\varepsilon_n^{-1}$ and $\|u\|>R_n$. By the construction of ξ, ξ is integrable over any paths and therefore we get the following lemma using the same argument as in [4].

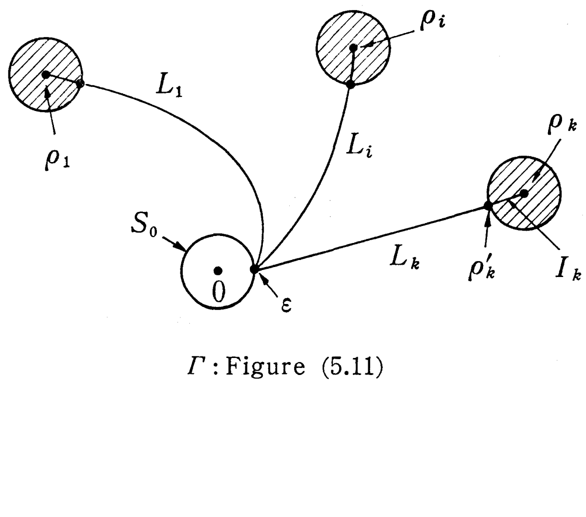
LEMMA (5.9). *Under the same assumption as in Lemma* (5.8), $g: C^n-g^{-1}(\Sigma)\to C-\Sigma$ *is a locally trivial fibration.*

Let $g(u)$ be as in Lemma (5.8) and let $Y^*=C^n-g^{-1}(0)$. Let $\mu^1, \cdots, \mu^p$ be the points of $S(g)\cap N^n$. Let $U=\{c=(c_1, \cdots, c_p)$; the polynomial $\sum_{j=1}^{p} c_j u^{\mu^j}$ is globally non degenerate with the support $S(g)\}$. By a standard argument, U is a non-empty Zariski open set of C^p. (Theorem 6.1 of [7] and Lemma (5.2).) In particular U is connected. Thus the following lemma says that Y^* is uniquely determined by $S(g)$ up to a diffeomorphism.

LEMMA (5.10). *Let* $g_t(u)$ $(0\leqq t\leqq 1)$ *be an analytic family of polynomials such that the corresponding coefficients stay in* U. *Then* (i) *the critical points of* g_t *in* $(C^*)^n$ *are contained in a compact set of* $(C^*)^n$, *independent of t, so that the corresponding critical values stay uniformly in a compact set of* C^*. (ii) $g_t^{-1}(\eta)$ $(0\leqq t\leqq 1)$ *are uniformly controlled by the spheres* S_r *in the sense of Lemma* (5.8). (iii) $C^n-g_0^{-1}(0)$ *is diffeomorphic to* $C^n-g_1^{-1}(0)$.

PROOF. (i) is derived from Lemma (5.7) because critical points of g_t in $(C^*)^n$ move continuously in t by Rouché's principle ([10]), (ii) is proved by the exact same argument as in Lemma (5.8). (iii) is an immediate consequence of (ii). (See also the proof of Lemma (6.5).) Q. E. D.

Now we have sufficient information to discuss the topology of Y^*. We may assume that the critical points of g in $(C^*)^n$ are all simple and let $\tilde{\Sigma}=\{\alpha^1, \cdots, \alpha^r\}$ be the critical points of g where $r=n!$ n-volume $S(g)$ and let $\Sigma^*=\{\rho_1, \cdots, \rho_{r'}\}$ be non-zero critical values of g. Take $\varepsilon>0$ so that the disks $D_j=\{\eta\in C; |\eta-\rho_j|\leqq\varepsilon\}$ $(j=0, 1, \cdots, r')$ are mutually disjoint where $\rho_0=0$. Take simple paths L_j which join $\varepsilon\in D_0$ and D_j so that $\Gamma:=S_0\cup(L_1\cup D_1)\cup\cdots\cup(L_{r'}\cup D_{r'})$ is contractible to S_0 where $S_0=\partial D_0=\{\eta \mid |\eta|=\varepsilon\}$. See Figure (5.11). By Lemma (5.9), Y^* is deformed into $g^{-1}(\Gamma)$. Let us first consider the space



Γ: Figure (5.11)

$g^{-1}(D_j)$. Let ρ'_j be the end point of L_j lying on ∂D_j and let I_j be the line segment $\overline{\rho_j\rho'_j}$. By virtue of the simplicity of α^k, we can take a local coordinate $w=(w_1, \cdots, w_n)$ in a neighborhood of $u=\alpha^k$ so that α^k corresponds to the origin of the coordinate w and $g(w)=w_1^2+\cdots+w_n^2+g(\alpha^k)$. Let $B_k(\delta)$ be a small disk $\{w\,;\|w\|\leqq\delta\}$. Assuming that δ is small enough and ε is sufficiently smaller than δ, we can see that (i) $S(\alpha^k):=\{w\,;w_1^2+\cdots+w_n^2=\rho'_j-\rho_j$ where $\rho_j=g(\alpha^k)$ and argument $(w_i)=$argument $(\rho'_j-\rho_j)$ for $i=1, \cdots, n\}$ is an embeded $(n-1)$-sphere in $g^{-1}(\rho'_j)\cap B_k(\delta)$ and (ii) $D(\alpha^k)=\{w\,;g(w)\in I_j$ and argument (w_i) $=$argument $(\rho'_j-\rho_j)$ for $i=1, \cdots, n\}$ is an embedded n-disk in $g^{-1}(I_j)$ which bounds $S(\alpha^k)$ in a canonical way.

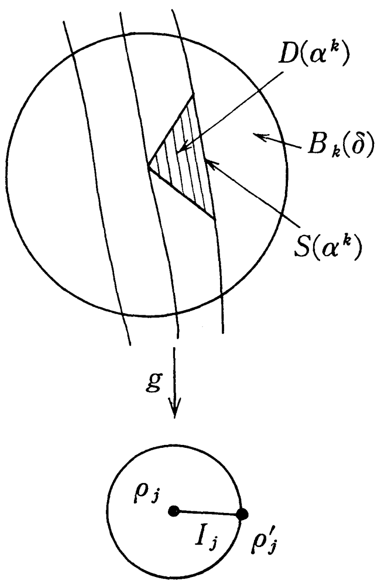


Figure (5.12)

By the standard argument for the topology of the isolated singularities of hypersurfaces (cf. [**10**]), $g^{-1}(D_j)$ is deformed into $g^{-1}(I_j)$ and $g^{-1}(I_j)$ has the homotopy type of the space $g^{-1}(\rho'_j)\cup(\cup D(\alpha^k))$ where the union is taken for k such that $g(\alpha^k)=\rho_j$. Let $E=g^{-1}(S_0)$. Because $g^{-1}(L_j)$ is diffeomorphic to $g^{-1}(\varepsilon)\times[0, 1]$, we obtain:

LEMMA (5.13). *Under the above notations, Y^* has the homotopy type of the space which is obtained from E by attaching $n!$ n-volume $S(g)$ n-cells $D(\alpha^k)$ along $S(\alpha^k)$.*

COROLLARY (5.13.1). *The Euler-Poincaré characteristic $\chi(Y^*)$ is $(-1)^n\cdot n!$ n-volume $S(g)$.*

This is an immediate consequence of $\chi(E)=0$ which is due to the fact that E is a fibration over the circle. This proves (i) of Theorem (5.3). Let $Y^*(\eta)$ be a generic fiber $g^{-1}(\eta)$ $(\eta\neq 0,\ \eta\notin\Sigma^*)$. Then

COROLLARY (5.13.2). $\pi_j(Y^*(\eta))\cong\pi_j(Y^*)$ *for* $j=2, \cdots, n-2$.

PROOF. Observe that $\pi_j(Y^*(\eta))=\pi_j(E)$ for any $j\neq 1$ and $\pi_j(E)=\pi_j(Y^*)$ for $j<n-1$. Thus the assertion is obvious.

REMARK (5.14). The above argument can be displaced by Morse theory using the Morse function $\xi(u):=\log|g(u)|$. The critical points of ξ are exactly $\alpha^1, \cdots, \alpha^r$ and the simplicity of α^j implies that α^j is a non-degenerate critical point of ξ having the Morse index n.

6. Proof of (ii) of Theorem (5.3).

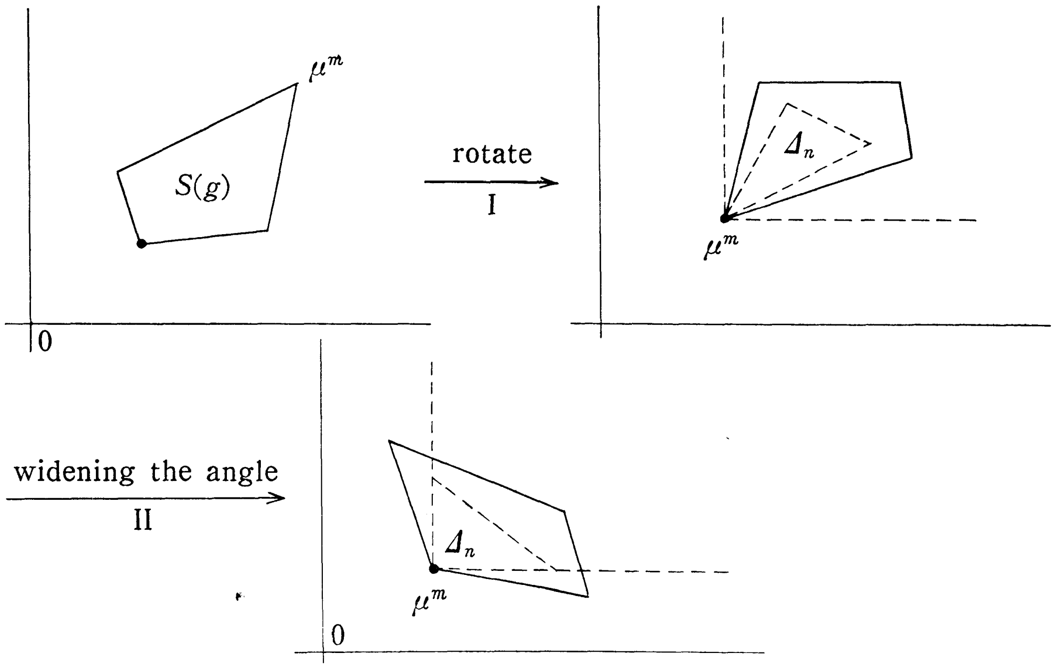
In this section, we prove (ii) of Theorem (5.3) by the induction on the number of the effective vertices of $S(g)$. Let $g(u)=\sum_{j=1}^{m} c_j u^{\mu^j}$ be as in Lemma (5.8). We assume that $\mu^1, \cdots, \mu^m$ are effective vertices of $S(g)$. The case that $m=n+1$ is already proved by Lemma (3.4) and the section 4. Thus we assume that $m>n+1$. By the assumption that $\dim S(g)=n$, we can find a vertex, say μ^m, of $S(g)$ so that the convex polyhedron in $\boldsymbol{R}^n$ which is spanned by $\mu^1, \cdots, \mu^{m-1}$ is of dimension n. We consider an analytic family $g_t(u):=g_0(u)+tc_m u^{\mu^m}$ $(0\leqq t\leqq 1)$ where $g_0(u)$ is $\sum_{j=1}^{m-1} c_j u^{\mu^j}$. We can assume that $g_t(u)$ is non-degenerate for each $0\leqq t\leqq 1$. Taking a suitable affine translation if necessary, we may suppose:

(6.1) $\Gamma_-(g_1)\subset\Gamma_-(g_0)$ and

(6.2) $g_0(u)$ and $g_t(u)$ $(t\neq 0)$ are regular

where $\Gamma_-(g_0)$ is the geometric cone of $\Gamma(g_t)$ and the origin of $\boldsymbol{R}^n$.

The following pictures illustrate the necessary transformation.



The procedure II is done in the following manner. Take a n-simplex Δ_n with vertices μ^m and $\nu^1, \cdots, \nu^n$ where ν^j $(j=1, \cdots, n)$ are contained in $Q^n \cap$ {the interior of $S(g)$}. Let A be the affine transformation over Q defined by $A\mu^m=\mu^m$ and $A\nu^j=\mu^m+e_j$ where $e_j=(0, \cdots, \overset{j}{\check{1}}, \cdots, 0)$. Taking a suitable integer r, we define an affine transformation over $\boldsymbol{Z}$ by $rA=\begin{bmatrix} r & & 0 \\ & \ddots & \\ 0 & & r \end{bmatrix}\cdot A$. The affine transformation defined by the composition of I and II satisfies the condition (6.1). After this process we may choose a suitable parallel translation to get an adequate transformation.

LEMMA (6.3). *If we choose nice coefficients* $\{c_j\}$, g_t *has* $n!$ n*-volume* $S(g_1)$ *simple critical points in* $(\boldsymbol{C}^*)^n$ *for* $t\neq 0$ *sufficiently small so that* (i) $n!$ n*-volume* $S(g_0)$ *of them converge to the simple critical points of* g_0 *in* $(\boldsymbol{C}^*)^n$ *and* (ii) *the rest converge to the origin when* t *goes to* 0.

PROOF. Remember the correspondence of the critical point g_t in $(\boldsymbol{C}^*)^n$ and the solution of $z\frac{\partial f_t}{\partial z_1}=\cdots=z_n\frac{\partial f_t}{\partial z_n}=0$ and $z_{n+1}\frac{\partial f_t}{\partial z_{n+1}}=1$ where $f_t(z)=g_t(z_1/z_{n+1}, \cdots, z_n/z_{n+1})\cdot z_{n+1}^d$ is a fixed homogeneous polynomial of degree d. Let Δ and Δ_0 be the respective Newton boundaries of f_t $(t\neq 0)$ and f_0. Then clearly $\Delta_0=S(g_0)$. Let $\nu^1, \cdots, \nu^m$ be the corresponding vertices of Δ $(\pi\nu^j=\mu^j)$. Let $\mathcal{S}=\{\Xi\,;\Xi$ is a closed face of $\Gamma(g_0)$ of dimension $n-1$ such that Ξ is not contained in $\Gamma(g_1)$ and let $\tilde{\mathcal{S}}$ be the set of the corresponding face of Δ_0. In the proof of Lemma (4.3) of [16], we have proved that for any $\tilde{\Xi}\in\tilde{\mathcal{S}}$ there are $(n+1)!$ $(n+1)$-volume $\tilde{\Xi}(\nu^m)$ solutions of (5.7.2; 1) for f_t which disappear in the infinity when $t\to 0$ and which are parametrized in a following Laurent series:

(i) $z(s)=(\beta_1 s^{a_1}+\cdots, \cdots, \beta_{n+1}s^{a_{n+1}}+\cdots)$ and $t=s^b$ where $a=(a_1, \cdots, a_{n+1})$ and b are uniquely determined up to the multiplication of a positive integer by $(a, \nu^k)=\sum_{j=1}^{n+1} a_j\nu_j^k=0$ for $\nu^k\in\tilde{\Xi}$ and $b=-(a, \nu^m)=-\sum_{j=1}^{n+1} a_j\nu_j^m>0$. (ii) $\beta=(\beta_1, \cdots, \beta_{n+1})$ moves in the solutions of (5.7.2; 1) for $f_{1,\tilde{\Xi}(\nu^m)}$ where $\tilde{\Xi}(\nu^m)$ is the cone of $\tilde{\Sigma}$ and ν^m. Remembering the correspondence of the solutions of (5.7.2; 1) and of (5.7.1), we have no difficulty to translate (i) and (ii) into the words of g_t. Let Ξ be the face of $\Gamma(g_0)$ in $\mathcal{S}$ which corresponds $\tilde{\Xi}$. Then there are $n!$ n-volume $\Xi(\mu^m)$ critical points of g_t which are parametrized in the following way.

(i)′ $u(s)=(\alpha_1 s^{p_1}+\cdots, \cdots, \alpha_n s^{p_n}+\cdots)$ and $t=s^b$ where $p=(p_1, \cdots, p_n)$ is $(a_1-a_{n+1}, \cdots, a_n-a_{n+1})$ and $b=-(a, \nu^m)>0$ and (ii)′ $\alpha=(\alpha_1, \cdots, \alpha_n)$ moves in the solutions of (5.7.1) for $g_{1,\Xi(\mu^m)}$.

The above condition (i) says that $\sum_{j=1}^{n} p_j\nu_j^k=-a_{n+1}d$ for $\nu^k\in\tilde{\Xi}$ and $\sum_{j=1}^{n} p_j\nu_j^m=$

$-a_{n+1}d-b$. Thus the hypersurface: $\sum_{j=1}^{n} p_j x_j = -a_{n+1}d$ contains Ξ and $b>0$ implies that $p_i>0$ for $i=1, 2, \cdots, n$.
See Figure (6.3.1).

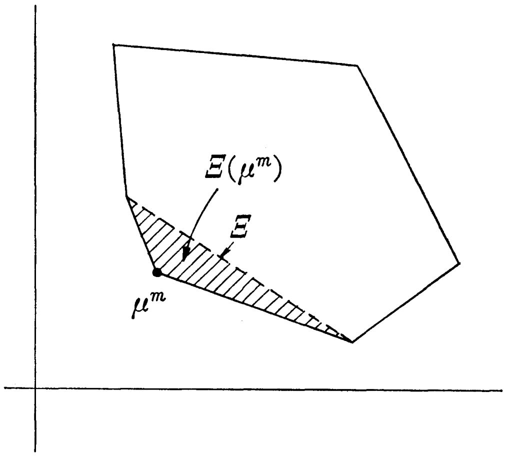


Figure (6.3.1)

LEMMA (6.4). *The family of hypersurfaces* $\{g_t(\eta)\}$ $(0 \leqq t \leqq 1)$ *are uniformly controlled by spheres near the infinity in the sense of Lemma* (5.8).

PROOF. The proof is almost parallel to the proof of Lemma (5.8). Assuming that the assertion is not true, we can use the Curve Selection lemma to find Laurent series:
$u(s)=(\alpha_1 s^{a_1}+\cdots, \cdots, \alpha_n s^{a_n}+\cdots)$, $t(s)=t_0 s^b+\cdots$ and $\lambda(s)=\lambda_0 s^c+\cdots$ such that (i) $\alpha=(\alpha_1, \cdots, \alpha_n)\in(\boldsymbol{C}^*)^n$, $\lambda_0\in\boldsymbol{C}^*$ and $0\leqq t(s)\leqq 1$ i. e. $b\geqq 0$ and (ii) $a_0:=$minimum $\{a_1, \cdots, a_n\}<0$ and (iii) $\varepsilon<|g_{t(s)}(u(s))|<\varepsilon^{-1}$ for some $0<\varepsilon<1$ and (iv) $\dfrac{\partial g_{t(s)}(u(s))}{\partial u_i}$ $=\lambda(s)\cdot\bar{u}_i(s)$ for $i=1, \cdots, n$.
$\lambda_0\neq 0$ is due to Lemma (6.3). Assume that $b=0$. i.e. $t(s)\to t_0\neq 0$. We get a contradiction to (iii) by exactly the same argument as in the proof of Lemma (5.8). Assume that $b>0$. Let $d(a)=$minimum $\{(a, \mu^1), \cdots, (a, \mu^m)\}$ and let $\Delta_a=\{\mu^i; (a, \mu^i)=d(a)\}$.

Assertion: μ^m is not contained in Δ_a. Assume this for a while and let $I=\{1\leqq i\leqq n; a_i=a_0\}$. By a similar argument as in Lemma (5.8), we get:

(6.4.1) $$d(a)=b+a_0 \quad \text{and}$$

(6.4.2) $$\alpha_i\frac{\partial g}{\partial u_i}(\alpha)=\begin{cases}\lambda_0|\alpha_i|^2 & \text{for } i\in I\\ 0 & \text{otherwise}.\end{cases}$$

Proceeding as the proof of Lemma (5.8), the case $d(a)=0$ gives a contradiction ord $\left\{\frac{dg_{t(s)}}{ds}(u(s))\right\}=-1$ to (iii) and the case $d(a)\neq 0$ induces a contradiction ord $\{g_{t(s)}(u(s))\}=d(a)$ again to (iii).

Now we prove the assertion. Let $P=\{\Xi\,;\,\Xi$ is a face of $\Gamma(g_1)$ of dimension $n-1$ containing $\mu^m\}$. For $\Xi\in P$, let $a_1(\Xi)x_1+\cdots+a_n(\Xi)x_n=d(\Xi)$ be the equation of the hyperplane of $\boldsymbol{R}^n$ containing Ξ such that $a_i(\Xi)$ $(i=1,\cdots,n)$ are positive integers and G.C.M. $(a_1(\Xi),\cdots,a_n(\Xi))=1$. Let P^* be the cone spanned by the vectors $a(\Xi)=(a_1(\Xi),\cdots,a_n(\Xi))$ for $\Xi\in P$ and let $P'=\{a\in\boldsymbol{R}^n\,;$ $(a,\mu^m)\leqq(a,\mu^j)$ for $j=1,\cdots,m-1\}$. By the definition of P^*, $P'\supset P^*$. Because P' is a convex set, we can see easily that $P'=P^*$. Assume that $\mu^m\in\Delta_a$. Then the definition of Δ_a implies $a\in P'$ hence $a\in P^*$. But this is impossible by (ii). Q.E.D.

Now we are ready to prove (ii) of Theorem (5.3). Take $\eta_0>0$ so that the absolute values of non-zero critical values of g_0 are greater than $2\eta_0$. By Lemma (6.3) we can take $\varepsilon>0$ so that for any critical point γ_t of g_t in $(\boldsymbol{C}^*)^n$ $(0\leqq t\leqq\varepsilon)$, $g_t(\gamma_t)\neq 0,\ \eta_0$, and γ_t is simple. Let $Y_i^*=(\boldsymbol{C}^*)^n-g_i^{-1}(0)$ and $Y_{i,\eta_0}^*=\{u\in(\boldsymbol{C}^*)^n\,;\,|g_i(u)|\geqq\eta_0\}$ for $i=0,\varepsilon$. Y_{0,η_0}^* is a deformation retract of Y_0^*. Let $W=\{(u,t)\in\boldsymbol{C}^n\times[0,\varepsilon]\,;\,|g_t(u)|\geqq\eta_0\}$. Let η_1 be a positive number such that the critical values of g_t $(0\leqq t\leqq\varepsilon)$ are contained in $D_{\eta_1}:=\{\eta\in\boldsymbol{C};\,|\eta|\leqq\eta_1\}$. Let $\pi_1: W\to\boldsymbol{C}\times[0,\varepsilon]$ be the map $(u,t)\mapsto(g_t(u),t)$ and let $\pi_2: W\to[0,\varepsilon]$ be the projection map. It is easy to see that (i) π_2 is a submersion and (ii) $\pi_1|_{\partial W\cup W_{\eta_1}}$ is a submersion where $W_{\eta_1}=W\cap\{|g_t(u)|\geqq\eta_1\}$. Thus we can construct a connection vector field v for π_2 such that (a) $d\pi_1(v(u,t))=\partial/\partial t$ for $(u,t)\in W_{\eta_1}\cup\partial W$ where $\partial/\partial t$ is the unit vector of $[0,\varepsilon]$ in the positive direction and (b) $v(u,t)$ is tangent to $S_{\|u\|}\times[0,\varepsilon]$ when $\|u\|$ is sufficiently large with respect to $|g_t(u)|$. Then it is obvious that v is completely integrable and therefore Y_{0,η_0}^* is diffeomorphic to Y_{ε,η_0}^* via the integration of v. On the other hand by the same argument as in Lemma (5.13), we can show without any difficulty that Y_ε^* is homotopically obtained by attaching $n!$ n-volume $(S(g_1)-S(g_0))$ n-cells to Y_{ε,η_0}^*. Thus we get:

LEMMA (6.5). *Y_ε^* has the homotopy type of the space which is obtained by attaching $n!$ n-volume $(S(g_1)-S(g_0))$ n-cells to Y_0^*.*

COROLLARY (6.5.1). *$\pi_i(Y_0^*)\to\pi_i(Y_\varepsilon^*)$ is bijective for $i<n-1$ and surjective for $i=n-1$.*

Now (ii) of Theorem (5.3) is a direct consequence of the inductive assumption and Corollary (6.5.1), Lemma (5.4) and Lemma (5.10). This also finishes the proof of Theorem (1.1).

PROOF OF COROLLARY (1.1.2). $(h^*)_*: H_1(F^*;\boldsymbol{Z})\to H_1(F^*;\boldsymbol{Z})$ is the identity map by Lemma (2.2). Thus $(h^*)^*: H^1(F^*;\boldsymbol{Z})\to H^1(F^*;\boldsymbol{Z})$ is also the identity map by the Universal Coefficient theorem. Because $H^j(F^*;\boldsymbol{Z})$ is generated by

$H^1(F^*;\boldsymbol{Z})$ as an algebra for $j<n$, the monodromy action on $H^j(F^*;\boldsymbol{Z})$ is trivial for $j<n$. Again by the Universal Coefficient theorem, $(h^*)_*:H_j(F^*;\boldsymbol{Z})\to H_j(F^*;\boldsymbol{Z})$ is the identity map for $j\leqq n-1$, proving (i). Let $F_j(F^*)$ be the fixed points of $(h^*)^j:F^*\to F^*$. Then $F^j(F^*)=\emptyset$ for $j\neq 0$ modulo d and $F_j(F^*)=F^*$ for $j=kd$ $(k\in\boldsymbol{N})$. Thus by the definition of [**10**],

$$\zeta(h^*;t)=\exp\sum_{k=1}^{\infty}\frac{t^k}{k}\chi(F_k(F^*))=\exp\sum_{j=1}^{\infty}\frac{t^{dj}}{dj}\chi(F^*)=(1-t^d)^{-\frac{\chi(F^*)}{d}}.$$

Q. E. D.

PROOF OF COROLLARY (1.2). Let $\nu^1,\cdots,\nu^m$ be the vertices of Δ, and let M be the subgroup of $\boldsymbol{Z}^{n+1}$ generated by $\nu^1,\cdots,\nu^m$. We can find a matrix $N\in SL(n+1,\boldsymbol{Z})$ such that N maps M into the subgroup $\boldsymbol{Z}^{r+1}=\{\nu\in Z^{n+1};\nu_j=0$ for $j>r+1\}$. Then F_N^* is diffeomorphic to F^* by φ_N and F_N^* clearly has the desired property. Now (i) of Corollary (1.2) is obvious and (ii) is proved by the exact same way as the proof of Corollary (1.1.2).

7. Topology of F.

Let $f(z_1,\cdots,z_{n+1})$ be a weighted homogeneous polynomial of type $(a_1,\cdots,a_{n+1};d)$ with a non-degenerate Newton boundary. In this section, we study the topology of the Milnor fiber $F=\{z\in\boldsymbol{C}^{n+1};f(z)=1\}$ using the results in §§ 1–6. For a non-empty subset $I\subset\{1,\cdots,n+1\}$, put $F_I^*=F\cap(\boldsymbol{C}^*)^I=\{z\in F;$ $z_i\neq 0$ if and only if $i\in I\}$. Then $\bigcup_I F_I^*$ gives a cannonical stratification of F. By the additivity of the Euler-Poincaré characteristic and by Theorem (1.1) we obtain the following theorem. (cf. Theorem (7.6) below):

THEOREM (7.1). $\chi(F)=(-1)^n\nu(\Delta(0))+1$ *where* $\nu(\Delta(0))$ *is the Newton number of* $\Delta(0)$.

For the definition of the Newton number, we refer to [7] or [**16**]. Recall that the monodromy map $h:F\to F$ is defined by $h(z)=\left(z_1\exp\frac{a_1}{d}2\pi\sqrt{-1},\cdots,z_{n+1}\exp\frac{a_{n+1}}{d}2\pi\sqrt{-1}\right)$. The zeta function of h can be computed directly along the definition of [**10**] using Theorem (1.1). To make the computation easier, we take the following viewpoint.

Let $\varphi:X\to X$ be a diffeomorphism of a differential manifold X. The zeta function of φ can be defined by an alternating product

$$\zeta(\varphi;t)=P_0(t)^{-1}\cdot P_1(t)\cdots P_n(t)^{(-1)^{n-1}}$$

where $n=\dim X$ and $P_i(X)$ is the determinant of the linear transformation $id-t\varphi_*:H_i(X;\boldsymbol{R})\to H_i(X;\boldsymbol{R})$. Assume that X_1 and X_2 are φ-invariant submanifolds of X such that $X=X_1\cup X_2$ and $X_1\cap X_2$ is also a smooth submanifold.

Let $\varphi_i=\varphi|X_i$ $(i=1, 2)$ and $\varphi_{12}=\varphi|X_1\cap X_2$. Then the zeta function has the following additivity property.

LEMMA (7.2).

$$\zeta(\varphi ; t)=\zeta(\varphi_1 ; t)\cdot\zeta(\varphi_2 ; t)/\zeta(\varphi_{12} ; t)$$

PROOF. This is an immediate consequence of the following exact sequences and commutative diagrams using a splitted short exact sequence technique.

$$\begin{tikzcd}
\cdots \arrow[r] & H_j(X_1\cap X_2 ; \boldsymbol{R}) \arrow[r] \arrow[d, "(\varphi_{12})_*"] \arrow[dr, phantom, "\circlearrowright"] & H_j(X_1 ; \boldsymbol{R})\oplus H_j(X_2 ; \boldsymbol{R}) \arrow[r] \arrow[d, "(\varphi_1)_*\oplus(\varphi_2)_*"] \arrow[dr, phantom, "\circlearrowright"] & H_j(X ; \boldsymbol{R}) \arrow[r] \arrow[d, "\varphi_*"] & \cdots \\
\cdots \arrow[r] & H_j(X_1\cap X_2 ; \boldsymbol{R}) \arrow[r] & H_j(X_1 ; \boldsymbol{R})\oplus H_j(X_2 ; \boldsymbol{R}) \arrow[r] & H_j(X ; \boldsymbol{R}) \arrow[r] & \cdots
\end{tikzcd}$$

Q. E. D.

By the definition of the monodromy map h and F_I^*, F_I^* is invariant by h and we put $h_I^*=h \mid F_I^*$. By the same argument as in the proof of Corollary (1.1.2), we get from Theorem (1.1) and Corollary (1.2):

LEMMA (7.3). $\zeta(h_I^* ; t)=(1-t^{d_I})^{-\frac{\chi(F_I^*)}{d_I}}$ *where* d_I *is defined by* d/b_I, b_I *being the greatest common divisor of* $\{a_i\}$, $i\in I$.

In the case of $\dim \Delta_I<|I|-1$ $(\Delta_I=\Delta\cap\boldsymbol{R}^I)$, the above lemma says $\zeta(h_I^* ; t)=1$ because $\chi(F_I^*)=0$ by Corollary (1.2). Now let $F(k)=\{z\in F ; z_i\neq 0 \text{ for } i\leqq k\}$ and $F_k(k-1)=F(k-1)-F(k)$. By the definition, $F^*=F(n+1)\subset F(n)\subset \cdots \subset F(1)\subset F(0)$ $=F$ and $F_k(k-1)$ is a codimension 1 submanifold of $F(k-1)$ and we can write $F_k(k-1)=\bigcup_J F_J^*$ where J moves in the subsets of $\{1, \cdots, n+1\}$ containing $\{1, 2, \cdots, k-1\}$ such that $J\not\ni k$. Of course $F_k(k-1)$ is an empty set if $f\,|_{z_k=0}=0$.

LEMMA (7.4). $F_k(k-1)$ *has a trivial normal bundle in* $F(k-1)$.

PROOF. As a polynomial function on a non-singular algebraic variety has finite critical values (Corollary (2.8) of [10]), we can take a positive number R so that F_I^* intersects the spheres S_r $(r\geqq R)$ transversely for each $I\subset\{1, \cdots, n+1\}$. Let $F'=F\cap B(R)$, $F(k-1)'=F(k-1)\cap B(R)$, $F_k(k-1)'=F_k(k-1)\cap B(R)$, etc. where $B(R)=\{z\in \boldsymbol{C}^{n+1} ; \|z\|\leqq R\}$. Because $F_I^{*\prime}$, $F_k(k-1)'$, $F(k-1)'$ and F' are deformation retracts of F_I^*, $F_k(k-1)$, $F(k-1)$ and F respectively, we need only prove that $F_k(k-1)'$ has a trivial normal bundle in $F(k-1)'$. Let π_k: $F'\to\boldsymbol{C}$ be the projection map $z\mapsto z_k$. Take $\varepsilon>0$ so that $\pi_k^{-1}(\eta)$ $(|\eta|\leqq\varepsilon)$ intersect transversely with $F_I^{*\prime}$ and $\partial F_I^{*\prime}=F_I^*\cap S_R$ for which $F_I^{*\prime}$ is not contained in

$\pi_k^{-1}(0)$. Now construct a smooth connection plane field ξ of $\pi_k:\pi_k^{-1}(D_\varepsilon)\to D_\varepsilon$ where $D_\varepsilon=\{\eta\in\boldsymbol{C};|\eta|\leqq\varepsilon\}$, satisfying (i) for $F_I^{*\prime}$ such that $F_I^{*\prime}\not\subset\pi_k^{-1}(0)$ and $z\in F_I^{*\prime}$, $\xi(z)\subset T_zF_I^{*\prime}$ and (ii) $\xi(z)\subset T_zS_R$ for $z\in S_R\cap F'$. By the integration of ξ, we get a trivialization $\varphi:\pi_k^{-1}(D_\varepsilon)\to\pi_k^{-1}(0)\times D_\varepsilon$ which preserves the closure of each stratum $F_I^{*\prime}$ such that $F_I^{*\prime}\not\subset\pi_k^{-1}(0)$ invariant. Thus $F(k-1)'$ is invariant by φ and the restriction of φ to $N:=\pi_k^{-1}(D_\varepsilon)\cap F(k-1)'$ gives a desired trivialization $\psi:N\to F_k(k-1)'\times D_\varepsilon$. Q. E. D.

As N and $F_k(k-1)'$ are invariant by the monodromy map h, the map $h_\psi:=\psi h\psi^{-1}:F_k(k-1)'\times D_\varepsilon\to F_k(k-1)'\times D_\varepsilon$ can be written as $h_\psi(z,\eta)=\left(\alpha_\eta(z),\exp\frac{a_k}{d}2\pi\sqrt{-1}\right)$ for $(z,\eta)\in F_k(k-1)'\times D_\varepsilon$ where α_η is a family of diffeomorphisms of $F_k(k-1)'$ parametrized smoothly by $\eta\in D_\varepsilon$: so that $\alpha_0=h|_{F_k(k-1)'}$ where $h|_{F_k(k-1)'}$ is the restriction of h to $F_k(k-1)'$. Then clearly $(\alpha_\eta)_*=(h|_{F_k(k-1)'})_*$ for $\eta\in D_\varepsilon$. Thus we get

COROLLARY (7.4.1).

(i) $\zeta(h|_N:t)=\zeta(h|_{F_k(k-1)'};t)$.

(ii) $\zeta(h|_{N-F_k(k-1)'};t)=1$.

PROOF. (i) is obvious. (ii) is derived from the following commutative diagrams where II is due to the Künneth's formula.

$$\begin{tikzcd}
H_j(N-F_k(k-1)') \arrow[r, "(h|_{N-F_k(k-1)})_*"] \arrow[d, "\cong"', "\psi_*"] \arrow[dr, phantom, "\mathrm{I}"] & H_j(N-F_k(k-1)') \arrow[d, "\cong"', "\psi_*"] \\
H_j(F_k(k-1)'\times(D_\varepsilon-0)) \arrow[r, "(h_\psi)_*"] \arrow[d] \arrow[dr, phantom, "\mathrm{II}"] & H_j(F_k(k-1)'\times(D_\varepsilon-0)) \arrow[d] \\
H_j(F_k(k-1)')\oplus H_{j-1}(F_k(k-1)') \arrow[r, "k_j\oplus k_{j-1}"] & H_j(F_k(k-1)')\oplus H_{j-1}(F_k(k-1)')
\end{tikzcd}$$

where k_j and k_{j-1} are induced homomorphisms by $h|_{F_k(k-1)'}$ on $H_j(F_k(k-1)')$ or $H_{j-1}(F_k(k-1)')$ respectively. Thus the assertion (ii) is clear from diagram II. Q. E. D.

Applying Lemma (7.2.) to $F(k-1)'=F(k)'\cup N$, we obtain the following inductive formula.

LEMMA (7.5).

$$\zeta(h|_{F(k-1)};t)=\zeta(h|_{F(k)};t)\cdot\zeta(h|_{F_k(k-1)};t).$$

Now we are ready to prove:

THEOREM (7.6).

$$\zeta(h;t)=\prod_{I\neq\emptyset}(1-t^{d_I})^{-\chi(F_I^*)/d_I}.$$

Here we assume that $\chi(\emptyset)=0$ *for the empty set.*

PROOF. We prove by the induction on $(n+1)-k$ that:

$$\zeta(h|_{F(k)};t)=\prod_{I\supset\{1,\cdots,k\}}(1-t^{d_I})^{-\chi(F_I^*)/d_I}. \tag{7.6.1}$$

Case $k=n+1$ is nothing but Lemma (7.3) for $I=\{1,\cdots,n+1\}$. Assume that (7.6.1) is true for $k\geqq s$. Then by Lemma (7.5) we get

$$\begin{aligned}\zeta(h|_{F(s-1)};t)&=\zeta(h|_{F(s)};t)\cdot\zeta(h|_{F_s(s-1)};t)\\&=\prod_{I\supset\{1,\cdots,s\}}(1-t^{d_I})^{-\chi(F_I^*)/d_I}\cdot\prod_{\substack{J\supset\{1,\cdots,s-1\}\\ J\not\ni s}}(1-t^{d_J})^{-\chi(F_J^*)/d_J}\\&=\prod_{I\supset\{1,\cdots,s-1\}}(1-t^{d_I})^{-\chi(F_I^*)/d_I},\end{aligned}$$

where the second equality is derived from the inductive assumption for $f|_{z_s=0}$. Q. E. D.

Now we consider the homological structure of F. Let $e(\Delta)$ be the rank of the kernel of $\rho_*:H_r(F^*)\to H_r((\boldsymbol{C}^*)^{n+1})$ where $\rho:F^*\to(\boldsymbol{C}^*)^{n+1}$ is the inclusion map and $r=\dim\Delta$. Then $e(\Delta)$ is $(n+1)!$ $(n+1)$-volume $\Delta(0)-1$ if $r=n$ and in the case of $r<n$, $e(\Delta)=(r+1)!$ $(r+1)$-volume $N\Delta(0)-1$ where N is as in the proof of Corollary (1.2).

LEMMA (7.8). (i) $\rho_*:H_j(F^*)\to H_j((\boldsymbol{C}^*)^{n+1})$ *is bijective for* $j<r$ *and surjective for* $j=r$. *For* $n\geqq j>r$, *the image of* ρ_* *is of dimension* $\binom{n+1}{j}-\binom{n-r}{j-r-1}$.

(ii) *The dimension of the kernel of* $\rho_*:H_j(F^*)\to H_j((\boldsymbol{C}^*)^{n+1})$ *is*

$$\begin{cases}0 & \textit{for}\ \ j<r\\ e(\Delta) & \textit{for}\ \ j=r\\ e(\Delta)\cdot\binom{n-r}{j-r} & \textit{for}\ \ n\geqq j>r.\end{cases}$$

PROOF. The case $r=n$ is an immediate consequence of Theorem (1.1). Assume $r<n$ and take a matrix $N\in SL(n+1;\boldsymbol{Z})$ as in the proof of Corollary (1.2). As $f_N(z)$ is a weighted homogeneous polynomial of variables $z_1,\cdots,z_{r+1}$, $F_N^*=G_N^*\times(\boldsymbol{C}^*)^{n-r}$ where $G_N^*=\{(z_1,\cdots,z_{r+1});f_N(z)=1\}$. Then the assertion is easily derived from the following diagram.

$$\begin{array}{ccc} H_j(G_N^* \times (C^*)^{n-r}) & \xrightarrow{(\varphi_N)_* \cong} & H_j(F^*) \\ \Big\downarrow (\rho_N)_* & \circlearrowleft & \Big\downarrow \rho_* \\ H_j((\boldsymbol{C}^*)^{n+1}) & \xrightarrow{(\varphi_N)_* \cong} & H_j((\boldsymbol{C}^*)^{n+1}) \end{array}$$

where ρ_N is the inclusion map and φ_N is defined as in § 3. Q. E. D.

Let $\Delta_i = \Delta \cap \{x_i = 0\}$ and let $I = \{1 \leqq i \leqq n+1 ; \Delta_i \neq \emptyset\}$.

LEMMA (7.9). *Assume that* $\dim \Delta > 1$. *Then* $\pi_1(F)$ *is a free abelian group of rank* $(n+1) - |I|$. *In particular,* F *is simply connected if and only if* $I = \{1, 2, \cdots, n+1\}$.

PROOF. Note that i is contained in I if and only if $F_i := \{z \in F ; z_i = 0\}$ is not empty. Let $J = \{1, \cdots, n+1\} - I$ and consider the commutative diagram

$$\begin{array}{ccc} \pi_1(F^*) & \xrightarrow{a} & \pi_1(F) \\ \Big\downarrow \rho_\# & & \Big\downarrow b \\ \pi_1((\boldsymbol{C}^*)^{n+1}) & \xrightarrow{c} & \pi_1((\boldsymbol{C}^*)^J) \end{array}$$

where b and c are homomorphisms induced by the respective projection maps. As c is clearly surjective, b is also surjective. Let e_i $(i = 1, \cdots, n+1)$ be the canonical generators of $\pi_1(F^*)$ as in § 2 such that $\rho_\# e_i = (0, \cdots, \overset{i}{\check{1}}, \cdots, 0)$. We have that $a(e_i) = 0$ for $i \in I$ by the definition of I. Thus $\pi_1(F)$ is generated by e_j $(j \in J)$ which can not have any relation by the surjectivity of b. Namely b is a bijection. Q. E. D.

Assume that $\Delta_{n+1} \neq \emptyset$ and let N be a tubular neighborhood of F_{n+1}^* as in Lemma (7.4). $(F_{n+1}^* = F_{n+1}(n))$. We identify ∂N with $F_{n+1}^* \times S^1$. Let $\rho_{n+1} : F_{n+1}^* \times S^1 \to (\boldsymbol{C}^*)^{n+1}$ be the cannonical map defined by $(z, \eta) \mapsto (z_1, \cdots, z_n, \eta)$ and let $k : F_{n+1}^* \times S^1 \to F^*$ be the inclusion map via the above identification. Then we have:

LEMMA (7.10). $k_* : H_j(F_{n+1}^* \times S^1) \to H_j(F^*)$ *is injective for* $j < \dim \Delta_{n+1}$. *For* $\dim \Delta_{n+1} \leqq j < \dim \Delta$, *the kernel of* $(\rho_{n+1})_* : H_j(F_{n+1}^* \times S^1) \to H_j((\boldsymbol{C}^*)^{n+1})$ *coincides with the kernel of* k_*.

PROOF. By the proof of Lemma (7.4), we have a commutative diagram

$$
\begin{tikzcd}
H_j(F^*_{n+1}\times S^1) \arrow[rr, "k_*"] \arrow[dr, "(\rho_{n+1})_*"'] & & H_j(F^*) \arrow[dl, "\rho_*"] \\
& H_j((\boldsymbol{C}^*)^{n+1}) &
\end{tikzcd}
$$

Thus the assertion is obvious by Lemma (7.8). Q. E. D.

For $\{i_1, \cdots, i_s\}\subset\{1, \cdots, n+1\}$, we define $\Delta_{i_1,\dots,i_s}$ by $\Delta\cap\{x_{i_j}=0\,;\, j=1, \cdots, s\}$. Now we have no difficulty in proving the following connectivity criterion.

THEOREM (7.11). *Assume that* $\dim\Delta_{i_1,\dots,i_j}>s-2j$ *for any* $j\geqq 0$ *and* $(i_1, \cdots, i_j)\subset\{1, \cdots, n+1\}$. *Then F is s-connected.* (Here $\dim\emptyset=-1$ by the definition.)

PROOF. The case $s=0$ or 1 is due to Theorem (1.1) and Corollary (1.2) and Lemma (7.9). Assume $s>1$. We consider the filtration

$$F^*\subset F(n)\subset F(n-1)\subset\cdots\subset F(1)\subset F.$$

First we consider the exact sequence:

$$A_n:\ \cdots\longrightarrow H_{j+1}(F(n), F^*)\xrightarrow{\delta_{j+1}} H_j(F^*)\longrightarrow H_j(F(n))\longrightarrow H_j(F(n), F^*)\longrightarrow\cdots.$$

Let N_{n+1} be a tubular neighborhood of F^*_{n+1} in $F(n)$ and we identify ∂N_{n+1} with $F^*_{n+1}\times S^1$ as in Lemma (7.10). By the excision property, we get

$$H_{j+1}(F(n), F^*)=H_{j+1}(N_{n+1}, \partial N_{n+1})=H_{j-1}(F^*_{n+1})\otimes\tilde{e}_{n+1}\subset H_{j+1}(\partial N_{n+1})$$

where $\tilde{e}_{n+1}$ is the canonical generator of $H_1(S^1)$. The last isomorphism is obtained from the homology exact sequence of the pair $(N_{n+1}, \partial N_{n+1})$ and the Künneth's formula. It is easy to see that $\delta_j=k_*\mid H_{j-1}(F^*_{n+1})\otimes\tilde{e}_{n+1}$ under the above identification where k is the composition of $F^*_{n+1}\times S^1\cong\partial N\hookrightarrow F^*$. Therefore δ_j is injective for $j<\dim\Delta_{n+1}+2$ and $H_j(F(n))=H_j(F^*)/k_*(H_{j-1}(F^*_{n+1})\otimes\tilde{e}_{n+1})$ for $j\leqq s$ by Lemma (7.10). Let $(\boldsymbol{C}^*)^k=\{z\in\boldsymbol{C}^{n+1}\,;\, z_i\neq 0,\ 1\leqq i\leqq k$ and $z_{k+1}=\cdots=z_{n+1}=0\}$ and let $p_k: F(k)\to(\boldsymbol{C}^*)^k$ be the canonical projection. Let e_i $(i=1, \cdots, n+1)$ be the canonical generators of $H_1((\boldsymbol{C}^*)^{n+1})$ corresponding to $(0, \cdots, \overset{i}{\check{1}}, \cdots, 0)$ under the identification $H_1((\boldsymbol{C}^*)^{n+1})=\boldsymbol{Z}^{n+1}$ via Künneth's formula. Then the image of $H_{j-1}(F^*_{n+1})\otimes\tilde{e}_{n+1}$ by $(\rho_{n+1})_*$ is the submodule generated by $e_{i_1}\otimes e_{i_2}\otimes\cdots\otimes e_{i_{j-1}}\otimes e_{n+1}$, $1\leqq i_1<i_2<\cdots<i_{j-1}\leqq n$ for $j\leqq s$. Thus $(p_n)_*: H_j(F(n))\to H_j((\boldsymbol{C}^*)^n)$ is bijective for $j\leqq s$.

Now we prove that $(p_k)_*: H_j(F(k))\to H_j((\boldsymbol{C}^*)^k)$ is bijective for $j\leqq s$ by the induction on $(n+1)-k$. Consider the exact sequence

$$A_{k-1}:\ \cdots\longrightarrow H_{j+1}(F(k-1), F(k))\xrightarrow{\delta_{j+1}} H_j(F(k))\longrightarrow H_j(F(k-1))\longrightarrow\cdots.$$

Let N_k be a trivial tubular neighborhood of $F_k(k-1)$ in $F(k-1)$ so that ∂N_k

is diffeomorphic to $F_k(k-1)\times S^1$. By the same argument as above, we have a natural isomorphism $H_{j+1}(F(k-1), F(k))\cong H_{j-1}(F_k(k-1))\otimes\tilde{e}_k$ which makes the following diagram commutative.

$$\begin{tikzcd}
H_{j+1}(F(k-1), F(k)) \arrow[rr, "\delta_{j+1}"] \arrow[dr, "\cong"'] \arrow[rr, phantom, shift right=4ex, "\circlearrowright"] & & H_j(F(k)) \\
& H_{j-1}(F_k(k-1))\otimes\tilde{e}_k \arrow[ur, "k_*"'] &
\end{tikzcd}$$

Here k is the inclusion map $F_k(k-1)\times S^1\cong\partial N_k\subset F(k)$. Let $q_k: F_k(k-1)\times S^1$ be the map defined by $(z, \eta)\mapsto(z_1, \cdots, z_{k-1}, \eta)$. By the proof of Lemma (7.4), the diagram

$$\begin{tikzcd}
H_j(F_k(k-1)\times S^1) \arrow[rr, "k_*"] \arrow[dr, "(q_k)_*"'] & & H_j(F(k)) \arrow[dl, "(p_k)_*"] \\
& H_j((\boldsymbol{C}^*)^k) &
\end{tikzcd}$$

is commutative. By the inductive hypothesis on $F_k(k-1)$, $(q_k)_*: H_{j-1}(F_k(k-1))\otimes\tilde{e}_k\to H_j((\boldsymbol{C}^*)^k)$ is injective for $j-1\leqq s-2$. Therefore we get from A_{k-1} that $H_j(F(k-1))=H_j(F(k))/\text{Image}(\delta_{j+1})$ and $(p_{k-1})_*: H_j(F(k-1))\to H_j((\boldsymbol{C}^*)^{k-1})$ is bijective for $j\leqq s$. Here we have used the fact that $(p_{k-1})_*$ in the above diagram is surjective for any $j\leqq s+1$ by the following commutative diagram:

$$\begin{tikzcd}
H_j(F(k-1)) \arrow[d, "(p_{k-1})_*"] & H_j(F(k)) \arrow[l] \arrow[d, "(p_k)_*"] \\
H_j((\boldsymbol{C}^*)^{k-1}) & H_j((\boldsymbol{C}^*)^k) \arrow[l]
\end{tikzcd}$$

Therefore $H_{j-1}(F_k(k-1))\to H_{j-1}((\boldsymbol{C}^*)^{k-1})$ is surjective for $j\leqq s$ by the induction's hypothesis on the dimension of the total space. This proves the assertion and Theorem (7.11) by the Hurewicz theorem. Q.E.D.

REMARK (7.12). The conditions in Theorem (7.11) is not a necessary condition. The necessary condition in the words of Δ is rather delicate and we do not dare to clarify it in this papper. Here are a few examples.

EXAMPLE 1. Let $f(z_1, \cdots, z_{n+1})=\sum_{i=1}^{n+1} z_1z_2\cdots\check{z}_i\cdots z_{n+1}$ $(n\geqq 2)$. Then F is simply connected and $H_2(F)=\boldsymbol{Z}$ if $n>2$. The Euler-Poincaré characteristic

$\chi(F)$ is $(-1)^n n$.

EXAMPLE 2. Let $f(z)=\sum_{1\leqq i_1<i_2<\cdots<i_r\leqq n+1} z_{i_1}z_{i_2}\cdots z_{i_r}$ and $r<n$. Then $\dim \varDelta=n$. $\dim \varDelta_i=n-1, \cdots, \dim \varDelta_{i_1 i_2\cdots i_{n-r}}=r$ and $\dim \varDelta_{i_1,\cdots,i_{n-r+1}}=0$. Thus by Theorem (7.11), F is $(2n-2r+1)$-connected for $r>n/2+1$. For $r\leqq n/2+1$, F is $(n-1)$-connected.

EXAMPLE 3. Let $f(z_1, \cdots, z_{2n})=z_1z_2+z_3z_4+\cdots+z_{2n-1}z_{2n}$. Then Theorem (7.11) says only that F is $(n-2)$-connected. However F is in fact $(2n-2)$-connected because F has the homotopy type of $(2n-1)$ dimensional sphere. Thus the condition in Theorem (7.11) is not a necessary condition.

Appendix.

PROOF OF LEMMA (3.2). Let $f(z)$ be as in Lemma (3.2). Let $\nu^1, \cdots, \nu^p$ be the integral points of $\varDelta$ and for each $c=(c_1, \cdots, c_p)\in \boldsymbol{C}^p$, we define a polynomial $f_c(z)$ by $\sum_{j=1}^{p} c_j z^{\nu^j}$. Let $U=\{c\in \boldsymbol{C}^p\,;\, \Gamma(f_c)=\varDelta$ and f_c is non-degenerate$\}$. As U is a non-empty Zariski open set (Theorem 6.1 of [7] and Appendix about the non-degeneracy condition [**16**]), we may only prove the following lemma.

LEMMA A. *Let $p(t)$ $(0\leqq t\leqq 1)$ be an analytic path in U and let $f_t(z)=f_{p(t)}(z)$. Then the hypersurfaces $f_t^{-1}(\eta)$ $(\eta\in \boldsymbol{C}^*)$ are uniformly controlled by the spheres S_r near the infinity in the sense of Lemma* (5.8). *In particular we have equivalences of the fibrations φ_1 and φ_2:*

$$\begin{tikzcd}
(\boldsymbol{C}^*)^{n+1}-f_0^{-1}(0) \arrow[rr, "\varphi_1\cong"] \arrow[dr, "f_0"] \arrow[dd, hook] & & (\boldsymbol{C}^*)^{n+1}-f_1^{-1}(0) \arrow[dl, "f_1"'] \arrow[dd, hook'] \\
& \boldsymbol{C}^* & \\
\boldsymbol{C}^n-f_0^{-1}(0) \arrow[rr, "\varphi_2\cong"] \arrow[ur, "f_0"] & & \boldsymbol{C}^{n+1}-f_1^{-1}(0) \arrow[ul, "f_1"']
\end{tikzcd}$$

PROOF. Assuming that the assertion is not true, we can use the Curve Selection lemma to find Laurent series $z(s)=(\alpha_1 s^{a_1}+\cdots, \cdots, \alpha_{n+1}s^{a_{n+1}}+\cdots)$, $t(s)=t_0 s^b+\cdots$ and $\lambda(s)=\lambda_0 s^c+\cdots$ $(0\leqq s\leqq 1)$ such that (i) $\alpha=(\alpha_1, \cdots, \alpha_{n+1})\neq 0$ and $a_0:=\text{minimum}\,\{a_1, \cdots, a_{n+1}\}<0$, (ii) $f_{t(s)}(z(s))$ converges to a non-zero constant. (iii) $\dfrac{\partial f_{t(s)}}{\partial z_j}(z(s))=\lambda(s)\bar{z}_j(s)$ for $j=1, \cdots, n+1$ and (iv) $0\leqq t(s)\leqq 1$. Let $d(a)=\text{minimum}\left\{\sum_{i=1}^{n+1} a_i x_i\,;\, x\in\varDelta\right\}$and let $\varDelta_a=\left\{x\in\varDelta\,;\, \sum_{i=1}^{n+1} a_i x_i=d(a)\right\}$. Let $I=\{1\leqq i$

$\leqq n+1\,;\,a_i=a_0\}$. By the non-degeneracy assumption of $f_{t(s)}$ we get

$$\alpha_i\frac{\partial f_{\Delta_a}}{\partial z_i}(\alpha)=\begin{cases}\lambda_0|\alpha_i|^2 & \text{for} \quad i\in I\\ 0 & \text{otherwise}\end{cases}$$

and $d(a)=c+2a_0$. Now the exact same argument as in the proof of Lemma (5.8) gives a contradiction to (ii). Let $W=\{(x,\ t)\in \boldsymbol{C}^{n+1}\times I\mid f_t(x)\neq 0\}$ and let $\pi: W\to I$ be the projection map. Then we can construct an integrable connection vector field v on W such that (i) $v(f_t(x))=0$ for any $(x,\ t)\in W$ and (ii) v is tangent to $W_J=\{(x,\ t)\in \boldsymbol{C}^J\times I\mid f_t(x)\neq 0\}$ for any $J\subset\{1, \cdots, n+1\}$. Then the assertion is obtained by the usual argument.

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