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Evolution equations associated with the subdifferentials

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§ 0. Introduction.

In this paper we consider the nonlinear evolution equation of the form

$$\text{(E)} \qquad du(t)/dt+\partial\phi^{t}(u(t))\ni f(t) \qquad 0\leqq t\leqq T\,,$$

in a real Hilbert space H. Here, for almost every $t\in[0, T]$, $\partial\phi^{t}$ is the subdifferential of a lower semicontinuous convex function ϕ^{t} from H into $]-\infty, \infty]$ ($\phi^{t}\not\equiv+\infty$).

Since Brézis [2] first treated the equation (E) in the case $\phi^{t}=\phi$ is independent of t, many authors have investigated the existence, uniqueness and regularity of solutions of (E). (See Attouch and Damlamian [1], Kenmochi [5], Maruo [6], Watanabe [8], Yamada [10], [11], etc.)

This paper establishes an existence, uniqueness theorem for strong solutions of (E) under relatively weak assumptions on the t-dependence of ϕ^{t} generalizing the results of [1], [5], [6], [8], [10] and [11]. We employ the method of Kenmochi [5], that is, we would like to approximate (E) by difference approximations with respect to the time. We also use the idea of Maruo [6] under these hypotheses to establish estimates for solutions of the approximation schemes. The main advance over [10, 11] is the relaxation of a hypothesis on the t-dependence of the ϕ^{t} from absolute continuity to bounded variation.

The contents of this paper are as follows. § 1 recalls the basic properties of a lower semicontinuous convex function ϕ. In § 2 we list the basic hypotheses and state the existence theorem for (E). § 3–7 comprise the proof of the theorem. § 3 shows the measurability of $\phi^{\cdot}(v(\cdot))$ for any strongly measurable function v. In § 4 we prepare some lemmas which play important roles in § 5. In § 5 we drive recursive inequalities for solutions of the approximation schemes and establish estimates for them. In § 6 we prove that the approximate solutions converge as the mesh of the partitions approaches zero. Then we get the local existence of the strong solution. In § 7 we prove the global existence of it.

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Notations.

H denotes a real Hilbert space with the inner product $(\cdot, \cdot)$ and the norm $|\cdot|$.

$C([0, T]; H]$ denotes the space of strongly continuous functions $u:[0, T] \to H$ with the norm $\|u\|=\max\{|u(t)|; 0\leqq t\leqq T\}$. $L^2(0, T; H)$ denotes the space of strongly measurable functions $v:]0, T[\to H$ such that

$$\|v\|_{L^2(0,T;H)}=\left(\int_0^T |v|^2 dt\right)^{1/2}<+\infty .$$

§ 1. Preliminaries.

In this section we collect some known results on the subdifferential of a convex function. For the proofs see Brézis [2], [3] or Watanabe [8].

Let ϕ be a lower semicontinuous convex function from H into $]-\infty, \infty]$, $\phi\not\equiv+\infty$. The *effective domain* $D(\phi)$ of ϕ is defined by

$$D(\phi)=\{u\in H; \phi(u)<+\infty\} .$$

For each $u\in D$ the set

$$\partial\phi(u)=\{w\in H; \phi(v)-\phi(u)\geqq(w, v-u), \text{ for all } v\in H\} ,$$

is called the *subdifferential* of ϕ at u and the domain of the subdifferential $\partial\phi$ is defined by $D(\partial\phi)=\{u\in D(\phi); \partial\phi \text{ is not empty}\}$. Then the subdifferential $\partial\phi$ is, by definition, *monotone* in H, i.e. $(v_1-v_2, u_1-u_2)\geqq 0$ if $v_i\in\partial\phi(u_i)$, $i=1, 2$. It is known that $\partial\phi$ is *maximal monotone* in H, i.e. $R(1+\lambda\partial\phi)=H$ for all $\lambda>0$.

Now for each $\lambda\in]0, 1[$ and $u\in H$ we define

$$J_\lambda u=(1+\lambda\partial\phi)^{-1}u , \tag{1.1}$$

$$\phi_\lambda(u)=\inf\left\{\phi(v)+\frac{1}{2\lambda}|u-v|^2; v\in H\right\}. \tag{1.2}$$

We can show that the infimum of (1.2) is always attained by the unique element $J_\lambda u$:

$$\phi_\lambda(u)=\phi(J_\lambda u)+\frac{1}{2\lambda}|u-J_\lambda u|^2. \tag{1.3}$$

Furthermore we have

$$\phi(J_\lambda u)\leqq\phi_\lambda(u)\leqq\phi(u) \quad \text{for any } u\in H, \tag{1.4}$$

$$\lim_{\lambda\to 0}\phi_\lambda(u)=\phi(u) \quad \text{for any } u\in H, \tag{1.5}$$

$$\lim_{\lambda\to 0} J_\lambda u=u \quad \text{for any } u\in D(\phi). \tag{1.6}$$

LEMMA 1.1. *For each $\lambda\in]0, 1[$, ϕ_λ is a Fréchet differentiable convex function on H and the Fréchet derivative $\partial\phi_\lambda$ of ϕ_λ is equal to the Yosida approximation $(\partial\phi)_\lambda=\frac{1}{\lambda}(1-J_\lambda)$ of $\partial\phi$. More precisely,*

$$0\leqq\phi_\lambda(v)-\phi_\lambda(u)-((\partial\phi)_\lambda(u), v-u)\leqq\frac{1}{\lambda}|v-u|^2 \tag{1.7}$$

holds for any $\lambda\in]0, 1[$ and $u, v\in H$.

REMARK 1.1. By the above lemma we shall write $\partial\phi_\lambda$ instead of $(\partial\phi)_\lambda=\lambda^{-1}(1-J_\lambda)$.

REMARK 1.2. By (1.1), we have $\frac{1}{\lambda}(u-J_\lambda u)\in\partial\phi(J_\lambda u)$:

$$\phi(v)-\phi(J_\lambda u)\geqq\frac{1}{\lambda}(u-J_\lambda u, v-J_\lambda u) \qquad \text{for all } v\in H. \tag{1.8}$$

§ 2. Statement of Theorem.

Let $T>0$ be fixed. We shall consider the problem under the following assumption (A).

(A.1) *There is a set $0\notin Z\subset[0, T]$ of zero measure such that ϕ^t is a lower semicontinuous convex function from H into $]-\infty, \infty]$ with the non-empty effective domain for each $t\in[0, T]-Z$.*

(A.2) *For any positive integer r there exist a constant $K_r>0$, an absolutely continuous function $g_r: [0, T]\to\boldsymbol{R}$ with $g_r'\in L^\beta(0, T)$ and a function of bounded variation $h_r: [0, T]\to\boldsymbol{R}$ such that if $t\in[0, T]-Z$, $x\in D(\phi^t)$ with $|x|\leqq r$ and $s\in[t, T]-Z$, then there exists an element $\tilde{x}\in D(\phi^s)$ satisfying*

$$\begin{cases} |\tilde{x}-x|\leqq|g_r(s)-g_r(t)|(\phi^t(x)+K_r)^\alpha, \\ \phi^s(\tilde{x})\leqq\phi^t(x)+|h_r(s)-h_r(t)|(\phi^t(x)+K_r), \end{cases}$$

where α is some fixed constant with $0\leqq\alpha\leqq 1$ and

$$\beta=\begin{cases} 2 & \text{if } 0\leqq\alpha\leqq\frac{1}{2}, \\ \frac{1}{1-\alpha} & \text{if } \frac{1}{2}\leqq\alpha\leqq 1. \end{cases}$$

REMARK 2.1. The assumption (A) implies, in particular, that for each positive integer r, there exists a positive constant K_r satisfying

$$\phi^t(x)+K_r\geqq 0 \tag{2.1}$$

for each $t\in[0, T]-Z$ and $x\in H$ with $|x|\leqq r$.

In fact, let $x\in H$ with $|x|\leqq r$ be fixed. Then if x is in $D(\phi^t)$, (2.1) follows from (A.2). If x is not in $D(\phi^t)$, then $\phi^t(x)=+\infty$. Therefore we have (2.1).

We now define a strong solution of (E).

DEFINITION 2.1. Let $u:[0, T]\to H$. Then u is called a *strong solution* of (E) on $[0, T]$ if (i) u is in $C([0, T]; H)$. (ii) u is strongly absolutely continuous on any compact subset of $]0, T[$ and (iii) $u(t)$ is in $D(\phi^t)$ for a.e. $t\in[0, T]$ and satisfies (E) for a.e. $t\in[0, T]$.

Then we have:

THEOREM. *Let the assumption* (A) *be satisfied. Then for each* $f\in L^2(0, T; H)$ *and* $u\in\overline{D(\phi^0)}$, *the equation* (E) *has a unique strong solution* u *on* $[0, T]$ *with* $u(0)=u_0$. *Moreover,* u *has the following properties.*

(i) *For all* $t\in]0, T]-Z$, $u(t)$ *is in* $D(\phi^t)$, *and* $\phi^t(u(t))$ *satisfies* $t\phi^t(u(t))\in L^\infty(0, T)$ *and* $\phi^t(u(t))\in L^1(0, T)$. *Furthermore for any* $0<\delta<T$, $\phi^t(u(t))$ *is of bounded variation on* $[\delta, T]-Z$.

(ii) *For any* $0<\delta<T$, u *is strongly absolutely continuous on* $[\delta, T]$, *and* $t^{1/2}du/dt\in L^2(0, T; H)$.

In particular, if $u_0\in D(\phi^0)$, *then* u *satisfies*

(i)′ *For all* $t\in[0, T]-Z$, $u(t)$ *is in* $D(\phi^t)$ *and* $\phi^t(u(t))$ *is of bounded variation on* $[0, T]-Z$.

(ii)′ u *is strongly absolutely continuous on* $[0, T]$ *and satisfies* $du/dt\in L^2(0, T; H)$.

REMARK 2.2. As for (i) and (i)′, see Definition 3.1.

REMARK 2.3. The assumption (A) is a generalization of [**1**], [**5**], [**6**], [**8**], [**10**] and [**11**]. That of Kenmochi [**5**] is reduced to the case when Z is an empty set, $\alpha=0$, g_r and h_r are Lipschitz continuous at that of Yamada [**10, 11**], to the case when Z is an empty set and h_r is absolutely continuous especially.

§ 3. Measurability of $\phi^\cdot(v(\cdot))$.

When we prove the theorem, we use the following lower semicontinuous convex function ϕ_0^T from $L^2(0, T; H)$ into $]-\infty, \infty]$ with the nonempty effective domain such that

$$\Phi_0^T(v)=\begin{cases}\int_0^T\phi^t(v(t))dt & \text{if } \phi^\cdot(v(\cdot))\in L^1(0, T), \\ +\infty & \text{otherwise.}\end{cases} \tag{3.1}$$

We shall show the measurability of $\phi^\cdot(v(\cdot))$ to guarantee the well-definedness in this section. The fact that the effective domain is not empty will be shown by Corollary 4.1 in the next section. First we prepare the fundamental lemmas for functions of bounded variation.

DEFINITION 3.1. Let f be a real-valued function defined on $[0, T]-N$, where N is a set of zero measure. We put

$$V_0^T f = \sup\{\sum_{k=1}^n |f(t_k)-f(t_{k-1})|\ ;$$
$$0 \leqq t_0 < t_1 < \cdots < t_n \leqq T\ ,\ \{t_j\}_{0\leq j\leq n} \subset [0, T]-N\}\ .$$

The extended real number $V_0^T f$ is called the total variation of f over $[0, T]-N$. If $V_0^T f < \infty$, then f is said to be of *bounded variation over* $[0, T]-N$.

REMARK 3.1. Let f be of bounded variation over $[0, T]-N$. Then

(i) The inequality $V_0^s f + V_s^t f \leqq V_0^t f$ holds for $0 \leqq s \leqq t \leqq T$,

(ii) The function $[0, T] \ni t \to V_0^t f \in \boldsymbol{R}$ is non-decreasing and bounded, where we define $V_t^t f = 0$ for any $t \in [0, T]$.

LEMMA 3.1. *Let $m(t)$ be a non-decreasing function on $[0, T]$ and $b(t)$ be a bounded function on $[0, T]-N$ satisfying the following inequality*

$$b(t)-b(s) \leqq m(t)-m(s) \tag{3.2}$$

for any $s, t \in [0, T]-N$ with $s \leqq t$. Then $b(t)$ is of bounded variation on $[0, T]-N$.

PROOF. We have

$$m(s)-b(s) \leqq m(t)-b(t)$$

by (3.2). Define $c(t) = m(t)-b(t)$. Then $c(t)$ is non-decreasing and bounded on $[0, T]-N$. So $b(t)$ is of bounded variation on $[0, T]-N$ since $b(t)$ is written $b(t) = m(t)-c(t)$. q. e. d.

COROLLARY 3.1. *Let $b(t)$ be of bounded variation on $[0, T]-N$. Then there exists a function of bounded variation $\tilde{b}(t)$ on $[0, T]$ such that $\tilde{b}(t) = b(t)$ for any $t \in [0, T]-N$.*

PROOF. Define $m : [0, T] \to \boldsymbol{R}$ by $m(t) = V_0^t b$, then $b(t)$ and $m(t)$ satisfy the assumption of Lemma. Set $c(t) = m(t)-b(t)$. Since $c(t)$ is non-decreasing and bounded on $[0, T]-N$, there exists a non-decreasing function $\tilde{c}$ on $[0, T]$ such that $\tilde{c}(t) = c(t)$ for any $t \in [0, T]-N$. Then $\tilde{b}(t) = m(t)-\tilde{c}(t)$ is a function which has the required properties. q. e. d.

Using the idea due to Attouch and Damlamian [**1**, Lemma 1], we get the following lemma. The proof is analogous to that of Yamada [**10**, Lemma 3.1] and is omitted.

LEMMA 3.2. *Let $\{\phi^t\}$ satisfy* (A). *Then there exist two positive constants C_1 and C_2 such that*

$$\phi^t(x) + C_1|x| + C_2 \geqq 0 \tag{3.3}$$

holds for all $t \in [0, T]-Z$ and $x \in H$.

For each $\lambda \in]0, 1[$ and $u \in H$ we now set $J_\lambda^t u = (1+\lambda\partial\phi^t)^{-1}u$ and $\phi_\lambda^t(u) = \phi^t(J_\lambda^t u) + (2\lambda)^{-1}|u - J_\lambda^t u|^2$. Then by Remark 1.1, we have $\partial\phi_\lambda^t = \lambda^{-1}(1-J_\lambda^t)$. Next we shall show that $\phi_\lambda^t(x)$ is of bounded variation on $[0, T]-Z$ with respect to t. The following lemma is essentially due to Yamada [**10**, Proposition 3.1].

LEMMA 3.3. *Let* $\{\phi^t\}$ *satisfy* (A). *Then*

(i) *There is a positive constant* C_3 *independent of* λ, t *and* x *such that*

$$|J_\lambda^t x| \leqq 2|x| + C_3 \quad \text{for} \quad t \in [0, T] - Z. \tag{3.4}$$

(ii) *For each* $\lambda \in]0, 1[$ *and* $x \in H$, $\phi_\lambda^t(x)$ *is of bounded variation on* $[0, T] - Z$.

PROOF. Let $\lambda \in]0, 1[$ and $x \in H$ be fixed. Then by (1.2) and (1.8) we have

$$\phi^t(v) + \frac{1}{2\lambda}|x - v|^2 \geqq \phi_\lambda^t(x) \tag{3.5}$$

and

$$\phi^t(v) - \phi^t(J_\lambda^t x) \geqq \frac{1}{\lambda}(x - J_\lambda^t x, v - J_\lambda^t x) \tag{3.6}$$

for all $t \in [0, T] - Z$ and $v \in H$. Next if $v_0 \in D(\phi^0)$ be fixed, then by the assumption (A) there exists a function on $[0, T] - Z$ such that

$$a(0) = v_0, \quad |a(t)| \leqq r_0 - 1 \quad \text{and} \quad \phi^t(a(t)) \leqq M_0 \tag{3.7}$$

for $t \in [0, T] - Z$ where r_0 and M_0 are positive integers. Hence taking $v = a(t)$ in (3.5) we obtain for $t \in [0, T] - Z$

$$\phi_\lambda^t(x) \leqq \phi^t(a(t)) + \frac{1}{2\lambda}|x - a(t)|^2 \leqq M_0 + \frac{1}{2\lambda}(|x| + r_0)^2, \tag{3.8}$$

which shows that $\phi_\lambda^t(x)$ is bounded in $t \in [0, T] - Z$. Also taking $v = a(t)$ in (3.6) we obtain

$$M_0 - \phi^t(J_\lambda^t x) \geqq \frac{1}{\lambda}|J_\lambda^t x|^2 - \frac{1}{\lambda}|J_\lambda^t x|(|x| + r_0) - \frac{1}{\lambda} r_0 |x|$$

for $t \in [0, T] - Z$. Hence using (3.3) we find that (3.4) holds.

Now we put $r = \sup\{|J_\lambda^t x|\,;\, t \in [0, T] - Z,\ \lambda \in]0, 1[\}$, which is finite by (3.4). Since $J_\lambda^s x \in D(\partial\phi^s) \subset D(\phi^s)$, by using assumption (A) again we see that, for each s, $t \in [0, T] - Z$ with $s < t$, there exists $w \in D(\phi^t)$ such that

$$\begin{cases} |w - J_\lambda^s x| \leqq |g_r(t) - g_r(s)|(\phi^s(J_\lambda^s x) + K_r)^\alpha, \\ \phi^t(w) \leqq \phi^s(J_\lambda^s x) + |h_r(t) - h_r(s)|(\phi^s(J_\lambda^s x) + K_r). \end{cases}$$

Therefore taking $v = w$ in (3.6) we obtain for s, $t \in [0, T] - Z$ with $s < t$

$$\begin{aligned} &|h_r(t) - h_r(s)|(\phi_\lambda^s(x) + K_r) + \phi^s(J_\lambda^s x) - \phi^t(J_\lambda^t x) \\ &\geqq \frac{1}{\lambda}(x - J_\lambda^t x, J_\lambda^s x - J_\lambda^t x) - |g_r(t) - g_r(s)||\partial\phi_\lambda^t(x)|(\phi_\lambda^s(x) + K_r)^\alpha, \end{aligned} \tag{3.9}$$

where we used (1.4). We note

$$\frac{1}{\lambda}(x - J_\lambda^t x, J_\lambda^s x - J_\lambda^t x) \geqq \frac{1}{2\lambda}(|x - J_\lambda^t x|^2 - |x - J_\lambda^s x|^2).$$

Consequently using (1.2) and (3.9) we obtain

$$|h_r(t)-h_r(s)|(\phi_\lambda^s(x)+K_r) + |g_r(t)-g_r(s)|\,|\partial\phi_\lambda^t(x)|(\phi_\lambda^s(x)+K_r)^\alpha \geqq \phi_\lambda^t(x)-\phi_\lambda^s(x) \tag{3.10}$$

for any s, $t\in[0, T]-Z$ with $s<t$. By Remark 1.1, (3.4) and (3.8), there exists positive constant $C_{x,\lambda}$ independent of t such that

$$|\partial\phi_\lambda^t(x)|\leqq C_{x,\lambda}, \quad \phi_\lambda^t(x)\leqq C_{x,\lambda} \tag{3.11}$$

for any $t\in[0, T]-Z$. Hence by (3.10) and (3.11)

$$\begin{aligned}\phi_\lambda^t(x)-\phi_\lambda^s(x) &\leqq |h_r(t)-h_r(s)|(C_{x,\lambda}+K_r)+|g_r(t)-g_r(s)|C_{x,\lambda}(C_{x,\lambda}+K_r)^\alpha \\ &\leqq D_{x,\lambda}\{V_0^t h_r+V_0^t g_r-(V_0^s h_r+V_0^s g_r)\}\,,\end{aligned} \tag{3.12}$$

where $D_{x,\lambda}=\max\{C_{x,\lambda}+K_r, C_{x,\lambda}(C_{x,\lambda}+K_r)^\alpha\}$. (3.12) implies (ii) by Lemma 3.1. q. e. d.

As a consequence of Lemma 3.3 we have the following lemma which we need.

LEMMA 3.4. *Let* $\{\phi^t\}$ *satisfy* (A) *and let* $v:[0, T]\to H$ *be a strongly measurable function. Then* $\phi^t(v(t))$ *is a measurable function on* $[0, T]$.

PROOF. Since $v:[0, T]\to H$ is strongly measurable, there exists a set $Z_1\subset[0, T]$ of zero measure such that $\{v(t);\ t\in[0, T]-Z_1\}$ is separable. Let $\{r_n\}_{n=1}^\infty$ be a dense subset of $\{v(t);\ t\in[0, T]-Z_1\}$. Then, by Lemma 3.3 and Corollary 3.1 $t\to\phi_\lambda^t(r_n)$ can be extended to a function of bounded variation on $[0, T]$ for each n. So the set of discontinuous points of $\phi_\lambda^t(r_n)$ is numerable. Denote it by D_n and define $Z_2=Z\cup Z_1\cup(\bigcup_{n=1}^\infty D_n)$. Then for any n, $[0, T]-Z_2\ni t\to\phi_\lambda^t(r_n)$ is continuous, and Z_2 is a set of zero measure.

Now, let ε be any given positive number. Choose a closed subset F_ε of $[0, T]$ such that $[0, T]-F_\varepsilon\supset Z_2$, the measure of $[0, T]-F_\varepsilon$ is not larger than ε and $v|F_\varepsilon$ is continuous on F_ε by Lusin's theorem. We shall show that $F_\varepsilon\ni t\to\phi_\lambda^t(v(t))\in \boldsymbol{R}$ is continuous. By (3.4),

$$|\partial\phi_\lambda^t(x)|=|\lambda^{-1}(x-J_\lambda^t x)|\leqq\lambda^{-1}(3|x|+C_3) \tag{3.13}$$

for all $t\in[0, T]-Z$ and $x\in H$. Hence using (1.7) and (3.13), we have

$$|\phi_\lambda^t(y)-\phi_\lambda^t(x)|\leqq\frac{1}{\lambda}(4|x|+|y|+C_3)|y-x|$$

for all $t\in[0, T]-Z$ and x, $y\in H$. Therefore

$$|\phi_\lambda^s(v(s))-\phi_\lambda^t(v(t))|$$

$$\leqq|\phi_\lambda^s(v(s))-\phi_\lambda^s(r_n)|+|\phi_\lambda^s(r_n)-\phi_\lambda^t(r_n)|+|\phi_\lambda^t(r_n)-\phi_\lambda^t(v(t))|$$

$$\leqq\frac{1}{\lambda}(4|v(s)|+|r_n|+C_3)|v(s)-r_n|+|\phi_\lambda^s(r_n)-\phi_\lambda^t(r_n)|$$

$$+\frac{1}{\lambda}(4|v(t)|+|r_n|+C_3)|v(t)-r_n|$$

$$\leqq\frac{2}{\lambda}(5\sup_{\cdot\in F_\varepsilon}|v(\cdot)|+C_3+|v(t)-r_n|)\{|v(s)-v(t)|+|v(t)-r_n|\}$$

$$+|\phi_\lambda^s(r_n)-\phi_\lambda^t(r_n)|.$$

Let $t\in F_\varepsilon$ be fixed. Then for any $\eta>0$, there exists n such that $|v(t)-r_n|<\eta$. Moreover there is $\delta(r_n)>0$ such that $s\in F_\varepsilon$ with $|s-t|<\delta$ implies $|\phi_\lambda^s(r_n)-\phi_\lambda^t(r_n)|<\eta$ and $|v(s)-v(t)|<\eta$. We have seen above the following: for each $\eta>0$, there exists $\delta>0$ such that

$$|\phi_\lambda^s(v(s))-\phi_\lambda^t(v(t))|\leqq\frac{2}{\lambda}(5\sup_{\cdot\in F_\varepsilon}|v(\cdot)|+C_3+\eta)\cdot 2\eta+\eta$$

for all $s\in F_\varepsilon$ with $|s-t|<\delta$.

Therefore $\phi_\lambda^\cdot(v(\cdot))$ is Lusin-measurable and so measurable. By (1.5), $\phi^t(v(t))=\lim_{\lambda\to 0}\phi_\lambda^t(v(t))$ for almost all $t\in[0, T]$. Hence $\phi^\cdot(v(\cdot))$ is measurable. q. e. d.

§ 4. Lemmas.

In this section we summarize some consequences of the assumption (A). The following lemma is well known and essential in obtaining necessary estimates.

LEMMA 4.1. *Let* $\{a_k\}_{k=0}^n$, $\{b_k\}_{k=0}^n$, $\{c_k\}_{k=0}^n$ *be sequences of non-negative numbers such that*

$$a_k\leqq(1+b_k)a_{k-1}+c_k\,,\qquad k=1, 2, \cdots, n\,.$$

Then (i) $a_k\leqq(a_0+\sum_{j=1}^k c_j)\exp\sum_{j=1}^k b_j\,,\qquad k=1, 2, \cdots, n\,.$

(ii) $a_k\leqq(a_1+\sum_{j=2}^k c_j)\exp\sum_{j=2}^k b_j\,,\qquad k=2, 3, \cdots, n\,.$

PROOF. (ii) follows (i). We prove (i). By the induction

$$a_k\leqq\{\Pi_{j=1}^k(1+b_j)\}\,a_0+\sum_{j=1}^k\{\Pi_{l=j+1}^k(1+b_l)\}\,c_j\,.$$

Therefore

$$a_k\leqq(\exp\sum_{j=1}^k b_j)a_0+(\exp\sum_{j=1}^k b_j)(\sum_{j=1}^k c_j)$$

$$=(\exp\sum_{j=1}^k b_j)(a_0+\sum_{j=1}^k c_j)\,.$$

q. e. d.

Since h_r in the assumption (A) is of bounded variation on $[0, T]$, the set of its discontinuous points is numerable. Denote it by D_r and define $Z_0=Z\cup\{\bigcup_{r=1}^\infty D_r\}$. Then Z_0 is a set of zero measure. We denote the *strong con-*

vergence by $\rightarrow$ and the *weak convergence* by $\xrightarrow{w}$.

LEMMA 4.2. *Assume* (A). *If* $x_n \in D(\phi^{t_n})$ *with* $|x_n| \leqq r$, $\phi^{t_n}(x_n) \leqq L$, t, $t_n \in [0, T]-Z_0$ *with* $t_n \leqq t$, $x_n \xrightarrow{w} x$ *in* H *and* $t_n \rightarrow t$ *as* $n \rightarrow \infty$ *where* r *and* L *are positive integers, then*

$$\phi^t(x) \leqq \liminf_{n\to\infty} \phi^{t_n}(x_n). \tag{4.1}$$

PROOF. Using (A), we can find $\tilde{x}_n \in D(\phi^t)$ for each n such that

$$|\tilde{x}_n - x_n| \leqq |g_r(t) - g_r(t_n)|(L+K_r)^\alpha$$

and

$$\phi^t(\tilde{x}_n) \leqq \phi^{t_n}(x_n) + |h_r(t) - h_r(t_n)|(L+K_r).$$

Since $\tilde{x}_n \xrightarrow{w} x$ in H and $\phi^t(\cdot)$ is lower semicontinuous in H, we have

$$\begin{aligned}\phi^t(x) &\leqq \liminf_{n\to\infty} \phi^t(\tilde{x}_n) \\ &\leqq \liminf_{n\to\infty} \phi^{t_n}(x_n) + (L+K_r) \lim_{n\to\infty} |h_r(t) - h_r(t_n)| \\ &= \liminf_{n\to\infty} \phi^{t_n}(x_n).\end{aligned}$$

q. e. d.

REMARK 4.1. If we replace Z_0 by Z in the above lemma, we have

$$\phi^t(x) \leqq \liminf_{n\to\infty} \phi^{t_n}(x_n) + (L+K_r) V_0^t h.$$

The following lemma is essentially due to Kenmochi [**5**, Lemma 3.3 and its Corollary]. The proof of it is a slight modification of that of Yamada [**9**, Lemma 3.2].

LEMMA 4.3. *Let* $\{\phi^t\}$ *satisfy* (A). *Then there exist positive constants* δ, r_0 *and* M *which have the following properties: for each* $t \in [0, T]-Z_0$ *there exists a strongly absolutely continuous function* v_t *on* $[t, \min\{t+\delta, T\}] \equiv I_{t,\delta}$ *such that*

(i) $|v_t(s)| \leqq r_0$ *for* $s \in I_{t,\delta}$,

(ii) $|v_t(s_2) - v_t(s_1)| \leqq (M+K_{r_0})^\alpha \int_{s_1}^{s_2} |g'_{r_0}| d\tau$ *for* $s_1, s_2 \in I_{t,\delta}$ *with* $s_1 \leqq s_2$,

(iii) $\phi^s(v_t(s)) \leqq M$ *for* $s \in I_{t,\delta} - Z_0$,

(iv) $\limsup_{s\downarrow t,\, s\in I_{t,\delta}-Z_0} \phi^s(v_t(s)) \leqq \phi^t(v_t(t))$.

PROOF. We assume $\frac{1}{2} \leqq \alpha \leqq 1$, noting that the proof also holds under $0 \leqq \alpha < \frac{1}{2}$ if we replace α and K_{r_0} by $\frac{1}{2}$ and $K_{r_0}+1$ respectively in the following proof. Let $v_0\in D(\phi^0)$ be fixed, then by assumption (A) there exists a function a on $[0, T]-Z$ satisfying (3.7). Let M and δ be constants defined by

$$M=(\exp V_0^T h_{r_0})(M_0+K_{r_0})-K_{r_0}, \tag{4.2}$$

$$\delta=2^{-1}\Big(\int_0^T |g_{r_0}'|^{1/(1-\alpha)}d\tau\Big)^{-(1-\alpha)/\alpha}(M+K_{r_0})^{-1}. \tag{4.3}$$

Note $Z\cup D_r\subset Z_0$. Fix any $t_0\in[0, T]-Z_0$. For simplicity we assume $t_0+\delta<T$. Choose δ_1 so that $t_0+\delta_1\in[0, T]-Z_0$ and $\delta\leqq\delta_1\leqq 2\delta$. There is a sequence of partitions $\{t_0=t_0^n<t_1^n<\cdots<t_n^n=t_0+\delta_1\}$ such that $\{t_k^n\}_{k=0}^n\subset[0, T]-Z_0$. Now, we are going to build a sequence $\{u_k^n\}_{k=0}^n$ as follows: Let $u_0^n=a(t_0)\equiv a_0$. When $u_{k-1}^n\in D(\phi^{t_{k-1}^n})$ with $|u_{k-1}^n|\leqq r_0$ is given, we choose $u_k^n\in D(\phi^{t_k^n})$ with $|u_k^n|\leqq r_0$ by using (A) so that

$$|u_k^n-u_{k-1}^n|\leqq|g_{r_0}(t_k^n)-g_{r_0}(t_{k-1}^n)|(\phi^{t_{k-1}^n}(u_{k-1}^n)+K_{r_0})^\alpha, \tag{4.4}$$

$$\phi^{t_k^n}(u_k^n)\leqq\phi^{t_{k-1}^n}(u_{k-1}^n)+|h_{r_0}(t_k^n)-h_{r_0}(t_{k-1}^n)|(\phi^{t_{k-1}^n}(u_{k-1}^n)+K_{r_0}). \tag{4.5}$$

This is well-defined by (4.2) and (4.3). Then we have

$$|u_k^n-a_0|\leqq 1, \ \phi^{t_k^n}(u_k^n)\leqq M, \qquad k=0, 1, \cdots, n. \tag{4.6}$$

Define a sequence $\{u_n\}_{n=1}^\infty$ of strongly absolutely continuous functions on $[t_0, t_0+\delta_1]$ such that

$$u_n(t)=\frac{t_k^n-t}{t_k^n-t_{k-1}^n}u_{k-1}^n+\frac{t-t_{k-1}^n}{t_k^n-t_{k-1}^n}u_k^n \qquad \text{if } t\in[t_{k-1}^n, t_k^n]. \tag{4.7}$$

By (4.4), (4.5), (4.6) and (4.7) we have

$$|u_n(t)|\leqq r_0 \qquad \text{for any } t_0\leqq t\leqq t_0+\delta_1, \tag{4.8}$$

$$|u_n(t)-u_n(s)|\leqq(M+K_{r_0})^\alpha\int_{s-\varepsilon^n}^{t+\varepsilon^n}|g_{r_0}'|d\tau \qquad \text{for } t_0\leqq s\leqq t\leqq t_0+\delta_1, \tag{4.9}$$

where $\varepsilon^n=\max\{|t_k^n-t_{k-1}^n|\,;\,1\leqq k\leqq n\}$.

Using (4.4) and Hörder's inequality, we have

$$\begin{aligned}\textstyle\sum_{k=1}^n\Big|\frac{u_k^n-u_{k-1}^n}{\varepsilon_k^n}\Big|^{1/(1-\alpha)}\varepsilon_k^n&=\textstyle\sum_{k=1}^n(\varepsilon_k^n)^{-\alpha/(1-\alpha)}|u_k^n-u_{k-1}^n|^{1/(1-\alpha)}\\ &\leqq\textstyle\sum_{k=1}^n(\varepsilon_k^n)^{-\alpha/(1-\alpha)}(\varepsilon_k^n)^{\alpha/(1-\alpha)}\Big(\int_{t_{k-1}}^{t_k}|g_{r_0}'|^{1/(1-\alpha)}dt\Big)(M+K_{r_0})^{\alpha/(1-\alpha)}\\ &=(M+K_{r_0})^{\alpha/(1-\alpha)}\Big(\int_{t_0}^{t_0+\delta_1}|g_{r_0}'|^{1/(1-\alpha)}dt\Big).\end{aligned}$$

Thus we get

$$\Big\{\int_{t_0}^{t_0+\delta_1}|u_n'|^{1/(1-\alpha)}dt\Big\}^{1-\alpha}\leqq(M+K_{r_0})^\alpha\Big\{\int_{t_0}^{t_0+\delta_1}|g_{r_0}'|^{1/(1-\alpha)}dt\Big\}^{(1-\alpha)} \tag{4.10}$$

by (4.7). Therefore the sequence $\{u_n\}$ and $\{u'_n\}$ are bounded in $L^2(t_0, t_0+\delta_1; H)$ by (4.8) and (4.10), noting $\frac{1}{2}\leqq\alpha\leqq 1$. There exist a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ and a strongly absolutely continuous function u such that

$$u_{n_j}\xrightarrow{w} u \quad \text{in } L^2(t_0, t_0+\delta_1; H), \tag{4.11}$$

$$u'_{n_j}\xrightarrow{w} u' \quad \text{in } L^2(t_0, t_0+\delta_1; H), \tag{4.12}$$

$$u_{n_j}(t)\xrightarrow{w} u(t) \quad \text{in } H \quad \text{for any } t\in[t_0, t_0+\delta_1]. \tag{4.13}$$

Therefore (i) and (ii) are given by (4.8), (4.9) and (4.13). We shall show the properties (iii) and (iv). Let $t\in[t_0, t_0+\delta_1]-Z_0$ be given. There is an interval $[t^n_{i(n)}, t^n_{i(n)+1}[$ such that $t\in[t^n_{i(n)}, t^n_{i(n)+1}[$ for any n where we define $t^n_{n+1}=T$. Set $t^n=t^n_{i(n)}$. By (4.9) we have

$$|u_n(t^n)-u_n(t)|\leqq(M+K_{r_0})^{\alpha}\int_{t^n-\varepsilon^n}^{t+\varepsilon^n}|g'_{r_0}|\,d\tau. \tag{4.14}$$

Therefore we have

$$u_{n_j}(t^{n_j})\xrightarrow{w} u(t) \quad \text{in } H \tag{4.15}$$

for any $t\in[t_0, t_0+\delta_1]$ as $j\to\infty$ by (4.13) and (4.14). From (4.5), we obtain

$$\phi^{t^{n_j}}(u(t^{n_j}))+K_{r_0}\leqq(\phi^{t_0}(u(t_0))+K_{r_0})\exp V^{t^{n_j}}_{t_0}\leqq(\phi^{t_0}(u_0)+K_{r_0})\exp V^{t}_{t_0}.$$

Using (4.15), Lemma 4.2 and the above inequality, we have

$$\phi^t(u(t))\leqq\liminf_{j\to\infty}\phi^{t^{n_j}}(u(t^{n_j}))\leqq(\phi^{t_0}(u_0)+K_{r_0})\exp V^t_{t_0}-K_{r_0}\leqq M \tag{4.16}$$

for $t\in[t_0, t_0+\delta_1]-Z_0$. This implies (iii). Since h_{r_0} is continuous at $t=t_0$, we have

$$\limsup_{t\downarrow t_0,\, t\in[t_0, t_0+\delta_1]-Z_0}\phi^t(u(t))\leqq\phi^{t_0}(u_0)$$

by (4.16). This implies (iv). q. e. d.

REMARK 4.2. If we do not need the property (iv) for $v_t(\cdot)$, we can replace Z_0 by Z in the statement of the above lemma. In fact, repeat the proof noting Remark 4.1.

COROLLARY 4.1. *Let* $\{\phi^t\}$ *satisfy* (A). *For each* $t\in[0, T]-Z_0$ *and each* $x\in D(\phi^t)$ *there is a function* $v\in L^2(0, T; H)$ *such that* $v(t)=x$, $\phi^{\cdot}(v(\cdot))\in L^1(0, T)$, v *is right-continuous at* t *and*

$$\limsup_{s\downarrow t,\, s\in[0, T]-Z_0}\phi^s(v(s))\leqq\phi^t(x).$$

PROOF. By assumption (A), for each $t\in[0, T]-Z$ and each $x\in D(\phi^t)$ there exists a function a_x on $[0, T]-Z$ such that

$$a_x(t)=x,\ |a_x(\cdot)|\leqq r_x-1 \quad \text{and} \quad \phi^{\cdot}(a_x(\cdot))\leqq M_x$$

for any $\cdot\in[0, T]-Z$ where r_x and M_x are positive integers. Repeat the proof of Lemma 4.3 replacing a by a_x, then we get a function $v_{t,x}$ instead of v_t satisfying the properties (i), (ii), (iii) and (iv) replacing δ, r_0 and M by δ_x, r_x and M_x respectively. Thus we can easily construct a function v.

q. e. d.

LEMMA 4.4. *Let* $\{\phi^t\}$ *satisfy* (A). *Suppose* $u\in L^2(0, T; H)$ *such that* $\phi^{\cdot}(u(\cdot))\in L^1(0, T)$ *and* $f\in L^2(0, T; H)$. *Then* $f\in\partial\Phi_0^T(u)$ *if and only if* $f(t)\in\partial\phi^t(u(t))$ *for a.e.* $t\in]0, T[$.

PROOF. Use Corollary 4.1, then we can show this lemma with a slight modification of the proof of [**5**, Proposition 1.1].

§ 5. The main estimates.

We assume that $\{\phi^t\}$ satisfy (A).

In the next section we prove the existence of a local strong solution by showing the convergence of solutions of certain problems approximating (E). As a step towards this goal we establish some estimates in the present section.

First by Remark 4.2 there are a positive constant M and family of $\{v_t; t\in[0, T]-Z\}$ of H-valued strongly absolutely continuous function v_t on $I_{t,\delta}=[t, \min\{t+\delta, T\}]$ satisfying the properties (i), (ii) and (iii) in Lemma 4.3. For simplicity we consider the problems approximating (E) on $[0, \delta]$. We denote by p the function v_0, and δ by T_1. Then we have

(5.1) $$|p(t)|\leqq C \qquad \text{for } t\in[0, T_1],$$

(5.2) $$|p(t_1)-p(t_2)|\leqq C\int_{t_1}^{t_2}|g_{r_0}'(\tau)|\,d\tau \qquad \text{for } t_1, t_2\in[0, T_1],$$

(5.3) $$\phi^t(p(t))\leqq C \qquad \text{for } t\in[0, T_1]-Z,$$

where $C=\max\{r_0, (M+K_{r_0})^\alpha, M\}$.

Let $f\in L^2(0, T; H)$, $u_0\in D(\phi^0)$. We may assume $T_1\in[0, T]-Z_0$, without loss of generality. There is a sequence of partition $P^n=\{0=t_0^n<t_1^n<\cdots<t_n^n=T_1\}$ such that $\{t_j^n\}_{j=1}^n\subset[0, T_1]-Z_0$ and $t_j-t_{j-1}\leqq t_{j-1}-t_{j-2}$ for $j=2, \cdots, n$. We propose to approximate (E) by the discrete problems

(5.4) $$\begin{cases} \dfrac{u_j^n-u_{j-1}^n}{t_j^n-t_{j-1}^n}+\partial\phi^{t_j^n}(u_j^n)\ni f_j^n, & j=1, 2, \cdots, n, \\ u_0^n=u_0\in D(\phi^0), \end{cases}$$

where

$$f_j^n=\frac{1}{t_j^n-t_{j-1}^n}\int_{t_{j-1}^n}^{t_j^n} f(\tau)d\tau\,, \qquad j=1, 2, \cdots, n\,.$$

Since the subdifferential is maximal monotone in H, $\{u_j^n\}_{j=0}^n$ is defined inductively by (5.4). Define the step sizes $\varepsilon_j^n=t_j^n-t_{j-1}^n$ for $j=1, 2, \cdots, n$. Then we have

$$\phi^{t_j^n}(x)-\phi^{t_j^n}(u_j^n)\geqq-\Big(\frac{u_j^n-u_{j-1}^u}{\varepsilon_j^n},\ x-u_j^n\Big)+(f_j^n,\ x-u_j^n) \tag{5.5}$$

for all $x\in H$ and $j=1, 2, \cdots, n$. Define $\varepsilon_0^n=\varepsilon_1^n$ and set $\varepsilon^n=\max\{\varepsilon_j^n\,;\ j=0, 1, \cdots, n\}$. From now on we assume ε^n is sufficiently small. Denote by C_k constants independent of f, u_0, $\phi^0(u_0)$, n and j. For simplicity we drop n if there is no fear of confusion.

LEMMA 5.1. *There is a positive constant* $M_1=M_1(\|f\|_{L^2(0,T;\,H)},\ |u_0|)$ *such that*

$$|u_j^n|\leqq M_1\,, \qquad \textit{for } 1\leqq j\leqq n\,, \tag{5.6}$$

$$\textstyle\sum_{j=1}^n \varepsilon_j^n\phi^{t_j^n}(u_j^n)\leqq M_1\,. \tag{5.7}$$

PROOF. We assume $\frac{1}{2}\leqq\alpha\leqq 1$, noting that the proof also holds under $0\leqq\alpha<\frac{1}{2}$ if we replace α by $\frac{1}{2}$ in the following proof. Denote $p(t_j^n)$ by p_j^n. Substituting p_j^n for x in (5.5), we see that

$$\phi^{t_j}(p_j)-\phi^{t_j}(u_j)\geqq-\Big(\frac{u_j-u_{j-1}}{\varepsilon_j},\ p_j-u_j\Big)+(f_j,\ p_j-u_j)=I+II \tag{5.8}$$

for $j=1, 2, \cdots, n$. Now we observe that

$$\begin{aligned}
I&=\varepsilon_j^{-1}(u_j-p_j-(u_{j-1}-p_{j-1})+p_j-p_{j-1},\ u_j-p_j)\\
&\geqq\varepsilon_j^{-1}\{|u_j-p_j|^2-|u_{j-1}-p_{j-1}|\,|u_j-p_j|-|p_j-p_{j-1}|\,|u_j-p_j|\}\\
&\geqq\varepsilon_j^{-1}\{|u_j-p_j|^2-2^{-1}|u_{j-1}-p_{j-1}|^2-2^{-1}|u_j-p_j|^2\\
&\qquad -2^{-1}\varepsilon_j^{-1}|p_j-p_{j-1}|^2-2^{-1}\varepsilon_j|u_j-p_j|^2\}\\
&\geqq\varepsilon_j^{-1}\Big\{2^{-1}|u_j-p_j|^2-2^{-1}|u_{j-1}-p_{j-1}|^2-2^{-1}\varepsilon_j|u_j-p_j|^2\\
&\qquad -C^2 2^{-1}\varepsilon_j^{-1}\Big[\int_{t_{j-1}}^{t_j}|g_{r_0}'|\,d\tau\Big]^2\Big\}
\end{aligned}$$

by (5.2). Let us estimate the final term. We have

$$\varepsilon_j^{-1}\Big[\int_{t_{j-1}}^{t_j}|g_{r_0}'|\,d\tau\Big]^2\leqq\varepsilon_j^{-1}\Big[\varepsilon_j^{\alpha}\Big\{\int_{t_{j-1}}^{t_j}|g_{r_0}'|^{1/(1-\alpha)}d\tau\Big\}^{1-\alpha}\Big]^2$$

$$=\varepsilon_j^{2\alpha-1}\left\{\int_{t_{j-1}}^{t_j}|g_{r_0}'|^{1/(1-\alpha)}d\tau\right\}^{2(1-\alpha)}$$

$$\leqq(2\alpha-1)\varepsilon_j+2(1-\alpha)\int_{t_{j-1}}^{t_j}|g_{r_0}'|^{1/(1-\alpha)}d\tau.$$

Therefore

$$I\geqq 2^{-1}\varepsilon_j^{-1}\left\{|u_j-p_j|^2-|u_{j-1}-p_{j-1}|^2-\varepsilon_j|u_j-p_j|^2 -C_3\varepsilon_j-C_3\int_{t_{j-1}}^{t_j}|g_{r_0}'|^{1/(1-\alpha)}d\tau\right\}. \tag{5.9}$$

$$II\geqq-|f_j||u_j-p_j|\geqq-2^{-1}|f_j|^2-2^{-1}|u_j-p_j|^2. \tag{5.10}$$

By (5.1), (5.3) and Lemma 3.2, we have

$$\phi^{t_j}(p_j)-\phi^{t_j}(u_j)\leqq C+C_1|u_j|+C_2=C+C_1|u_j-p_j|+C_1|p_j|+C_2 \leqq C_4+2^{-1}|u_j-p_j|^2. \tag{5.11}$$

From (5.8), (5.9), (5.10) and (5.11), we deduce

$$|u_j-p_j|^2\leqq(1-3\varepsilon_j)^{-1}\left\{|u_{j-1}-p_{j-1}|^2+C_5\left(\varepsilon_j+\int_{t_{j-1}}^{t_j}|g_{r_0}'|^{1/(1-\alpha)}d\tau+\varepsilon_j|f_j|^2\right)\right\}$$

$$\leqq(1+4\varepsilon_j)|u_{j-1}-p_{j-1}|^2+C_6\left\{\varepsilon_j+\int_{t_{j-1}}^{t_j}|g_{r_0}'|^{1/(1-\alpha)}d\tau+\int_{t_{j-1}}^{t_j}|f|^2d\tau\right\}.$$

Applying Lemma 4.1, we have

$$|u_j-p_j|^2\leqq\left\{|u_0-p_0|^2+C_6\left(t_j+\int_0^{t_j}|g_{r_0}'|^{1/(1-\alpha)}d\tau+\int_0^{t_j}|f|^2d\tau\right)\right\}e^{4t_j}. \tag{5.12}$$

Return to (5.8) and using (5.9) and (5.10),

$$\varepsilon_j\phi^{t_j}(u_j)\leqq 2^{-1}\left\{-|u_j-p_j|^2+|u_{j-1}-p_{j-1}|^2+C_7\varepsilon_j +C_3\int_{t_{j-1}}^{t_j}|g_{r_0}'|^{1/(1-\alpha)}d\tau+\int_{t_{j-1}}^{t_j}|f|^2d\tau+2\varepsilon_j|u_j-p_j|^2\right\}. \tag{5.13}$$

Combining (5.12) and (5.13), we find

$$\sum_{i=1}^{j}\varepsilon_i\phi^{t_i}(u_i)\leqq C_8\left\{|u_0-p_0|^2+t_j+\int_0^{t_j}|g_{r_0}'|^{1/(1-\alpha)}d\tau+\int_0^{t_j}|f|^2d\tau\right\}. \tag{5.14}$$

q. e. d.

COROLLARY 5.1. *There is a positive constant* $L_0=C_8+|\phi^0(u_0)|$ *such that*

$$\sum_{i=0}^{j-1}\varepsilon_i^n\phi^{t_i^n}(u_i^n)\leqq L_0\left\{t_j^n+\int_0^{t_j^n}|g_{r_0}'|^\beta d\tau+\int_0^{t_j^n}|f|^2d\tau+|u_0-p_0|^2\right\} \tag{5.15}$$

for $1\leqq j\leqq n$.

PROOF. By definition $\varepsilon_0^n=\varepsilon_1^n$. Then we have

$$\varepsilon_0^n \phi^{t_0^n}(u_0^n)=\varepsilon_1^n \phi^0(u_0) \leqq t_j^n |\phi^0(u_0)| \tag{5.16}$$

for $1 \leqq j \leqq n$. Hence we have (5.15) by (5.14) and (5.16). q. e. d.

From now on we take $r=[\sqrt{M_1}+|u_0|+1]$ in the assumption (A). We may assume $\phi^{t_j^n}(u_j^n)+K_r \geqq 1$ for $0 \leqq j \leqq n$ without loss of generality by noting (A). For simplicity let g, h and K denote g_r, h_r and K_r respectively. We prepare some simple lemma which is useful for estimates.

LEMMA 5.2. *There is a positive constant C_9 such that*

$$\frac{1}{\varepsilon_j^n}|g(t_j^n)-g(t_{j-1}^n)|^2(\phi^{t_{j-1}^n}(u_{j-1}^n)+K)^{2\alpha}$$
$$\leqq C_9\left[\varepsilon_{j-1}^n(\phi^{t_{j-1}^n}(u_{j-1}^n)+K)+\int_{t_{j-1}^n}^{t_j^n}|g'|^\beta d\tau\right](\phi^{t_{j-1}^n}(u_{j-1}^n)+K).$$

PROOF. We assume $\frac{1}{2} \leqq \alpha \leqq 1$, noting that the proof is simpler if $0 \leqq \alpha < \frac{1}{2}$. By using Hörder's and Minkowski's inequalities,

$$\varepsilon_j^{-1}|g(t_j)-g(t_{j-1})|^2(\phi^{t_{j-1}}(u_{j-1})+K)^{2\alpha}$$
$$\leqq \varepsilon_j^{-1}\left(\int_{t_{j-1}}^{t_j}|g'|d\tau\right)^2(\phi^{t_{j-1}}(u_{j-1})+K)^{2\alpha-1}(\phi^{t_{j-1}}(u_{j-1})+K)$$
$$\leqq \{\varepsilon_j(\phi^{t_{j-1}}(u_{j-1})+K)\}^{2\alpha-1}\left\{\int_{t_{j-1}}^{t_j}|g'|^{1/(1-\alpha)}d\tau\right\}^{2(1-\alpha)}(\phi^{t_{j-1}}(u_{j-1})+K)$$
$$\leqq \left[(2\alpha-1)\left\{\frac{\varepsilon_j}{\varepsilon_{j-1}}\varepsilon_{j-1}(\phi^{t_{j-1}}(u_{j-1})+K)\right\}+2(1-\alpha)\int_{t_{j-1}}^{t_j}|g'|^{1/(1-\alpha)}d\tau\right]$$
$$\times(\phi^{t_{j-1}}(u_{j-1})+K)$$
$$\leqq \left[(2\alpha-1)\varepsilon_{j-1}(\phi^{t_{j-1}}(u_{j-1})+K)+2(1-\alpha)\int_{t_{j-1}}^{t_j}|g'|^{1/(1-\alpha)}d\tau\right]$$
$$\times(\phi^{t_{j-1}}(u_{j-1})+K).$$
q. e. d.

The following lemma is the most important. The central idea of the proof is due to that of Maruo [6, Lemma 2.2].

LEMMA 5.3. *There is a positive constant $L_1=L_1(\|f\|_{L^2(0,T;H)}, |u_0|, \phi^0(u_0))$ such that*

$$\sum_{i=1}^n \frac{1}{\varepsilon_i^n}|u_i^n-u_{i-1}^n|^2 \leqq L_1, \tag{5.17}$$

$$\phi^{t_j^n}(u_j^n) \leqq L_1 \quad \textit{for } 1 \leqq j \leqq n. \tag{5.18}$$

PROOF. By (A) there exist $\tilde{u}_j$ for each u_{j-1}, $1 \leqq j \leqq n$, such that

$$\begin{cases} |\tilde{u}_j-u_{j-1}| \leqq |g(t_j)-g(t_{j-1})|(\phi^{t_{j-1}}(u_{j-1})+K)^\alpha, \\ \phi^{t_j}(\tilde{u}_j) \leqq \phi^{t_{j-1}}(u_{j-1})+|h(t_j)-h(t_{j-1})|(\phi^{t_{j-1}}(u_{j-1})+K). \end{cases} \tag{5.19}$$

Taking $\tilde{u}_j$ as x in (5.5), we have

$$\phi^{t_j}(\tilde{u}_j)-\phi^{t_j}(u_j)\geqq -\varepsilon_j^{-1}(u_j-u_{j-1},\ \tilde{u}_j-u_j)+(f_j,\ \tilde{u}_j-u_j)=I+II. \tag{5.20}$$

$$\begin{aligned} I&=\varepsilon_j^{-1}(u_j-u_{j-1},\ u_j-u_{j-1}+u_{j-1}-\tilde{u}_j)\\ &\geqq 2^{-1}\varepsilon_j^{-1}\{|u_j-u_{j-1}|^2-|u_{j-1}-\tilde{u}_j|^2\}\\ &\geqq 2^{-1}\varepsilon_j^{-1}|u_j-u_{j-1}|^2-2^{-1}\varepsilon_j^{-1}|g(t_j)-g(t_{j-1})|^2(\phi^{t_{j-1}}(u_{j-1})+K)^{2\alpha}\end{aligned} \tag{5.21}$$

by (5.19).

$$\begin{aligned} \mathrm{II}&=(f_j,\ \tilde{u}_j-u_{j-1}+u_{j-1}-u_j)\\ &\geqq -|f_j|\,|\tilde{u}_j-u_{j-1}|-|f_j|\,|u_{j-1}-u_j|\\ &\geqq -2^{-1}\varepsilon_j|f_j|^2-2^{-1}\varepsilon_j^{-1}|\tilde{u}_j-u_{j-1}|^2-\varepsilon_j|f_j|^2-4^{-1}\varepsilon_j^{-1}|u_{j-1}-u_j|^2\\ &\geqq -2^{-1}3\varepsilon_j|f_j|^2-4^{-1}\varepsilon_j^{-1}|u_j-u_{j-1}|^2\\ &\qquad -2^{-1}\varepsilon_j^{-1}|g(t_j)-g(t_{j-1})|^2(\phi^{t_{j-1}}(u_{j-1})+K)^{2\alpha}\end{aligned} \tag{5.22}$$

by (5.19). Using (5.19), (5.21) and (5.22) in (5.20), we have

$$\begin{aligned} &\phi^{t_{j-1}}(u_{j-1})+|h(t_j)-h(t_{j-1})|(\phi^{t_{j-1}}(u_{j-1})+K)-\phi^{t_j}(u_j)\\ &\quad\geqq 4^{-1}\varepsilon_j^{-1}|u_j-u_{j-1}|^2-2^{-1}3\varepsilon_j|f_j|^2-\varepsilon_j^{-1}|g(t_j)-g(t_{j-1})|^2(\phi^{t_{j-1}}(u_{j-1})+K)^{2\alpha}.\end{aligned}$$

From this inequality and Lemma 5.2 we get

$$\begin{aligned} &4^{-1}\varepsilon_j^{-1}|u_j-u_{j-1}|^2+(\phi^{t_j}(u_j)+K)\\ &\quad\leqq(1+|h(t_j)-h(t_{j-1})|)(\phi^{t_{j-1}}(u_{j-1})+K)+\frac{3}{2}\int_{t_{j-1}}^{t_j}|f|^2d\tau\\ &\qquad+C_9\left[\varepsilon_{j-1}(\phi^{t_{j-1}}(u_{j-1})+K)+\int_{t_{j-1}}^{t_j}|g'|^\beta d\tau\right](\phi^{t_{j-1}}(u_{j-1})+K).\end{aligned} \tag{5.23}$$

Therefore

$$\begin{aligned} &\phi^{t_j}(u_j)+K\\ &\quad\leqq\left\{\phi^0(u_0)+K+\frac{3}{2}\int_0^{t_j}|f|^2d\tau\right\}\\ &\qquad\times\exp\left\{V_0^{t_j}h+C_9\int_0^{t_j}|g'|^\beta d\tau+C_9\textstyle\sum_{i=1}^{j}\varepsilon_{i-1}(\phi^{t_{i-1}}(u_{i-1})+K)\right\}\\ &\quad\leqq\left\{\phi^0(u_0)+K+\frac{3}{2}\int_0^{t_j}|f|^2d\tau\right\}\end{aligned} \tag{5.24}$$

$$\times\exp\left\{V_0^{t_j}h+C_9(1+L_0)\int_0^{t_j}|g'|^\beta d\tau\right.$$

$$\left.+C_9L_0\left[t_j+\int_0^{t_j}|f|^2d\tau+|u_0-p_0|^2\right]\right\}$$

by Lemma 4.1 and Corollary 5.1. We set

$$L_3'=\exp\left\{V_0^Th+C_9(1+L_0)\int_0^T|g'|^\beta d\tau\right.\tag{5.25}$$

$$\left.+C_9L_0\left[T+\int_0^T|f|^2d\tau+|u_0-p_0|^2\right]\right\}.$$

By (5.24) and (5.25), we have

$$\phi^{t_j}(u_j)\leqq\phi^0(u_0)+\frac{3}{2}\int_0^{t_j}|f|^2d\tau\tag{5.26}$$

$$+L_3'\left(\phi^0(u_0)+K+\frac{3}{2}\int_0^{t_j}|f|^2d\tau\right)$$

$$\times\left\{V_0^{t_j}h+C_9(1+L_0)\int_0^{t_j}|g'|^\beta d\tau+C_9L_0\left[t_j+\int_0^{t_j}|f|^2d\tau+|u_0-p_0|^2\right]\right\}.$$

So, there is a positive constant $L_4'=L_4'(\|f\|_{L^2(0,T;H)}, |u_0|, \phi^0(u_0))$ such that

$$\phi^{t_j}(u_j)\leqq L_4'.\tag{5.27}$$

Returning to (5.23) and using (5.27), we obtain

$$\sum_{i=1}^n\frac{1}{\varepsilon_i}|u_i-u_{i-1}|^2\leqq L_5. \qquad \text{q. e. d.}$$

The following corollary is trivial from (5.26).

COROLLARY 5.2. *There is a positive constant $L_2=L_2(|u_0|, \phi^0(u_0))$ such that*

$$\phi^{t_j^n}(u_j^n)\leqq\phi^0(u_0)+L_2\left\{t_j^n+V_0^{t_j^n}h+\int_0^{t_j^n}|g'|^\beta d\tau+\int_0^{t_j^n}|f|^2d\tau+|u_0-p_0|^2\right\}.$$

LEMMA 5.4. *There is a positive constant $M_2=M_2(\|f\|_{L^2(0,T;H)}, |u_0|)$ such that*

$$\sum_{j=2}^n\frac{1}{\varepsilon_j^n}t_j^n|u_j^n-u_{j-1}^n|^2\leqq M_2,\tag{5.28}$$

$$t_j^n\phi^{t_j^n}(u_j^n)\leqq M_2,\qquad 1\leqq j\leqq n.\tag{5.29}$$

PROOF. Multiplying (5.23) t_j^n, we have

$$4^{-1}\varepsilon_j^{-1}t_j|u_j-u_{j-1}|^2+t_j(\phi^{t_j}(u_j)+K)\tag{5.30}$$

$$\leqq\left\{1+|h(t_j)-h(t_{j-1})|+C_9\int_{t_{j-1}}^{t_j}|g'|^\beta d\tau\right.$$

$$+C_9\varepsilon_{j-1}(\phi^{t_{j-1}}(u_{j-1})+K)\Big\}t_j(\phi^{t_{j-1}}(u_{j-1})+K)$$
$$+\frac{3}{2}t_j\int_{t_{j-1}}^{t_j}|f|^2d\tau.$$

By the way

$$(5.31)\qquad \begin{aligned} &t_j(\phi^{t_{j-1}}(u_{j-1})+K)\\ &=(t_j-t_{j-1})(\phi^{t_{j-1}}(u_{j-1})+K)+t_{j-1}(\phi^{t_{j-1}}(u_{j-1})+K)\\ &=\varepsilon_j\varepsilon_{j-1}^{-1}\varepsilon_{j-1}(\phi^{t_{j-1}}(u_{j-1})+K)+t_{j-1}(\phi^{t_{j-1}}(u_{j-1})+K)\\ &\leqq\varepsilon_{j-1}(\phi^{t_{j-1}}(u_{j-1})+K)+t_{j-1}(\phi^{t_{j-1}}(u_{j-1})+K).\end{aligned}$$

By (5.30) and (5.31), we have

$$(5.32)\qquad \begin{aligned} &4^{-1}\varepsilon_j^{-1}t_j|u_j-u_{j-1}|^2+t_j(\phi^{t_j}(u_j)+K)\\ &\leqq\Big\{1+|h(t_j)-h(t_{j-1})|+C_9\int_{t_{j-1}}^{t_j}|g'|^\beta d\tau+C_9\varepsilon_{j-1}(\phi^{t_{j-1}}(u_{j-1})+K)\Big\}\\ &\quad\times\{t_{j-1}(\phi^{t_{j-1}}(u_{j-1})+K)+\varepsilon_{j-1}(\phi^{t_{j-1}}(u_{j-1})+K)\}\\ &\quad+\frac{3}{2}t_j\int_{t_{j-1}}^{t_j}|f|^2d\tau.\end{aligned}$$

for $2\leqq j\leqq n$. Using Lemma 4.1 (ii) and Lemma 5.1, we obtain

$$\begin{aligned} t_j(\phi^{t_j}(u_j)+K)\leqq\Big\{&M_1+KT+\frac{3}{2}T\int_0^T|f|^2d\tau\\ &+(M_1+KT)\exp\Big[V_0^Th+C_9\int_0^T|g'|^\beta d\tau+C_9M_1+C_9TK\Big]\Big\}\\ &\times\exp\Big\{V_0^Th+C_9\int_0^T|g'|^\beta d\tau+C_9M_1+C_9TK\Big\}.\end{aligned}$$

Thus we have (5.29). Return to (5.30) and use Lemma 5.1 and (5.29), then we get (5.28). q. e. d.

REMARK 5.1. As easily checked, we see the following: M_1 and M_2 can be chosen so as to be bounded on each bounded set of $\|f\|_{L^2(0,T;H)}$ and $|u_0|$, and also L_1 and L_2 can be chosen so as to be bounded on each bounded set of $\|f\|_{L^2}$, $|u_0|$ and $\phi^0(u_0)$.

Now, define step functions u^n and $\nabla^n u^n$ for each n as follows:

$$u^n(t)=u_j^n \quad\text{and}\quad \nabla^n u^n(t)=\frac{1}{\varepsilon_j^n}(u_j^n-u_{j-1}^n)\qquad \text{if } t\in I_j^n$$

for $j=1, 2, \cdots, n$, where $I_1^n=[0, t_1^n]$ and $I_j^n=]t_{j-1}^n, t_j^n]$ for $j=2, \cdots, n$.

LEMMA 5.5. *For any* $s, t\in[0, T_1]$ *we have*

$$|u^n(t)-u^n(s)|\leqq\sqrt{(|t-s|+2\varepsilon^n)L_1}\,.$$

PROOF. Let $s\in I_j^n$, $t\in I_k^n$ and $j\leqq k$. Then by Lemma 5.3,

$$\begin{aligned}|u^n(t)-u^n(s)|&=|u_k^n-u_j^n|\\&=|\textstyle\sum_{i=j+1}^k(u_i^n-u_{i-1}^n)|\\&\leqq\Big\{\textstyle\sum_{i=j+1}^k\dfrac{1}{\varepsilon_i^n}|u_i^n-u_{i-1}^n|^2\Big\}^{1/2}\{\textstyle\sum_{i=j+1}^k\varepsilon_i^n\}^{1/2}\\&\leqq\sqrt{(|t-s|+2\varepsilon^n)L_1}\,.\end{aligned}$$

q. e. d.

§6. Local existence of strong solutions.

In this section also we assume that the condition (A) is satisfied and show that a suitable subsequence of $\{u^n\}_{n=1}^\infty$ constructed in the previous section converges to a strong solution on $[0, T_1]$.

By Lemma 5.1 and Lemma 5.3 the sequence $\{u^n\}_{n=1}^\infty$ and $\{\nabla^n u^n\}_{n=1}^\infty$ are bounded in $L^2(0, T_1; H)$, so there is a subsequence $\{n_k\}$ of $\{n\}$ such that

(6.1) $$u^{n_k}\xrightarrow{w}u \qquad \text{in } L^2(0, T_1; H),$$

and

(6.2) $$\nabla^{n_k}u^{n_k}\xrightarrow{w}\bar{u} \qquad \text{in } L^2(0, T_1; H),$$

as $k\to\infty$ for some $u, \bar{u}\in L^2(0, T_1; H)$. For simplicity we denote these subsequences $\{u^{n_k}\}$ and $\{\nabla^{n_k}u^{n_k}\}$ by $\{u^n\}$ and $\{\nabla^n u^n\}$ again respectively.

We can show the following three lemmas with a slight modification of the proof of Kenmochi [**5**, Lemmas 5.1, 5.2, and 5.4].

LEMMA 6.1. *$\bar{u}=u'$ in $L^2(0, T; H)$.*

LEMMA 6.2. (a) *u is a strongly absolutely continuous function on $[0, T_1]$ such that $u(0)=u_0$.*

(b) *There is a subsequence $\{u^{n_k}\}$ of $\{u^n\}$ sucn that $u^{n_k}(t)\xrightarrow{w}u(t)$ in H for all $t\in[0, T_1]$ as $k\to\infty$.*

LEMMA 6.3. $\liminf_{k\to\infty}\int_0^{T_1}(\nabla^{n_k}u^{n_k}, u^{n_k})d\tau\geqq\int_0^{T_1}(u', u)d\tau.$

For simplicity we denote the subsequence $\{n_k\}$ by $\{n\}$ again.

Now define a function $\phi_n^t(x)=\phi^{t_j^n}(x)$ if $t\in I_j^n$ and $x\in H$, and a function $\Phi_n: L^2(0, T; H)\to]-\infty, \infty]$ by

$$\Phi_n(v)=\begin{cases}\int_0^{T_1}\phi_n^t(v(t))dt & \text{if } v\in D(\Phi_n),\\ +\infty & \text{otherwise},\end{cases}$$

where $D(\Phi_n)=\{v\in L^2(0, T_1; H); \phi_n^\cdot(v(\cdot))\in L^1(0, T_1)\}$. Denote $\Phi_0^{T_1}$ of (3.1) by Φ. Then we can show the following lemma with a slight modification of the proof of Kenmochi [5, Lemma 3.4], noting our Lemma 4.3 and that h is continuous on $[0, T_1]-Z_0$.

LEMMA 6.4. *For each* $v\in D(\Phi)$ *there exists a sequence* $\{v^n\}\subset L^2(0, T_1; H)$ *such that* $v^n\in D(\Phi_n)$, $v^n\to v$ *in* $L^2(0, T_1; H)$ *as* $n\to\infty$ *and*

$$\limsup_{n\to\infty}\Phi_n(v^n)\leqq\Phi(v).$$

LEMMA 6.5. $\Phi(u)\leqq\liminf_{n\to\infty}\Phi_n(u^n)\leqq M_1$.

PROOF. We observe from Lemma 5.3 that

$$M_1\geqq\Phi_n(u^n)=\sum_{i=1}^n\varepsilon_i^n\phi^{t_i^n}(u_i^n)\geqq\sum_{i=1}^{n-1}\varepsilon_i^n\phi^{t_i^n}(u_i^n)-\varepsilon_n^n L_1$$
$$=\int_0^T\phi_n^t(w^n(t))dt-\varepsilon_n^n L_1,$$

where

$$\phi_n^t(\cdot)=\begin{cases}0 & \text{if } t\in I_1^n,\\ \phi^{t_{j-1}^n}(\cdot) & \text{if } t\in I_j^n \quad \text{for } 2\leqq j\leqq n,\end{cases}$$

and

$$w^n(t)=\begin{cases}0 & \text{if } t\in I_1^n,\\ u_{j-1}^n & \text{if } t\in I_j^n \quad \text{for } 2\leqq j\leqq n.\end{cases}$$

Using Lemma 4.2, then for each $t\in[0, T_1]-Z_0$ we have

$$\phi^t(u(t))\leqq\liminf_{n\to\infty}\phi_n^t(w^n(t)).$$

Hence by Fatou's lemma we have the lemma. q.e.d.

LEMMA 6.6.

$$\Phi(v)-\Phi(u)\geqq\int_0^{T_1}\left(f-\frac{du}{dt}, v-u\right)dt, \quad \textit{for any } v\in D(\Phi). \tag{6.3}$$

PROOF. We observe from (5.5) that

$$\phi_n^t(w(t))-\phi_n^t(u^n(t))\geqq(f^n(t)-\nabla^n u^n(t), w(t)-u^n(t)) \quad \text{for a.e. } t\in]0, T_1[, \tag{6.4}$$

for all $w\in D(\Phi_n)$, where $f^n(t)=f_j^n$ for $t\in I_j^n$ $(1\leqq j\leqq n)$. Using Lemma 6.4 for each $v\in D(\Phi)$ we can find a sequence $\{v^n\}$ such that $v^n\in D(\Phi_n)$, $v^n\to v$ in $L^2(0, T_1; H)$ as $n\to\infty$ and $\limsup_{n\to\infty}\Phi_n(v^n)\leqq\Phi(v)$. Taking v^n for w in (6.4) and integrating the both sides of (6.4) over $[0, T_1]$, we get

$$\Phi_n(v^n)-\Phi_n(u^n)\geqq\int_0^{T_1}(f^n-\nabla^n u^n, v^n-u^n)dt.$$

Let $n\to\infty$ in this inequality. Then noting Lemmas 6.3, 6.5 and the fact that $f^n\to f$ in $L^2(0, T_1; H)$, we obtain (6.3). q.e.d.

LEMMA 6.7. *Let u_1 and u_2 be two strong solutions of* (E). *Then*

$$|u_1(t)-u_2(t)|\leqq|u_1(s)-u_2(s)|$$

holds for $0\leqq s\leqq t\leqq T$.

PROOF. See e.g. Watanabe [8, Lemma 4.1].

LEMMA 6.8. *u is a unique strong solution of* (E) *on $[0, T_1]$ with the initial data $u_0\in D(\phi^0)$ such that $u(t)\in D(\phi^t)$ for $t\in[0, T_1]-Z$.*

PROOF. By Lemma 6.6 we have $f-u'\in\partial\Phi(u)$. Use Lemma 4.4, then $f(t)-u'(t)\in\partial\phi^t(u(t))$ a.e. $t\in]0, T_1[$. The fact $u(0)=u_0$ follows from Lemma 6.2 (a). The uniqueness is given by Lemma 6.7. The last part follows from the next lemma. q.e.d.

LEMMA 6.9. *Set $L_3=L_1+L_2+K$. Then for any $t\in[0, T_1]-Z$,*

$$\phi^t(u(t))\leqq\phi^0(u_0)+L_3\left\{t+V_0^t h+\int_0^t|g'|^\beta d\tau+\int_0^t|f|^2 d\tau+|u_0-p_0|^2\right\}. \tag{6.5}$$

PROOF. Let t be any point in $[0, T_1]-Z$ and take a sequence $\{t_j^n\}\subset[0, T_1]-Z_0$ such that $t_j^n\uparrow t$ as $n\to\infty$. Then clearly, $u_j^n\overset{w}{\to}u(t)$ as $n\to\infty$, so that by Remark 4.1

$$\phi^t(u(t))\leqq\liminf_{n\to\infty}\phi^{t_j^n}(u_j^n)+(L_1+K)V_0^t h.$$

From this together with Corollary 5.2, we infer the required inequality.

q.e.d.

LEMMA 6.10. *u has the following properties: For any $s, t\in[0, T_1]-Z$ with $s\leqq t$,*

$$\phi^t(u(t))-\phi^s(u(s))\leqq L_4\left\{t-s+V_s^t h+\int_s^t|g'|^\beta d\tau+\int_s^t|f|^2 d\tau+|u_0-p_0|^2\right\} \tag{6.6}$$

where L_4 is a positive constant depending only on $\|f\|_{L^2(0,T;H)}$, $|u_0|$ and $\phi^0(u_0)$.

PROOF. By Lemma 6.9 we have $u(t)\in D(\phi^t)$ for $t\in[0, T_1]-Z$. Let s be any point in $]0, T_1]-Z$. Then it is easy to see that the restriction of u to $[s, T_1]$ is a unique strong solution of (E) on $[s, T_1]$. Furthermore, by Lemmas 4.2 and 5.3,

$$\phi^s(u(s))\leqq L_1+(L_1+K)V_0^T h.$$

Therefore, taking s as the initial time and $u(s)$ as the initial data and repeating the same argument as in §5 and §6, we obtain from Lemma 6.9 that for each $t\in[s, T_1]-Z$,

$$\phi^t(u(t))-\phi^s(u(s))\leqq\tilde{L}_3\left\{t-s+V_s^t h+\int_s^t|g'|^\beta d\tau+\int_s^t|f|^2 d\tau+|u_0-p_0|^2\right\},$$

where $\tilde{L}_3=L_3(\|f\|_{L^2(0,T;H)}, |u(s)|, \phi^s(u(s)))$ is a positive constant and, as was noticed in Remark 5.1, $L_3=L_1+L_2+K$ can be chosen as to be bounded on each bounded set of three variables. Hence, if we put $L_4=\sup\{L_3(\|f\|_{L^2(0,T;H)}, |u(s)|, \phi^s(u(s)))\,;\ s\in]0, T_1]-Z\}$, then (6.6) holds for L_4. q.e.d.

LEMMA 6.11. (i) $\int_0^{T_1}\left|\frac{du}{d\tau}\right|^2 d\tau \leqq L_1$,

(ii) $\int_0^{T_1}\left|\sqrt{\tau}\frac{du}{d\tau}\right|^2 d\tau \leqq M_2$,

(iii) $t\phi^t(u(t))\leqq M_2$, *for* $t\in[0, T]-Z_0$.

PROOF. (i) follows from Lemmas 5.3, 6.1, and (6.2). We can show (ii) with a slight modification of the proof of Kenmochi [**5**, Lemma 5.8]. (iii) follows from Lemmas 4.2, 5.4 and 6.2 (b). q.e.d.

§ 7. Global existence of strong solutions.

In this section we give the proof of Theorem. Let $f\in L^2(0, T; H)$, $u_0\in \overline{D(\phi^0)}$. Under the assumption (A), ϕ^t may not be defined at $t=T$, so we need a slight device.

I. *First we shall prove the theorem when* $u_0\in D(\phi^0)$.

Let S be any point in $]0, T[-Z_0$. We construct a strong solution of (E) on $[0, S]$. Let δ be the same number as in Lemma 4.3, and choose a partition $\{0=T_0<T_1<\cdots<T_m=S\}$ of $[0, S]$ such that $\{T_k\}_{k=1}^m\subset[0, T]-Z_0$ and $\max\{|T_k-T_{k-1}|\,;1\leqq k\leqq m\}<\delta$. Then by virtue of Lemma 6.8, we can find H-valued functions u_k on $[T_{k-1}, T_k]$, $k=1, \cdots, m$, such that each u_k is a strong solution of (E) on $[T_{k-1}, T_k]$, with the initial data $u_{k-1}(T_{k-1})$, where $u_0(0)=u_0$. Putting $u(t)=u_k(t)$ for $t\in[T_{k-1}, T_k]$, $k=1, 2, \cdots, m$, we clearly see that u is a strong solution of (E) on $[0, S]$.

We shall show some estimates on $[0, S]$ for u and $\phi^t(u)$. Repeat the proof of Lemma 4.3 taking $u(t)$ instead of $a(t)$. Then there exist positive constants $\bar{\delta}$, $\bar{r}$ and $\bar{M}$ which have the following properties: for each $t\in[0, T]-Z_0$ there exists a strongly absolutely continuous function $\bar{v}_t$ on $[t, \min\{t+\bar{\delta}, T\}]\equiv I_{t,\bar{\delta}}$ such that

$$|\bar{v}_t(s)|\leqq\bar{r} \qquad \text{for } s\in I_{t,\bar{\delta}},$$

$$|\bar{v}_t(s_2)-\bar{v}_t(s_1)|\leqq(\bar{M}+K_{\bar{r}})^{\alpha}\int_{s_1}^{s_2}|g_{\bar{r}}'|\,d\tau, \qquad \text{for } s_1, s_2\in I_{t,\bar{\delta}} \text{ with } s_1\leqq s_2,$$

$$\phi^s(\bar{v}_t(s))\leqq\bar{M} \qquad \text{for } s\in I_{t,\bar{\delta}}-Z.$$

Taking $\bar{v}_t(\cdot)$ instead of $v_t(\cdot)$ and repeating the same arguments in § 5 and § 6, we obtain the following inequalities from Lemmas 6.10 and 6.11.

$$\phi^t(u(t))-\phi^s(u(s))\leqq L_5\left\{t-s+V_s^t h+\int_s^t|g'|^\beta d\tau+\int_s^t|f|^2 d\tau\right\} \tag{7.1}$$

for $s,\ t\in[0,\ S]-Z$ with $s\leqq t$,

$$\int_0^S\left|\frac{du}{d\tau}\right|^2 d\tau\leqq L_5, \tag{7.2}$$

$$\int_0^S\left|\sqrt{\tau}\,\frac{du}{d\tau}\right|^2 d\tau\leqq M_3, \tag{7.3}$$

$$t\phi^t(u(t))\leqq M_3 \qquad \text{for } t\in[0,\ S]-Z_0, \tag{7.4}$$

where L_5 is a positive constant depending only on $\|f\|_{L^2}$, $|u_0|$ and $\phi^0(u_0)$, and M_3 is a positive constant depending only on $|u_0|$ and $\|f\|_{L^2(0,T;H)}$. Furthermore L_5 and M_3 are independent on S.

Now, we construct the strong solution of (E) on $[0,\ T]$. Denote by $u(\cdot\,;S)$ the strong solution of (E) on $[0,\ S]$ with the initial data $u(0\,;S)=u_0$. Define $\bar{u}\in C([0,\ T[\,;H)$ such that

$$\bar{u}(t)=u(t\,;S) \qquad \text{for } t\leqq S<T.$$

This is well-defined by the uniqueness of the strong solution, and we have

$$\frac{d\bar{u}}{dt}\in L^2(0,\ T\,;H)$$

by (7.2). Therefore $\bar{u}$ can be extended to a strongly absolutely continuous function $\bar{u}_1$ on $[0,\ T]$ by defining

$$\bar{u}_1(t)=u_0+\int_0^t\frac{d\bar{u}}{d\tau}d\tau \qquad \text{for } t\in[0,\ T].$$

Denote $\bar{u}_1$ by u again. Then u is a strong solution of (E) on $[0,\ T]$. By (7.2), (7.3), (7.4) and Lemma 6.5 we have

$$\int_0^T\left|\frac{du}{dt}\right|^2 dt\leqq L_5, \tag{7.5}$$

$$\int_0^T\left|\sqrt{t}\,\frac{du}{dt}\right|^2 dt\leqq M_3, \tag{7.6}$$

$$t\phi^t(u(t))\leqq M_3, \qquad \text{for } t\in[0,\ T]-Z_0, \tag{7.7}$$

$$\int_0^T\phi^t(u(t))\leqq dt\leqq M_3. \tag{7.8}$$

From (7.1)

$$\phi^t(u(t))-\phi^s(u(s))\leqq L_5\left\{t-s+V_s^t h+\int_s^t|g'|^\beta d\tau+\int_s^t|f|^2 d\tau\right\}, \tag{7.9}$$

for $s,\ t\in[0,\ T]-Z$ with $s\leqq t$. Using (7.9) and Lemma 3.1, we can show the property (i)$'$. (ii)$'$ follows from (7.5).

II. *We can show Theorem when* $u_0 \in \overline{D(\phi^0)}$ with a slight modification of the argument in Yamada [**10**, p. 506], noting the estimates (7.6), (7.7) and (7.8).

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