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Compact minimal submanifolds of a sphere with positive Ricci curvature

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1. Introduction.

Let M be an n-dimensional simply connected compact orientable submanifold minimally immersed in an $(n+p)$-dimensional sphere of constant curvature 1. The pinching problem with respect to the scalar curvature of M [6] [1] and the sectional curvature of M [4] [8] have been studied. In this note, we shall prove a pinching theorem with respect to the Ricci curvature of M. Some examples are:

EXAMPLE 1. In general, let $S^{q}(r)$ denote a q-dimensional sphere in R^{q+1} with radius r. Let m and n be positive integers such that $m<n$ and let $M_{m,n-m}=S^{m}\left(\sqrt{\frac{m}{n}}\right)\times S^{n-m}\left(\sqrt{\frac{n-m}{n}}\right)$. We imbed $M_{m,n-m}$ into $S^{n+1}=S^{n+1}(1)$ as follows. Let (u, v) be a point of $M_{m,n-m}$, where u (resp. v) is a vector in R^{m+1} (resp. R^{n-m+1}) of length $\sqrt{\frac{m}{n}}$ $\left(\text{resp. } \sqrt{\frac{n-m}{n}}\right)$. We can consider (u, v) as a unit vector in $R^{n+2}=R^{m+1}\times R^{n-m+1}$. It is easily shown that $M_{m,n-m}$ is a minimal submanifold of S^{n+1}. Furthermore from the fact the first eigenvalue of the Laplacian of $M_{m,n-m}$ is n and the dimension of the eigenspace is $n+2$, we can prove the following.

Let χ be a minimal immersion of $M_{m,n-m}$ into S^{n+p} such that the immersion is full, i.e. $\chi(M_{m,n-m})$ is not contained in a linear subspace of R^{n+p+1}. Then $p=1$ and the immersion is rigid. The Ricci curvature of $M_{m,n-m}$ varies between $\frac{n(m-1)}{m}$ and $\frac{n(n-m-1)}{n-m}$.

EXAMPLE 2. We can define a minimal immersion of an n-dimensional complex projective space $P^{n}_{2n/(n+1)}$ with holomorphic sectional curvature $\frac{2n}{n+1}$ into $S^{n(n+2)-1}$ such that the usual coordinate functions of $R^{n(n+2)}$ are all independent hermitian harmonic functions of degree 1 on $P^{n}_{2n/(n+1)}$.

N. R. Wallach proved in [7] that if χ is a minimal immersion of $P^n_{2n/(n+1)}$ into S^{n+p} such that the immersion is full, then $p=n^2-1$ and the immersion is rigid. The Ricci curvature of $P^n_{2n/(n+1)}$ is equal to n.

THEOREM. *Let M be an n-dimensional simply connected compact orientable minimal submanifold immersed in S^{n+p} such that the immersion is full. If $n \geqq 4$ and the Ricci curvature of $M \geqq n-2$, then M is either S^n (totally geodesic), $M_{m,m}$ in S^{n+1} $(n=2m)$ or $P^2_{4/3}$ in S^7.*

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2. Preliminaries.

Let M be an n-dimensional Riemannian manifold isometrically immersed in an $(n+p)$-dimensional space form $\tilde{M}$ of constant curvature $\tilde{c}$. We denote by ∇(resp. $\tilde{\nabla}$) the covariant differentiation of M (resp. $\tilde{M}$). Then the second fundamental form σ of the immersion is given by

$$\sigma(X, Y)=\tilde{\nabla}_X Y-\nabla_X Y$$

and it satisfies $\sigma(X, Y)=\sigma(Y, X)$. We choose a local field of orthonormal frames $e_1, \cdots, e_n, e_{\tilde{1}}, \cdots, e_{\tilde{p}}$ in $\tilde{M}$ in such a way that, restricted to M, $e_1, \cdots, e_n$ are tangent to M. With respect to the frame field of M chosen above, let $\omega^1, \cdots, \omega^n, \omega^{\tilde{1}}, \cdots, \omega^{\tilde{p}}$ be the field of dual frames. Then the structure equations of M are given by(*)

$$d\omega^A=-\sum\omega^A_B\wedge\omega^B, \qquad \omega^A_B+\omega^B_A=0, \tag{2.1}$$

$$d\omega^A_B=-\sum\omega^A_C\wedge\omega^C_B+\tilde{c}\omega^A\wedge\omega^B. \tag{2.2}$$

Restricting these forms to M, we have the structure equations of the immersion

$$\omega^\alpha=0 \tag{2.3}$$

$$\omega^\alpha_i=\sum h^\alpha_{ij}\omega^j, \qquad h^\alpha_{ij}=h^\alpha_{ji} \tag{2.4}$$

$$d\omega^i=-\sum\omega^i_j\wedge\omega^j, \qquad \omega^i_j+\omega^j_i=0 \tag{2.5}$$

$$d\omega^i_j=-\sum\omega^i_k\wedge\omega^k_j+\Omega^i_j, \qquad \Omega^i_j=\frac{1}{2}\sum R^i_{jkl}\omega^k\wedge\omega^l \tag{2.6}$$

$$R^i_{jkl}=\tilde{c}(\delta^i_k\delta_{jl}-\delta^i_l\delta_{jk})+\sum(h^\alpha_{ik}h^\alpha_{jl}-h^\alpha_{il}h^\alpha_{jk}). \tag{2.7}$$

(*) We use the following convention on the ranges of indices unless otherwise stated: $A, B, C=1, \cdots, n, \tilde{1}, \cdots, \tilde{p}$: $i, j, k, l=1, \cdots, n$ $\alpha, \beta, \gamma=\tilde{1}, \cdots, \tilde{p}$.

The second fundamental form σ and h_{ij}^{α} are related by

$$\sigma(e_i, e_j)=\sum h_{ij}^{\alpha}e_{\alpha}\,. \tag{2.8}$$

Define h_{ijk}^{α} by

$$\sum h_{ijk}^{\alpha}\omega^{k}=dh_{ij}^{\alpha}-\sum h_{ik}^{\alpha}\omega_{j}^{k}-\sum h_{kj}^{\alpha}\omega_{i}^{k}+\sum h_{ij}^{\beta}\omega_{\beta}^{\alpha}\,. \tag{2.9}$$

Then from (2.2), (2.3) and (2.4) we have

$$h_{ijk}^{\alpha}=h_{ikj}^{\alpha}\,. \tag{2.10}$$

Then second fundamental form σ is said to be parallel if $h_{ijk}^{\alpha}=0$ for all i, j, k, α. The second fundamental form σ satisfies a differential equation. In fact we have the following.

LEMMA 2.1 ([6]).

$$\frac{1}{2}\Delta(\sum h_{ij}^{\alpha}h_{ij}^{\alpha})=\sum h_{ijk}^{\alpha}h_{ijk}^{\alpha}-\sum(\sum_{k}(h_{ik}^{\alpha}h_{kj}^{\beta}-h_{jk}^{\alpha}h_{ik}^{\beta}))^{2}-\sum h_{ij}^{\alpha}h_{ij}^{\beta}h_{kl}^{\alpha}h_{kl}^{\beta}+n\tilde{c}\sum h_{ij}^{\alpha}h_{ij}^{\alpha}\,,$$

where Δ denotes the Laplacian.

3. Lemmas.

In general, for a matrix $A=(a_{ij})$ we denote by $N(A)$ the square of the norm of A, i.e. $N(A)=\sum a_{ij}^{2}$. Clearly, $N(A)=N(T^{-1}AT)$ for any orthogonal matrix T. Now we have

$$\sum(\sum_{k}(h_{ik}^{\alpha}h_{kj}^{\beta}-h_{jk}^{\alpha}h_{ki}^{\beta}))^{2}=\sum N(A_{\alpha}A_{\beta}-A_{\beta}A_{\alpha})\,,$$

where $A_{\alpha}=(h_{ij}^{\alpha})$.

LEMMA 3.1. *$(n\times n)$-symmetric matrix $(\delta_{jl}-\sum_{i,\alpha}h_{ij}^{\alpha}h_{il}^{\alpha})$ is positive semi definite. In particular*

$$1-\sum_{i,\alpha}h_{ij}^{\alpha}h_{ij}^{\alpha}\geqq 0 \qquad \text{for each } j, \tag{1}$$

$$n\geqq\|\sigma\|^{2}\,, \tag{2}$$

where $\|\sigma\|^{2}=\sum h_{ij}^{\alpha}h_{ij}^{\alpha}$.

PROOF. From Gauss equation (2.7) and the fact the immersion is minimal, we obtain

$$S(e_j, e_l)=(n-1)\delta_{jl}-\sum_{i,\alpha}h_{ij}^{\alpha}h_{il}^{\alpha}\,,$$

where S denotes the Ricci tensor of M. From the assumption of the theorem, $S(e_j, e_l)-(n-2)\delta_{jl}=\delta_{jl}-\sum_{i,\alpha}h_{ij}^{\alpha}h_{il}^{\alpha}$ is the (j, l) entry of a positive semi definite symmetric matrix. Q. E. D.

LEMMA 3.2. *For each α*

$$\sum_{\beta} N(A_\alpha A_\beta - A_\beta A_\alpha) \leqq 4N(A_\alpha) - 4N(A_\alpha^2).$$

In particular, we have

$$\sum_{\beta,\gamma} N(A_\gamma A_\beta - A_\beta A_\gamma) \leqq 4\|\sigma\|^2 - \sum_{\gamma} 4N(A_\gamma^2).$$

PROOF. Let $\lambda_1^\alpha, \cdots, \lambda_n^\alpha$ be the eigenvalues of A_α. By a simple calculation, we obtain

$$\sum_{\beta} N(A_\alpha A_\beta - A_\beta A_\alpha) = \sum_{\beta,i,l} (h_{il}^\beta)^2(\lambda_i^\alpha - \lambda_l^\alpha)^2 = \sum_{\beta\neq\alpha,i,l} (h_{il}^\beta)^2(\lambda_i^\alpha - \lambda_l^\alpha)^2.$$

Since $(\lambda_i^\alpha - \lambda_l^\alpha)^2 \leqq 2((\lambda_i^\alpha)^2 + (\lambda_l^\alpha)^2)$, we obtain

$$\sum_{\beta} N(A_\alpha A_\beta - A_\beta A_\alpha) \leqq \sum_{\beta\neq\alpha,i,l} 2(h_{il}^\beta)^2((\lambda_i^\alpha)^2 + (\lambda_l^\alpha)^2) = 4\sum_{\beta\neq\alpha,i,l} (h_{il}^\beta)(\lambda_l^\alpha)^2.$$

From Lemma 3.1 (1)

$$1 - (\lambda_l^\alpha)^2 \geqq \sum_{\beta\neq\alpha,i} (h_{il}^\beta)^2 \qquad \text{for each } l.$$

Hence we obtain

$$\sum_{\beta} N(A_\alpha A_\beta - A_\beta A_\alpha) \leqq 4\sum_{l} (1 - (\lambda_l^\alpha)^2)(\lambda_l^\alpha)^2 = 4N(A_\alpha) - 4N(A_\alpha^2).$$

Q. E. D.

LEMMA 3.3.

$$N(A_\alpha^2) \geqq \frac{N(A_\alpha)^2}{n} \qquad \textit{for each } \alpha.$$

The equality holds if and only if A_α^2 is proportional to the identity.

PROOF. Let $\lambda_1^\alpha, \cdots, \lambda_n^\alpha$ be the eigenvalues of A_α. Then

$$nN(A_\alpha^2) - (N(A_\alpha))^2 = n\sum_{i} (\lambda_i^\alpha)^4 - (\sum_{i} (\lambda_i^\alpha)^2)^2 = \sum_{i,j} ((\lambda_i^\alpha)^2 - (\lambda_j^\alpha)^2)^2.$$

The equality holds if and only if $(\lambda_1^\alpha)^2 = \cdots = (\lambda_n^\alpha)^2$. Q. E. D.

4. Proof of theorem.

We set $S_{\alpha\beta} = \sum_{i,j} h_{ij}^\alpha h_{ij}^\beta$. Then $(p\times p)$-matrix $(S_{\alpha\beta})$ is symmetric and can de diagonalized for a suitable choice of a basis $e_{\tilde{1}}, \cdots, e_{\tilde{p}}$ at each point so that

$$\sum h_{ij}^\alpha h_{ij}^\beta h_{kl}^\alpha h_{kl}^\beta = \sum_{\alpha} N(A_\alpha)^2.$$

From Lemma 2.1, 3.1 (2) and 3.3, we obtain

$$\frac{1}{2}(\Delta\|\sigma\|^2) \geqq \sum h_{ijk}^\alpha h_{ijk}^\alpha + n\|\sigma\|^2 - 4\|\sigma\|^2 + 4\sum N(A_\alpha^2) - \sum N(A_\alpha)^2$$

$$\geqq \sum h_{ijk}^\alpha h_{ijk}^\alpha + (n-4)\|\sigma\|^2 + \frac{4}{n}\sum N(A_\alpha)^2 - \sum N(A_\alpha)^2$$

$$
\begin{aligned}
&=\sum h^{\alpha}_{ijk}h^{\alpha}_{ijk}+(n-4)\|\sigma\|^2-\frac{(n-4)}{n}\sum N(A_\alpha)^2\\
&\begin{cases} =\sum h^{\alpha}_{ijk}h^{\alpha}_{ijk}\geqq 0 & \text{for } n=4\\ \geqq \sum h^{\alpha}_{ijk}h^{\alpha}_{ijk}+\dfrac{(n-4)}{n}\|\sigma\|^2(n-\|\sigma\|^2)\geqq 0\,, & \text{for } n\geqq 5\end{cases}
\end{aligned}
$$

at each point. Since M compact and orientable, we obtain that $\sum h^{\alpha}_{ijk}h^{\alpha}_{ijk}=0$. Furthermore if $n\geqq 5$, we obtain that $\|\sigma\|^2(n-\|\sigma\|^2)=0$. If M is not totally geodesic, then $\|\sigma\|^2=n$. Hereafter we consider the case where M is not totally geodesic. Since the second fundamental form σ is parallel, M is locally symmetric. Since the equality of Lemma 3.3 holds, the eigenvalues of each A_α can be written as $\overbrace{\lambda^\alpha, \cdots \lambda^\alpha}^{m}$, $\overbrace{-\lambda^\alpha, \cdots, \lambda^\alpha}^{m}$. If $A_\alpha=0$ for some α, then from the fact that the second fundamental form σ is parallel and a result of J. Erbacher, the immage of M is contained in some hypersphere of S^{n+p}. This contradicts the assumption that the immersion is full. From the above and the equality of Lemma 3.2 holds, we have the equality of Lemma 3.1 (1). This proves that

$$S=(n-2)g\,,$$

where g denotes the metric tensor of M.

CASE $n\geqq 5$. Since $\sum N(A_\alpha)^2=\|\sigma\|^4=(\sum N(A_\alpha))^2$, we obtain that $(p-1)$ A_α's must be zero so that $p=1$. Since $p=1$ and $\|\sigma\|^2=n$, a result of [1] implies that M must be $M_{m,n-m}$. Furthermore $S=(n-2)g$ shows that $M=M_{m,m}$.

CASE $n=4$. Since M is simply connected and locally symmetric with $S=2g$, from [5], M must be $S^2\left(\sqrt{\frac{1}{2}}\right)\times S^2\left(\sqrt{\frac{1}{2}}\right)$, $P^2_{4/3}$ or $S^4\left(\sqrt{\frac{3}{2}}\right)$. From [2], if $S^4(r)$ is minimally immersed in S^{4+p}, $r=\sqrt{\frac{s(s+3)}{4}}$ for some positive integer s. $S^4\left(\sqrt{\frac{3}{2}}\right)$ can not be immersed in S^{4+p}. Q.E.D.

REMARK. Although we can prove the theorem without use of the result of [5], it is somewhat more complicated. Furthermore we can prove the following.

Let M be an n-dimensional minimal submanifold immersed in S^{n+p} such that the immersion is full. If $n\geqq 4$, the Ricci curvature of $M\geqq n-2$ and the scalar curvature of M is constant, then M is locally either S^n (totally geodesic), $M_{m,m}$ in S^{n+1} $(n=2m)$ or $P^2_{4/3}$ in S^7.

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