J. Math. Soc. Japan
Vol. 31, No. 2, 1979

# Squeezing deformations in Schottky spaces

By Hiro-o YAMAMOTO

(Received March 24, 1977)
(Revised Aug. 7, 1978)

It is well known that any compact Riemann surface can be represented by a Schottky group. The so-called Schottky space, the set of all normalized marked Schottky groups of genus $p$, has been investigated by many authors. On the other hand, the concept of squeezing deformations of Riemann surfaces is powerful to treat boundaries of spaces of Kleinian groups.

If a compact Riemann surface $S$ of genus $p$ is squeezed with respect to a homotopically independent set $\{\alpha_i\}_{i=1}^q$ of loops on $S$, then there is a path in the Schottky space of genus $p$ which tends to the boundary of the space. The aim of this paper is to study squeezing deformations of Riemann surfaces by investigating the behavior of the path in the Schottky space.

In §1 we shall state the definition of squeezing deformations. Some useful properties in the later discussions are proved in §2. In §4 we define the Schottky space $\mathcal{S}_p$ of genus $p$ and the boundary of $\mathcal{S}_p$, and classify the boundary points of $\mathcal{S}_p$ along the line due to Bers and Chuckrow. §5 is devoted to prove the main Theorem 5.1 which asserts that, if a compact Riemann surface of genus $p$ is squeezed with respect to a homotopically independent set $\{\alpha_i\}_{i=1}^q$ of $A$-cycles, then the path in $\mathcal{S}_p$ corresponding to the deformation tends to a cusp. We shall show another main Theorem 6.1 in §6 which states that, if a compact Riemann surface of genus $p$ is squeezed with respect to a homotopically independent set $\{\alpha_i\}_{i=1}^q$ of $B$-cycles, then the path in $\mathcal{S}_p$ corresponding to the deformation tends to a node.

The above theorems correspond to those in Abikoff [1] which concern with squeezing deformations in the Teichmüller space of a finitely generated Fuchsian groups of the first kind.

The author is indebted to Professors Oikawa, Kuroda and Taniguchi for their suggestions and continual encouragement during his preparation of this paper.

## §1. The definition of squeezing deformations.

Let $\Gamma$ be a finitely or an infinitely generated Kleinian group acting on the extended complex plane $\hat{C}$ and let $\Omega(\Gamma)$ be the region of discontinuity of $\Gamma$.

Throughout this paper, a Kleinian group means a non-elementary Kleinian group.

We denote by $\Delta$ a quasicomponent of $\Gamma$, that is, a (connected) domain contained in $\Omega(\Gamma)$ and invariant under $\Gamma$. Necessarily the boundary of $\Delta$ contains at least three points. The quotient surface $S=\Delta/\Gamma$ has a natural complex structure so that the natural projection $\pi: \Delta \rightarrow S$ is holomorphic.

Consider a set $\{\mu_t\}_{t\in[0,1)}$ of Beltrami differentials for $\Gamma$, that is, $\mu_t$ is a measurable function with support in $\Omega(\Gamma)$ satisfying $\|\mu_t\|_\infty<1$ and $\mu_t(\gamma(z))\overline{\gamma'(z)}/\gamma'(z)=\mu(z)$ a. e. for any $\gamma\in\Gamma$. Then there exists a unique quasiconformal homeomorphism $F_t$ of $\hat{\boldsymbol{C}}$ onto itself which satisfies

$$\frac{\partial F_t(z)}{\partial \bar{z}}=\mu_t(z)\frac{\partial F_t(z)}{\partial z} \quad \text{on } \hat{\boldsymbol{C}} \text{ and fixes } 0,\ 1 \text{ and } \infty.$$

Clearly $F_t$ is compatible with $\Gamma$ and takes $\Gamma$ into a Kleinian group $\Gamma_t=F_t\circ\Gamma\circ F_t^{-1}$. By setting $\hat{\mu}_t(w)=\mu_t(\hat{\pi}(w))\overline{\hat{\pi}'(w)}/\hat{\pi}'(w)$ on the universal covering surface $U=\{w: |w|<1\}$ of $\Delta$, where $\hat{\pi}: U\rightarrow\Delta$ is a natural projection, we see that there exists a unique quasiconformal homeomorphism $\hat{F}_t$ of $U$ onto itself which satisfies the equation $\dfrac{\partial\hat{F}_t(w)}{\partial\bar{w}}=\hat{\mu}_t(w)\dfrac{\partial\hat{F}_t(w)}{\partial w}$ on $U$ and keeps the points 0 and 1 invariant (Ahlfors-Bers [4]). Further, $\hat{F}_t$ is compatible with a Fuchsian model $\hat{\Gamma}$ of $\Gamma$, which is the set of all Möbius transformations satisfying $\hat{\pi}\circ\hat{\gamma}=\gamma\circ\pi$ for some $\gamma\in\Gamma$, and takes $\hat{\Gamma}$ into a Fuchsian model $\hat{\Gamma}_t=\hat{F}_t\circ\hat{\Gamma}\circ\hat{F}_t^{-1}$ of $\Gamma_t$. Then there exists a quasiconformal homeomorphism $f_t$ of $S$ onto $S_t=\Delta_t/\Gamma_t$, $\Delta_t=F_t(\Delta)$, such that the following diagram

$$\begin{array}{ccc}
U & \xrightarrow{\hat{F}_t} & U_t=\{w: |w|<1\} \\
\hat{\pi}\downarrow & & \downarrow\hat{\pi} \\
\Delta & \xrightarrow{F_t} & \Delta_t=F_t(\Delta) \\
\pi\downarrow & & \downarrow\pi_t \\
S=\Delta/\Gamma & \xrightarrow{f_t} & S_t=\Delta_t/\Gamma_t
\end{array}$$

is commutative. Here the vertical arrows represent natural projections. For $t\in[0,1)$, we denote by $\hat{\rho}_{U_t}(w)|dw|$, $\rho_{\Delta_t}(z)|dz|$ and $\rho_{S_t}(\zeta)|d\zeta|$ the Poincaré metrics on $U_t$, $\Delta_t$ and $S_t$, respectively.

For $t\in[0,1)$ and for two points $P$ and $Q$ on the Riemann surface $S_t$, we denote by $d_{S_t}(P,Q)$ the non-Euclidean distance measured by the Poincaré metric $\rho_{S_t}(\zeta)|d\zeta|$ on $S_t$. For an analytic curve $\sigma$ on $S_t$ we denote by $l_{S_t}(\sigma)$ the non-Euclidean length of $\sigma$ measured by $\rho_{S_t}(\zeta)|d\zeta|$.

Let $\Delta'$ be the domain obtained from $\Delta$ by deleting all the fixed points of elliptic elements in $\Gamma$. Now let $\sigma$ be a loop on the subsurface $S'=\Delta'/\Gamma$ of $S$ with a suitable initial and terminal point $P$. Letting $z_0\in\Delta'$ be a point mapped onto $P$ by $\pi: \Delta\to S$, we can lift $\sigma$ to a curve $\tilde{\sigma}$ starting from $z_0$. The end point of $\tilde{\sigma}$ is $\gamma(z_0)$, where $\gamma\in\Gamma$ is uniquely determined by $\sigma$ and $z_0$. The conjugacy class of $\gamma$ in $\Gamma$ depends only on $\sigma$ and we denote by $\tau(\sigma)$ the conjugacy class. By the same manner as above, we determine a conjugacy class $\hat{\tau}(\sigma)$ in $\hat{\Gamma}$ for $\sigma$.

A set of oriented analytic simple curves $\alpha_1, \alpha_2, \cdots$ on $S'$ is called homotopically independent, if the following conditions are satisfied;

(i) $\alpha_i\cap\alpha_j=\emptyset,\ i\neq j$,

(ii) every $\alpha_i$ bounds neither a disc nor a punctured disc on $S'$ and

(iii) $\alpha_i$ and $(\alpha_j)^{\pm1}$ $(i\neq j)$ are not freely homotopic.

Let $\{\alpha_i\}_{i=1}^q$ be a homotopically independent finite set of loops on $S'$. For every $i$ we can find a doubly connected domain $E_i$ $(\supset\alpha_i)$ on $S'$ surrounded by simple loops such that $\bar{E}_i\cap\bar{E}_j=\emptyset, i\neq j$, where $\bar{E}_i$ is the closure of $E_i$.

Let $\{\mu_t\}_{t\in[0,1)}$ be a set of Beltrami differentials satisfying

(A) $\mu_0=0$,

(B) $\mu_t$ has the support in $\pi^{-1}(\bigcup_{i=1}^{q} E_i)$,

(C) there exists a simple loop $\tilde{\alpha}_i$ in $E_i$ freely homotopic to $\alpha_i$ such that $\lim_{t\to1} l_{S_t}(f_t(\tilde{\alpha}_i))=0,\ i=1, \cdots, q$,

(D) $\mu_t$ converges to a measurable function $\mu$ almost everywhere such that, for any compact subset $L$ of $S-\bigcup_{i=1}^{q}\tilde{\alpha}_i$, $|\mu(z)|<d_L<1$ for almost all $z\in\pi^{-1}(L)$, where $d_L$ is a constant depending only on $L$, and

(E) each component of $f_t(E_i-\tilde{\alpha}_i)$ converges to a domain conformal to a punctured disc, $i=1, \cdots, q$.

Then we shall call the deformation of $S$ obtained by letting $t$ tend to 1 the squeezing deformation of $S$ with respect to $\{\alpha_i\}_{i=1}^q$.

We note that $\{\tilde{\alpha}_i\}_{i=1}^q$ is a homotopically independent set of loops. Our definition of squeezing deformations is essentially same as the one introduced by Bers [6] except a slight modification (D) and (E), in other word, on any compact subset of $S-\bigcup_{i=1}^{q}\tilde{\alpha}_i$, $f_t$ is $K$-quasiconformal uniformly with respect to the parameter $t\in[0, 1)$ of deformation. This fact plays an important role in the proof of Theorem 5.1. We remark that neither $\{E_i\}_{i=1}^q$ nor $\{\tilde{\alpha}_i\}_{i=1}^q$ can be determined uniquely by $\{\alpha_i\}_{i=1}^q$, but this is not essential. Note that $f_0$, $F_0$ and $\hat{F}_0$ are the identity maps and that $\hat{\Gamma}_0, \Gamma_0, U_0, \Delta_0$ and $S_0$ coincide with $\hat{\Gamma}, \Gamma, U, \Delta$ and $S$, respectively.

Now we shall show the existence of the squeezing deformation. For each $i$, we can find a conformal homeomorphism $\phi_i$ of $E_i$ onto $D_i=\{\zeta_i: a_i^2<|\zeta_i|<1\}$.

We note that $\phi_i(\alpha_i)$ is freely homotopic to a circle $d_i=\{\zeta_i : |\zeta_i|=a_i\}$ on $D_i$ and the loop $\tilde{\alpha}_i=\phi_i^{-1}(d_i)$ is in $E_i$ and is freely homotopic to $\alpha_i$.

For $t\in[0, 1)$, consider a quasiconformal homeomorphism $f_{i,t}$ of $D_i$ onto the annulus $D_{i,t}=\left\{\xi_i : \frac{a_i^2(1-t)^2}{(1-a_it)^2}<|\xi_i|<1\right\}$:

$$f_{i,t}(\zeta_i)=\begin{cases} \dfrac{r_i-a_it}{1-a_it}e(\theta_i), & a_i\leqq|\zeta_i|=r_i<1 \\ \dfrac{a_ir_i(1-t)^2}{(1-a_it)(a_i-r_it)}e(\theta_i), & a_i^2<|\zeta_i|=r_i<a_i, \end{cases}$$

where $e(\theta)=\exp(\sqrt{-1}\,\theta)$ and $\zeta_i=r_ie(\theta_i)$. It should be noted that

$$\sup\{|\tilde{\mu}_{i,t}(\zeta_i)| : t\in[0, 1),\ a_i<b_i<|\zeta_i|<1\}<\frac{a_i}{2b_i-c_i}<1 \quad \text{and}$$

$$\sup\{|\tilde{\mu}_{i,t}(\zeta_i)| : t\in[0, 1),\ a_i^2<|\zeta_i|<c_i<a_i\}<\frac{c_i}{2a_i-c_i}<1,$$

$$\text{where} \quad \tilde{\mu}_{i,t}(\zeta_i)=\frac{\partial f_{i,t}(\zeta_i)}{\partial\bar{\zeta}_i}\Big/\frac{\partial f_{i,t}(\zeta_i)}{\partial\zeta_i}.$$

Denoting by $\zeta$ a local parameter on $S$, we define the Beltrami differential $\tilde{\mu}_t$ as follows:

$$\tilde{\mu}_t(\zeta)=\begin{cases} \tilde{\mu}_{i,t}(\phi_i(\zeta))\overline{\phi_i'(\zeta)}/\phi_i'(\zeta) & \text{on } \bigcup_{i=1}^{q} E_i, \\ 0 & \text{elsewhere}. \end{cases}$$

Next we define the Beltrami differential $\mu_t$ for $\Gamma$ as follows:

$$\mu_t(z)=\begin{cases} \tilde{\mu}_t(\pi(z))\overline{\pi'(z)}/\pi'(z) & \text{on } \pi^{-1}(\bigcup_{i=1}^{q} E_i), \\ 0 & \text{elsewhere}. \end{cases}$$

The set $\{\mu_t\}_{t\in[0,1)}$ gives the desired squeezing deformation. In fact, we see obviously that $\{\mu_t\}_{t\in[0,1)}$ satisfies (A), (B), (D) and (E). We can also show that $\{\mu_t\}_{t\in[0,1)}$ satisfies (C) in a similar way to that of Bers [6].

## §2. Properties of squeezing deformations.

First we shall prove the following.

LEMMA 2.1. *Let $\Gamma$ be a function group with a quasicomponent $\Delta$. Let $\{\alpha_i\}_{i=1}^q$ be a homotopically independent set of loops on $S'=\Delta'/\Gamma$. If $S$ is squeezed with respect to $\{\alpha_i\}_{i=1}^q$, then*

$$\lim_{t\to1}(\operatorname{trace} F_t\circ\gamma_i\circ F_t^{-1})^2=4 \qquad \textit{for} \quad \gamma_i\in\tau(\alpha_i),\ i=1, \cdots, q.$$

PROOF. We denote by $\rho_{\Delta^*}(z)|dz|$ the Poincaré metric on $\Delta^*=\boldsymbol{C}-\{0, 1\}$. Note that there exists a positive constant $M$ satisfying $\rho_{\Delta^*}(z)>M$ for all $z\in\{z: 2\leqq|z|\leqq 4\}$.

It suffices to prove the lemma in the case $q=1$. We write $\tilde{\alpha}$ and $\gamma$ instead of $\tilde{\alpha}_1$ and $\gamma_1$, respectively, and set $\gamma_t=F_t\circ\gamma\circ F_t^{-1}$ for $t\in[0, 1)$. We may assume that $\gamma_t$ is of the form $(a_t, 0; 0, a_t^{-1})$ and that $\Delta_t$ does not contain the point $z=1$, because the square of the trace of a Möbius transformation is invariant under the conjugation.

Let $\Sigma_t$ be a path in $\Delta_t$ joining the repelling fixed point 0 to the attracting fixed point $\infty$ of $\gamma_t$ such that $\pi_t(\Sigma_t)=f_t(\tilde{\alpha})$. Let $z_t^*$ be a point on $\Sigma_t$ with $|z_t^*|=3$ and let $\Sigma_t^*$ be the subarc of $\Sigma_t$ with end points $z_t^*$ and $\gamma_t(z_t^*)$. Clearly we have

$$
\begin{aligned}
l_{S_t}(f_t(\tilde{\alpha}))&=\int_{f_t(\tilde{\alpha})}\rho_{S_t}(\zeta_t)|d\zeta_t| \\
&=\int_{\Sigma_t^*}\rho_{\Delta_t}(z_t)|dz_t|\geqq\int_{\Sigma_t^*}\rho_{\Delta^*}(z)|dz| \\
&\geqq M\int_{\Sigma_t^{**}}|dz|,
\end{aligned}
$$

where $\Sigma_t^{**}=\Sigma_t^*\cap\{z: 2\leqq|z|\leqq 4\}$. By the definition of squeezing deformation, $\int_{\Sigma_t^{**}}|dz|$ must tend to zero as $t\to 1$, or equivalently, $\gamma_t(z_t^*)$ approaches to $z_t^*$ as $t\to 1$. This is the desired conclusion.

REMARK. Bers [6] proved Lemma 2.1 in the case where $\Gamma$ is a Fuchsian group by using the method of extremal length. By using Bers' argument, Sato [23] also proved Lemma 2.1 in the case where $\Gamma$ is a Schottky group. We can see that Bers' argument is applicable for any Kleinian group.

The following important lemma is due to Jørgensen [13].

LEMMA 2.2. *Let $g$ and $h$ be two Möbius transformations which generate a Kleinian group. Then the inequality*

$$|(\mathrm{trace}\, g)^2-4|+|\mathrm{trace}\, g\circ h\circ g^{-1}\circ h^{-1}-2|\geqq 1$$

*is satisfied.*

By using this lemma, we prove the following

PROPOSITION 2.3. *Let $\{\alpha_i\}_{i=1}^q$ be a homotopically independent set of loops on $S'$. Let $\beta$ be a loop on $S'$ which bounds neither a disc nor a punctured disc such that the intersection number of $\beta$ and some $\alpha_i$ is not zero. If $S$ is squeezed with respect to $\{\alpha_i\}_{i=1}^q$, then*

$$\lim_{t\to 1} l_{S_t}(f_t(\beta))=\infty .$$

PROOF. For brevity we consider only the case $q=1$ and we write $\tilde{\alpha}$ instead of $\tilde{\alpha}_1$. Let $\hat{\gamma}\in\hat{\tau}(\alpha)$ and $\hat{\delta}\in\hat{\tau}(\beta)$ satisfy the condition that the axis of $\hat{\gamma}$ and that of $\hat{\delta}$ have a common point. Denote by $\kappa_1\,(>1)$ and $\kappa_2\,(>1)$ the multipliers of $\hat{\gamma}$ and of $\hat{\delta}$, respectively. Obviously we have

$$\begin{aligned} l_{S_t}(f_t(\tilde{\alpha}))&\geqq\log\kappa_1\\ &=\log\frac{(\mathrm{trace}\,\hat{\gamma}_t)^2-2+((\mathrm{trace}\,\hat{\gamma}_t)^2((\mathrm{trace}\,\hat{\gamma}_t)^2-4))^{1/2}}{2}\end{aligned}$$

and

$$\begin{aligned} l_{S_t}(f_t(\beta))&\geqq\log\kappa_2\\ &=\log\frac{(\mathrm{trace}\,\hat{\delta}_t)^2-2+((\mathrm{trace}\,\hat{\delta}_t)^2((\mathrm{trace}\,\hat{\delta}_t)^2-4))^{1/2}}{2},\end{aligned}$$

where $\hat{\gamma}_t=\hat{F}_t\circ\hat{\gamma}\circ\hat{F}_t^{-1}$ and $\hat{\delta}_t=\hat{F}_t\circ\hat{\delta}\circ\hat{F}_t^{-1}$. By the definition of squeezing deformation we see that $(\mathrm{trace}\,\hat{\gamma}_t)^2$ tends to 4 as $t\to 1$. To complete the proof of our proposition, it suffices to show that $(\mathrm{trace}\,\hat{\delta}_t)^2$ tends to infinity as $(\mathrm{trace}\,\hat{\gamma}_t)^2\to 4$.

Throughout the proof of our proposition, we assume that the Fuchsian model $\hat{\Gamma}_t$ of $\Gamma_t$ acts on the upper half plane $H_t=\{w:\mathrm{Im}\,w>0\}$ and that $\hat{\gamma}_t$ is of the form $(a_t, 0\,;0, a_t^{-1})\,(a_t>1)$. We denote by $\xi_t$ and $\xi_t'$ the attracting and the repelling fixed points of $\hat{\delta}_t$, respectively, and by $k_t^2\,(>1)$ the multiplier of $\hat{\delta}_t$. Then $\hat{\delta}_t$ is of the form

$$\begin{pmatrix} \dfrac{\xi_t k_t-\xi_t' k_t^{-1}}{\xi_t-\xi_t'} & \dfrac{-\xi_t\xi_t' k_t+\xi_t\xi_t' k_t^{-1}}{\xi_t-\xi_t'}\\[2ex] \dfrac{k_t-k_t^{-1}}{\xi_t-\xi_t'} & \dfrac{-\xi_t' k_t+\xi_t k_t^{-1}}{\xi_t-\xi_t'}\end{pmatrix}.$$

As the axis of $\hat{\gamma}$ intersects that of $\hat{\delta}$, we see that the axis of $\hat{\gamma}_t$ intersects that of $\hat{\delta}_t$. Hence $\xi_t\xi_t'<0$. So we may assume $\xi_t\xi_t'=-1$ and we have

$$\mathrm{trace}\,\hat{\gamma}_t\circ\hat{\delta}_t\circ\hat{\gamma}_t^{-1}\circ\hat{\delta}_t^{-1}=2-\frac{(k_t-k_t^{-1})^2(a_t-a_t^{-1})^2}{(\xi_t-\xi_t')^2}$$

and

$$(\xi_t-\xi_t')^2=\xi_t^2+2+\xi_t'^2>2\,.$$

Therefore, we see

$$|\mathrm{trace}\,\hat{\gamma}_t\circ\hat{\delta}_t\circ\hat{\gamma}_t^{-1}\circ\hat{\delta}_t^{-1}-2|<\frac{1}{2}(a_t-a_t^{-1})^2(k_t-k_t^{-1})^2\,.$$

This inequality and Lemma 2.2 imply

$$1-(a_t-a_t^{-1})^2<\frac{1}{2}(a_t-a_t^{-1})^2(k_t-k_t^{-1})^2\,.$$

Using our assumption, we see

$$\lim_{t\to 1}(a_t - a_t^{-1})^2 = \lim_{t\to 1} |(\mathrm{trace}\,\hat{\gamma}_t)^2 - 4| = 0\,.$$

Hence we can conclude that $k_t$ tends to infinity as $t \to 1$. Thus $(\mathrm{trace}\,\hat{\delta}_t)^2 = (k_t + k_t^{-1})^2$ also tends to infinity as $t \to 1$, which proves our assumption.

In the above proof, the condition $\lim_{t\to 1} l_{S_t}(f_t(\tilde{\alpha})) = 0$ is essential rather than the assumption that $S$ is squeezed with respect to $\alpha$. Hence we can obtain the following.

PROPOSITION 2.4. *Let $\alpha$ and $\beta$ be simple loops on a Riemann surface $S$ such that the intersection number $\alpha$ and $\beta$ is not zero. Assume that both $\alpha$ and $\beta$ bound neither a disc nor a punctured disc. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of quasiconformal homeomorphisms of $S$ such that $l_{f_n(S)}(f_n(\alpha))$ tend to zero as $n \to \infty$. Then $\lim_{n\to\infty} l_{f_n(S)}(f_n(\beta)) = \infty$.*

REMARK. Proposition 2.3 and Proposition 2.4 can be also obtained by a theorem of Keen [**14**].

## § 3. Schottky spaces.

A Kleinian group $\Gamma$ is called a marked Schottky group of genus $p$ with standard generators $\gamma_1, \cdots, \gamma_p$, if there exist $2p$ Jordan curves $C_1, C_1', \cdots, C_p, C_p'$ surrounding a $2p$-ply connected domain $R$ such that $\gamma_j(R) \cap R = \emptyset$ and $\gamma_j(C_j') = C_j$ for every $j$. This marked Schottky group is denoted by $\Gamma = \langle \gamma_1, \cdots, \gamma_p \rangle$. The ordinary set $\Omega(\Gamma)$ of a Schottky group $\Gamma$ is connected. Maskit [**17**] proved that a Kleinian group $\Gamma$ is a Schottky group if and only if $\Gamma$ is a finitely generated free group and is purely loxodromic. From now on, we denote by $\Gamma$ a Schottky group, and assume $p \geqq 2$.

Let $\Gamma = \langle \gamma_1, \cdots, \gamma_p \rangle$ be a marked Schottky group of genus $p$. We denote by $Q_{\mathrm{norm}}(\Gamma)$ the set of all quasiconformal homeomorshisms of $\hat{\boldsymbol{C}}$ onto itself being compatible with $\Gamma$ which keep 0, 1 and $\infty$ invariant. The Schottky space $\mathcal{S}_p(\Gamma)$ attached to $\Gamma$ is defined as the set

$$\mathcal{S}_p(\Gamma) = \{\langle F\circ\gamma_1\circ F^{-1}, \cdots, F\circ\gamma_p\circ F^{-1}\rangle : F \in Q_{\mathrm{norm}}(\Gamma)\}\,.$$

As is well known, for $F$ and $\tilde{F}\ (\neq F)$ belonging to $Q_{\mathrm{norm}}(\Gamma)$, a marked Schottky group $\langle F\circ\gamma_1\circ F^{-1}, \cdots, F\circ\gamma_p\circ F^{-1}\rangle$ may coincide with the marked Schottky group $\langle \tilde{F}\circ\gamma_1\circ\tilde{F}^{-1}, \cdots, \tilde{F}\circ\gamma_p\circ\tilde{F}^{-1}\rangle$. We define that a sequence of Möbius transformations $\{(a_n, b_n; c_n, d_n)\}_{n=1}^{\infty}$ converges to a Möbius transformation $(a, b; c, d)$ as $a_n \to a$, $b_n \to b$, $c_n \to c$, $d_n \to d$ $(n \to \infty)$. We topologize $\mathcal{S}_p(\Gamma)$ by requiring the marked Schottky group $\langle F_n\circ\gamma_1\circ F_n^{-1}, \cdots, F_n\circ\gamma_p\circ F_n^{-1}\rangle$ converges to a marked Schottky group $\langle g_1, \cdots, g_p\rangle$, if, for every $j$, $F_n\circ\gamma_j\circ F_n^{-1}$ converges to $g_j$ as $n \to \infty$.

Let $\Gamma^* = \langle \gamma_1^*, \cdots, \gamma_p^* \rangle$ be another marked Schottky group of genus $p$. Then there exists a quasiconformal homeomorphism $F^*$ of $\hat{\boldsymbol{C}}$ onto itself satisfying

$\gamma_j^*=F^*\circ\gamma_j\circ(F^*)^{-1}$, $j=1, \cdots, p$ (See Bers [9]). Hence the Schottky space $\mathcal{S}_p(\Gamma^*)$ attached to $\Gamma^*$ is homeomorphic to $\mathcal{S}_p(\Gamma)$. Therefore, in what follows, we fix a marked Schottky group $\Gamma=\langle\gamma_1, \cdots, \gamma_p\rangle$ such that the attracting fixed points of $\gamma_1, \gamma_2$ and $\gamma_1\circ\gamma_2$ are 0, 1 and $\infty$, respectively. We write merely $\mathcal{S}_p$ instead of $\mathcal{S}_p(\Gamma)$ and we call $\mathcal{S}_p$ the Schottky space of genus $p$.

Marden [17] defined the Schottky space $\mathcal{S}_p^1$ of genus $p$ as follows:

$$\mathcal{S}_p^1=\{\langle F\circ\gamma_1\circ F^{-1}, \cdots, F\circ\gamma_p\circ F^{-1}\rangle : F\in Q(\Gamma)\},$$

where $Q(\Gamma)$ is the set of all quasiconformal homeomorphism of $\hat{C}$ onto itself being compatible with $\Gamma$. Obviously we see that $\mathcal{S}_p^1$ is a $3p$-dimensional complex manifold. Chuckrow [10] defined the Schottky space $\mathcal{S}_p^2$ as the quotient space $\mathcal{S}_p^1/\sim$, where $\sim$ is an equivalent relation defined in the following way: $\langle\gamma_1, \cdots, \gamma_p\rangle \sim \langle\gamma_1^*, \cdots, \gamma_p^*\rangle$ if there exists a Möbius transformation $h$ satisfying $\gamma_j^* = h\circ\gamma_j\circ h^{-1}$ for every $j$. Hence we can see that $\mathcal{S}_p^2$ is a $(3p-3)$-dimensional complex manifold. Clearly, $\mathcal{S}_p$ is homeomorphic to $\mathcal{S}_p^2$ and is also $(3p-3)$-dimensional complex manifold. In the later of this paper, we shall be concerned with $\mathcal{S}_p$.

Let $\Psi$ be a homeomorphism of the set Möb of all Möbius transformations into the 3-dimensional complex projective space $P_3(C)$ transforming $(a, b\,; c, d)$ to $(a, b, c, d)$. We denote by $\mathcal{L}$ the set of all loxodromic transformations of the form $(a, b\,; c, d)$, $c\neq 0$. Denoting by $\overline{\Psi(\mathcal{L})}$ the closure of $\Psi(\mathcal{L})$ in $P_3(C)$, we set $\xi(x)=\lim_{m\to\infty}\xi(x_n)$ and $\xi'(x)=\lim_{n\to\infty}\xi'(x_n)$ for $x\in\overline{\Psi(\mathcal{L})}-\Psi(id)$, where $\{x_n\}_{n=1}^{\infty}$ is a sequence of points in $\Psi(\mathcal{L})$ converging to $x$ and $\xi(x_n)$ (resp. $\xi'(x_n)$) is the attracting (resp. the repelling) fixed point of $\Psi^{-1}(x_n)$. We note that $\xi(x)$ and $\xi'(x)$ are well defined, because the fixed points of $\Psi^{-1}(x_n)=(a_n, b_n\,; c_n, d_n)$ are $(d_n-a_n\pm((a_n+d_n)^2-4)^{1/2})c_n^{-1}=-4b_n(d_n-a_n\mp((a_n+d_n)^2-4)^{1/2})^{-1}$. Let $\Phi$ be the homeomorphism of $S_p$ into the $p$-times product $P_3(C)^p$ of $P_3(C)$ transforming $\langle g_1, \cdots, g_p\rangle$ to $(\Psi(g_1), \cdots, \Psi(g_p))$. We define the boundary $\partial\mathcal{S}_p$ of $\mathcal{S}_p$ as $\overline{\Phi(\mathcal{S}_p)}-\Phi(\mathcal{S}_p)$, where $\overline{\Phi(\mathcal{S}_p)}$ is the closure of $\Phi(\mathcal{S}_p)$ in $P_3(C)^p$. Points in $\partial\mathcal{S}_p$ are classified as follows (See Bers [8] and Chuckrow [10]):

The boundary point $(x_1, \cdots, x_p)$ of $\mathcal{S}_p$ belongs to the category (I), if $\Psi^{-1}(x_j)$ is a Möbius transformation for all $j$ and if the group $G$ generated by $\Psi^{-1}(x_1), \cdots, \Psi^{-1}(x_p)$ contains a parabolic transformation. In this case, the point $(x_1, \cdots, x_p)\in\partial\mathcal{S}_p$ is also called a cusp.

The boundary point $(x_1, \cdots, x_p)$ of $\mathcal{S}_p$ belongs to the category (II), if $\Psi^{-1}(x_j)$ is a Möbius transformation for all $j$ and if the group $G$ generated by $\Psi^{-1}(x_1), \cdots, \Psi^{-1}(x_p)$ does not contain a parabolic transformation.

The boundary point $(x_1, \cdots, x_p)$ of $\mathcal{S}_p$ belongs to the category (III), if the following three conditions are satisfied:

(i) the set $J=\{j\in\{1, \cdots, p\} : x_j\notin\Psi(\text{Möb})\}$ is not empty,

(ii) the set $\{\Phi^{-1}(x_k): k \in \{1, \cdots, p\}-J\}$ of Möbius transformations generates a Schottky group of genus $p-$(the number of elements of $J$), which may be reduced to the trivial group consisting of only the identity, and

(iii) any two points in the set $\{\xi(x_1), \xi'(x_1), \cdots, \xi(x_p), \xi'(x_p)\}$ are different from each other. In this case the boundary point $(x_1, \cdots, x_p)$ of $\mathcal{S}_p$ is also called a node.

The boundary point of $\mathcal{S}_p$ belonging to none of (I), (II) and (III) belongs to the category (IV).

Here we shall state some definitions. After Chuckrow [11], we call a Kleinian group $G$ an extended Schottky group with standard generators $g_1, \cdots, g_p, \tilde{g}_1, \cdots, \tilde{g}_r$ and with defining curves $C_1, C_1', \cdots, C_p, C_p', \tilde{C}_1, \tilde{C}_1', \cdots, \tilde{C}_r, \tilde{C}_r'$, if the following are satisfied:

(i) those curves are disjoint Jordan curves, except that, for every $k$, $\tilde{C}_k$ and $\tilde{C}_k'$ have only one common point $\xi_k$, and surround a $(2p+r)$-ply connected domain $R$ such that

$$g_j(R) \cap R = \tilde{g}_k(R) \cap R = \emptyset,\ j=1, \cdots, p;\ k=1, \cdots, r,$$

(ii) $g_j(C_j')=C_j$, $j=1, \cdots, p$, and $\tilde{g}_k(\tilde{C}_k')=\tilde{C}_k$, $k=1, \cdots, r$, and

(iii) $\tilde{g}_k$ is a parabolic transformation with the fixed point $\xi_k$.

Clearly an extended Schottky group is a function group and does not contain an elliptic transformation. Note that, if $r=0$ in the above definition of an extended Schottky group $G$, then $G$ is a Schottky group.

In general, a Kleinian group $G$ is called to be conformally extendable (Marden [16]), if every type-preserving isomorphism between $G$ and some other group $G^*$ induced by a conformal homeomorphism of the ordinary set of $G$ onto that of $G^*$ is, in fact, induced by a Möbius transformation.

A Keinian group $G$ is called to be geometrically finite, if it has a finite sided Dirichlet fundamental polyhedra (see Marden [16] and Beardon-Maskit [5]).

The following is due to Beardon-Maskit [5], Marden [16] and Maskit [19]. (See also Maskit [20]).

LEMMA 3.1. *Exteded Schottky groups are conformally extendable and geometrically finite, and the limit sets of extended Schottky groups have zero area.*

## § 4. Limits of Möbius transformations.

For a loxodromic transformation $g$, a pair $(C, C')$ of mutually disjoint Jordan curves $C$ and $C'$ is called a pair of defining curves of $g$, if $g(C')=C$ and $g(R) \cap R = \emptyset$ for the doubly connected domain $R$ surrounded by $C$ and $C'$. For a parabolic transformation $g$, a pair $(C, C')$ of Jordan curves $C$ and $C'$, which meet only at the fixed point of $g$, is called a pair of defining curves of $g$, if $g(C')=C$ and $g(R) \cap R = \emptyset$ for the simply connected domain $R$ surrounded by $C$ and $C'$.

Clearly, for a parabolic or a loxodromic transformation $g$, a pair of defining curves of $g$ exists but cannot be determined uniquely.

LEMMA 4.1. Let $\gamma=(a, b; c, d)$ $(c\neq 0)$ be a loxodromic transformation and let $(C, C')$ be a pair of defining curves of $\gamma$ such that the domain surrounded by $C$ and $C'$ contains the point $\infty$. Then $C$ separates the repelling fixed point of $\gamma$ from $\infty$ and $C'$ separates the attracting fixed point of $\gamma$ from $\infty$. Moreover, if Area $X$ denotes the Euclidean area of a measurable set $X$ in $\boldsymbol{C}$, if $[\Sigma]$ denotes the bounded domain surrounded by a closed Jordan curve $\Sigma$ on $\boldsymbol{C}$, and if $I(\gamma)$ is the isometric circle of $\gamma$, then

$$\mathrm{Area}\,[C]+\mathrm{Area}\,[C']\geqq \mathrm{Area}\,[I(\gamma)]+\mathrm{Area}\,[I(\gamma^{-1})]\,.$$

PROOF. The first assertion is obvious. To show the second assertion, set $D=[C]$, $D'=[C']$, $E=[I(\gamma)]$ and $E'=[I(\gamma^{-1})]$. By a well known property of isometric circles (see Ford [12], Theorem 17 at page 25) we have

$$\begin{aligned}&\mathrm{Area}\,D+\mathrm{Area}\,D'\\ &=\mathrm{Area}\,(D\cap E)+\mathrm{Area}\,(D-E)+\mathrm{Area}\,(D'\cap E')+\mathrm{Area}\,(D'-E')\\ &\geqq \mathrm{Area}\,(D\cap E)+\mathrm{Area}\,(E'-D')+\mathrm{Area}\,(D'\cap E')+\mathrm{Area}\,(E-D)\\ &=\mathrm{Area}\,E+\mathrm{Area}\,E',\end{aligned}$$

which is the required.

LEMMA 4.2. *Let* $\{\gamma_n\}_{n=1}^{\infty}$ *be a sequence of loxodromic transformation satisfying the following conditions:*

(i) $\gamma_n$ *is of the form* $(a_n, b_n; c_n, d_n)$ *such that* $a_nd_n-b_nc_n=1$ *and* $c_n\neq 0$,

(ii) *there exist disjoint three points* $z_1$, $z_2$ *and* $z_3$ *in* $\boldsymbol{C}$ *such that, for every* $i$, $\gamma_n(z_i)\to w_i$ *as* $n\to\infty$, *where* $w_i\neq w_j\,(i\neq j)$,

(iii) *for an arbitrary* $n$, *there exists a pair* $(C_n, C_n')$ *of defining curves of* $\gamma_n$ *such that* $C_n$ *and* $C_n'$ *are contained in a closed disc* $\{z: |z|\leqq M\}$ *in* $\boldsymbol{C}$ *and such that* $C_n$ *and* $C_n'$ *surround an unbounded domain, where* $M$ *is independent of* $n$, *and*

(iv) $\lim_{n\to\infty} a_n$, $\lim_{n\to\infty} b_n$, $\lim_{n\to\infty} c_n$ and $\lim_{n\to\infty} d_n$

*exist in* $\hat{\boldsymbol{C}}$.

*Then* $\{\gamma_n\}_{n=1}^{\infty}$ *converges to a Möbius transformation* $g=(a, b; c, d)$ $(ad-bc=1, c\neq 0)$ *as* $n\to\infty$.

PROOF. Assume that $c=\lim_{n\to\infty} c_n$ is infinity. Then the radius $|c_n|^{-1}$ of the isometric circle $I(\gamma_n)$ of $\gamma_n$ tends to zero as $n\to\infty$. Hence two points, for instance, $z_1$ and $z_2$ among $z_1$, $z_2$ and $z_3$, are in the exterior of $I(\gamma_n)$. So $\gamma_n(z_1)$ and $\gamma_n(z_2)$ are in the interior of the isometric circle $I(\gamma_n^{-1})$ of $\gamma_n^{-1}$, whose radius $|c_n|^{-1}$ tends to zero as $n\to\infty$. This implies $w_1=w_2$, which contradicts (ii).

Therefore, $c$ must be finite.

Next, assume $c=0$. Then the Euclidean area $4\pi|c_n|^{-2}$ of $[I(\gamma_n)]\cup[I(\gamma_n^{-1})]$ tends to infinity as $n\rightarrow\infty$. By Lemma 4.1 we observe that Area $[C_n]$ $+$Area$[C_n']$ also tends to infinity, where Area $[C]$ denotes the Euclidean area of the bounded domain surrounded by a Jordan curve $c$. This contradicts our assumption (iii). Thus $c$ must not be zero.

By (iii) we have $|a_n\cdot c_n^{-1}|=|\gamma_n(\infty)|\leqq M$ and $|d_n\cdot c_n^{-1}|=|\gamma_n^{-1}(\infty)|\leqq M$, $n=1, 2, \cdots$. So we see that $a=\lim_{n\to\infty} a_n$ and $d=\lim_{n\to\infty} d_n$ are finite. Hence

$$b=\lim_{n\to\infty} b_n=\lim_{n\to\infty}(a_nd_n-1)\cdot c_n^{-1}$$

is also finite, and we complete the proof of our lemma.

For our study of limits of Schottky groups, the following result due to Chuckrow [**10**] and Jørgensen [**13**] is fundamental.

LEMMA 4.3. *Let $\Gamma_n=\langle\gamma_{1,n},\cdots,\gamma_{p,n}\rangle$ be a marked Schottky group, $n=1, 2, \cdots$. Assume that, for every $j$, $\{\gamma_{j,n}\}_{n=1}^{\infty}$ converges to a Möbius transformation $g_j$ as $n\rightarrow\infty$. Then none of $g_1,\cdots,g_p$ is the identity.*

Now we prove one more lemma.

LEMMA 4.4. *Let $\{\gamma_n\}_{n=1}^{\infty}$ be a sequence of loxodromic transformations converging to a parabolic transformation $g$. Let $(C_n, C_n')$ be a pair of defining curves of $\gamma_n$ such that $C_n$ and $C_n'$ surround an unbounded domain. Assume the existence of compact sets $K$ and $\tilde{K}$ in* **C** *satisfying the following conditions:*

(i) *the interior of $K$ contains $\tilde{K}$,*

(ii) *the boundary of $K$ is a Jordan curve $\Sigma$, and*

(iii) *$\tilde{K}$ contains $C_n$ and $C_n'$ for every $n$.*

*Then there exists a pair $(C, C')$ of defining curves of $g$ such that $C$ and $C'$ are contained in $K$.*

PROOF. As $\gamma_n(z)$ is in $\tilde{K}$ for any $z\in\Sigma$ and for any $n$, we see that $g(z)$ is in $\tilde{K}$ for any $z\in\Sigma$. Hence Jordan curves $\Sigma$ and $g(\Sigma)$ are disjoint each other. Let $h$ be a Möbius transformation such that $h\circ g\circ h^{-1}=(1, 1; 0, 1)$. We note that the compact set $h(K)$ contains the point $\infty$ as its interior point and that the boundary of $h(K)$ is the Jordan curve $h(\Sigma)$. Let $z_1=x_1+\sqrt{-1}\,y_1$ be a point on $h(\Sigma)$ satisfying $y_1\geqq y$ for any point $z=x+\sqrt{-1}\,y\in h(\Sigma)$. If such a $z_1$ is not uniquely determined, we cut away a sufficiently small piece of $K$ so that, for the resulting compact set $K^*$ and the boundary Jordan curve $\Sigma^*$ of $K^*$, there exists a point $z_1^*=x_1^*+\sqrt{-1}\,y_1^*$ on $h(\Sigma^*)$ satisfying $y_1^*>y$ for any $z=x+\sqrt{-1}\,y\in h(\Sigma^*)$.

From now on, we write merely $z_1$, $y_1$, $h(K)$ and $h(\Sigma)$ instead of $z_1^*$, $y_1^*$, $h(K^*)$ and $h(\Sigma^*)$, respectively. Similarly, let $z_2=x_2+\sqrt{-1}\,y_2$ be the point on $h(\Sigma)$ such that $y_2<y$ for any point $z=x+\sqrt{-1}\,y\in h(\Sigma)$. Then $h(\Sigma)$ is divided into two Jordan arcs by $z_1$ and $z_2$. Take one of them containing the point

$z_3=x_3+\sqrt{-1}\,y_3$ such that $x_3\leqq x$ for any $z=x+\sqrt{-1}\,y\in h(\Sigma)$ and denote it by $\sigma_3$.

Let $\sigma_1=\{z=x+\sqrt{-1}\,y\in \boldsymbol{C}: x=x_1, y\geqq y_1\}$ and $\sigma_2=\{z=x+\sqrt{-1}\,y\in \boldsymbol{C}: x=x_2, y\leqq y_2\}$. Then $\sigma_1$, $\sigma_2$, $\sigma_3$ and the point $\infty$ make a Jordan curve $\sigma'$ in $h(K)$. Set $\sigma=h\circ g\circ h^{-1}(\sigma')$.

Now we shall show that $\sigma$ and $\sigma'$ have only one common point $\infty$. Obviously, $\sigma_i\cap h\circ g\circ h^{-1}(\sigma_i)=\emptyset$, $i=1, 2$. As $\sigma_3$ is contained in $h(\Sigma)$, we see that $h\circ g\circ h^{-1}(\sigma_3)$ is contained in $h\circ g(\Sigma)$. Moreover, $h(\Sigma)$ does not meet $h\circ g(\Sigma)$, since $\Sigma$ does not meet $g(\Sigma)$ as stated already. Therefore, $\sigma_3$ and $h\circ g\circ h^{-1}(\sigma_3)$ are disjoint each other. Finally, by the manner of choosing $z_1$ and $z_2$, we have $\sigma_i\cap h\circ g\circ h^{-1}(\sigma_j)=\emptyset$, $i\neq j$. Set $C=h^{-1}(\sigma)$ and $C'=h^{-1}(\sigma')$. Then we can immediately observe that $(C, C')$ is the desired pair of defining curves of $g$.

## § 5. Squeezing deformations with respect to $A$-cycles.

In this section, recalling notations and conventions stated in the previous sections, we shall prove the following.

THEOREM 5.1. *Let $\Gamma=\langle\gamma_1, \cdots, \gamma_p\rangle$ be a marked Schottky group of genus $p$ and let $\{\alpha_i\}_{i=1}^q$ be a homotopically independent set of loops on $S=\Omega(\Gamma)/\Gamma$ such that $\tau(\alpha_i)\,(i=1, \cdots, q)$ contains an element of standard generators of $\Gamma$. If $S$ is squeezed with respect to $\{\alpha_i\}_{i=1}^q$, then the point $P_t=(\Psi(F_t\circ\gamma_1\circ F_t^{-1}), \cdots, \Psi(F_t\circ\gamma_p\circ F_t^{-1}))$ in $\Phi(\mathcal{S}_p)$ approaches to a cusp $Q=(x_1, \cdots, x_p)$ and the group $G$ generated by $g_1=\Psi^{-1}(x_1), \cdots, g_p=\Psi^{-1}(x_p)$ satisfies the following:*

(i) *if $\gamma_j\in \bigcup_{i=1}^q \tau(\alpha_i)$, then $g_j$ is parabolic and $G$ is an extended Schottky group,*

(ii) *$G$ is geometrically finite,*

(iii) *the limit set $\Lambda(G)$ of $G$ has area zero, and*

(iv) *$\Omega(G)/G$ represents a Riemann surface homeomorphic to $S-\bigcup_{i=1}^q \alpha_i$.*

REMARK. Every $\alpha_i$ is a so-called $A$-cycle.

Before proving Theorem 5.1, we mention that the set $\Omega_\delta=\Omega(\Gamma)-\pi^{-1}(\bigcup_{i=1}^q E_i)$ is connected. Let $R$ be the fundamental domain for $\Gamma$ surrounded by $C_1$, $C_1'$, $\cdots$, $C_p$ and $C_p'$, as we stated in § 3. Let $\Sigma_j$ be a closed Jordan curve in $\Omega_\delta\cap R$ which separates $\{C_j, C_j'\}$ from the set $(\bigcup_{k\neq j}(C_k\cup C_k'))$ and satisfies $\Sigma_j\cap\Sigma_k=\emptyset$, $j\neq k$. By Lemma 3.1 it suffices to prove (i) and (iv) of Theorem 5.1.

LEMMA 5.2. *Under the same assumptions as in Theorem 5.1, there exists a sequence $\{(F_{t_m}\circ\gamma_1\circ F_{t_m}^{-1}, \cdots, F_{t_m}\circ\gamma_p\circ F_{t_m}^{-1})\}_{m=1}^\infty$ in $\{P_t\}_{t\in[0,1)}\subset\Phi(\mathcal{S}_p)$ approaching to a cusp $Q=(g_1, \cdots, g_p)$, which satisfies* (i) *and* (iv) *in Theorem* 5.1.

PROOF. We recall that $F_t$ is a quasiconformal homeomorphism of $\hat{\boldsymbol{C}}$ onto itself which leaves 0, 1 and $\infty$ fixed. Clearly the restriction $F_t^\delta$ of $F_t$ on $\Omega_\delta$ is holomorphic and univalent. Since $F_t^\delta$ cannot take the values 0, 1 and $\infty$, we see that $\{F_t^\delta\}_{t\in[0,1)}$ is a normal family. So we find a sequence $\{F_{t_m}^\delta\}_{m=1}^\infty$ in $\{F_t^\delta\}_{t\in[0,1)}$ converging to a holomorphic function $\mathfrak{F}^\delta$ locally uniformly in $\Omega_\delta$.

By taking a subsequence, again denoted by $\{F_{t_m}\circ\gamma_j\circ F_{t_m}^{-1}\}_{m=1}^\infty$, of $\{F_{t_m}\circ\gamma_j\circ F_{t_m}^{-1}\}_{m=1}^\infty$ and by noting Lemma 4.2, we see that $F_{t_m}\circ\gamma_j\circ F_{t_m}^{-1}$ tends to a Möbius transformation $g_j$ for every $j$. By Lemma 2.1 and by Lemma 4.3, we see that $g_i$ is parabolic for $i\in\hat{I}=\{j\in\{1, \cdots, p\} : \gamma_j\in\bigcup_{i=1}^q \tau(\alpha_i)\}$.

As $\mathfrak{F}^\delta$ is holomorphic and univalent on an open set containing $C_j\cup C_j'$ for $j\in\{1, \cdots, p\}-\hat{I}$, we see that $(\mathcal{C}_j, \mathcal{C}_j')$ is a pair of defining curves of $g_j$ for $j\in\{1, \cdots, p\}-\hat{I}$, where $\mathcal{C}_j=\mathfrak{F}^\delta(C_j)$ and $\mathcal{C}_j'=\mathfrak{F}^\delta(C_j')$. By Lemma 4.4, we can find a pair $(\mathcal{C}_j, \mathcal{C}_j')$ of defining curves of a parabolic transformation $g_j$ for every $j\in\hat{I}$, such that, for any $j$, $\mathfrak{F}^\delta(\Sigma_j)$ separates the set $\{\mathcal{C}_j, \mathcal{C}_j'\}$ from the set $\bigcup_{k\neq j}\{\mathcal{C}_k, \mathcal{C}_k'\}$. Thus we have the desired conclusion by a theorem of Maskit [19].

Next we shall define a mapping of $\Omega(\Gamma)-\pi^{-1}(\bigcup_{i=1}^q \tilde{\alpha}_i)$ into $\hat{\boldsymbol{C}}$, which is identical with $\mathfrak{F}^\delta$ on $\Omega_\delta$. Let $\delta_i^k$ be a doubly connected domain on $S=\Omega(\Gamma)/\Gamma$ such that $\delta_i^k$ contains $\tilde{\alpha}_i$ $(k=1, 2, \cdots)$ and such that $\delta_i^k\supset\delta_i^l$ $(k<l)$ and $\bigcap_{k=1}^\infty \delta_i^k=\tilde{\alpha}_i$ for every $i$. We may assume $\delta_i^k\cap\delta_j^k=\emptyset\,(i\neq j)$ for every $k$. We put $\Omega_{\delta(k)}=\Omega(\Gamma)-\pi^{-1}(\bigcup_{i=1}^q \delta_i^k)$ and let $F_t^{\delta(k)}$ be the restriction of $F_t$ on $\Omega_{\delta(k)}$ for $t\in[0, 1)$. As mentioned in § 2, $F_t^{\delta(k)}$ is quasiconformal uniformly with respect to $t\in[0, 1)$ for any $k$. We denote by $K(k, t)$ the maximal dilatation of $F_t^{\delta(k)}$ in $\Omega_{\delta(k)}$ and set $K_k=\sup_{t\in[0,1)} K(k, t)$. Let $\{F_{t_m}^\delta\}_{m=1}^\infty$ be the sequence of holomorphic and univalent functions converging to a holomorphic and univalent function $\mathfrak{F}^\delta$ obtained in the proof of Lemma 5.2. Let $\check{U}=\{w : |w|<1\}$ be the universal covering surface of $\Omega_{\delta(1)}$ with the natural projection $\check{\pi}$. Let $\check{F}_m$ be the quasiconformal homeomorphism of $\check{U}$ onto $\check{U}$ keeping 0 and 1 invariant and satisfying the equation $\dfrac{\partial\check{F}_m(w)}{\partial\bar{w}}=\check{\mu}_m(w)\dfrac{\partial\check{F}_m(w)}{\partial w}$, where $\check{\mu}_m(w)\overline{\check{\pi}'(w)}/\check{\pi}'(w)=\mu_{t_m}\,|\,\Omega_{\delta(1)}(z)$. Then clearly the following diagram

$$\begin{array}{ccc} \check{U} & \xrightarrow{\check{F}_m} & \check{U} \\ \check{\pi}\Big\downarrow & & \Big\downarrow\check{\pi}_m \\ \Omega_{\delta(1)} & \xrightarrow{F_{t_m}^{\delta(1)}} & F_{t_m}^{\delta(1)}(\Omega_{\delta(1)}) \end{array}$$

is commutative, where $\check{\pi}_m$ is the natural projection. In the similar way as in the proof of Lemma 4.2, we can find a subseqence, denoted again $\{\check{\pi}_m\}_{m=1}^\infty$, of $\{\check{\pi}_m\}_{m=1}^\infty$ converging to a holomorphic function $\check{\pi}_*$ such that $\check{\pi}_*(U)$ is a (non-empty) domain. By the definition of squeezing deformation $\dfrac{\partial\check{F}_m(w)}{\partial\bar{w}}\Big/\dfrac{\partial\check{F}_m(w)}{\partial w}$

converges to a measurable function $\check{\mu}$ with $\|\check{\mu}\|_\infty < d_1 < 1$ almost everywhere. So the sequence $\{\check{F}_m\}_{m=1}^{\infty}$ converges to a $K_1$-quasiconformal homeomorphism $\check{\mathcal{F}}_*$ uniformly on $U$ (Ahlfors-Bers [4]). Therefore, $F_{i_m}^{\delta(1)}$ converges to $K_1$-quasiconformal homeomorphism $\mathcal{F}^{\delta(1)}$ of $\Omega_{\delta(1)}$ as $m \to \infty$. Repeating this procedure, we can construct a $K_k$-quasiconformal homeomorphism $F^{\delta(k)}$ of $\Omega_{\delta(k)}$. Obviously $\mathcal{F}^{\delta(k)} = \mathcal{F}^{\delta(l)}$ on $\Omega_{\delta(k)}$ for $k<l$. By setting $\mathcal{F} = \mathcal{F}^{\delta(k)}$ on $\Omega_{\delta(k)}$ for every $k$, we define a mapping of $\Omega(\Gamma) - \pi^{-1}(\bigcup_{i=1}^{q} \tilde{\alpha}_i)$ into $\hat{\boldsymbol{C}}$.

LEMMA 5.3. *The mapping $\mathcal{F}$ constructed above is a homeomorphism of $\Omega(\Gamma)$ $-\pi^{-1}(\bigcup_{i=1}^{q} \tilde{\alpha}_i)$ onto $\Omega(G)$, where $G$ is the extended Schottky group obtained in Lemma* 5.2.

PROOF. For any $z_1$ and $z_2$ in $\Omega(\Gamma) - \pi^{-1}(\bigcup_{i=1}^{q} \tilde{\alpha}_i)$ there exists a number $k$ such that $z_1$ and $z_2$ are in $\Omega_{\delta(k)}$. So $\mathcal{F}(z_1) = \mathcal{F}^{\delta(k)}(z_1)$ is not identical with $\mathcal{F}(z_2)$ $= \mathcal{F}^{\delta(k)}(z_2)$. Recalling the construction of $\mathcal{F}$, we see that $\mathcal{F}$ gives a local homeomorphism of $\Omega(\Gamma) - \pi^{-1}(\bigcup_{i=1}^{q} \tilde{\alpha}_i)$, so we observe that $\mathcal{F}$ is a homeomorphism of $\Omega(\Gamma) - \pi^{-1}(\bigcup_{i=1}^{q} \tilde{\alpha}_i)$ into $\hat{\boldsymbol{C}}$.

Consider a point $z \in \Omega(\Gamma) - \pi^{-1}(\bigcup_{i=1}^{q} \tilde{\alpha}_i)$. Then there exists a number $k$ such that $z \in \Omega_{\delta(k)}$. Taking a neighborhood $V$ $(\subset \mathcal{F}(\Omega_{\delta(k)}))$ of $\mathcal{F}(z)$ suitably and recalling the construction of $\mathcal{F}$, we see that $\#\{g \in G : g \circ \mathcal{F}(z) \in V\} = \#\{\gamma \in \Gamma : \gamma(z) \in \mathcal{F}^{-1}(V)\}$, where $\# A$ denotes the number of elements in the set $A$. Since $z \in \Omega_{\delta(k)} \subset \Omega(\Gamma)$, we see $\#\{g \in G : g \circ \mathcal{F}(z) \in V\} = \#\{\gamma \in \Gamma : \gamma(z) \in \mathcal{F}^{-1}(V)\} < \infty$. We observe that $\mathcal{F}(z)$ is in $\Omega(G)$ and that $\mathcal{F}(\Omega(\Gamma) - \pi^{-1}(\bigcup_{i=1}^{q} \tilde{\alpha}_i)) \subset \Omega(G)$.

Now we will show $\mathcal{F}(\Omega(\Gamma) - \pi^{-1}(\bigcup_{i=1}^{q} \tilde{\alpha}_i)) = \Omega(G)$. We can regard the sequence $\{\mathcal{F}^{\delta(k)}(\Omega_{\delta(k)})/G\}_{k=1}^{\infty}$ of Riemann surfaces as an increasing sequence with respect to inclusion relation and can see that $\mathcal{F}(\Omega)(\Gamma) - \pi^{-1}(\bigcup_{i=1}^{q} \tilde{\alpha}_i))/G = \bigcup_{k=1}^{\infty} \mathcal{F}^{\delta(k)}(\Omega_{\delta(k)})/G$ is a compact Riemann surface of genus $p-q$ with $2q$ points removed by the definition of squeezing deformation. From Lemma 5.2 we see that $\Omega(G)/G$ is also a Riemann surface as stated above. Therefore, we see $\mathcal{F}(\Omega(\Gamma) - \pi^{-1}(\bigcup_{i=1}^{q} \alpha_i)) = \Omega(G)$. Thus we have our lemma.

Now we stand at the place to complete the proof of Theorem 5.1. Let $G$ and $G_*$ be extended Schottky groups obtained in Lemma 5.2 as limits of some sequences of points in $\{(F_t \circ \gamma_1 \circ F_t^{-1}, \cdots, F_t \circ \gamma_p \circ F_t^{-1})\}_{t \in [0,1)} \subset \Phi(\mathcal{S}_p)$. Our final task is to show that $G$ coincides with $G_*$. Let $\mathcal{F}$ (resp. $\mathcal{F}_*$) be the homeomorphism of $\Omega(\Gamma) - \pi^{-1}(\bigcup_{i=1}^{q} \tilde{\alpha}_i)$ onto $\Omega(G)$ (resp. $\Omega(G_*)$) obtained as stated above.

Then $G$ coincides with $\mathcal{F}\circ\Gamma\circ\mathcal{F}^{-1}$ as a group acting on $\Omega(G)$ and $G_*$ is identical with $\mathcal{F}_*\circ\Gamma\circ\mathcal{F}_*^{-1}$ as a group acting on $\Omega(G_*)$. Hence $G_*$ is equal to $\mathcal{F}_*\circ\mathcal{F}^{-1}\circ G\circ\mathcal{F}\circ\mathcal{F}_*^{-1}$ as a group acting on $\Omega(G_*)$. By calculating the Beltrami coeficient, we see that $B=\mathcal{F}\circ\mathcal{F}_*^{-1}$ is holomorphic and univalent on $\Omega(G_*)$. By Lemma 3.1, $B$ is a Möbius transformation.

Let $\{\beta_n\}_{n=1}^{\infty}$ be the set of Jordan curves in $\Omega(G_*)$ such that $\beta_n$ separates 0 from 1 and $\infty$ and such that the Euclidean diameter of $\beta_n$ tends to zero as $k\to\infty$. The existence of $\{\beta_n\}_{n=1}^{\infty}$ is obvious, since $G_*$ is quasiconformally equivalent to an extended Schottky group preserving the upper half plane. For any $n$ we can find an integer $k$ such that $\mathcal{F}_*^{\delta(k)}(\Omega_{\delta(k)})$ contains $\beta_n$ and we see that $B(\beta_n)=\mathcal{F}^{\delta(k)}\circ(\mathcal{F}_*^{\delta(k)})^{-1}(\beta_n)$ separates 0 from 1 and $\infty$. Since $B$ is a Möbius transformation, the Euclidean diameter of $B(\beta_n)$ tends to zero as $n\to\infty$. Let $z_n$ be a point on $\beta_n$ for every $n$. Then the above fact shows

$$|B(0)|=\lim_{n\to\infty}|B(z_n)|\leqq\lim_{n=\infty}\sup_{w_n\in\beta_n}|B(w_n)|=0\,,$$

which means $B(0)=0$. Similarly we observe that $B$ fixes 1 and $\infty$, and hence $B$ is the identity. Now we have $G=G_*$ and complete the proof of Theorem 5.1.

## § 6. Squeezing deformations with respect to $B$-cycles.

Let $\Gamma=\langle\gamma_1,\cdots,\gamma_p\rangle$ be a marked Schottky group of genus $p$ convented in § 4 and let $\{\alpha_i\}_{i=1}^{q}$ be a homotopically independent set of loops on $S=\Omega(\Gamma)/\Gamma$ such that, for every $i$, $\alpha_i$ is freely homotopic to the projection of one of $C_1, C_1', \cdots, C_p$ and $C_p'$. We may assume $\pi(C_1)=\alpha_1$. Since the length $l_{F_t(\Omega(\Gamma))}(F_t(C_1))$ of $F_t(C_1)$ tends to zero, the Euclidean area of the bounded domain surrounded by $F_t(C_1)$ tends to zero as $t\to 1$. The similar fact holds for $F_t(C_t')$. By Lemma 4.1, we can see that the radius of the isometric circle of $F_t\circ\gamma_1\circ F_t^{-1}$ tends to zero as $t\to 1$. By the same manner as in the proof of Theorem 5.1 we have the following.

THEOREM 6.1. *If $S$ is squeezed with respect to $\{\alpha_i\}_{i=1}^{q}$, then the point $(\Psi(F_t\circ\gamma_1\circ F_t^{-1}),\cdots,\Psi(F_t\circ\gamma_p\circ F_t^{-1}))$ in $\Phi(\mathcal{S}_p)$ approaches to a node $(x_1,\cdots,x_p)\in\partial\mathcal{S}_p$ and $\Omega(H)/H-\pi_H(\xi(x_1),\xi'(x_1),\cdots,\xi(x_p),\xi'(x_p))$ is homeomorphic to $S-\bigcup_{i=1}^{q}\alpha_i$, where $H$ is the Schottky group generated by all Möbius transformations among $\Psi^{-1}(x_1),\cdots,\Psi^{-1}(x_p)$ and $\pi_H$ is the natural projection of $\Omega(H)$ to $\Omega(H)/H$, respectively.*

## § 7. Remarks.

Theorem 5.1 asserts that $\Omega(G)/G$ is homeomorphic to $\Omega(\Gamma)/\Gamma-\{\alpha_i\}_{i=1}^{q}$. It seems very natural. Theorem 6.1 seems also natural in the same sense. In general, let $\{\alpha_i\}_{i=1}^q$ any homotopically independent set of loops on $S=\Omega(\Gamma)/\Gamma$. If $S$ is squeezed with respect to $\{\alpha_i\}_{i=1}^q$, then we have a path $\{(F_t\circ\gamma_1\circ F_t^{-1}, \cdots, F_t\circ\gamma_p\circ F_t^{-1})_{t\in[0,1)}$ in $\Phi(\mathcal{S}_p)$ and there exists a sequence $\{t_m\}_{m=1}^\infty$ such that $(F_{t_m}\circ\gamma_1\circ F_{t_m}^{-1}, \cdots, F_{t_m}\circ\gamma_p\circ F_{t_m}^{-1})$ approaches to a boundary point $Q=(x_1, \cdots, x_p)$ of $\mathcal{S}_p$. Now, there is a question: What does $Q$ represent? From the view point of Theorem 5.1 and Theorem 6.1, one might expect that $Q$ represents a union of Riemann surfaces homeomorphic to $\Omega(\Gamma)/\Gamma-\{\alpha_i\}_{i=1}^q$. But, as the following example shows, there is a case where $Q$ does not represent a union of Riemann surfaces homeomorphic to $\Omega(\Gamma)/\Gamma-\bigcup_{i=1}^q \alpha_i$.

Let $\Gamma=\langle\gamma_1, \cdots, \gamma_p\rangle$ be the marked Schottky group convented in §4. Let $C$ be a simple loop in $\Omega(\Gamma)$ which separates loops $C_1$, $C_1'$ and the point 0 from loops $C_2$, $C_2'$, $\cdots$, $C_p$, $C_p'$ and the points 1, $\infty$. If $\Omega(\Gamma)/\Gamma$ is squeezed with respect to $\pi(C)$, then we have a path

$$\{\Psi(F_t\circ\gamma_1\circ F_t^{-1}), \cdots, \Psi(F_t\circ\gamma_p\circ F_t^{-1})\}_{t\in[0,1)} \quad \text{in} \quad \Phi(\mathcal{S}_p).$$

There exists a sequence $\{t_m\}_{m=1}^\infty$ in $[0, 1)$ such that $(F_{t_m}\circ\gamma_1\circ F_{t_m}^{-1}, \cdots, F_{t_m}\circ\gamma_p\circ F_{t_m}^{-1})$ approaches to a boundary point $Q\in\partial\mathcal{S}_p$. Since $F_{t_m}(C)$ shrinks to the point 0, we see that $F_{t_m}\circ\gamma_1\circ F_{t_m}^{-1}$ cannot tend to a Möbius transformation by Lemma 4.1 and that both the attracting and the repelling fixed points of $F_{t_m}\circ\gamma_1\circ F_{t_m}^{-1}$ tend to the point 0 as $m\to\infty$. Therofore, $Q$ is neither a cusp nor a node. We may say that $Q$ does not represent a union of Riemann surfaces homeomorphic to $\Omega(\Gamma)/\Gamma-\pi(C)$ and represents only one part of $\Omega(\Gamma)/\Gamma-\pi(C)$.

## References

[1] W. Abikoff, On boundaries of Teichmüller spaces and on Kleinian groups III, Acta Math.

[2] L. Ahlfors, Lectures on quasiconformal mappings, Van Nostrand, New York, 1966.

[3] L. Ahlfors, Conformal invariants; topics in geometric function theory, McGrow-Hill, New York, 1973.

[4] L. Ahlfors and L. Bers, Riemann's mapping theorem for variable metrics, Ann. of Math., **72** (1960), 385-404.

[5] A. Beardon and B. Maskit, Limit points of Kleinian groups and finite-sided fundamental polyhedra, Acta Math., **132** (1974), 1-12.

[6] L. Bers, On boundaries of Teichmüller spaces and on Kleinian groups I, Ann. of Math., **91** (1970), 570-600.

[7] L. Bers, Spaces of Kleinian groups, in Maryland Conferences in Several Complex Variables I, Maryland, 1970, Lecture Notes in Math., **155**, Springer-Verlag, New York.

[8] L. Bers, Spaces of degenerating Riemann surfaces, in Discontinuous Groups and Riemann Surfaces, Ann. of Math. Studies, **79**, Princeton Univ. Press, Princeton, 1971.

[9] L, Bers, Automorphic forms for Schottky groups, Advances in Math., **16** (1975), 332-361.

[10] V. Chuckrow, On Schottky groups with applications to Kleinian groups, Ann. of Math., **88** (1968), 47-61.
[11] V. Chuckrow, Subgroups and automorphisms of extended Schottky type groups, Trans. Amer. Math. Soc., **150** (1970), 121-129.
[12] L. Ford, Automorphic functions, 2nd., Chelsea, New York, 1951.
[13] T. Jørgensen, On discrete groups of Möbius transformations, Amer. J. Math., **98** (1976), 739-749.
[14] L. Keen, Collars on Riemann surfaces, in Discontinuous Groups and Riemann Surfaces, Ann. of Math. Studies, **79**, Princeton Univ. Press, Princeton, 1971.
[15] I. Kra, On spaces of Kleinian groups, Comm. Math. Helv., **47** (1972), 53-69.
[16] A. Marden, The geometry of finitely generated Kleinian groups, Ann. of Math., **99** (1974), 383-462.
[17] A. Marden, Schottky groups and circles, in Contributions to Analysis: A collected paper dedicated to Lipman Bers, Academic Press, New York, 1974.
[18] B. Maskit, A characterization of Schottky groups, J. Analyse Math., **19** (1967), 227-230.
[19] B. Maskit, On Klein's Combination Theorem III, in Advances in the Theory of Riemann Surfaces, Ann. of Math. Studies, **66**, Princeton Univ. Press, Princeton, 1971.
[20] B. Maskit, Good and bad Kleinian groups, in a Crash Course on a Kleinian Groups, Lecture Notes in Math., **400**, Springer-Verlag, New York.
[21] B. Maskit, Self-maps on Kleinian groups, Amer. J. Math., **93** (1971), 840-856.
[22] H. Sato, Cusps on boundaries of Teichmüller spaces, Nagoya Math. J., **60** (1976), 81-92.
[23] H. Sato, On boundaries of Schottky spaces, Nagoya Math. J., **62** (1976), 97-124.

Hiro-o YAMAMOTO
Mathematical Institute
Tohoku University
Sendai, Japan