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On the least positive eigenvalue of the Laplacian for compact group manifolds

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§ 1. Introduction.

Let M be an n-dimensional compact connected manifold. For every Riemannian metric g on M, let $-\Delta_g$ be the Laplace-Beltrami operator acting on smooth functions on M. Let $\lambda_1(g)$ be the least positive eigenvalue of Δ_g. M. Berger ([1] p. 138) posed the problem: Does there exist a positive constant $k(M)$ such that

$$\lambda_1(g)\,\mathrm{vol}\,(M, g)^{2/n} \leqq k(M)\,,$$

for every Riemannian metric g on M? J. Hersch [5] showed that if M is diffeomorphic with the 2-dimensional sphere S^2, then for every Riemannian metric g on M,

$$\lambda_1(g)\,\mathrm{area}\,(S^2, g) \leqq 8\pi\,.$$

The equality holds if and only if (S^2, g) is the canonical sphere.

In the present paper, let M be a compact connected. Lie group. Let us consider the problem: Does there exist a positive constant $k(M)$ such that

$$\lambda_1(g)\,\mathrm{vol}\,(M, g)^{2/n} \leqq k(M)$$

for every *left invariant* Riemannian metric g on M? For this problem we claim (cf. theorem 4) the following: *The only compact Lie group M which has a positive answer for this problem is a torus* $\boldsymbol{T}^n$, that is, *if the compact connected Lie group M has a non-trivial commutator subgroup, then there exists a family of left invariant Riemannian metrics* $g(t)\,(0<t<\infty)$ *on M such that* $\lim_{t\to\infty}\lambda_1(g(t))=\infty$, $\lim_{t\to 0}\lambda_1(g(t))=0$ *and* $\mathrm{vol}\,(M, g(t))$ *is constant in t*. In particular, since $SU(2)$ (resp. $SO(3)$) is diffeomorphic with S^3 (resp. $P^3(\boldsymbol{R})$), the above shows that M. Berger's conjecture is negative for S^3 and $P^3(\boldsymbol{R})$. It is known (cf. [1]) that, for a torus $\boldsymbol{T}^n$, there exists a positive constant $k(\boldsymbol{T}^n)$ such that $\lambda_1(g)\,\mathrm{vol}\,(\boldsymbol{T}^n, g)^{n/2} \leqq k(\boldsymbol{T}^n)$ for every left invariant Riemannian metric g on $\boldsymbol{T}^n$.

In § 2, we shall express the Laplace-Beltrami operator on a connected Lie group in term of the left invariant vector fields. In § 3, we shall give an estimation for $\lambda_1(g)$ for every left invariant Riemannian metric on a compact connected Lie group. In § 4, we shall prove theorem 4 and in § 5, we shall calculate $\lambda_1(g)$ for some Riemannian metrics on $SU(2)$ and $SO(3)$.

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§ 2. The Laplace-Beltrami operator for a left invariant Riemannian metric.

2.1. Let M be an n-dimensional connected Lie group, and let $\mathfrak{m}$ be the Lie algebra of all left invariant vector fields on M. Let $(,)$ be an inner product on $\mathfrak{m}$. Let g be a Riemannian metric on M defined by

$$g_m(X_m, Y_m)=(X, Y), \qquad X, Y\in\mathfrak{m} \tag{2.1}$$

where X_m, Y_m are the tangent vectors at $m\in M$ corresponding to X, Y. Let $-\Delta_g$ be the Laplace-Beltrami operator on M, that is

$$\Delta_g f=-\mathrm{Trace}_g\,(\mathrm{Hess}\,(f)) \tag{2.2}$$

for a smooth function f on M. Here let $\mathrm{Hess}\,(f)=\nabla(df)$ be the Hessian of f, where ∇ is the covariant derivation with respect to the Riemannian metric g on M. The Hessian $h=\mathrm{Hess}\,(f)$ is a covariant symmetric 2-tensor on M, and so let $\mathrm{Trace}_g\,(h)=\sum_{i,j=1}^{n} g^{ij}h_{ij}$, where (g^{ij}) is the inverse matrix of (g_{ij}) and (g_{ij}), (h_{ij}) are the components of g, h with respect to the local coordinate $(x_1, \cdots, x_n)$. Then it is known (cf. [1] p. 135) that

$$\Delta_g f=-\sum_{i,j=1}^{n} g^{ij}\left(\frac{\partial^2 f}{\partial x_i\partial x_j}-\sum_{k=1}^{n}\Gamma_{ij}^{k}\frac{\partial f}{\partial x_k}\right), \tag{2.3}$$

where Γ_{ij}^k is the Christoffel symbol of the Riemannian connection for g. Moreover we have the following theorem.

THEOREM 1. *Let M be a connected Lie group and $\mathfrak{m}$ be the Lie algebra of M. Then we have, for a smooth function f,*

$$\Delta_g f=-\sum_{i=1}^{n} Y_i^2(f)+\sum_{i=1}^{n}\mathrm{Trace}\,(\mathrm{ad}\,(Y_i))Y_i(f), \tag{2.4}$$

where $\{Y_i\}_{i=1}^n$ is an orthonormal basis of $\mathfrak{m}$ with respect to $(,)$ and Trace (ad(X)) *is the trace of an endomorphism* ad(X) *of $\mathfrak{m}$. For $X\in\mathfrak{m}$,*

$$Xf(m)=\left[\frac{d}{dt}f(m\exp\,(tX))\right]_{t=0}.$$

PROOF. Since the left translations L_x, $x\in M$, are isometries with respect to g, the operator Δ_g satisfies $\Delta_g\circ L_x(f)=L_x\circ\Delta_g(f)$ for a smooth function f

(cf. [4] p. 387), and so we may prove (2.4) at the identity e of M. Take the local coordinate $(x_1, \cdots, x_n)$ around e defined by the mapping $\exp(\sum_{i=1}^{n} x_i Y_i) \mapsto (x_1, \cdots, x_n)$. For $x=\exp(X)$, $X\in\mathfrak{m}$, we have (cf. [4] p. 95)

$$\left(\frac{\partial}{\partial x_i}\right)_x = \exp_{*X}(Y_i) = L_{x_{*_e}} \circ \sum_{n=0}^{\infty}(-\mathrm{ad}(X))^n(Y_i)/(n+1)\,!$$

where $\exp_{*X}$(resp. $L_{x_{*_e}}$) is the differential of the exponential mapping (resp. the translation L_x) at $X\in\mathfrak{m}$ (resp. the identity e). Then

$$g_{ij}(x) = \Big(\sum_{n=0}^{\infty}(-\mathrm{ad}(X))^n(Y_i)/(n+1)\,!\,,\ \sum_{n=0}^{\infty}(-\mathrm{ad}(X))^n(Y_j)/(n+1)\,!\Big)\,. \tag{2.5}$$

Hence we have

$$g_{ij}(e)=\delta_{ij}\,, \qquad Y_k g_{ij}(e) = -(c_{ki}^j + c_{kj}^i)/2\,, \tag{2.6}$$

where we put $[Y_i, Y_j] = \sum_{k=1}^{n} c_{ij}^k Y_k$ $(1\leqq i,\ j\leqq n)$. In fact,

$$\begin{aligned} Y_k g_{ij}(e) &= \left[\frac{d}{dt} g_{ij}(\exp(tY_k))\right]_{t=0} \\ &= \left[\frac{d}{dt}\Big(\sum_{n=0}^{\infty}(-t\,\mathrm{ad}(Y_k))^n(Y_i)/(n+1)\,!\,,\ \sum_{n=0}^{\infty}(-t\,\mathrm{ad}(Y_k))^n(Y_j)/(n+1)\,!\Big)\right]_{t=0} \\ &= (([-Y_k, Y_i], Y_j) + (Y_i, [-Y_k, Y_j]))/2\,. \end{aligned}$$

Therefore we have

$$\Gamma_{ij}^k(e) = (c_{ki}^j + c_{kj}^i)/2\,, \tag{2.7}$$

in paticular $\Gamma_{ii}^k(e) = c_{ki}^i$. For we have

$$\begin{aligned} \Gamma_{ij}^k(e) &= \left(\frac{\partial g_{kj}}{\partial x_i}(e) + \frac{\partial g_{ik}}{\partial x_j}(e) - \frac{\partial g_{ij}}{\partial x_k}(e)\right)\Big/2 \\ &= (Y_i g_{kj}(e) + Y_j g_{ik}(e) - Y_k g_{ij}(e))/2 \\ &= (c_{ki}^j + c_{kj}^i)/2 \end{aligned}$$

by (2.6) and $c_{ki}^j + c_{ik}^j = 0$. Since $\frac{\partial^2 f}{\partial x_i^2}(e) = \left[\frac{d^2}{dtds} f(\exp((s+t)Y_i))\right]_{s=t=0} = Y_i^2 f(e)$, we obtain

$$\begin{aligned} \Delta_g f(e) &= -\sum_{i=1}^{n} Y_i^2 f(e) + \sum_{i,k=1}^{n} c_{ki}^i Y_k f(e) \\ &= -\sum_{i=1}^{n} Y_i^2 f(e) + \sum_{k=1}^{n} \mathrm{Trace}\,(\mathrm{ad}(Y_k)) Y_k f(e)\,. \end{aligned}$$

Q. E. D.

The following is immediate from the above.

COROLLARY 1. *Let M be a connected Lie group and $\mathfrak{m}$ be the Lie algebra of M. We assume $\mathfrak{m}$ is a unimodular algebra, that is* Trace(ad(X))=0 *for every $X\in\mathfrak{m}$. Then we have, for a smooth function f,*

$$\Delta_g f=-\sum_{i=1}^{n} Y_i^2(f), \tag{2.8}$$

where $\{Y_i\}_{i=1}^n$ is an orthonormal basis of $\mathfrak{m}$ with respect to $(\ ,\ )$.

2.2. Let M be a connected Lie group and let $\mathfrak{m}$ be the Lie algebra of M. Let $\mathcal{M}$ be the set of all inner product on $\mathfrak{m}$. Fix an element $(\ ,\ )_0$ in $\mathcal{M}$. Let g_0 be the corresponding left invariant Riemannian metric on M. Let $\mathcal{M}_0$ be the set of all inner products on $\mathfrak{m}$ which induce the left invariant Riemannian metrics with the same volume element Ω as g_0. To avoid mere changes of scale, we consider only metrics induced from elements in $\mathcal{M}_0$.

Let $\{X_i\}_{i=1}^n$ be an orthonormal basis for $(\ ,\ )_0$. For $A\in SL(n, \boldsymbol{R})$, let $Y_j=\sum_{k=1}^{n} A_{kj}X_k$ $(1\leqq j\leqq n)$. Then there exists a unique inner product $(\ ,\ )\in\mathcal{M}_0$ for which $\{Y_j\}_{j=1}^n$ is an orthonormal basis. For the left invariant Riemannian metric g on M induced from $(\ ,\ )$, we notice that

$$\Delta_g f=-\sum_{k,m=1}^{n}(A\,{}^tA)_{km}X_kX_mf+\sum_{k,m=1}^{n}(A\,{}^tA)_{km}\,\mathrm{Trace}\,(\mathrm{ad}(X_k))X_m(f), \tag{2.9}$$

in particular, if $\mathfrak{m}$ is a unimodular Lie algebra,

$$\Delta_g f=-\sum_{k,m=1}^{n}(A\,{}^tA)_{km}X_kX_mf \tag{2.9$'$}$$

for every smooth function f on M.

REMARK. Let M be a compact Lie group. For the M-bi-invariant Riemannian metric g_0, the above corollary 1 is well-known (cf. [8]).

§ 3. The case of a compact Lie group I.

3.1. The notations in § 2 are preserved. In this part, we prepare, for the Peter-Weyl theorem (cf. [7], [8]), some notations. Let M be a compact connected Lie group.

We take an Ad(M)-invariant inner product on $\mathfrak{m}$ as the fixed one $(\ ,\ )_0$ on $\mathfrak{m}$ in 2.2.

Let T be a maximal torus in M and M_1 be the connected component of the commutator subgroup of M. Put l=dim.(T), p=rank(M_1) and n=dim.(M). Let Δ be the root system of the complexification $\mathfrak{m}^c$ of $\mathfrak{m}$ with respect to the Lie algebra $\mathfrak{t}$ of T, that is the set of non-zero elements α of the dual space $\mathfrak{t}^*$ of $\mathfrak{t}$ such $\mathfrak{m}_\alpha^c=\{E\in\mathfrak{m}^c\,;\ [H, E]=\sqrt{-1}\alpha(H)E$ for any $H\in\mathfrak{t}\}$ is not zero. We introduce a lexicographic order $>$ of $\mathfrak{t}^*$ and fix it once for all. Let Δ^+ be the all positive roots. Let $\Pi=\{\alpha_1, \cdots, \alpha_p\}$ ($p=\text{rank}(M_1)$) be the fundamental system of Δ with respect to this order $>$. For $\lambda\in\mathfrak{t}^*$, define an element H_λ in $\mathfrak{t}$ by $(H_\lambda, H)_0=\lambda(H)$ for each $H\in\mathfrak{t}$. We define an inner product $(\ ,\ )_0$ in $\mathfrak{t}^*$ by $(\lambda, \lambda')_0=(H_\lambda, H_{\lambda'})_0$ for λ, λ' in $\mathfrak{t}^*$. Let $\Gamma=\{H\in\mathfrak{t}; \exp(H)=e\}$. Let I be the set of all M-integral forms on $\mathfrak{t}$:

$$I=\{\lambda\in\mathfrak{t}^*;\ \lambda(H)\in 2\pi\boldsymbol{Z}\quad\text{for all } H\in\Gamma\}.$$

Put

$$D=\{\lambda\in I;\ (\lambda, \alpha_i)_0\geqq 0\ (1\leqq i\leqq p)\}.$$

In this paper, a finite dimensional unitary representation of M is called a *representation* of M. For a representation (π, V) of M, we use the same letter for its differential representation of $\mathfrak{m}$ and the extension to $\mathfrak{m}^C$, that is

$$\pi(X)v=\left[\frac{d}{dt}\pi(\exp(tX))v\right]_{t=0},\ \pi(X+\sqrt{-1}Y)v=\pi(X)v+\sqrt{-1}\pi(Y)v$$

for $X, Y\in\mathfrak{m}$ and $v\in V$. For $\mu\in\mathfrak{t}^*$, put $V^\mu=\{v\in V; \pi(H)v=\sqrt{-1}\mu(H)v$ for each $H\in\mathfrak{t}\}$. If $V^\mu\neq(0)$, μ (resp. V^μ) is called a *weight* (resp. a *weight space*) for the representation (π, V) of M. Then $V=\sum_{\mu\in I}V^\mu$ (finite direct sum). This decomposition is orthogonal with respect to the M-invariant hermitian inner product $((\ ,\ ))$ on V. Notice that

$$\pi(H)\pi(E_{\pm\alpha})v=\sqrt{-1}(\mu(H)\pm\alpha(H))\pi(E_{\pm\alpha})v,\ v\in V^\mu. \tag{3.1}$$

The set I coincides with the set of all the weights of the representations of M. The maximal element among the weights of the representation (π, V) in the order $>$ in $\mathfrak{t}^*$ is called the highest weight of (π, V). The set D coincides with the set of all highest weights of the representations of M. Since an irreducible representation of M is uniquely determined, up to equivalence, by its highest weight, there exists a bijection from D onto the set of equivalence classes of irreducible representations of M. We choose, once for all, an irreducible unitary representation (π^λ, V_λ) with the highest weight λ for each $\lambda\in D$. Put $d_\lambda=\dim. V_\lambda$.

3.2. The Peter-Weyl theorem. Let $C^\infty(M)$ be the set of all $\boldsymbol{C}$-valued infinitely differentiable functions on M. We define a hermitian inner product $((\ ,\ ))$ on $C^\infty(M)$ by

$$((f_1, f_2))=\text{vol}(M)^{-1}\int_M f_1(x)\overline{f_2(x)}\Omega(x) \tag{3.2}$$

for $f_1, f_2\in C^\infty(M)$. Here vol(M) is the volume of M for the volume element Ω.

The $\boldsymbol{C}$-linear space $C^\infty(M)$ is an M-module defined by $r_x f(y)=f(yx)$, $x, y\in M$, $f\in C^\infty(M)$. For $X\in\mathfrak{m}$, let $Xf(x)=\left[\frac{d}{dt}f(x\exp(tX))\right]_{t=0}$. For an irreducible representation (π^λ, V_λ), $\lambda \in D$, let θ^λ be the M-submodule of $C^\infty(M)$ spanned by M-submodules of $C^\infty(M)$ isomorphic with V_λ as an M-module. Let $(\pi^\lambda \otimes \mathbf{1}, V_\lambda \otimes V_\lambda^*)$ be the representation of M defined by $\pi^\lambda \otimes \mathbf{1}(x)(v \otimes \xi) = \pi^\lambda(x)v \otimes \xi$ for $x \in M$, $v \in V_\lambda$ and $\xi \in V_\lambda^*$ where V_λ^* is the dual space of V_λ. Define a homomorphism Φ_λ of $V_\lambda \otimes V_\lambda^*$ into $C^\infty(M)$ by

$$\Phi_\lambda(v \otimes \xi)(x) = \xi(\pi^\lambda(x)v), \quad x \in M, \quad v \in V_\lambda \quad \text{and} \quad \xi \in V_\lambda^* .$$

Then Φ_λ is an M-homomorphism, that is

$$r_x \circ \Phi_\lambda = \Phi_\lambda \circ (\pi^\lambda \otimes \mathbf{1})(x)$$

for all $x \in M$. The Peter-Weyl theorem can be stated as follows:

THEOREM (Peter-Weyl). 1) *For any $\lambda \in D$, the homomorphism Φ_λ defines an isomorphism of $V_\lambda \otimes V_\lambda^*$ onto θ^λ. Let $\{v_i\}_{i=1}^{d_\lambda}$ be an orthonormal basis of V_λ with respect to the M-invariant hermitian inner product $((\ ,\ ))$. Put $\pi_{ij}^\lambda(x) = ((\pi^\lambda(x)v_j, v_i))$ $(1 \leq i, j \leq d_\lambda)$. Then $\{\sqrt{d_\lambda}\pi_{ij}^\lambda\}_{i,j=1}^{d_\lambda}$ is an orthonormal basis of θ^λ with respect to* $((\ ,\ ))$ (3.2).

2) *If $\lambda, \lambda' \in D$, $\lambda \neq \lambda'$, then θ^λ and $\theta^{\lambda'}$ are orthogonal with respect to $((\ ,\ ))$. We have the following decomposition: $C^\infty(M) = \sum_{\lambda \in D} \theta^\lambda$, that is, for $f \in C^\infty(M)$, f can be expanded by*

$$f = \sum_{\lambda \in D} d_\lambda \sum_{i,j=1}^{d_\lambda} ((f, \pi_{ij}^\lambda))\pi_{ij}^\lambda$$

in the sense of the uniform convergence on M or the L^2-convergence with respect to $((\ ,\ ))$.

3.3. In this part, for each $(\ ,\ ) \in \mathcal{M}_0$, we calculate $\lambda_1(g)$ for the Riemannian metric g induced from $(\ ,\ )$, making use of the Peter-Weyl theorem.

REMARK. Let $C_{\boldsymbol{R}}^\infty(M)$ be the set of all $\boldsymbol{R}$-valued infinitely differentiable functions on M. We notice that the least positive eigenvalue $\lambda_1(g)$ of Δ_g on $C_{\boldsymbol{R}}^\infty(M)$ coincides the least positive eigenvalue of the extension of Δ_g on $C^\infty(M)$ (cf. [3] p. 141).

THEOREM 2. *The above notations are preserved. Let $(\ ,\ )$ be any element in $\mathcal{M}_0$. Then the least positive eigenvalue $\lambda_1(g)$ of Δ_g for the Riemannian metric g induced from $(\ ,\ )$ is given as follows:*

$$\lambda_1(g) = \min_{\lambda \in D - (0)} . \min_{v \in V_\lambda, ((v,v))=1} . ((\pi^\lambda(C_g)v, v)),$$

where $\pi^\lambda(C_g) = -\sum_{i=1}^n \pi^\lambda(Y_i)^2$ for an orthonormal basis $\{Y_i\}_{i=1}^n$ of $\mathfrak{m}$ with respect to $(\ ,\ )$.

LEMMA 1. *The assumptions in Theorem 2 are preserved. For an irreducible representation (π^λ, V_λ), $\lambda \in D$, the operator Δ_g satisfies*

(3.3) $$\Delta_g \circ \Phi_\lambda(v\otimes\xi)=\Phi_\lambda \circ (\pi^\lambda(C_g)v\otimes\xi)$$

for all $v\in V_\lambda$, $\xi\in V_\lambda^*$.

PROOF. For $X\in\mathfrak{m}$, $v\in V_\lambda$ and $\xi\in V_\lambda^*$, we have

$$X(\Phi_\lambda(v\otimes\xi))(x)=\left[\frac{d}{dt}\Phi_\lambda(v\otimes\xi)(x\exp(tX))\right]_{t=0}$$

$$=\left[\frac{d}{dt}\xi(\pi^\lambda(x\exp(tX)v))\right]_{t=0}$$

$$=\xi(\pi^\lambda(x)\pi^\lambda(X)v)=\Phi_\lambda(\pi^\lambda(X)v\otimes\xi)(x).$$

Thus, due to corollary 1 in § 2, lemma 1 is immediate. Q. E. D.

From lemma 1, the operator Δ_g preserves θ^λ for each $\lambda\in D$. So we investigate the eigenfunctions of Δ_g in θ^λ. The M-invariance of $((\ ,\ ))$ implies $((\pi^\lambda(X)u, v))=-((u, \pi^\lambda(X)v))$ for $X\in\mathfrak{m}$, and we have

$$((\pi^\lambda(C_g)u, v))=((u, \pi^\lambda(C_g)v))$$

for all $u, v\in V_\lambda$. Then for each $\lambda\in D-(0)$, there exist positive numbers λ_i ($1\leqq i\leqq d_\lambda$) and an orthonormal basis $\{v_i\}_{i=1}^{d_\lambda}$ of V_λ such that $\pi^\lambda(C_g)v_i=\lambda_i v_i$. We choose the dual basis $\{\xi_i\}_{i=1}^{d_\lambda}$ of V_λ^* for $\{v_i\}_{i=1}^{d_\lambda}$. Then

$$\Phi_\lambda(v_j\otimes\xi_i)(x)=\xi_i(\pi^\lambda(x)v_j)=((\pi^\lambda(x)v_j, v_i))=\pi^\lambda_{ij}(x)$$

for each $x\in M$ and

$$\Delta_g(\pi^\lambda_{ij})=\Delta_g\Phi_\lambda(v_j\otimes\xi_i)=\Phi_\lambda(\pi^\lambda(C_g)v_j\otimes\xi_i)=\lambda_j\Phi_\lambda(v_j\otimes\xi_i)=\lambda_j\pi^\lambda_{ij}$$

by lemma 1. Therefore the functions $\{\sqrt{d_\lambda}\pi^\lambda_{ij}\}_{i,j=1}^{d_\lambda}$ form an orthonormal basis of θ^λ with respect to $((\ ,\ ))$ (the Peter-Weyl theorem) and also eigenfunctions of Δ_g for the eigenvalues λ_i. Then the least positive eigenvalue of Δ_g on θ^λ coincides with $\min.\{\lambda_1, \cdots, \lambda_{d_\lambda}\}=\min\limits_{v\in V_\lambda, ((v,v))=1}((\pi^\lambda(C_g)v, v))$. Thus we obtain

LEMMA 2. *For* $\lambda\in D-(0)$, *the least positive eigenvalue of* Δ_g *on* θ^λ *is given by*

$$\min_{v\in V_\lambda, ((v,v))=1}((\pi^\lambda(C_g)v, v)).$$

Its multiplicity coincides d_λ *times the multiplicity of the least positive eigenvalue of* $\pi^\lambda(C_g)$ *on* V_λ.

Lemma 2 and the Peter-Weyl theorem imply theorem 2.

3.4. Making use of theorem 2, we give an estimation of $\lambda_1(g)$.

We choose an orthonormal basis of $\mathfrak{m}$ with respect to the inner product $(\ ,\ )_0$ as follows: For $\alpha\in\Delta$, let E_α be a root vector such that $\tau E_\alpha=E_{-\alpha}$, $(E_\alpha, E_{-\alpha})_0=1$ and $[E_\alpha, E_{-\alpha}]=\sqrt{-1}H_\alpha$. Here the mapping τ is the conjugation of $\mathfrak{m}^C$ with respect to $\mathfrak{m}$. For $\alpha\in\Delta^+$, put $U_\alpha=E_\alpha+E_{-\alpha}$, $V_\alpha=\sqrt{-1}(E_\alpha-E_{-\alpha})$ belonging to $\mathfrak{m}$. Let $\{H_i\}_{i=1}^{l}$ be an orthonormal basis of $\mathfrak{t}$ with respect to $(\ ,\ )_0$. Then

$$\left\{\frac{1}{\sqrt{2}}U_\alpha,\ \frac{1}{\sqrt{2}}V_\alpha,\ H_i\,;\ \alpha\in\Delta^+,\ 1\leqq i\leqq l\right\} \tag{3.4}$$

is an orthonormal basis of $\mathfrak{m}$ with respect to $(\ ,\ )_0$.

For $A\in SL(n, \boldsymbol{R})$, $(n=\dim. M)$, let g be the left invariant Riemannian metric on M induced from the inner product for A as in 2.2. Then, due to (2.9′), we can express $\pi^\lambda(C_g)$ as follows:

$$\begin{aligned}\pi^\lambda(C_g)=&-\frac{1}{2}\sum_{\alpha,\beta\in\Delta^+}(A\,{}^tA)_{\alpha\beta}(\pi^\lambda(E_\alpha)+\pi^\lambda(E_{-\alpha}))(\pi^\lambda(E_\beta)+\pi^\lambda(E_{-\beta}))\\ &+\frac{1}{2}\sum_{\alpha,\beta\in\Delta^+}(A\,{}^tA)_{\bar\alpha\bar\beta}(\pi^\lambda(E_\alpha)-\pi^\lambda(E_{-\alpha}))(\pi^\lambda(E_\beta)-\pi^\lambda(E_{-\beta}))\\ &-\frac{\sqrt{-1}}{2}\sum_{\alpha,\beta\in\Delta^+}(A\,{}^tA)_{\alpha\bar\beta}(\pi^\lambda(E_\alpha)+\pi^\lambda(E_{-\alpha}))(\pi^\lambda(E_\beta)-\pi^\lambda(E_{-\beta}))\\ &-\frac{\sqrt{-1}}{2}\sum_{\alpha,\beta\in\Delta^+}(A\,{}^tA)_{\bar\alpha\beta}(\pi^\lambda(E_\alpha)-\pi^\lambda(E_{-\alpha}))(\pi^\lambda(E_\beta)+\pi^\lambda(E_{-\beta}))\\ &-\frac{1}{\sqrt{2}}\sum_{\alpha\in\Delta^+,i}(A\,{}^tA)_{\alpha i}((\pi^\lambda(E_\alpha)+\pi^\lambda(E_{-\alpha}))\pi^\lambda(H_i)\\ &\qquad\qquad+\pi^\lambda(H_i)(\pi^\lambda(E_\alpha)+\pi^\lambda(E_{-\alpha})))\\ &-\frac{\sqrt{-1}}{\sqrt{2}}\sum_{\alpha\in\Delta^+,i}(A\,{}^tA)_{\bar\alpha i}((\pi^\lambda(E_\alpha)-\pi^\lambda(E_{-\alpha}))\pi^\lambda(H_i)\\ &\qquad\qquad+\pi^\lambda(H_i)(\pi^\lambda(E_\alpha)-\pi^\lambda(E_{-\alpha})))\\ &-\sum_{i,j=1}^{l}(A\,{}^tA)_{ij}\pi^\lambda(H_i)\pi^\lambda(H_j)\,.\end{aligned} \tag{3.5}$$

Now let $v_\lambda\in V_\lambda$ be the highest weight vector with $((v_\lambda, v_\lambda))=1$. Then, due to (3.1), we have $\pi^\lambda(E_\alpha)v_\lambda=0$ $(\alpha\in\Delta^+)$ and $\pi^\lambda(E_\alpha)\pi^\lambda(E_{-\alpha})v_\lambda=\pi^\lambda([E_\alpha, E_{-\alpha}])v_\lambda=-\lambda(H_\alpha)v_\lambda$. Thus we have

$$\begin{aligned}((\pi^\lambda(C_g)v_\lambda, v_\lambda))=&\frac{1}{2}\sum_{\alpha\in\Delta^+}((A\,{}^tA)_{\alpha\alpha}+(A\,{}^tA)_{\bar\alpha\bar\alpha})\lambda(H_\alpha)\\ &+\sum_{i,j=1}^{l}(A\,{}^tA)_{ij}\lambda(H_i)\lambda(H_j)\,.\end{aligned}$$

Therefore, together with theorem 2, the least positive eigenvalue $\lambda_1(g)$ of Δ_g on $C^\infty(M)$ is estimated as follows:

$$\lambda_1(g)\leqq\min_{\lambda\in D-(0)}\left[\frac{1}{2}\sum_{\alpha\in\Delta^+}((A\,{}^tA)_{\alpha\alpha}+(A\,{}^tA)_{\bar\alpha\bar\alpha})\lambda(H_\alpha)\right. \tag{3.6}$$

$$+\sum_{i,j=1}^{l}(A\,{}^tA)_{ij}\lambda(H_i)\lambda(H_j)\Big].$$

Since $((A\,{}^tA)_{ij})_{1\leq i,j\leq l}$ is a symmetric positive definite matrix of degree l,

$$\sum_{i,j=1}^{l}(A\,{}^tA)_{ij}\lambda(H_i)\lambda(H_j)\leqq\Big(\sum_{i=1}^{l}(A\,{}^tA)_{ii}\Big)\Big(\sum_{i=1}^{l}\lambda(H_i)^2\Big)$$

$$=\Big(\sum_{i=1}^{l}(A\,{}^tA)_{ii}\Big)(\lambda,\lambda)_0.$$

Hence by (3.6), we have

$$\lambda_1(g)\leqq\min_{\lambda\in D-(0)}\Big[\frac{1}{2}\sum_{\alpha\in\Delta^+}((A\,{}^tA)_{\alpha\alpha}+(A\,{}^tA)_{\bar{\alpha}\bar{\alpha}})\lambda(H_\alpha)+\sum_{i=1}^{l}(A\,{}^tA)_{ii}(\lambda,\lambda)_0\Big]$$

$$\leqq\mathrm{Trace}\,(A\,{}^tA)\min_{\lambda\in D-(0)}\Big[\sum_{\alpha\in\Delta^+}\lambda(H_\alpha)+(\lambda,\lambda)_0\Big]$$

$$=\mathrm{Trace}\,(A\,{}^tA)\lambda_1(g_0).$$

Therefore we have theorem 3.

THEOREM 3. *Let M be an n-dimensional compact connected Lie group. Let g_0 be the M-bi-invariant Riemannian metric. For any $A\in SL(n,\boldsymbol{R})$, let g be the left invariant Riemannian metric on M induced from A as in* 2.2. *Then the least positive eigenvalue $\lambda_1(g)$ of Δ_g has the following estimation:*

$$\lambda_1(g)\leqq\lambda_1(g_0)\,\mathrm{Trace}\,(A\,{}^tA).$$

§ 4. The case of a compact Lie group II.

4.1. First, let $M=\boldsymbol{T}^n$ be an n-dimensinal torus, that is a compact connected abelian Lie group. Let $\mathfrak{t}$ be the Lie algebra of $\boldsymbol{T}^n$. Put $\Gamma=\{H\in\mathfrak{t}\,;\,\exp(H)=e\}$. Then there exists a basis $\{H_i\}_{i=1}^n$ of $\mathfrak{t}$ such that $\Gamma=\{\sum_{i=1}^{n}n_iH_i\,;\,n_i\in\boldsymbol{Z}\ (1\leqq i\leqq n)\}$. Let $(\ ,\ )_0$ be the inner product of $\mathfrak{t}$ defined by $(H_i,H_j)_0=\delta_{ij}\ (1\leqq i,j\leqq n)$. For $A\in SL(n,\boldsymbol{R})$, put $H_j'=\sum_{i=1}^{n}A_{ij}H_i$. Let $(\ ,\ )$ be the inner product defined by $(H_i',H_j')=\delta_{ij}\,(1\leqq i,j\leqq n)$. Then the Laplace-Beltrami operator Δ (resp. Δ_0) on $\boldsymbol{T}^n$ corresponding the Riemannian metric induced from $(\ ,\ )$ (resp. $(\ ,\ )_0$) is given by

$$\Delta f=-\sum_{i,j=1}^{n}(A\,{}^tA)_{ij}H_iH_j(f)\ \ (\text{resp. }\Delta_0f=-\sum_{i=1}^{n}H_i^2(f))$$

for a smooth function f on $\boldsymbol{T}^n$. Let Γ^* be the dual lattice of Γ in $\mathfrak{t}^*$, that is $\Gamma^*=\{\sum_{i=1}^{n}m_i\lambda_i\,;\,m_i\in\boldsymbol{Z}\ (1\leqq i\leqq n)\}$ where $\{\lambda_i\}_{i=1}^n$ is the dual basis of $\mathfrak{t}^*$ for $\{H_i\}_{i=1}^n$. Then the Peter-Weyl theorem says that each $f\in C^\infty(\boldsymbol{T}^n)$ can be expanded by

$$f=\sum_{\eta\in\Gamma^*}((f,\chi_\eta))\chi_\eta .$$

Here $((f_1, f_2))=\int_{T^n} f_1(x)\overline{f_2(x)}\Omega(x)$ where Ω is the volume element corresponding to the Riemannian metric on $\boldsymbol{T}^n$ induced from $(\ ,\ )$ or $(\ ,\ )_0$. The function χ_η for $\eta=\sum_{i=1}^n m_i\lambda_i\in\Gamma^*$ is given by $\chi_\eta(x)=\exp(2\pi\sqrt{-1}\sum_{i=1}^n x_i m_i)$ $(x=\exp(\sum_{i=1}^n x_i H_i)\in\boldsymbol{T}^n)$. Then the spectrum of Δ on $C^\infty(\boldsymbol{T}^n)$ is

$$\begin{aligned}\mathrm{Spec}(\Delta)&=\{4\pi^2\sum_{i,j=1}^n (A\,{}^tA)_{ij}m_i m_j\,;\ m_i\in\boldsymbol{Z}\,(1\leqq i\leqq n)\}\\ &=\{4\pi^2(H, H)_0\,;\ H\in\Gamma(A)\}.\end{aligned}$$

Here $\Gamma(A)$ is the lattice in $\mathfrak{t}$ spanned by $\{H_j(A)\}_{j=1}^n$, $H_j(A)=\sum_{k=1}^n A_{jk}H_k$. That is, Spec (Δ) coincides with the spectrum of the Laplace-Beltrami operator for the Riemannian metric on the flat torus $\mathfrak{t}/\Gamma(A)$ induced from the inner product $(\ ,\ )_0$ on $\mathfrak{t}$. Since $A\in SL(n, \boldsymbol{R})$, the volume of the flat torus $\mathfrak{t}/\Gamma(A)$ is constant in A. Then there exists (cf. [1], [2]) a positive constant $k(\boldsymbol{T}^n)$ such that the least positive eigenvalue $\lambda_1(\Delta)$ of Δ is bounded upper by $k(\boldsymbol{T}^n)$ for all $A\in SL(n, \boldsymbol{R})$. If $n=2$, then the so called equilateral torus attains the maximum of $\lambda_1(\Delta)$ among the flat tori $\mathfrak{t}/\Gamma(A)$, $A\in SL(2, \boldsymbol{R})$.

4.2. Conversely, we obtain

THEOREM 4. *Let M be a compact connected Lie group. We assume M has the non-trivial commutator subgroup, that is the commutator Lie subalgebra $\mathfrak{m}_1$ of $\mathfrak{m}$ is not zero. Then there exists a family of left invariant Riemannian metrics $g(t)$ $(0<t<\infty)$ on M such that $\lim_{t\to\infty}\lambda_1(g(t))=\infty$, $\lim_{t\to 0}\lambda_1(g(t))=0$ and* vol$(M, g(t))$ *is constant in t.*

PROOF. Put $r=\#\Delta^+>0$ and $p=\mathrm{rank}(M_1)$. Let $\mathfrak{z}$ be the center of $\mathfrak{m}$ and let Z_0 be the connected subgroup of M corresponding to $\mathfrak{z}$. Put $q=\dim.\mathfrak{z}$. Then $\mathfrak{t}=\mathfrak{t}_1+\mathfrak{z}$ $(\mathfrak{t}_1=\mathfrak{t}\cap\mathfrak{m}_1)$ and $\mathfrak{m}=\mathfrak{m}_1+\mathfrak{z}$. Choose an orthonormal basis $\{H_i\}_{i=1}^l$ of $\mathfrak{t}$ with respect to $(\ ,\ )_0$ such that $\{H_i\}_{i=1}^p$ spans $\mathfrak{t}_1$ and $\{H_{p+i}\}_{i=1}^q$ spans $\mathfrak{z}$. Let A be a diagonal matrix in $SL(n, \boldsymbol{R})$ which has the components:

$$\begin{cases} A_{\alpha\alpha}=A_{\bar\alpha\bar\alpha}=t^{1/2}\,,\ \alpha\in\Delta^+\,,\\ A_{ii}=t^{-(n-p)/2p}\,,\ 1\leqq i\leqq p\,,\\ A_{p+i,\,p+i}=t^{1/2}\,,\ 1\leqq i\leqq q\,,\end{cases}$$

for any positive number t. Let $(\ ,\ )$ be the inner product on $\mathfrak{m}$ induced from A. Then the decomposition $\mathfrak{m}=\mathfrak{m}_1+\mathfrak{z}$ is orthogonal with respect to this $(\ ,\ )$. Let $\tilde{g}(t)$ (resp. $g(t)$, $g_1(t)$, $g_0(t)$) be the left invariant Riemannian metric on the compact Lie group $M_1\times Z_0$ (resp. M, M_1, Z_0). Let $\Delta_{\tilde{g}(t)}$ (resp. $\Delta_{g(t)}$, $\Delta_{g_1(t)}$, $\Delta_{g_0(t)}$) be the corresponding Laplace-Beltrami operator on $M_1\times Z_0$ (resp. M, M_1, Z_0). Let $\lambda_1(\tilde{g}(t))$ (resp. $\lambda_1(g(t))$, $\lambda_1(g_1(t))$, $\lambda_1(g_0(t))$) be the least positive eigenvalue of $\Delta_{\tilde{g}(t)}$ (resp. $\Delta_{g(t)}$, $\Delta_{g_1(t)}$, $\Delta_{g_0(t)}$). Then

LEMMA 3. *The above notations are preserved. We have*

$$\lambda_1(g(t))\geqq\min.\{\lambda_1(g_1(t)),\ \lambda_1(g_0(t))\}=\min.\{\lambda_1(g_1(t)),\ t\lambda_1(g_0(1))\}.$$

PROOF. Since the covering mapping $M_1\times Z_0\ni(m, z)\mapsto mz\in M$ is the Riemannian covering mapping of $(M_1\times Z_0, \tilde{g}(t))$ onto $(M, g(t))$, due to [3] p. 145, $\mathrm{Spec}(\Delta_{g(t)})\subset\mathrm{Spec}(\Delta_{\tilde{g}(t)})$. So $\lambda_1(g(t))\geqq\lambda_1(\tilde{g}(t))$. Due to [3] p. 144,

$$\mathrm{Spec}(\Delta_{\tilde{g}(t)})=\{\lambda+\lambda'\ ;\ \lambda\in\mathrm{Spec}(\Delta_{g_1(t)}),\ \lambda'\in\mathrm{Spec}(\Delta_{g_0(t)})\}.$$

So,

$$\lambda_1(\tilde{g}(t))=\min.\ \{\lambda_1(g_1(t)),\ \lambda_1(g_0(t))\}. \qquad \text{Q.E.D.}$$

Now for an irreducible representation $(\pi^{\lambda}, V_{\lambda})$, $\lambda\in D$, of M, we have, due to (3.5),

$$\begin{aligned}(4.1)\qquad \pi^{\lambda}(C_{g(t)})&=-t\sum_{\alpha\in\Delta^+}(\pi^{\lambda}(E_{\alpha})\pi^{\lambda}(E_{-\alpha})+\pi^{\lambda}(E_{-\alpha})\pi^{\lambda}(E_{\alpha}))\\ &\quad -t^{-(n-p)/p}\sum_{i=1}^{p}\pi^{\lambda}(H_i)^2-t\sum_{i=1}^{q}\pi^{\lambda}(H_{p+i})^2\\ &=t\pi^{\lambda}(C_{g(1)})+(t^{-(n-p)/p}-t)\Big(-\sum_{i=1}^{p}\pi^{\lambda}(H_i)^2\Big).\end{aligned}$$

Here $\pi^{\lambda}(C_{g(1)})=-\sum_{\alpha\in\Delta^+}(\pi^{\lambda}(E_{\alpha})\pi^{\lambda}(E_{-\alpha})+\pi^{\lambda}(E_{-\alpha})\pi^{\lambda}(E_{\alpha}))-\sum_{i=1}^{l}\pi^{\lambda}(H_i)^2$ is the Casimir operator for $(\pi^{\lambda}, V_{\lambda})$. Then it is known (cf. [7]) that

$$\pi^{\lambda}(C_{g(1)})v=(\lambda+2\delta,\ \lambda)_0 v \qquad \text{for all} \quad v\in V_{\lambda}$$

where $\delta=\frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$. Hence we have

$$\begin{aligned}(4.2)\qquad &\min_{v\in V_{\lambda},\,((v,v))=1}((\pi^{\lambda}(C_{g(t)})v,\ v))\\ &=\min_{v\in V_{\lambda},\,((v,v))=1}\Big[t(\lambda+2\delta,\ \lambda)_0+(t^{-(n-p)/p}-t)\Big(\Big(-\sum_{i=1}^{p}\pi^{\lambda}(H_i)^2 v,\ v\Big)\Big)\Big].\end{aligned}$$

Therefore 1) if $t\leqq 1$ (i.e. $t^{-(n-p)/p}-t\geqq 0$), then the right hand side of (4.2) is

$$(4.3)\qquad t(\lambda+2\delta,\ \lambda)_0+(t^{-(n-p)/p}-t)\min_{v\in V_{\lambda},\,((v,v))=1}\Big(\Big(-\sum_{i=1}^{p}\pi^{\lambda}(H_i)^2 v,\ v\Big)\Big).$$

2) if $t\geqq 1$ (i.e. $t^{-(n-p)/p}-t\leqq 0$), then the one of (4.2) is

$$(4.3')\qquad t(\lambda+2\delta,\ \lambda)_0+(t^{-(n-p)/p}-t)\max_{v\in V_{\lambda},\,((v,v))=1}\Big(\Big(-\sum_{i=1}^{p}\pi^{\lambda}(H_i)^2 v,\ v\Big)\Big).$$

LEMMA 4. *For an irreducible representation* (π^λ, V_λ), $\lambda \in D$ *of* M,

$$\min_{v\in V_\lambda, ((v,v))=1} ((-\sum_{i=1}^{p} \pi^\lambda(H_i)^2 v, v)) = \min.\{(\mu_1, \mu_1)_0 ;\ \mu, \text{ weight of } V_\lambda\},$$

and

$$\max_{v\in V_\lambda, ((v,v))=1} ((-\sum_{i=1}^{p} \pi^\lambda(H_i)^2 v, v)) = \max.\{(\mu_1, \mu_1)_0 ;\ \mu, \text{ weight of } V_\lambda\}.$$

Here for $\mu \in \mathfrak{t}^*$, *let* $\mu_1 \in \mathfrak{t}_1^*$ *be the restriction to* $\mathfrak{t}_1$ *of* μ *and* $(\mu_1, \mu_1)_0$ *be the inner product on* $\mathfrak{t}_1$ *induced from* $(\ ,\ )_0$.

PROOF. Corresponding to the decomposition $V_\lambda = \sum_{\mu\in I} V^\mu$, for every $v \in V_\lambda$, put $v = \sum_{\mu\in I, i=1,\cdots,\dim. V^\mu} a_{\mu,i} v_{\mu,i}$, $a_{\mu,i} \in \boldsymbol{C}$ where $\{v_{\mu,i}\}_{i=1,\cdots,\dim. V^\mu}$ is an orthonormal basis of V^μ with respect to $((\ ,\ ))$ for each weight μ of V_λ. Then

$$(4.4)\qquad ((-\sum_{i=1}^{p} \pi^\lambda(H_i)^2 v, v)) = \sum_{\mu\in I, i=1,\cdots,\dim. V^\mu} |a_{\mu,i}|^2 (\sum_{i=1}^{p} \mu(H_i)^2)$$

$$= \sum_{\mu\in I, i=1,\cdots,\dim. V^\mu} |a_{\mu,i}|^2 (\mu_1, \mu_1)_0 .$$

Then the maximum (resp. minimum) of (4.4) with the condition

$$((v, v)) = \sum_{\mu\in I, i=1,\cdots,\dim. V^\mu} |a_{\mu,i}|^2 = 1 ,$$

coincides with the desired result. Q. E. D.

The case of 1); $t \leqq 1$. There exists $\lambda_0 \in D-(0)$ such that V_{λ_0} has a non-zero 0-weight vector v_0, that is $\pi^{\lambda_0}(H)v_0 = 0$, for all $H \in \mathfrak{t}$. Then (4.3) coincides $t(\lambda_0 + 2\delta, \lambda_0)_0$. Hence

$$\lambda_1(g(t)) \leqq t(\lambda_0 + 2\delta, \lambda_0)_0 .$$

Therefore we have $\lim_{t\to 0} \lambda_1(g(t)) = 0$.

The case of 2); $t \geqq 1$. Let D_1 be the set of all highest weights of M_1 with respect to the order on $\mathfrak{t}_1$ induced from the one on $\mathfrak{t}$. Then for an irreducible representation (π^λ, V_λ) of M_1, $\lambda \in D_1$, by (4.3) and lemma 4, we obtain

$$(4.5)\qquad \max_{v\in V_\lambda, ((v,v))=1} ((\pi^\lambda(C_{g_1(t)})v, v)) = t(\lambda + 2\delta, \lambda)_0 + (t^{-(n-p)/p} - t)$$

$$\times \max.\{(\mu, \mu)_0 ;\ \mu, \text{ weight of } V_\lambda\}$$

where $\pi^\lambda(C_{g_1(t)}) = -\sum_{i=1}^{n-q} \pi^\lambda(Y_i)^2$ for an orthonormal basis $\{Y_i\}_{i=1}^{n-q}$ of $\mathfrak{m}_1$ with respect to the inner product on $\mathfrak{m}_1$ corresponding to A. It is noticed (cf. [6] 19-06) that for every irreducible representation (π^λ, V_λ) of M_1,

$$(\mu, \mu)_0 \leqq (\lambda, \lambda)_0$$

for all weight μ of V_λ. In fact, for each weight μ of V_λ, there exist an orthogonal transformation S with respect to $(\ ,\ )_0$ and a weight ν of V_λ such that $\mu=S(\nu)$ and $\nu\in D_1$. Every weight ν of V_λ can be expressed as $\nu=\lambda-\sum_{i=1}^{p}m_i\alpha_i$, $m_i\geqq 0\ (1\leqq i\leqq p)$. Then

$$(\lambda,\lambda)_0=(\nu+\sum_{i=1}^{p}m_i\alpha_i,\ \nu+\sum_{i=1}^{p}m_i\alpha_i)_0$$

$$=(\nu,\nu)_0+2(\nu,\ \sum_{i=1}^{p}m_i\alpha_i)_0+(\sum_{i=1}^{p}m_i\alpha_i,\ \sum_{i=1}^{p}m_i\alpha_i)\geqq(\nu,\nu)_0=(\mu,\mu)_0\,.$$

Then the right hand side of (4.5) is greater than $t(\lambda, 2\delta)_0$. Hence we have

$$\lambda_1(g_1(t))\geqq t\min_{\lambda\in D_1-(0)}(2\delta,\lambda)_0\,.$$

We notice that $C_1=\min_{\lambda\in D_1-(0)}(2\delta,\lambda)_0=\min.\{(2\delta,\lambda_1)_0,\cdots,(2\lambda,\lambda_p)_0\}>0$. Therefore by lemma 3,

$$\lambda_1(g(t))\geqq t\min.\{\lambda_1(g_0(1)),\ C_1\},$$

that is $\lim_{t\to\infty}\lambda_1(g(t))=\infty$. Q.E.D.

§5. The case of $SU(2)$ and $SO(3)$.

In this section, we investigate the behavior of $\lambda_1(g)$ for left invariant Riemannian metrics on $SU(2)$ and $SO(3)$.

Let $M=SU(2)=\{x\in SL(2,\boldsymbol{C})\,;\,x^t\bar{x}=I\}$, and $\mathfrak{m}=\mathfrak{su}(2)=\left\{\begin{pmatrix}\sqrt{-1}\,\theta & \alpha\\ -\bar{\alpha} & -\sqrt{-1}\,\theta\end{pmatrix};\alpha\in\boldsymbol{C},\ \theta\in\boldsymbol{R}\right\}$ isomorphic with $\mathfrak{so}(3)$ as a Lie algebra. Let $(X,Y)_0=-B(X,Y)=-4\,\mathrm{Trace}\,(XY)$ for $X,Y\in\mathfrak{m}$ where B is the Killing form on $\mathfrak{m}$. Let $\mathfrak{t}=\left\{\begin{pmatrix}\sqrt{-1}\,\theta & 0\\ 0 & -\sqrt{-1}\,\theta\end{pmatrix};\ \theta\in\boldsymbol{R}\right\}$. Let $\Delta^+=\{\alpha\}$, $\alpha\,;\,\mathfrak{t}\ni\begin{pmatrix}\sqrt{-1}\,\theta & 0\\ 0 & -\sqrt{-1}\,\theta\end{pmatrix}\mapsto 2\theta\in\boldsymbol{R}$. Put $E_\alpha=\begin{pmatrix}0 & \sqrt{-1}/2\\ 0 & 0\end{pmatrix}$, $E_{-\alpha}=\begin{pmatrix}0 & 0\\ \sqrt{-1}/2 & 0\end{pmatrix}$ and $H_\alpha=\frac{\sqrt{-1}}{4}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. Then $(\alpha,\alpha)_0=(H_\alpha,H_\alpha)_0=1/2$. The element $\frac{1}{\sqrt{2}}U_\alpha=\frac{1}{\sqrt{2}}\begin{pmatrix}0 & \sqrt{-1}/2\\ \sqrt{-1}/2 & 0\end{pmatrix}$, $\frac{1}{\sqrt{2}}V_\alpha=\frac{1}{\sqrt{2}}\begin{pmatrix}0 & -1/2\\ 1/2 & 0\end{pmatrix}$ and $H_1=\sqrt{2}H_\alpha=\frac{\sqrt{-1}}{2\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ form the basis of $\mathfrak{m}$ as in (3.4). Put $H_\alpha{}^*=\begin{pmatrix}\sqrt{-1} & 0\\ 0 & -\sqrt{-1}\end{pmatrix}$. Then $\Gamma=\{H\in\mathfrak{t}\,;\,\exp(H)=e\}=2\pi\boldsymbol{Z}\,H_\alpha{}^*=2\pi\boldsymbol{Z}\begin{pmatrix}\sqrt{-1} & 0\\ 0 & -\sqrt{-1}\end{pmatrix}$, $I=\{\lambda\in\mathfrak{t}^*\,;\,\lambda(H_\alpha{}^*)\in\boldsymbol{Z}\}=\{l\alpha\,;\,l\in 1/2\boldsymbol{Z}\}$ and $D=\{\lambda\in I\,;\,(\lambda,\alpha)_0\geqq 0\}=\{l\alpha\,;\,l\in 1/2\boldsymbol{Z},\ l\geqq 0\}$. In case of $M=SO(3)$, under the identification of $\mathfrak{su}(2)$ with $\mathfrak{so}(3)$, $D=\{l\alpha\,;\,l\in\boldsymbol{Z},\ l\geqq 0\}$.

For $A\in SL(3, \boldsymbol{R})$, let $\Delta_{g(A)}$ be the Laplace-Beltrami operator on $SU(2)$ (resp. $SO(3)$) of the left invariant Riemannian metric $g(A)$ on $SU(2)$ (resp. $SO(3)$) induced from A as in 2.2. Then the estimation (3.6) of the least positive eigenvalue $\lambda_1(A)$ of $\Delta_{g(A)}$ on $SU(2)$ (resp. $SO(3)$) becomes

$$\lambda_1(A)\leqq \min_{l\in 1/2\boldsymbol{Z},\, l>0}\left[\frac{l}{4}((A\,{}^tA)_{\alpha\alpha}+(A\,{}^tA)_{\bar{\alpha}\bar{\alpha}})+\frac{l^2}{2}(A\,{}^tA)_{11}\right], \tag{5.1}$$

$$\left(\text{resp. } \lambda_1(A)\leqq \min_{l\in \boldsymbol{Z},\, l>0}\left[\frac{l}{4}((A\,{}^tA)_{\alpha\alpha}+(A\,{}^tA)_{\bar{\alpha}\bar{\alpha}})+\frac{l^2}{2}(A\,{}^tA)_{11}\right]\right).$$

Let $A=\begin{pmatrix} A_{\alpha\alpha} & 0 & 0 \\ 0 & A_{\bar{\alpha}\bar{\alpha}} & 0 \\ 0 & 0 & A_{11}\end{pmatrix}\in SL(3, \boldsymbol{R})$ such that $A_{\alpha\alpha}=A_{\bar{\alpha}\bar{\alpha}}=t^{1/2}$, $A_{11}=t^{-1}$ $(t>0)$. Let $g(t)$ be the Riemannian metric on $SU(2)$ ($SO(3)$) induced from A. For an irreducible representation (π^λ, V_λ), $\lambda=l\alpha\in D$, of $SU(2)$ ($SO(3)$), (4.1) becomes

$$\pi^\lambda(C_{g(t)})=t\pi^\lambda(C_{g(1)})+(t^{-2}-t)(-\pi^\lambda(H_1)^2)\,. \tag{5.2}$$

For $v\in V_\lambda$, $\pi^\lambda(C_{g(1)})v=\frac{1}{2}l(l+1)v$. Let v_λ be the highest weight vector of V_λ. Put $v_l=v_\lambda$, $v_{l-1}=\pi^\lambda(E_{-\alpha})v_l$, $v_{l-2}=\pi^\lambda(E_{-\alpha})v_{l-1}, \cdots$. Then the $(2l+1)$ elements $\{v_l, v_{l-1}, \cdots, v_{1-l}, v_{-l}\}$ of V_λ form a basis of V_λ such that

$$\pi^\lambda(H_\alpha)v_m=\frac{\sqrt{-1}}{2}mv_m\ (m=l, l-1, \cdots, 1-l, -l)\,.$$

The $(2l+1)$ elements $m\alpha$ $(m=l, l-1, \cdots, 1-l, -l)$ of $\mathfrak{t}^*$ form all weights of V_λ with the weight vector v_m. Put $w_m=b_m v_m$, $b_m\in\boldsymbol{R}$ to be normalized with respect to $((\,,\,))$. Then

$$\pi^\lambda(C_{g(t)})w_m=\frac{t}{2}l(l+1)w_m+(t^{-2}-t)\frac{m^2}{2}w_m\,. \tag{5.3}$$

Hence for (π^λ, V_λ),

$$\min_{v\in V_\lambda,\, ((v,v))=1}((\pi^\lambda(C_{g(t)})v, v)) \tag{5.4}$$

$$=\frac{t}{2}l(l+1)+\min_{\sum_{m=-l}^{l}|c_m|^2=1,\, c_m\in\boldsymbol{C}}\left[\left(\sum_{m=-l}^{l}|c_m|^2m^2\right)\left(\frac{t^{-2}-t}{2}\right)\right]$$

$$=\frac{t}{2}l(l+1)+\begin{cases}\dfrac{t^{-2}-t}{2}\max\limits_{\sum_{m=-l}^{l}|c_m|^2=1,}\left(\sum\limits_{m=-l}^{l}|c_m|^2m^2\right) & (t\geqq 1)\,, \\[2ex] \dfrac{t^{-2}-t}{2}\min\limits_{\sum_{m=-l}^{l}|c_m|^2=1,}\left(\sum\limits_{m=-l}^{l}|c_m|^2m^2\right) & (t\leqq 1)\,.\end{cases}$$

Here $\max.(\sum_{m=-l}^{l}|c_m|^2m^2)=l^2$ and $\min.(\sum_{m=-l}^{l}|c_m|^2m^2)=\begin{cases} 0 & (l\in \boldsymbol{Z}), \\ \frac{1}{4} & (l\notin \boldsymbol{Z}). \end{cases}$

Therefore we have

THEOREM 5. 1) *Let $M=SU(2)$, diffeomorphic with S^3. For $A\in SL(3, \boldsymbol{R})$,*

$$\lambda_1(A)\leqq\frac{1}{8}\mathrm{Trace}\,(A\,{}^tA).$$

There exists a family of left invariant Riemannian metrics $g(t)$ $(t>0)$ on $SU(2)$ such that

$$\lambda_1(g(t))=\begin{cases} \frac{t}{4}+\frac{t^{-2}}{8} & \left(t>\frac{1}{\sqrt[3]{6}}\right) \text{ with the multiplicity } 4, \\ \frac{1}{\sqrt[3]{6}} & \left(t=\frac{1}{\sqrt[3]{6}}\right) \text{ with the multiplicity } 7, \\ t & \left(t<\frac{1}{\sqrt[3]{6}}\right) \text{ with the multiplicity } 3. \end{cases}$$

2) *Let $M=SO(3)$, diffeomorphic with $P^3(\boldsymbol{R})$. For $A\in SL(3, \boldsymbol{R})$,*

$$\lambda_1(A)\leqq\frac{1}{4}((A\,{}^tA)_{\alpha\alpha}+(A\,{}^tA)_{\bar{\alpha}\bar{\alpha}})+\frac{1}{2}(A\,{}^tA)_{11}.$$

There exists a family of left invariant Riemannian metrics $g(t)$ $(t>0)$ on $SO(3)$ such that

$$\lambda_1(g(t))=\begin{cases} \frac{t}{2}+\frac{t^{-2}}{2} & (t>1) \text{ with the multiplicity } 6, \\ 1 & (t=1) \text{ with the multiplicity } 9, \\ t & (t<1) \text{ with the multiplicity } 3. \end{cases}$$

§ 6. The case of $SU(2)$.

Following [1] p. 146, we prepare some notions. For a Riemannian manifold (M, g) with the involutive isometry σ, that is $\sigma^2=id$, let

$$C^+=\{f\in C^\infty(M)\,;\, f\circ\sigma=f\},\ C^-=\{f\in C^\infty(M)\,;\, f\circ\sigma=-f\}.$$

Put $C_\lambda^+=C^+\cap C_\lambda^\infty(M)$ and $C_\lambda^-=C^-\cap C_\lambda^\infty(M)$ where $C_\lambda^\infty(M)$ is the eigenspace for the eigenvalue λ of Δ_g in $C^\infty(M)$. Let

$$\lambda_1^+(g)=\inf.\{\lambda>0\,;\, C_\lambda^+\neq 0\},\ \lambda_1^-(g)=\inf.\{\lambda>0\,;\, C_\lambda^-\neq 0\}.$$

Then M. Berger has conjectured (cf. [1] p. 146)

$$\lambda_1^-(g)\leqq\lambda_1^+(g). \tag{6.1}$$

In this section, we investigate this problem in case of $SU(2)\cong S^3$. We preserve the notations in § 5. Let $x_0=\exp(4\pi H_\alpha)=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\in SU(2)$. Let σ be the left translation of $SU(2)$ by x_0. Then σ coincides the mapping $S^3\ni p\mapsto \tilde{p}\in S^3$ (the antipodal point of p) under the identification $SU(2)\ni\begin{pmatrix}\alpha & \beta\\ -\bar{\beta} & \bar{\alpha}\end{pmatrix}\mapsto(\alpha, \beta)\in \{(\alpha, \beta)\in \boldsymbol{C}^2;\ |\alpha|^2+|\beta|^2=1\}\cong S^3$. Then the Riemannian metric $g(t)$ $(0<t<\infty)$ in § 5 satisfies

$$\sigma^* g(t)=g(t) \tag{6.2}$$

due to the left invariance of $g(t)$. For this isometry σ of $(SU(2), g(t))$ $(0<t<\infty)$, we calculate $\lambda_1^+(g(t))$ and $\lambda_1^-(g(t))$ as follows:

THEOREM 6. *Let $M=SU(2)\cong S^3$. The above situations are preserved. Then we obtain*

$$\lambda_1^+(g(t))=\begin{cases}\dfrac{t}{2}+\dfrac{t^{-2}}{2} & (1\leqq t<\infty),\\ t & (0<t<1),\end{cases}$$

$$\lambda_1^-(g(t))=\frac{t}{4}+\frac{t^{-2}}{8} \quad (0<t<\infty).$$

Thus, in particular, we have

$$\lambda_1^+(g(t))<\lambda_1^-(g(t)) \qquad \left(0<t<\frac{1}{\sqrt[3]{6}}\right),$$

which implies M. Berger's conjecture (6.1) *is negative for* S^3.

PROOF. For $\lambda=l\alpha\in D$ $(l\in 1/2\boldsymbol{Z})$, the matrix elements $\pi^\lambda_{mm'}$ (cf. 3.2) of (π^λ, V_λ) satisfy

$$\begin{cases}\pi^\lambda_{mm'}\circ\sigma=\pi^\lambda_{mm'} & \text{if } m\in\boldsymbol{Z}\ (m=l, l-1, \cdots, 1-l, -l),\\ \pi^\lambda_{mm'}\circ\sigma=-\pi^\lambda_{mm'} & \text{if } m\notin\boldsymbol{Z}\ (m=l, l-1, \cdots, 1-l, -l).\end{cases}$$

Because we have $\pi^\lambda_{mm'}\circ\sigma(x)=((\pi^\lambda(x_0x)w_{m'}, w_m))=((\pi^\lambda(x)w_{m'}, \pi^\lambda(\exp(-4\pi H_\alpha))w_m))$ $=\exp(\sqrt{-1}\,2\pi m)\pi^\lambda_{mm'}(x)$. Then we have

$$\begin{cases}(1) \text{ if } \lambda=l\alpha\ (l\in\boldsymbol{Z}), \text{ all } \pi^\lambda_{mm'} \text{ belong to } C^+,\\ (2) \text{ if } \lambda=l\alpha\ (l\notin\boldsymbol{Z}), \text{ all } \pi^\lambda_{mm'} \text{ belong to } C^-.\end{cases}$$

Moreover, since $\Delta_{g(t)}(\pi^\lambda_{mm'})=\lambda_{m'}\pi^\lambda_{mm'}$ where $\pi^\lambda(C_{g(t)})w_{m'}=\lambda_{m'}w_{m'}$ (cf. proof of lemma 2), all $\pi^\lambda_{mm'}$ satisfy

$$\Delta_{g(t)}\pi^\lambda_{mm'}=\left\{\frac{t}{2}l(l+1)+(t^{-2}-t)\frac{(m')^2}{2}\right\}\pi^\lambda_{mm'} \tag{6.3}$$

for all $\lambda=l\alpha$ $(l\in 1/2\boldsymbol{Z})$, by (5.3).

Thus, together (1) and (2), we have

$$\begin{cases} \lambda_1^+(g(t))=\min_{l\in Z} \min_{m'=l,l-1,\cdots,1-l,-l}\left\{\frac{t}{2}l(l+1)+(t^{-2}-t)\frac{(m')^2}{2}\right\}, \\ \lambda_1^-(g(t))=\min_{l\notin Z,\, l\in 1/2Z}\min_{m'=l,l-1,\cdots,1-l,-l}\left\{\frac{t}{2}l(l+1)+(t^{-2}-t)\frac{(m')^2}{2}\right\}. \end{cases}$$

In case of $l\in \boldsymbol{Z}$, we have

$$\min_{m'=l,l-1,\cdots,1-l,-l}\left\{\frac{t}{2}l(l+1)+(t^{-2}-t)\frac{(m')^2}{2}\right\}=\begin{cases} \frac{t}{2}l+\frac{t^{-2}}{2}l^2 & (1\leqq t<\infty), \\ t & (0<t<1). \end{cases}$$

Then

$$\lambda_1^+(g(t))=\begin{cases} \frac{t}{2}+\frac{t^{-2}}{2} & (1\leqq t<\infty), \\ t & (0<t<1). \end{cases}$$

In case of $l\notin \boldsymbol{Z}$, $l\in 1/2\boldsymbol{Z}$, we have

$$\min_{m'=l,l-1,\cdots,1-l,-l}\left\{\frac{t}{2}l(l+1)+(t^{-2}-t)\frac{(m')^2}{2}\right\}=\begin{cases} \frac{t}{2}l+\frac{t^{-2}}{2}l^2 & (1\leqq t<\infty), \\ \frac{t}{2}l(l+1)+\frac{t^{-2}-t}{8} & (0<t<1). \end{cases}$$

Then

$$\lambda_1^-(g(t))=\frac{t}{4}+\frac{t^{-2}}{8} \quad (0<t<\infty).$$

Q. E. D.

Added in proof. Recently S. Tanno [The first eigenvalue of the Laplacian on spheres (to appear)] showed *M. Berger's conjecture in introduction is negative for all odd dimensional spheres.* From its content one sees that the one in § 6 is also negative for all odd dimensional spheres.

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