

## A difference inequality and its application to nonlinear evolution equations

By Mitsuhiro NAKAO

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### Introduction.

Let  $H$  be a real Hilbert space and  $V, W$  be real Banach spaces with  $V \subset W \subset H$ . We assume  $V$  is dense in  $W$  and  $H$ , and the natural injections from  $V$  into  $W$  and from  $W$  into  $H$  are continuous. We identify  $H$  with its dual  $H^*$  (i.e.,  $V \subset W \subset H \subset W^* \subset V^*$ ). Let us consider the nonlinear evolution equations

$$u''(t) + B(t)u'(t) + A(t)u(t) = f(t) \quad (\text{a})$$

and

$$B(t)u'(t) + A(t)u(t) = f(t) \quad (\text{b})$$

where  $A(t)$  is the Fréchet derivative of a functional  $F_{A(t)}(u)$  on  $V$  and  $B$  is a bounded operator from  $W$  to  $W^*$ .

Recently in [6], the author has discussed the decay property of solutions of the equations (a) and (b) in the case  $A(t)$  and  $B(t)$  are independent of  $t$ . There the problem is reduced to the difference inequality of the form

$$\sup_{s \in [t, t+1]} \phi(s)^{1+r} \leq C(\phi(t) - \phi(t+1)) + \delta(t) \quad (r \geq 0)$$

where  $C$  is a positive constant,  $\phi(t)$  is a nonnegative function on  $R^+ = [0, \infty)$  and  $\delta(t)$  is a function tending to 0 as  $t \rightarrow \infty$ . The decay property of  $\phi(t)$  as  $t \rightarrow \infty$  has been discussed in [4] and [5] with applications to the wave equation with a nonlinear dissipative term.

In this paper we first treat a more general difference inequality

$$\sup_{s \in [t, t+1]} \phi(s)^{1+r} \leq C(1+t)^\alpha (\phi(t) - \phi(t+1)) + \delta(t)$$

where  $\alpha, r$  are constant with  $0 \leq \alpha \leq 1, r \geq 0$ . Next, the result for the above inequality is applied to the investigation of the asymptotic behaviour of the solutions of (a) and (b). As is in [6], the equation (a) will be treated in detail, while we give a brief discussion of eq. (b), because the latter is simpler.

A typical example which our result is applicable to is the nonlinear generalized Euler-Poisson-Darboux equation

$$\frac{\partial^2}{\partial t^2} u - \Delta u + t^{-\theta} \rho \left( \frac{\partial}{\partial t} u \right) + \beta(x, u) = f \quad \text{on } [t_0, \infty) \times \Omega$$

( $t_0 > 0$ ,  $0 \leq \theta \leq 1$ ) with boundary condition  $u|_{\partial \Omega} = 0$ , where  $\Omega$  is a bounded domain in  $n$ -dimensional Euclidean space  $R^n$ . This equation seems to be interesting because the effect of the dissipation weakens on and on as  $t \rightarrow \infty$ , which differs from usual dissipative wave equations. For a special case  $\theta = 1$ ,  $\rho \left( \frac{\partial}{\partial t} u \right) = \frac{\partial}{\partial t} u$ , the growth property of solutions have been investigated by Levine [1]. Further examples are also given in the last section.

### 1. Difference inequality.

In this section we prove:

THEOREM 1. Let  $\phi(t)$  be a nonnegative function on  $[0, \infty)$  satisfying

$$\sup_{s \in [t, t+1]} \phi(s)^{1+\alpha} \leq C_0(1+t)^r (\phi(t) - \phi(t+1)) + g(t) \quad (1.1)$$

where  $C_0$  is a positive constant and  $g(t)$  is a nonnegative function. Then we have:

(i) If  $\alpha > 0$ ,  $r = 1$  and  $\lim_{t \rightarrow \infty} (\log t)^{1+\frac{1}{\alpha}} g(t) = 0$

then

$$\phi(t) \leq C_1 (\log(1+t))^{-\frac{1}{\alpha}},$$

(ii) if  $\alpha > 0$ ,  $0 \leq r < 1$  and  $\lim_{t \rightarrow \infty} t^{(1-r)(1+1/\alpha)} g(t) = 0$

then

$$\phi(t) \leq C_2 t^{-(1-r)/\alpha},$$

(iii) if  $\alpha = 0$ ,  $r = 1$  and  $g(t) \leq \text{const. } t^{-\theta-1}$

then

$$\phi(t) \leq C_3 (1+t)^{-\theta'} \quad \text{with } \theta' = \min(C_0^{-1}, \theta)$$

(iv) if  $\alpha = 0$ ,  $0 \leq r < 1$  and  $g(t) \leq \text{const. } t^{-\theta} \exp \left[ -\frac{1}{(C_0+1)(1-r)} (t+1)^{1-r} \right]$  with

$\theta > 1$ , then

$$\phi(t) \leq C_4 \exp \left\{ -\frac{1}{(C_0+1)(1-r)} t^{1-r} \right\}.$$

In the above,  $C_i$  ( $i=1, 2, \dots$ ) are constants depending on  $\phi(0)$  and other known constants.

PROOF. The basic idea of the proof is the same as that of the lemma in [5] where the case  $r=0$  is treated. We give the proofs of (i)-(iv) separately.

Proof of (i). We set

$$\phi(t) = \phi(t) + \nu(\log(1+t))^{-1/\alpha}$$

where  $\nu$  is a positive to be determined later. Then by (1.1) we have

$$\begin{aligned} \sup_{s \in [t, t+1]} \phi(s)^{1+\alpha} &\leq 2^{1+\alpha} \{ \max_{s \in [t, t+1]} \phi(s)^{1+\alpha} + \nu^{1+\alpha} (\log(1+t))^{-(1+\alpha)/\alpha} \} \\ &\leq \text{const.} \{ (1+t)(\phi(t) - \phi(t+1)) + I_1(t) \} \end{aligned} \tag{1.2}$$

where

$$\begin{aligned} I_1(t) &= (1+t) \{ \nu(\log(t+2))^{-1/\alpha} - \nu(\log(t+1))^{-1/\alpha} \} + g(t) \\ &\quad + \nu^{1+\alpha} (\log(t+1))^{-(1+\alpha)/\alpha}. \end{aligned}$$

We shall show  $I_1(t) \leq 0$  if  $t$  is sufficiently large.

$$\begin{aligned} I_1(t) &= \nu(\log(t+1))^{-1-\frac{1}{\alpha}} \left[ (1+t) \log(t+1) \left\{ \left( \frac{\log(t+2)}{\log(t+1)} \right)^{-\frac{1}{\alpha}} - 1 \right\} \right. \\ &\quad \left. + \frac{1}{\nu} (\log(t+1))^{1+\frac{1}{\alpha}} g(t) + \nu^\alpha \right]. \end{aligned}$$

Since

$$\begin{aligned} \left( \frac{\log(t+2)}{\log(t+1)} \right)^{-\frac{1}{\alpha}} - 1 &= \left( \frac{\log(t+2) - \log(t+1)}{\log(t+1)} + 1 \right)^{-\frac{1}{\alpha}} - 1 \\ &\leq -(2\alpha)^{-1} (\log(t+2) - \log(t+1)) (\log(t+1))^{-1}, \end{aligned}$$

for large  $t$ , we have

$$\begin{aligned} I_1(t) &\leq \nu(\log(t+1))^{-1-\frac{1}{\alpha}} \{ -(2\alpha)^{-1} (1+t) \log((t+2)/(t+1)) \\ &\quad + \nu^{-1} (\log(t+1))^{1+\frac{1}{\alpha}} g(t) + \nu^\alpha \} \quad \text{for large } t. \end{aligned}$$

Now, by the assumption on  $g(t)$ , the second term in the bracket of the right hand side tends to 0 as  $t \rightarrow \infty$ , and moreover we see easily

$$\lim_{t \rightarrow \infty} (1+t) \log\left(\frac{t+2}{t+1}\right) = 1.$$

Therefore, there exists  $T_1 > 0$  such that if  $t \geq T_1$  we have

$$I_1(t) \leq \nu(\log(t+1))^{-1-\frac{1}{\alpha}} \left( -(4\alpha)^{-1} + \frac{1}{2} \nu^\alpha \right) < 0$$

where we choose  $\nu$  as

$$\nu < (2\alpha)^{-1/\alpha}.$$

Thus for  $t > T_1$  and small  $\nu$  we have

$$\sup_{s \in [t, t+1]} \phi(s)^{1+\alpha} \leq C_0 (1+t) (\phi(t) - \phi(t+1)). \tag{1.3}$$

Setting  $\phi(s)^{-\alpha} = w(s)$ , we have

$$\begin{aligned}
w(t) - w(t+1) &= \int_0^1 \frac{d}{d\theta} (\theta\phi(t) + (1-\theta)\phi(t+1))^{-\alpha} d\theta \\
&= -\alpha \int_0^1 \{\theta\phi(t) + (1-\theta)\phi(t+1)\}^{-1-\alpha} d\theta (\phi(t) - \phi(t+1)) \\
&\leq -\alpha C_0^{-1} (1+t)^{-1}.
\end{aligned} \tag{1.4}$$

Therefore, for the integer  $n$  with  $n+T_1 \leq t < n+1+T_1$ ,

$$\begin{aligned}
w(t-n) - w(t) &\leq -\alpha C_0^{-1} \sum_{i=0}^{n-1} \frac{1}{(t-i)} \\
&\leq -\alpha C_0^{-1} \int_0^{n-1} \frac{1}{t-x} dx \\
&\leq -\alpha C_0^{-1} (\log t - \log(t+1-n))
\end{aligned}$$

and hence

$$w(t) \geq \inf_{s \in [0,1]} w(s) + \alpha C_0^{-1} \log t - \alpha C_0^{-1} \log(T_1+2)$$

which yields immediately

$$\phi(t) < \phi(t) \leq \left\{ \inf_{s \in [T_1, T_1+1]} w(s) + \alpha C_0^{-1} \log t - \alpha C_0^{-1} \log(T_1+2) \right\}^{-\frac{1}{\alpha}}. \tag{1.5}$$

For  $t \leq T_1$  we see easily by (1.1)

$$\phi(t) \leq \max \{g(t), (C_0\phi(0) + g(0))^{1/(1+\alpha)}\}. \tag{1.6}$$

The estimates (1.5) and (1.6) imply readily (i).

Proof of (ii). At this time we set

$$\psi(t) = \phi(t) + \nu t^{-(1-r)/\alpha}.$$

Then as in (1.2) we have

$$\sup_{s \in [t, t+1]} \psi(s)^{1+\alpha} \leq 2^{1+\alpha} \{C_0(1+t)^r (\psi(t) - \psi(t+1)) + I_2(t)\} \tag{1.7}$$

where

$$\begin{aligned}
I_2(t) &= \nu t^{-(1-r)/\alpha} \left\{ C_0(1+t)^r \left( \left( \frac{1+t}{t} \right)^{-(1-r)/\alpha} - 1 \right) \right. \\
&\quad \left. + \nu^{-1} t^{(1-r)/\alpha} g(t) + \nu^\alpha t^{-(1-r)} \right\}.
\end{aligned}$$

Using the assumption on  $g(t)$  and the inequality

$$\left( \frac{t+1}{t} \right)^{-(1-r)/\alpha} - 1 \leq -\frac{(1-r)}{2\alpha} t^{-1} \quad \text{for large } t$$

we have

$$\begin{aligned}
I_2(t) &\leq \nu t^{-(1-r)/\alpha + r - 1} \left\{ -C_0(1+t)^r (1-r)/2\alpha + \frac{1}{2} \nu^\alpha \right\} \\
&< 0 \quad \text{for large } t,
\end{aligned}$$

where we choose  $\nu$  sufficiently small.

Thus there exists  $T_2 > 0$  such that if  $t \geq T_2$

$$\sup_{s \in [t, t+1]} \phi(s)^{1+\alpha} \leq 2^{1+\alpha} C_0 (1+t)^r (\phi(t) - \phi(t+1))$$

and therefore as is in the proof of (i)

$$\begin{aligned} w(t-n) - w(t) &\leq -\alpha C_0'^{-1} \int_0^{n-1} \frac{1}{(t-x)^r} dx \quad (C_0' \equiv 2^{1+\alpha} C_0) \\ &\leq \alpha C_0'^{-1} (1-r)^{-1} \{(t-n+1)^{1-r} - t^{1-r}\} \end{aligned}$$

for any positive integer  $n$ , which proves (ii) immediately.

Proofs of (iii) and (iv). The proofs of (iii) and (iv) are almost the same, and we give only the one of (iv). By (1.1) we have

$$\phi(t+1) \leq \frac{C_0(1+t)^r}{C_0(1+t)^r + 1} \phi(t) + g(t)$$

and hence, by induction,

$$\begin{aligned} \phi(t+1) &\leq \prod_{i=0}^n \frac{C_0(t+1-i)^r}{C_0(t+1-i)^r + 1} \phi(t-n) + \sum_{j=0}^n \prod_{i=0}^j \frac{C_0(t+1-i)^r}{C_0(t+1-i)^r + 1} g(t-j) \\ &\equiv I_1 + I_2. \end{aligned}$$

Fix the integer  $n$  such that  $n \leq t < n+1$ . Then, we have easily

$$\begin{aligned} \log(I_1) &\leq -\sum_{i=0}^n \frac{1}{C_0(t+1-i)^r + 1} + \sup_{s \in [0, 1]} \log \phi(s) \\ &\leq -\int_0^n \frac{1}{C_0(t+1-x)^r + 1} dx + \sup_{s \in [0, 1]} \log \phi(s) \end{aligned}$$

(we may assume  $\sup_{s \in [0, 1]} \phi(s) > 0$ )

$$\begin{aligned} &\leq -\frac{1}{(C_0+1)(1-r)} (1+t)^{1-r} + \frac{1}{(C_0+1)(1-r)} (t+1-n)^{1-r} \\ &\quad + \log \{C_0 \phi(0) + g(0)\}^{1/(1+\alpha)}. \end{aligned}$$

Thus we have

$$I_1 \leq C_5 \exp\left\{-\frac{1}{(C_0+1)(1-r)} (1+t)^{1-r}\right\}.$$

In a similar way,

$$\begin{aligned} I_2 &\leq \sum_{j=0}^{n-1} \exp\left\{-\int_0^j \frac{1}{C_0(1+t-x)^r + 1} dx\right\} \cdot g(t-j) \\ &\leq C_6 \exp\left\{-\frac{1}{(C_0+1)(1-r)} (1+t)^{1-r}\right\}. \end{aligned}$$

The proof is now complete.

## 2. Nonlinear evolution equation.

In this section we apply the results in §2 to nonlinear evolution equations. We discuss the decay property of solutions of the equations:

$$u''(t) + B(t)u'(t) + A(t)u(t) = f(t) \quad (\text{a})$$

and

$$B(t)u'(t) + A(t)u(t) = f(t), \quad (\text{b})$$

where  $A(t)$ ,  $B(t)$  are nonlinear operators mentioned in the introduction.

The pairing of  $V^*$  and  $V$  is denoted by  $(\cdot, \cdot)$ . We make the following assumptions.

A<sub>1</sub>.  $A(t)$  and  $F_{A(t)}$  satisfy the conditions

$$k_0(1+t)^{-\alpha} \|u\|_V^p \leq k_1 F_{A(t)}(u) \leq (A(t)u, u) \quad \text{for } u \in V$$

where  $k_0, k_1, \alpha, p$  are constants such that  $k_0, k_1 > 0$ ,  $\alpha \geq 0$ , and  $p > 1$ .

A<sub>2</sub>. For each  $u \in V$ ,  $F_{A(t)}(u)$  is differentiable in  $t$  and

$$0 \leq -\frac{d}{dt} F_{A(t)}(u) \leq \rho(t) F_{A(t)}(u)$$

with a function  $\rho(t)$  tending to 0 as  $t \rightarrow \infty$ .

(For simplicity we write  $F_{A(t)}(u)$  for  $\frac{d}{dt} F_{A(t)}(u)$ .)

A<sub>3</sub>.  $B(t)$  satisfies

$$k_2(1+t)^{-\theta_0} \|u\|_W^{r+2} \leq (B(t)u, u)$$

and

$$\|B(t)u\|_{W^*} \leq k_3((1+t)^{-\theta_1} \|u\|_W^{r+1} + (1+t)^{-\theta_2} \|u\|_W)$$

where

$$k_2, k_3 > 0, r, \theta_0 \geq 0, \theta_1, \theta_2, \quad \text{are all constants.}$$

A<sub>4</sub>.  $f(\cdot) \in L_{\text{loc}}^{(r+2)/(r+1)}(R^+; W^*)$  ( $R^+ = [0, \infty)$ )

with  $r$  in  $A_3$ .

REMARK 1. Even for the case of  $A(t)$ ,  $B(t)$  being independent of  $t$  our assumptions here are somewhat simpler and weaker than those in [6].

Now we give the definition of the solutions of (a) and (b).

DEFINITION 1. A  $V$ -valued function  $u(t)$  on  $R^+$  is said to be a solution of (a) if

$$\begin{aligned} u &\in C(R^+; V), \quad u' \in C(R^+; H) \cap L_{\text{loc}}^{r+2}(R^+; W), \\ u'' &\in L_{\text{loc}}^1(R^+; V^*) \end{aligned}$$

and the equation (a) is valid in  $V^*$  for almost all  $t \in R^+$ .

DEFINITION 2. A  $V$ -valued function  $u(t)$  on  $R^+$  is said to be a solution of (b) if

$$u \in C(R^+; V), \quad u' \in L_{loc}^{+2}(R^+; W)$$

and (b) is valid in  $V^*$  for a.e.  $t \in R^+$ .

Let  $u(t)$  be a solution of (a). Then we have formally, for  $t_1, t_2 \in R^+$ ,

$$\begin{aligned} & \frac{1}{2} \|u'(t_2)\|_H^2 + F_{A(t_2)}(u(t_2)) - \int_{t_1}^{t_2} F_{A'(t)}(u(t)) dt + \int_{t_1}^{t_2} (B(t)u'(t), u'(t)) dt \\ & = \frac{1}{2} \|u'(t_1)\|_H^2 + F_{A(t_1)}(u(t_1)) + \int_{t_1}^{t_2} (f(t), u'(t)) dt \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \int_{t_1}^{t_2} (A(t)u(t), u(t)) dt & = \int_{t_1}^{t_2} \{-(B(t)u'(t), u(t)) + (f(t), u(t))\} dt \\ & + \int_{t_1}^{t_2} \|u'(t)\|_H^2 dt + (u'(t_1), u(t_1)) - (u'(t_2), u(t_2)). \end{aligned} \tag{2.2}$$

Also, if  $u(t)$  is a solution of (b) we have formally

$$\begin{aligned} & F_{A(t_2)}(u(t_2)) - \int_{t_1}^{t_2} F_{A'(t)}(u(t)) dt + \int_{t_1}^{t_2} (B(t)u'(t), u'(t)) dt \\ & = F_{A(t_1)}(u(t_1)) + \int_{t_1}^{t_2} (f(t), u'(t)) dt \end{aligned} \tag{2.3}$$

and

$$\int_{t_1}^{t_2} (A(t)u(t), u(t)) dt = \int_{t_1}^{t_2} \{-(B(t)u'(t), u(t)) + (f(t), u(t))\} ds. \tag{2.4}$$

In what follows we consider only the solutions which satisfy (2.1)–(2.2) or (2.3)–(2.4). If  $A(t)$  is a linear operator, all the solutions of (a) and (b) satisfy (2.1)–(2.2) and (2.3)–(2.4), respectively, under some additional natural conditions on spaces (see Strauss [10]). In general we do not know if the above energy equalities hold or not. However, even if these equalities are not valid, in many important cases, solution  $u$  can be obtained as a (weak or weak\*) limit function of the approximate solutions  $u_m(t)$  ( $m=1, 2, 3, \dots$ ) which satisfy the above equalities (see Lions [2], Lions and Strauss [3], Tsutsumi [11] etc.), and the results of this section remain valid for such solutions. Under the assumptions mentioned above we shall derive a difference inequality concerning the energy of  $u(t)$ .

Let  $u(t)$  be a solution of (a) which satisfies (2.1)–(2.2). Then, by the assumptions  $A_3$  and (2.1), we have

$$\begin{aligned}
 k_2 \int_t^{t+1} (1+s)^{-\theta_0} \|u'(s)\|_{W^*}^{r+2} ds &\leq \int_t^{t+1} (B(s)u'(s), u'(s)) ds \\
 &\leq E(u(t)) - E(u(t+1)) \\
 &\quad + \int_t^{t+1} \|f(s)\|_{W^*} \|u'(s)\|_W ds
 \end{aligned}
 \tag{2.5}$$

where we set

$$E(u(t)) = \frac{1}{2} \|u'(t)\|_H^2 + F_{A(t)}(u(t)).$$

Since

$$\begin{aligned}
 &\int_t^{t+1} \|f(s)\|_{W^*} \|u'(s)\|_W ds \\
 &\leq \left(\frac{r+1}{r+2}\right) \left(\frac{2}{r+2}\right)^{1/(r+1)} k_2^{-1/(r+1)} \int_t^{t+1} (1+s)^{\theta_0/(r+1)} \|f(s)\|_{W^*}^{(r+2)/(r+1)} ds \\
 &\quad + \frac{1}{2} k_2 \int_t^{t+1} (1+s)^{-\theta_0} \|u'(s)\|_{W^*}^{r+2} ds,
 \end{aligned}$$

we have from (2.5)

$$\begin{aligned}
 &k_2 \int_t^{t+1} (1+s)^{-\theta_0} \|u'(s)\|_{W^*}^{r+2} ds \\
 &\leq 2\{E(u(t)) - E(u(t+1))\} + \text{const.} \int_t^{t+1} (1+s)^{\theta_0/(r+1)} \|f(s)\|_{W^*}^{(r+2)/(r+1)} ds \\
 &\equiv k_2 D(t)^{r+2}.
 \end{aligned}
 \tag{2.6}$$

Therefore there exist  $t_1 \in [t, t + \frac{1}{4}]$  and  $t_2 \in [t + \frac{3}{4}, t + 1]$  such that

$$\|u'(t_i)\|_W \leq 4^{1/(r+2)} (1+t)^{\theta_0/(r+2)} D(t) \quad (i=1, 2). \tag{2.7}$$

Thus, by the equation (2.2) with above  $t_i$ , we have

$$\begin{aligned}
 \int_{t_1}^{t_2} (A(s)u(s), u(s)) ds &\leq \int_{t_1}^{t_2} \{ |(B(s)u'(s), u(s))| + |(f(s), u(s))| \} ds \\
 &\quad + \int_{t_1}^{t_2} \|u'(s)\|_H^2 ds + \sum_{i=1}^2 |(u'(t_i), u(t_i))| \\
 &\leq \text{const.} \int_{t_1}^{t_2} \{ (1+s)^{-\theta_1} \|u'(s)\|_{W^*}^{r+1} + (1+s)^{-\theta_2} \|u'(s)\|_W \\
 &\quad + \|f(s)\|_{W^*} \} ds \max_{s \in [t, t+1]} \|u(s)\|_W \\
 &\quad + \text{const.} \{ (1+t)^{\theta_0/(r+2)} D(t) \max_{s \in [t, t+1]} \|u(s)\|_H + (1+t)^{2\theta_0/(r+2)} D(t)^2 \}
 \end{aligned}$$



$$\begin{aligned} &\leq \text{const.} \{ (1+t)^{-\theta_1+\theta_0(r+1)/(r+2)} D(t)^{r+1} + (1+t)^{-\theta_2+\theta_0/(r+2)} D(t) \\ &\quad + \delta(t) + (1+t)^{\theta_0/(r+2)} D(t) \} (1+t)^{\alpha/p} \max_{s \in [t, t+1]} E(u(s))^{1/p} \\ &\quad + (1+t)^{2\theta_0/(r+2)} D(t)^2 \} \end{aligned} \tag{2.8}$$

where we set

$$\delta(t) = \left( \int_t^{t+1} \|f(s)\|_{W^*}^{(r+2)/(r+1)} ds \right)^{(r+1)/(r+2)}.$$

From (2.6), (2.8) and  $A_1$ , we see that there exists a time  $t^* \in [t_1, t_2]$  such that

$$\begin{aligned} E(u(t^*)) &\leq \text{const.} [ (1+t)^{2\theta_0/(r+2)} D(t)^2 + \{ (1+t)^{-\theta_1+\theta_0(r+1)/(r+2)} D(t)^{r+1} \\ &\quad + (1+t)^{-\theta_2+\theta_0/(r+2)} D(t) + \delta(t) + (1+t)^{\theta_0/(r+2)} D(t) \} (1+t)^{\alpha/p} \\ &\quad \times \max_{s \in [t, t+1]} E(u(s))^{1/p} ] \end{aligned} \tag{2.9}$$

and therefore by (2.1) we have

$$\begin{aligned} \max_{s \in [t, t+1]} E(u(s)) &\leq E(u(t^*)) + \int_t^{t+1} \{ (B(s)u'(s), u'(s)) \\ &\quad + |F_{A'(s)}(u(s))| + |(f(s), u'(s))| \} ds \\ &\leq E(u(t^*)) + \text{const.} \{ (1+t)^{-\theta_1+\theta_0} D(t)^{r+2} \\ &\quad + (1+t)^{-\theta_2+2\theta_0/(r+2)} D(t)^2 + \delta(t)(1+t)^{\theta_0/(r+2)} D(t) \} \\ &\quad + \rho(t) \max_{s \in [t, t+1]} E(u(s)). \end{aligned} \tag{2.10}$$

Since  $\rho(t) \rightarrow 0$  as  $t \rightarrow \infty$ , we see from (2.9) and (2.10) that there exists  $T > 0$  such that if  $t > T$ ,

$$\begin{aligned} \max_{s \in [t, t+1]} E(u(s)) &\leq \text{const.} [ (1+t)^{2\theta_0/(r+2)} D(t)^2 + (1+t)^{-\theta_1+\theta_0} D(t)^{r+2} \\ &\quad + (1+t)^{-\theta_2+2\theta_0/(r+2)} D(t)^2 + \delta(t)(1+t)^{\theta_0/(r+2)} D(t) \\ &\quad + (1+t)^{\alpha/(p-1)} \{ (1+t)^{-\theta_1+\theta_0(r+1)/(r+2)} D(t)^{r+1} \\ &\quad + (1+t)^{-\theta_2+\theta_0/(r+2)} D(t) + \delta(t) + (1+t)^{\theta_0/(r+2)} D(t) \}^{p/(p-1)} ]. \end{aligned} \tag{2.11}$$

Here we assume

$$\delta(t) \leq \text{const.} (1+t)^{-\lambda_0} \tag{2.12}$$

where  $\lambda_0$  is a constant satisfying the following condition :

$$\lambda_0 \geq \max(- (r+1)\theta_1/(r+2) + \theta_0, - (r+1)\theta_2/2 + \theta_0, \alpha/p - \theta_1 + \theta_0, \alpha(r+1)/p + \theta_0) \tag{2.13}$$

Then we can see easily from (2.11) that  $E(u(t))$  is bounded on  $R^+$ . Indeed, if  $E(u(t)) \leq E(u(t+1))$  for some  $t \geq T$  we have

$$D(t) \leq \text{const.} (1+t)^{\theta_0/(r+1)(r+2)} \delta(t)^{1/(r+1)}$$

and hence, by (2.11) and (2.12),

$$\begin{aligned} \max_{s \in [t, t+1]} E(u(s)) &\leq \text{const.} \{ (1+t)^{2(-\lambda_0+\theta_0)/(r+1)} \\ &\quad + (1+t)^{(-\lambda_0-\theta_1(r+1)/(r+2)+\theta_0)(r+2)/(r+1)} \\ &\quad + (1+t)^{2(-\lambda_0-(r+1)\theta_2/2+\theta_0)/(r+1)} \\ &\quad + (1+t)^{(-\lambda_0+\theta_0/(r+2))(r+2)/(r+1)} \\ &\quad + (1+t)^{(-\lambda_0+\alpha/p-\theta_1+\theta_0)p/(p-1)} \\ &\quad + (1+t)^{p(-\lambda_0+\alpha(r+1)/p-(r+1)\theta_2+\theta_0)/(p-1)(r+1)} \\ &\quad + (1+t)^{(-\lambda_0+\alpha/p)p/(p-1)} \\ &\quad + (1+t)^{(-\lambda_0+\alpha(r+1)/p+\theta_0)p/((p-1)(r+1))} \} \\ &\leq M (= \text{const.}) < +\infty \quad (\text{by (2.13)}). \end{aligned}$$

Therefore we have for  $t \geq T$ ,

$$E(u(t)) \leq \max( \max_{s \in [T, T+1]} E(u(s)), M ).$$

However, by a similar inequality as (2.6), we know

$$E(u(t)) \leq E(u(0)) + \text{const.} \int_0^{T+1} (1+s)^{\theta_0/(r+1)} \|f(s)\|_{W^*}^{(r+2)/(r+1)} ds$$

for  $t \leq T+1$ , and we obtain

$$E(u(t)) \leq C_6 < +\infty \quad (2.14)$$

where  $C_6$  is a constant depending on  $E(u(0))$ ,  $M$  and  $T$ . In what follows  $C_i$  ( $i=7, 8, \dots$ ) denote constants depending on  $E(u(0))$  and other known constants. By (2.11) and (2.14) we have easily, for  $t \geq T$ ,

$$\max_{s \in [t, t+1]} E(u(s)) \leq C_7 \{ (1+t)^{\tau_1} D(t)^{\tau_2} + \delta(t)^2 + (1+t)^{\alpha/(p-1)} \delta(t)^{p/(p-1)} \} \quad (2.15)$$

where

$$\begin{aligned} \tau_1 &= \max \{ 2\theta_0/(r+2), -\theta_1+\theta_0, -\theta_2+2\theta_0/(r+2), \\ &\quad \alpha/(p-1)+p(-\theta_1+\theta_0(r+1)/(r+2))/(p-1), \\ &\quad \alpha/(p-1)+p(-\theta_2+\theta_0/(r+2))/(p-1), \alpha/(p-1)+p\theta_0/((r+2)(p-1)) \}, \\ \tau_2 &= \min(2, p/(p-1)). \end{aligned}$$

Now, recalling the definition of  $D(t)$ , we have from (2.15)

$$\begin{aligned} \max_{s \in [t, t+1]} E(u(s))^{(r+2)/\tau_2} &\leq C_8 \{ (1+t)^{\tau_1(r+2)/\tau_2} \{ (E(u(t)) - E(u(t+1))) \\ &\quad + g_0(t) \} \quad (t \geq T) \end{aligned} \quad (2.16)$$

where

$$g_0(t) = (1+t)^{\theta_0/(r+1)+\tau_1(r+2)/\tau_2} \delta(t)^{(r+2)/(r+1)} + \delta(t)^{2(r+2)/\tau_2} + (1+t)^{\alpha(r+2)/((p-1)\tau_2)} \delta(t)^{p(r+2)/(p-1)\tau_2}. \tag{2.17}$$

Thus the required difference inequality concerning the energy of the solution  $u(t)$  of the equation (a) has been derived. In a similar manner we obtain for a solution  $u(t)$  of the equation (b):

$$\max_{s \in [t, t+1]} F_{A(s)}(u(s))^{(r+2)/\tau_2} \leq C_9 \{ (1+t)^{\tau'_1(r+2)/\tau_2} (F_{A(t)}(u(t)) - F_{A(t+1)}(u(t+1)) + g_0(t)) \} \tag{2.18}$$

where

$$\begin{aligned} \tau'_1 = \max \{ & 2\theta_0/(r+2), -\theta_1 + \theta_0, -\theta_2 + 2\theta_0/(r+2), \\ & \alpha/(p-1) + p(-\theta_1 + \theta_0(r+1)/(r+2))/(p-1), \\ & \alpha/(p-1) + p(-\theta_2 + \theta_0/(r+2))/(p-1) \}. \end{aligned}$$

Applying the results in section 2 to the above inequalities we obtain the following theorems.

**THEOREM 2.** *Suppose  $A_1$ - $A_4$ . Let  $u(t)$  be a solution of (a), satisfying (2.1) and (2.2). Then it holds that*

(i) *if  $(r+2)/\tau_2 - 1 > 0$ ,  $\tau_1(r+2)/\tau_2 = 1$  and*

$$\lim_{t \rightarrow \infty} (\log t)^{1+\tau_2/(r+2)-\tau_2} g_0(t) = 0,$$

*then we have*

$$E(u(t)) \leq C_{10} (\log(1+t))^{-\tau_2/(r+2)-\tau_2};$$

(ii) *if  $(r+2)/\tau_2 - 1 > 0$ ,  $0 \leq \tau_1(r+2)/\tau_2 < 1$  and*

$$\lim_{t \rightarrow \infty} t^{1-\tau_1(r+2)/\tau_2 + (1+\tau_2/(r+2)-\tau_2)} g_0(t) = 0$$

*then we have*

$$E(u(t)) \leq C_{11} (1+t)^{-(\tau_2-\tau_1(r+2))/(r+2)-\tau_2};$$

(iii) *if  $(r+2)/\tau_2 - 1 = 0$ ,  $\tau_1(r+2)/\tau_2 = 1$  (i. e.  $\tau_1 = 1$ ) and*

$$g_0(t) \leq \text{const.} (1+t)^{-\eta-1} \quad (\eta > 0)$$

*then*

$$E(u(t)) \leq C_{12} (1+t)^{-\eta'} \quad (\eta' = \min(C_8^{-1}, \eta));$$

(iv) *if  $(r+2)/\tau_2 - 1 = 0$ ,  $0 \leq \tau_1 < 1$  and*

$$g_0(t) \leq \text{const.} \exp\{-(t+1)^{1-r'}\}$$

*with some  $r' < \tau_1$ , then*

$$E(u(t)) \leq C_{13} \exp\left\{-\frac{t^{1-\tau_1}}{(C_8+1)(1-\tau_1)}\right\}.$$

THEOREM 3. Suppose  $A_1$ - $A_4$ . Let  $u(t)$  be a solution of (b), satisfying (2.3) and (2.4). Then the assertion of Theorem 2 holds with  $\tau_1$  and  $E(u(t))$  replaced by  $\tau'_1$  and  $F_{A(t)}(u(t))$ , respectively.

REMARK 2. The condition (2.12) is automatically fulfilled if  $g_0(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

The following corollaries are special cases of above theorems.

COROLLARY 1. Suppose  $A_1$ - $A_4$  are valid with  $p=2$ ,  $\theta_0=\theta_1=\theta_2$  and  $\alpha=0$ . Let  $u(t)$  be a solution of (a) satisfying (2.1), (2.2). Then it holds that  $\tau_1=2\theta_0/(r+2)$ ,  $\tau_2=2$  and

$$(i) \text{ if } r>0, \theta_0=1 \text{ and } \lim_{t \rightarrow \infty} (\log t)^{1+2/r} g_1(t)=0$$

$$(g_1(t) \equiv (1+t)^{\theta_0(r+1)} \delta(t)^{(r+2)/(r+1)})$$

then

$$E(u(t)) \leq C'_{10} (\log(1+t))^{-2/r};$$

$$(ii) \text{ if } r>0, 0 \leq \theta_0 < 1 \text{ and } \lim_{t \rightarrow \infty} t^{(1-\theta_0)(1+2/r)} g_1(t)=0$$

then

$$E(u(t)) \leq C'_{11} (1+t)^{-2(1-\theta_0)/r};$$

$$(iii) \text{ if } r=0, \theta_0=1 \text{ and } g_1(t) \leq \text{const. } (1+t)^{-\eta-1} \quad (\eta > 0)$$

then

$$E(u(t)) \leq C'_{12} (1+t)^{-\eta'}$$

for some  $\eta' > 0$ ;

$$(iv) \text{ if } r=0, 0 \leq \theta_0 < 1 \text{ and } g_1(t) \leq \text{const. } \exp\{-(1+t)^{1-\theta_0}\} \quad (\theta_0 < \theta_0), \text{ then}$$

$$E(u(t)) \leq C'_{13} \exp\{-C_{14} t^{1-\theta_0}\}.$$

COROLLARY 2. Suppose  $A_1$ - $A_4$  are valid with  $r=\theta_0=\theta_1=0$ ,  $p \geq 2$  and let  $u(t)$  be a solution of (b) satisfying (2.3) and (2.4). Then we have

$$\tau'_1 = \alpha/(p-1), \quad \tau_2 = p/(p-1)$$

and

$$(i) \text{ if } p>2, 2\alpha=p \text{ and } \lim_{t \rightarrow \infty} (\log t)^{1+p/(p-2)} (1+t) \delta(t)^2 = 0$$

then

$$F_{A(t)}(u(t)) \leq C''_{10} (\log(1+t))^{-p/(p-2)};$$

$$(ii) \text{ if } p>2, 0 \leq 2\alpha/p < 1 \text{ and } \lim_{t \rightarrow \infty} t^{(p-1-\alpha)/(p-2)} \delta(t) = 0$$

then

$$F_{A(t)}(u(t)) \leq C''_{11} (1+t)^{-(p-2\alpha)/(p-2)};$$

(iii) if  $p=2$ ,  $2\alpha=p$  (i. e.  $\alpha=1$ ) and  $\delta(t)\leq \text{const.}(1+t)^{-\eta-1}$  ( $\eta>0$ ),

then

$$F_{A(t)}(u(t))\leq C'_{12}(1+t)^{-\eta'}$$

for some  $\eta'>0$ ;

(iv) if  $p=2$ ,  $0\leq\alpha<1$  and

$$\delta(t)\leq \text{const. exp}\{-(t+1)^{1-\alpha'}\} \text{ for some } \alpha'<\alpha,$$

then

$$F_{A(t)}(u(t))\leq C'_{13} \exp\{-C_{15}t^{1-\alpha'}\}.$$

REMARK 3. In some concrete problems we can find precise values of  $\theta'$ ,  $\eta'$ ,  $C_{14}$  and  $C_{15}$  (cf. example 2 in section 3).

REMARK 4. If  $\theta_0=\alpha=0$  in (iv) in corollaries 2, 3, the conditions on  $\delta(t)$  can be replaced by

$$\delta(t)\leq \text{const. exp}\{-\lambda(t+1)\} \quad (\lambda>0).$$

### 3. Some examples.

Here we give some typical examples. We begin with a simple ordinary differential equation.

EXAMPLE 1. Consider the equation

$$\ddot{x}(t)+(t+1)^{-\theta}\rho(\dot{x}(t))+\beta(x(t))=f(t) \quad (t\geq 0). \tag{3.1}$$

Let  $\rho$  and  $\beta$  be continuous on  $R$  and satisfy the following conditions

$$k_0|s|^p\leq k_1\int_0^s\beta(\eta)d\eta\leq\beta(s)s \quad (p\geq 2)$$

and

$$k_2|s|^{r+2}\leq\rho(s)s\leq k_3(1+|s|^r)|s|^2.$$

Moreover we assume, for simplicity,  $f(t)$  is continuous on  $[0, \infty)$  and  $|f(t)|\rightarrow 0$  as  $t\rightarrow\infty$ .

Then all of the assumptions  $A_1$ - $A_4$  are satisfied with  $\alpha=0$ ,  $\rho(t)\equiv 0$ ,  $\theta_0=\theta_1=\theta_2=\theta$ . The equations (2.1) and (2.2) are of course fulfilled with the usual solutions of (3.1). In this case  $\tau_1$ ,  $\tau_2$  and  $g_0(t)$  in Theorem 2 become

$$\begin{aligned} \tau_1 &= \frac{2\theta}{r+2}, \quad \tau_2 = \frac{p}{p-1} \quad \text{and} \\ g_0(t) &= (1+t)^{\theta/(r+1)}\delta(t)^{(r+2)/(r+1)} + \delta(t)^{2(r+2)(p-1)/p} \\ &\quad + \delta(t)^{r+2}, \end{aligned}$$

respectively. Thus, for example, if  $(p-1)(r+2)>p$ ,  $2(p-1)\theta<p$  and

$$\lim_{t \rightarrow \infty} t^{(1-2(p-1)\theta/p)(1+p/(p-1)(r+2)-p)} g_0(t) = 0,$$

we have

$$|\dot{x}(t)|^2 + |x(t)|^p \leq \text{const.} (1+t)^{-(p-2(p-1)\theta)/((p-1)(r+2)-p)}.$$

The equation (3.1) with  $\theta=0$  has been investigated in detail in [8].

EXAMPLE 2. Consider the generalized Euler-Poisson-Darboux equation

$$\begin{cases} \frac{\partial^2}{\partial t^2} u - \Delta u + (1+t)^{-\theta} \rho\left(x, \frac{\partial}{\partial t} u\right) + \beta(x, u) = f(x, t) & \text{on } \Omega \times [0, \infty) \\ u|_{\partial\Omega} = 0 \end{cases} \tag{3.2}$$

where  $\Omega$  is a bounded domain in  $R^n$  and  $\partial\Omega$  its boundary. Let  $\rho(x, s)$  and  $\beta(x, s)$  be measurable on  $\Omega \times (-\infty, \infty)$  and continuous in  $s$  for each  $x$ , and satisfy the following conditions

$$\rho(x, 0) = 0, \quad k_0 |s_1 - s_2|^{r+2} \leq (\rho(x, s_1) - \rho(x, s_2))(s_1 - s_2) \leq k_1 (1 + |s_1| + |s_2|)^r |s_1 - s_2|^2$$

and

$$k_2 |s|^{r+2} \leq \beta(x, s) \leq k_3 |s|^{r+2}$$

for some  $k_0, k_1, k_2 > 0$  and  $\alpha, r$  satisfying

$$0 \leq \eta, r < \frac{4}{n-2} \quad \text{if } n \geq 3 \quad \text{and} \quad 0 \leq \eta, r < \infty \quad \text{if } n = 1, 2.$$

Let  $f(t) \in L^2_{\text{loc}}([0, \infty); L^2(\Omega))$  and  $(u_0, u_1) \in \dot{H}_1 \times L^2(\Omega)$ . Then the problem (3.2) admits a unique solution such that

$$u(t) \in C(R^+; \dot{H}_1), \quad u'(t) \in C(R^+; L^2(\Omega)) \cap L^2_{\text{loc}}(R^+; L^{r+2}(\Omega))$$

and  $(u(0), \frac{\partial}{\partial t} u(0)) = (u_0, u_1)$  (see [3], [10]). Thus we can take  $H = L^2(\Omega)$ ,  $W = L^{(r+2)}(\Omega)$ ,  $V = \dot{H}_1$ ,  $A = -\Delta + \beta(x, \cdot)$  and  $B(\cdot) = (1+t)^{-\theta} \rho(x, \cdot)$ . The equations (2.1)-(2.2) are known to be valid for such solutions. In this case we have

$$\tau_1 = \frac{2\theta}{r+2}, \quad p = \tau_2 = 2 \quad \text{and} \quad g_0(t) = (1+t)^{\theta/(r+1)} \delta(t)^{(r+2)/(r+1)} + \delta(t)^{r+2}.$$

Therefore the conclusion of Corollary 1 are applied to this equation. For a special case  $\rho\left(x, \frac{\partial}{\partial t} u\right) = \frac{\partial}{\partial t} u$  and  $\theta=1$ , we can obtain more precise result. Indeed, for this case, we have, for  $\forall \varepsilon > 0$ ,

$$\max_{s \in [t, t+1]} E(u(t)) \leq (32S + 4 + \varepsilon)(t+1) \{E(u(t)) - E(u(t+1))\} + \text{const.} (t+1)^2 \delta(t)^2$$

for  $t \geq T_\varepsilon$

where  $T_\varepsilon$  is a large time depending on  $\varepsilon$ , and  $S$  is the Sobolev constant :

$$\|u\|_{L^2} \leq S \|u\|_{\dot{H}_1} \quad \text{for } u \in \dot{H}_1.$$

Thus we obtain

$$E(u(t)) \leq C(\varepsilon)(1+t)^{-(32S+4+\varepsilon)^{-1}}$$

provided that  $g_i(t) = o(t^{-(32S+4+\varepsilon)^{-1}})$  as  $t \rightarrow \infty$ .

Moreover, for a special case, e. g.,  $n=3, \theta=1, \rho(x, \frac{\partial}{\partial t} u) = -\frac{\partial}{\partial t} u, \beta(x, u) = u^3$ , we can discuss the decay of classical solutions (cf. Sather [9], Nakao [7]). But the details are omitted.

EXAMPLE 3. We consider a first order equation :

$$\begin{cases} \frac{\partial}{\partial t} u - C(t) \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( \left| \frac{\partial}{\partial x_i} u \right|^{p-2} \frac{\partial}{\partial x_i} u \right) + \beta(x, u) = f(x, t) & \text{on } \Omega \times [0, \infty) \\ u|_{\partial\Omega} = 0 & (p \geq 2) \end{cases} \quad (3.3)$$

where  $\Omega$  is a domain as in (3.2) and  $\beta(x, u)$  a nonlinear function continuous in  $u$  for each  $x \in \Omega$  and measurable in  $x$  for each  $u \in R$ , satisfying

$$0 \leq \beta(x, u)u \leq \text{const.}(1 + |u|^q)|u| \quad \text{and} \quad (\beta(x, u) - \beta(x, v))(u - v) \geq 0$$

with

$$0 \leq q < \frac{np}{n-p} - 1 \quad \text{if } n > p \quad \text{and} \quad 0 \leq q < \infty \quad \text{if } 1 \leq n \leq p$$

and  $C(t)$  is a differentiable function such that

$$k_0|t+1|^{-\alpha} \leq C(t) \leq k_1|t+1|^{-\alpha} \quad \text{and} \quad |C'(t)| \leq \text{const.}|t+1|^{-\alpha-\varepsilon} \quad (\varepsilon > 0).$$

In this case we can take  $W = H = L^2(\Omega), V = W_0^{1,p}$  and .

$$F_{A(t)}(u) = \frac{C(t)}{p} \int_{\Omega} \sum_{i=1}^n \left| \frac{\partial}{\partial x_i} u \right|^p dx + \int_{\Omega} \int_0^{u(x)} \beta(x, s) ds dx.$$

For any initial value  $u_0 \in W_0^{1,p}$ , the problem (3.3) admits a unique solution  $u(t) \in L_{loc}^{\infty}(R^+, W_0^{1,p})$  with  $u(0) = u_0$  given as a weak\* limit of approximate solutions  $\{u_m(t)\}$  which satisfy the equalities (2.3), (2.4) (see [2]). Thus, we conclude that the result of Corollary 2 is valid for the solutions of (3.3).

EXAMPLE 4. Consider the equation :

$$\begin{cases} \frac{\partial^2}{\partial t^2} u - C_0(t) \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( \left| \frac{\partial}{\partial x_i} u \right|^{p-2} \frac{\partial}{\partial x_i} u \right) - C_1(t) \Delta \frac{\partial}{\partial t} u = f & \text{on } \Omega \times R^+ \\ u|_{\partial\Omega} = 0. \end{cases}$$

In this case we can take  $V = W_0^{1,p}, W = \dot{H}_1$  and  $H = L^2(\Omega)$ . For the existence of solution see [11]. Under appropriate conditions on  $C_0(t)$  and  $C_1(t)$ , we can discuss the decay property of solutions for (3.4). But we omit the details.

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Mitsuhiro NAKAO  
Department of Mathematics  
College of General Education  
Kyushu University  
Fukuoka 810, Japan