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Homeomorphisms on a three dimensional handle

By Mitsuyuki OCHIAI

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McMillan proved that any two sets of generators for $\pi_1(H)$ are equivalent for an orientable handle H. We extend his result to the non-orientable case. These results may be interesting in view of non-orientable Heegaard diagrams of closed 3-manifolds, in particular $P^2\times S^1$ which has that of genus two. All manifolds considered are to be triangulated. All embeddings and homeomorphisms are to be piecewise linear.

DEFINITION. Let H be a compact connected 3-manifold. We say that H is an orientable or non-orientable handle with genus n respectively when H is homeomorphic to $D_1^2\times S^1 \sharp \cdots \sharp D_n^2\times S^1$ or $M_1^2\times I \sharp \cdots \sharp M_n^2\times I$ where D_i^2 is a 2-disk, S^1 is a 1-sphere, M_i^2 is a Mobius band, I is a unit interval and $\sharp$ is a disk sum (boundary connected sum).

Note that $D^2\times S^1 \sharp M^2\times I$ is homeomorphic to $M^2\times I \sharp M^2\times I$.

DEFINITION. Let H be a handle with genus n and $J_1, \cdots, J_n$ mutually disjoint simple closed curves on ∂H. We say that $\{J_k\}_{k=1}^n$ is a system of generators for $\pi_1(H)$ when S is connected and the inclusion homomorphism $\pi_1(S)\rightarrow\pi_1(H)$ is onto where $S=\partial H-\bigcup_{k=1}^n \mathring{N}(J_k, \partial H)$ and $N(J_k, \partial H)$ is a regular neighborhood of J_k's in ∂H. (Compare the definition in [3].)

DEFINITION. Let $\{J_i\}_{i=1}^n$, $\{\tilde{J}_k\}_{k=1}^n$ be two systems of generators for $\pi_1(H)$. We say that $\{J_i\}_{i=1}^n$ is equivalent to $\{\tilde{J}_k\}_{k=1}^n$ when there is a homeomorphism of H onto H throwing the elements of $\{J_k\}_{k=1}^n$ onto those of $\{\tilde{J}_i\}_{i=1}^n$.

DEFINITION. Let M be a compact 3-manifold. We say that M is irreducible when any 2-sphere embedded in M bounds a 3-cell in M.

Hereafter let M be a compact connected 3-manifold such that ∂M is non-empty.

DEFINITION. Let L be a simple closed curve in M. Then the curve L is said to be orientable (resp. non-orientable) if $N(L, M)$ is homeomorphic to $D^2\times S^1$ (resp. $M^2\times I$ where M^2 is a Mobius band).

LEMMA 1. *If M is irreducible and $\pi_1(M)$ is n-free, then ∂M is connected.*

PROOF. The proof is by induction on the rank of $\pi_1(M)$. If $\pi_1(M)=\{0\}$, then each component of ∂M is a 2-sphere and so all the 2-spheres bound 3-cells.

Hence ∂M is only one 2-sphere and connected. We assume that the lemma is true when the rank of $\pi_1(M)$ is not greater than $(n-1)$. Then we will verify that the lemma is true when $\pi_1(M)$ is n-free. At first, let F be one of the components of ∂M. Then the inclusion homomorphism $\pi_1(F)\to\pi_1(M)$ has a non-trivial kernel, since $\pi_1(M)$ is n-free and $\pi_1(F)$ is not, by Nielsen-Schreier theorem [4]. By Loop theorem [6] and Dehn's lemma [5], there is a proper 2-disk D in M such that ∂D is not homotopic to zero in F, Then two cases happen.

Case (1). Suppose that ∂D does not separate F into two components. Then there is a simple closed curve L in F such that $L\cap\partial D$ is only one point. Let $N(L\cup D, M)$ be a regular neighborhood of $L\cup D$ in M. It is easy to see that $N(L\cup D, M)$ is an orientable or non-orientable handle. And $M=M_1\#N(L\cup D, M)$ where $M_1=\overline{M-N(L\cup D, M)}$. It is trivial that M_1 is irreducible and ∂M_1 is non-empty. By van Kampen [2], we have a following fact, $\pi_1(M)=\pi_1(M_1)*\pi_1(N(L\cup D, M))=\pi_1(M_1)*Z$. By Nielsen-Schreier theorem [4], $\pi_1(M_1)$ is also $(n-1)$-free. It follows that ∂M_1 is connected by induction. Hence ∂M is also connected.

Case (2). Suppose that ∂D separates F into two components. Then we will verify that D separates M into two components. For the purpose, suppose that D does not separate M into two components. Let E be $\overline{M-N(D, M)}$. Then E is connected and irreducible. By van Kampen [2], $\pi_1(M)=\pi_1(E)*Z$. It follows that $\pi_1(E)$ is $(n-1)$-free by Nielsen-Schreier theorem [4]. Hence ∂E is connected by induction but this contradicts that ∂D separates F into two components. Thus D separates M into two components M_1, M_2. By van Kampen [2], $\pi_1(M)=\pi_1(M_1)*\pi_1(M_2)$. Since ∂D is not homotopic to zero in F, $\pi_1(M_i)$ $(i=1, 2)$ is non-trivial. And so $\pi_1(M_i)$ $(i=1, 2)$ is m-free and $m<n$ by Nielsen-Schreier theorem [4]. It is easy to see that M_i $(i=1, 2)$ is irreducible and ∂M_i $(i=1, 2)$ is non-empty. Hence ∂M_i $(i=1, 2)$ is connected by induction. Since $M=M_1\# M_2$, ∂M is also connected. The proof is complete.

Now let M be satisfy the same conditions as in Lemma 1 and D a properly embedded 2-disk in M such that ∂D is not homotopic to zero in ∂M. Then we have;

COROLLARY 1.1. *If ∂D separates ∂M into two components, then D separates M into two components.*

It is clear that next Theorem 1 follows from Lemma 1.

THEOREM 1. *If M is irreducible and $\pi_1(M)$ is n-free, then M is an orientable or non-orientable handle with genus n.*

Note that ∂M is non-empty. (Compare Theorem 32.1 in [5] and Lemma in [3].)

For the time being let H be a handle with genus one and $\{J_k\}_{k=1}^m$ mutually disjoint simple closed curves in ∂H such that $S=\partial H-\bigcup_{k=1}^m \mathring{N}(J_k, \partial H)$ is connected ($m\geqq 1$). Then we have;

LEMMA 2. *If the inclusion homomorphism $\pi_1(S)\rightarrow\pi_1(H)$ is onto, then $m=1$ and J_1 generates $\pi_1(H)$ and is non-orientable when H is non-orientable.*

PROOF. It is trivial if H is orientable. Thus let H be non-orientable. Since S is connected, all of $\{J_k\}_{k=1}^m$ are non-orientable by Lickorish [**1**] and $m=1$ because of the inclusion homomorphism $\pi_1(S)\rightarrow\pi_1(H)$ being onto. The proof is complete.

Next let H be an orientable handle with genus n and $\{J_i\}_{i=1}^n$, $\{\tilde{J}_k\}_{k=1}^n$ be any two systems of generators for $\pi_1(H)$. Then the following lemma follows from McMillan's method.

LEMMA 3. *$\{J_i\}_{i=1}^n$ is equivalent to $\{\tilde{J}_k\}_{k=1}^n$.*

PROOF. Let d be the natural homeomorphism from H onto H^*, a disjoint copy of H. Then form the compact 3-manifold M by identifying points which correspond under $d/S=S^*$. Since the inclusion homomorphism $\pi_1(S)\rightarrow\pi_1(H)$ is onto, the inclusion homomorphism $\pi_1(H)\rightarrow\pi_1(M)$ is also onto by van Kampen [**2**]. It is also one-to-one since the identifying map is the natural homeomorphism of H. Hence $\pi_1(M)$ is n-free. Next each of H and M is embedded in a compact 3-manifold, which is simply connected, constructed by attaching n fat disks ($D^2\times I$) along $\partial H-\mathring{S}$. Apparently such a 3-manifold has an embedded 2-sphere in which S is contained and so it is simply connected. Consequently McMillan's method can apply to our lemma. (Compare the proof of Theorem in [**3**].) The proof is complete.

Hereafter suppose that H is a non-orientable handle with genus n and $J_1, \cdots, J_m$ ($m\geqq 1$) are mutually disjoint simple closed curves in ∂H such that $S=\partial H-\bigcup_{i=1}^m \mathring{N}(J_i, \partial H)$ is connected and the inclusion homomorphism $\pi_1(S)\rightarrow\pi_1(H)$ is onto. Now let D be a properly embedded 2-disk in H such that ∂D is contained in S and is not homotopic to zero in ∂H.

LEMMA 4. *If $\partial H-\partial D$ is connected, then $S-\partial D$ is also connected.*

PROOF. We may assume that at least one of J_k's is orientable. Now let ∂D separate S into two components S_1, S_2 respectively. Then there is a simple closed curve L in ∂H such that $L\cap\bigcup_{k=1}^m J_k=L\cap J_1$ is only one point and $\partial D\cap L$ is also only one point where J_1 is an orientable loop in $\{J_k\}_{k=1}^m$. Since the inclusion homomorphism $\pi_1(S)\rightarrow\pi_1(H)$ is onto, there is a loop $\tilde{L}$ in S such that $\tilde{L}$ is homotopic to L in H. And so the intersection number (mod 2) between $\tilde{L}$ and D is 1 since $L\cap D$ consists only one point. Consequently two boundaries of $N(\partial D, \partial H)$ is connected by an arc in $S-\mathring{N}(\partial D, \partial H)$. But it contradicts that ∂D separates S into two components. Hence $S-\partial D$ is connected.

LEMMA 5. *If at least one of* $\{J_k\}_{k=1}^m$ *is non-orientable, then there are two handles* H_1, H_2 *such that* $H=H_1 \# H_2$, H_1 *is a handle of genus one whose boundary contains* J_1 *where* J_1 *is a non-orientable loop in* $\{J_k\}_{k=1}^m$ *and is a system of generators for* $\pi_1(H_1)$, *and that* H_2 *is a handle of genus* $(n-1)$ *whose boundary contains* $\{J_k\}_{k=2}^m$ *which is a system of generators for* $\pi_1(H_2)$.

PROOF. We prove the lemma by induction of the genus of H. At first it is trivial by Lemma 2 when the genus of H is one. We assume that the lemma is true when the genus of H is less than n. Then we prove that the lemma is also true when the genus of H is n. As in the proof of Lemma 3, form the compact 3-manifold M. Then at least one component of ∂M is a Klein bottle K since J_1 is non-orientable. We recall that $\pi_1(M)$ is also n-free. Consider the inclusion homomorphism $\pi_1(K) \to \pi_1(M)$. Since $\pi_1(M)$ is n-free but $\pi_1(K)$ is not, the kernel of the inclusion homomorphism is non-trivial. By Loop theorem [**6**] and Dehn's lemma [**5**], there is a 2-disk D in M such that $D \cap \partial M = D \cap K = \partial D$ and ∂D is not homotopic to zero in K. We may assume from Lemma 1 in Lickorish [**1**] that ∂D is $\partial N(J_1, \partial H)$, where K contains J_1, or a meridian circle of K. Then first case does not happen, since $\pi_1(M)$ is free. By the general position argument, $D \cap S$ consist of only one arc and simple closed curves. If all the simple closed curves are homotopic to zero in ∂H, then they are also homotopic to zero in S because of S being connected. Thus there is a 2-disk $\tilde{D}$ such that $\partial\tilde{D}=\partial D$ and $\tilde{D} \cap S$ is only one arc. Then $\tilde{D} \cap H = E$ is a 2-disk and $E \cap \partial H = \partial E$, $E \cap \bigcup_{k=1}^m J_k = E \cap J_1$ and $E \cap J_1$ is only one point. Let $N(E \cup J_1, H)$ be a regular neighborhood of $E \cup J_1$ in H. Then $N(E \cup J_1, H)$ is a non-orientable handle with genus one. We set $H_1 = \overline{H - N(E \cup J_1, H)}$, then $H = H_1 \# N(E \cup J_1, H)$ and J_1 is contained in $N(E \cup J_1, H)$, in which it is a system of generators for $\pi_1(N(E \cup J_1, H))$. It is easy to see that H_1 is a handle with genus $(n-1)$ by Theorem 1. Next if $D \cap S$ contain at least a simple closed curve which is not homotopic to zero in ∂H, then there is a 2-disk E in H (or H^*) such that $E \cap \partial H = \partial E$, $E \cap \bigcup_{k=1}^m J_k = \emptyset$ and that ∂E is not homotopic to zero in ∂H. Then two cases happen.

Case (1). Suppose that ∂E separates ∂H into two components. Then by Corollary 1.1 E separates H into two components H_1, H_2. By Theorem 1, H_1, H_2 are handles with positive genus. (Since ∂E is not homotopic to zero in ∂H.) Thus $H = H_1 \# H_2$ and J_1 is contained in ∂H_1 or ∂H_2. Let ∂H_1 contain J_1 and $S_i = \partial H_i - \bigcup_{\alpha_i} \mathring{N}(J_{\alpha_i}, \partial H_i)$ where $\{J_k\}_{k \in \alpha_1} \cup \{J_d\}_{d \in \alpha_2} = \{J_i\}_{i=1}^m$. Then S_i $(i=1, 2)$ is connected and H_i $(i=1, 2)$ is a retract of H. Thus the inclusion homomorphism $\pi_1(S_i) \to \pi_1(H_i)$ $(i=1, 2)$ is onto. Since the genus of H_1 is less than n, by induction there is a non-orientable handle with genus one such that its boundary contains J_1.

Case (2). Suppose that $\partial H-\partial E$ is connected. Then by Lemma 4 $S-\partial E$ is connected. Hence there is a simple closed curve c on S which intersects ∂E with only one point, and which has no intersections with $\{J_i\}_{i=1}^m$. Let $N(E\cup c, H)$ be a regular neighborhood of $E\cup c$ in H. Thus $H=H_1 \# N(E\cup c, H)$ where $H_1=\overline{H-N(E\cup c, H)}$. By Theorem 1, H_1 is a handle such that J_1 is contained in ∂H_1. Since H_1 is a retract of H, the inclusion homomorphism $\pi_1(S_1)\to\pi_1(H_1)$ is onto where $S_1=\partial H_1-\bigcup_{k=1}^{m} N(J_k, \partial H_1)$. Since the genus of H_1 is less than n, by induction there is a handle with genus one such that its boundary contains J_1. (Note that case (2) does not happen if $m=n$.) The proof is complete.

LEMMA 6. Let $\{J_k\}_{k=1}^n$ be a system of generators for $\pi_1(H)$. Then at least one of $\{J_k\}_{k=1}^n$ is non-orientable.

PROOF. Since the inclusion homomorphism $\pi_1(S)\to\pi_1(H)$ is onto, S is non-orientable. Now we may assume that all of $\{J_k\}_{k=1}^n$ are orientable. Then S is embedded in a 2-sphere since S is connected, the Euler characteristics of ∂H is $2-2n$ and all of $\{J_k\}_{k=1}^n$ are orientable. It contradicts that S is non-orientable. The proof is complete.

Finally we have the following theorem.

MAIN THEOREM 2. *Let H be a non-orientable handle with genus n and $\{J_k\}_{k=1}^n$, $\{\tilde{J}_i\}_{i=1}^n$ two systems of generators for $\pi_1(H)$ both of which contain the same number of orientable loops. Then $\{J_k\}_{k=1}^n$ is equivalent to $\{\tilde{J}_i\}_{i=1}^n$.*

PROOF. We prove the theorem by induction of the genus of H. At first, it is trivial by Lemma 2 when the genus of H is one. We assume that the lemma is true when the genus of H is less than n. Then we prove that the lemma is also true when the genus of H is n. Let J_1 (resp. $\tilde{J}_1$) be a non-orientable loop in $\{J_k\}_{k=1}^n$ (resp. $\{\tilde{J}_i\}_{i=1}^n$) by Lemma 6. Then it follows from Lemma 5 that $M=M_1\# M_2=\tilde{M}_1\#\tilde{M}_2$, M_1 (resp. $\tilde{M}_1$) is a non-orientable handle of genus one such that J_1 (resp. $\tilde{J}_1$) is a system of generators for $\pi_1(H_1)$ (resp. $\pi_1(\tilde{H}_1)$), and M_2 (resp. $\tilde{M}_2$) is a handle of genus $(n-1)$ such that $\{J_k\}_{k=2}^n$ (resp. $\{\tilde{J}_i\}_{i=2}^n$) is a system of generators for $\pi_1(M_2)$ (resp. $\pi_1(\tilde{M}_2)$). Then two cases happen by the assumption in the theorem that both of $\{J_k\}_{k=1}^n$ and $\{\tilde{J}_i\}_{i=1}^n$ contain the same number of orientable loops. Case (1) is that H_2, $\tilde{H}_2$ are orientable and Case (2) is that H_2, $\tilde{H}_2$ are non-orientable. Then there is a homeomorphism h_2 of H_2 onto $\tilde{H}_2$ throwing the elements of $\{J_k\}_{k=2}^n$ onto those of $\{\tilde{J}_i\}_{i=2}^n$, by Lemma 3 in Case (1) and by induction in Case (2). Let h_1 be a homeomorphism of H_1 onto $\tilde{H}_1$ throwing J_1 onto $\tilde{J}_1$. Then we can find a homeomorphism, which extends both h_1, h_2, of H onto H throwing the elements of $\{J_k\}_{k=1}^n$ onto the elements of $\{\tilde{J}_i\}_{i=1}^n$ (see the last part of the proof in Theorem in [3]). This completes the proof.

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Mitsuyuki OCHIAI
Department of Mathematics
Faculty of Science
Tokyo Institute of Technology
Tokyo, Japan