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On the fundamental group of the complement of certain plane curves

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§ 0. Notations.

Throughout this paper, we use the following notations.

$\boldsymbol{Z}$: the integers or an infinite cyclic group

$(n_1, n_2, \cdots, n_k)$: the greatest common divisor of $n_1, n_2, \cdots, n_k$

G, G_1, G_2: groups

$Z(G)$: the center of G

$D(G)$: the commutator group of G

$G_1 * G_2, G_1 * G_2 * G$: the free product of G_1 and G_2 or of G_1, G_2 and G respectively

$\boldsymbol{Z}_p$: a cyclic group of order p

$F(p)$: a free group of rank p

$\{e\}$: the trivial group

e: the unit element

$\left.\begin{matrix} G(p\,;q) \\ G(p\,;q\,;r) \end{matrix}\right\}$: special groups. See the definitions in § 2.

§ 1. Introduction and statement of results.

Let C be an irreducible curve in the projective space $\boldsymbol{P}^2$ and let G be the fundamental group of the complement of C. So far known, we have only two cases: (I) G is infinite and the commutator group $D(G)$ is a free group of a finite rank (Zariski [8]; Oka [6]). (II) G is a finite group (Zariski [8]).

We do not know whether this is true or not in general. The purpose of this paper is to give a theorem which says that, for a certain case, we have only the case (I). Namely let

$$C: \prod_{j=1}^{l}(Y-\beta_j Z)^{\nu_j} - \prod_{i=1}^{m}(X-\alpha_i Z)^{\lambda_i}=0 \tag{1.1}$$

where X, Y and Z are homogenous coordinates of $\boldsymbol{P}^2$ and

(1.2) $$n=\sum_{j=1}^{l}\nu_j=\sum_{i=1}^{m}\lambda_i$$

is the degree of the curve C and $\{\alpha_i\}$ $(i=1, 2, \cdots, m)$ or $\{\beta_j\}$ $(j=1, 2, \cdots, l)$ are mutually distinct complex numbers respectively. C is not necessarily irreducible. Let $\nu=(\nu_1, \nu_2, \cdots, \nu_l)$ and $\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_m)$. The result is this:

THEOREM (1.3). *Assume that the singular points of C are contained in the intersection of lines: $\prod_{j=1}^{l}(Y-\beta_jZ)^{\nu_j}=\prod_{i=1}^{m}(X-\alpha_iZ)^{\lambda_i}=0$. Then the fundamental group $\pi_1(\boldsymbol{P}^2-C)$ is isomorphic to the group $G(\nu;\lambda;n/\nu)$. Therefore the structure of the group $\pi_1(\boldsymbol{P}^2-C)$ is decided by three integers n, ν, λ.*

As a corollary of Theorem (1.3) and Theorem (2.12), one obtains

COROLLARY (1.4). (i) *The center $Z(\pi_1(\boldsymbol{P}^2-C))$ contains a cyclic group $\boldsymbol{Z}_a$ such that $\boldsymbol{Z}_a\cap D(\pi_1(\boldsymbol{P}^2-C))=\{e\}$ where a is defined by the integer $ns/\lambda\cdot\nu$, $s=(\nu, \lambda)$.*

(ii) *The quotient group of $\pi_1(\boldsymbol{P}^2-C)$ by $\boldsymbol{Z}_a$ is isomorphic to*

$$\boldsymbol{Z}_{\nu/s}*\boldsymbol{Z}_{\lambda/s}*F(s-1).$$

(iii) *Therefore the commutator group $D(\pi_1(\boldsymbol{P}^2-C))$ is isomorphic to $D(\boldsymbol{Z}_{\nu/s}*\boldsymbol{Z}_{\lambda/s}*F(s-1))$. In the case of $s=1$ (i.e. C is irreducible), this is isomorphic to $F((\nu-1)(\lambda-1))$.*

As for the geometric meaning of $D(\pi_1(\boldsymbol{P}^2-C))$, we refer to Oka [4]. Note that $\boldsymbol{Z}_a$ is equal to the center $Z(\pi_1(\boldsymbol{P}^2-C))$ if $\pi_1(\boldsymbol{P}^2-C)$ is not abelian.

§ 2. Combinatorial group theory.

In this section, we consider a certain group theoretical problems which we encounter in the process of the calculation of the fundamental group.

DEFINITION (2.1). Let p and q be positive integers. A group $G(p;q)$ is defined by

(2.2) $$G(p;q)=\langle\omega, a_i\ (i\in\boldsymbol{Z});\ \omega=a_{p-1}a_{p-2}\cdots a_0, R_1, R_2\rangle$$

where

(2.3) $$R_1 \text{ (Periodicity)}: a_i=a_{i+q} \quad \text{for } i\in\boldsymbol{Z}$$

and

(2.4) $$R_2 \text{ (Conjugacy)}: a_{i+p}=\omega a_i\omega^{-1} \quad \text{for } i\in\boldsymbol{Z}.$$

(This group appears as a local fundamental group. See § 3.)

PROPOSITION (2.5). *Let $r=(p, q)$ and let $q_1=q/r$. Then ω^{q_1} is contained in the center $Z(G(p;q))$.*

PROOF. Let $p_1=p/r$. Then

$$\begin{aligned} a_i &= a_{i+p_1q} && \text{by (2.3)} \\ &= \omega^{q_1} a_i \omega^{-q_1} && \text{by (2.4)}. \end{aligned}$$

This says that ω^{q_1} is contained in $Z(G(p\,;q))$.

PROPOSITION (2.6).

$$\omega = a_i a_{i-1} \cdots a_{i-p+1} \quad \textit{for any } i \in \boldsymbol{Z}.$$

PROOF. This is proved by the two-sided induction on i starting at $i=p-1$. Assume that this is true for i. Then

$$\begin{aligned} a_{i+1}a_i \cdots a_{i-p+2} &= \omega a_{i-p+1}\omega^{-1}\cdot a_i a_{i-1} \cdots a_{i-p+1}\cdot a_{i-p+1}^{-1} && \text{by (2.4)} \\ &= \omega \\ a_{i-1}a_{i-2} \cdots a_{i-p} &= a_i^{-1}\cdot a_i a_{i-1} \cdots a_{i-p+1}\cdot \omega^{-1} a_i \omega && \text{by (2.4)} \\ &= \omega\,. \end{aligned}$$

Note that we need only (2.4) and $\omega=a_{p-1}a_{p-2}\cdots a_0$ to prove the above proposition.

Now let $q_1, q_2, \cdots, q_m$ be positive integers and let

(2.7) $\quad G(p\,;\{q_1, q_2, \cdots, q_m\}) = \langle \omega, a_i\ (i \in \boldsymbol{Z})\,;\ \omega = a_{p-1}a_{p-2}\cdots a_0,\ R_1',\ R_2 \rangle$

where R_2 is as before ((2.4)) and

(2.8) $\quad R_1'\colon\ a_i = a_{i+q_j} \quad \text{for } i \in \boldsymbol{Z} \text{ and } 1 \leqq j \leqq m\,.$

PROPOSITION (2.9). *$G(p\,;\{q_1, q_2, \cdots, q_m\})$ is isomorphic to $G(p\,;q)$ for $q=(q_1, q_2, \cdots, q_m)$.*

PROOF. We can write $q=k_1q_1+k_2q_2+\cdots+k_mq_m$ for some $k_1, \cdots, k_m \in \boldsymbol{Z}$. Then by (2.8) we get

(2.10) $\quad a_{i+q} = a_i \quad \text{for } i \in \boldsymbol{Z}\,.$

On the other side, (2.10) clearly implies (2.8).

DEFINITION (2.11). Let r be a positive integer. We define a group $G(p\,;q\,;r)$ by

$$G(p\,;q\,;r) = \langle \omega, a_i\ (i \in \boldsymbol{Z})\,;\ \omega = a_{p-1}a_{p-2}\cdots a_0,\ R_1,\ R_2,\ \omega^r = e \rangle$$

where R_1 and R_2 are as before ((2.3), (2.4)).

By the definition, $G(p\,;q\,;r)$ is a quotient group of $G(p, q)$. As is stated in Theorem (1.3), $G(p\,;q\,;r)$ appears as a global fundamental group. The following theorem describes the structure of $G(p;q;r)$.

THEOREM (2.12). *Let $s=(p, q)$ and $a=(q/s, r)$. Then we have*

(i) *The center $Z(G(p;q;r))$ contains the cyclic group $\boldsymbol{Z}_{r/a}$ generated by ω^a and $\boldsymbol{Z}_{r/a}\cap D(G(p;q;r))=\{e\}$.*

$$\left(\begin{array}{l}\textit{The latter is equivalent to that the composite homomorphism}\\ \qquad \boldsymbol{Z}_{r/a} \hookrightarrow G(p;q;r) \longrightarrow G(p;q;r)/D(G(p;q;r)) \\ \textit{is injective.}\end{array}\right)$$

(ii) *The quotient group $G(p;q;r)/\boldsymbol{Z}_{r/a}$ is isomorphic to $\boldsymbol{Z}_{p/s} * \boldsymbol{Z}_a * F(s-1)$.*

PROOF. Let $H_1(G(p;q;r))=G(p;q;r)/D(G(p;q;r))$ (the abelianization). Then it is easy to see that $H_1(G(p;q;r))$ is an abelian group generated by $\bar{a}_0, \bar{a}_1, \cdots, \bar{a}_{s-1}, \bar{\omega}$ and they have two relations:

$$\text{(i)}\quad r\cdot(p/s)\cdot\sum_{i=0}^{s-1}\bar{a}_i=0 \qquad \text{and} \qquad \text{(ii)}\quad \bar{\omega}=(p/s)\cdot\sum_{i=0}^{s-1}\bar{a}_i$$

where $\bar{g}$ is the equivalence class of $g\in G(p;q;r)$ in $H_1(G(p;q;r))$. This homological consideration proves that ω^a is an element of order r/a and $\boldsymbol{Z}_{r/a}\cap D(G(p;q;r))=\{e\}$ where $\boldsymbol{Z}_{r/a}$ is the cyclic group generated by ω^a. Write $a=k_1(q/s)+k_2r$. Then

$$\omega^a=(\omega^{q/s})^{k_1} \text{ by the relation } \omega^r=e.$$

Therefore by Proposition (2.5) ω^a is contained in $Z(G(p;q;r))$. The quotient group $\tilde{G}(p;q;r)\equiv G(p;q;r)/\boldsymbol{Z}_{r/a}$ is represented by

$$\tilde{G}(p;q;r)=\langle \omega, a_i\ (i\in\boldsymbol{Z});\ R_0, R_1, R_2, R_3\rangle$$

where

$$\begin{aligned} &R_0:\ \omega=a_{p-1}a_{p-2}\cdots a_0 \\ &R_1:\ a_i=a_{i+q} && \text{for } i\in\boldsymbol{Z} \\ &R_2:\ a_{i+p}=\omega a_i\omega^{-1} && \text{for } i\in\boldsymbol{Z} \\ &R_3:\ \omega^a=e. \end{aligned}$$

We can write $s=p_1p+q_1q$ for some $p_1, q_1\in\boldsymbol{Z}$. Then

$$\begin{aligned} a_{i+s}&=a_{i+p_1p} && \text{by } R_1 \\ &=\omega^{p_1}a_i\omega^{-p_1} && \text{by } R_2. \end{aligned}$$

Namely we get

$$a_{i+s}=\omega^{p_1}a_i\omega^{-p_1} \quad \text{for } i\in\boldsymbol{Z}. \tag{2.13}$$

By (2.13) and R_0,

$$\omega=\omega^{(p/s-1)p_1}a_{s-1}\omega^{-(p/s-1)p_1}\cdot\omega^{(p/s-1)p_1}a_{s-2}\omega^{-(p/s-1)p_1}\cdots a_{s-1}a_{s-2}\cdots a_0$$
$$=\omega(\omega^{-p_1}a_{s-1}a_{s-2}\cdots a_0)^{p/s} \qquad \text{by } R_3.$$

Namely we get

$$(\omega^{-p_1}a_{s-1}a_{s-2}\cdots a_0)^{p/s}=e. \tag{2.14}$$

Conversely R_3, (2.13) and (2.14) imply R_0, R_1, R_2:

$$R_0: a_{p-1}a_{p-2}\cdots a_0=\omega^{(p/s-1)p_1}a_{s-1}\omega^{-(p/s-1)p_1}\omega^{(p/s-1)p_1}a_{s-2}\omega^{-(p/s-1)p_1}$$
$$\cdots a_{s-1}\cdots a_0 \qquad \text{by (2.13)}$$
$$=\omega(\omega^{-p_1}a_{s-1}\cdots a_0)^{p/s} \qquad \text{by } R_3$$
$$=\omega \qquad \text{by (2.14)}$$

$$R_1: a_{i+q}=a_{i+s(q/s)}$$
$$=\omega^{(q/s)p_1}a_i\omega^{-(q/s)p_1} \qquad \text{by (2.13)}$$
$$=a_i \qquad \text{by } R_3$$
$$R_2: a_{i+p}=\omega^{(p/s)p_1}a_i\omega^{-(p/s)p_1} \qquad \text{by (2.13)}$$
$$=\omega^{1-(q/s)q_1}a_i\omega^{-1+(q/s)q_1}$$
$$=\omega a_i\omega^{-1} \qquad \text{by } R_3.$$

Thus we get the representation

$$\tilde{G}(p;q;r)=\langle\omega, a_i\ (i\in\boldsymbol{Z});\ R_3, (2.13), (2.14)\rangle.$$

By the elimination of generators, one gets:

$$\tilde{G}(p;q;r)=\langle\omega, a_0, a_1, \cdots, a_{s-1};\ (2.14), R_3\rangle. \tag{2.15}$$

Taking ω, a_0, a_1, $\cdots$, a_{s-2} and $b\equiv\omega^{-p_1}a_{s-1}a_{s-2}\cdots a_0$ as generators, we can rewrite (2.15) as

$$\tilde{G}(p;q;r)=\langle\omega, a_0, a_1, \cdots, a_{s-2}, b;\ \omega^a=b^{p/s}=e\rangle.$$

Therefore one obtains the desired isomorphism

$$\tilde{G}(p;q;r)\cong\boldsymbol{Z}_{p/s}*\boldsymbol{Z}_a*F(s-1),$$

completing the proof.

COROLLARY (2.16). *$G(p;q;r)$ is abelian if and only if*

(i) *$s=1$ and $a=1$ i.e. $(p,q)=1$ and $(q,r)=1$*

or

(ii) $p=1$

or

(iii) $s=2$, $a=1$ *and* $p=2$.

Namely we get:

$$G(p\,;q\,;r)\cong\begin{cases}\boldsymbol{Z}_{pr} & \textit{if } (p,q)=1,\ (q,r)=1\\ \boldsymbol{Z}_r & \textit{if } p=1\\ \boldsymbol{Z}\oplus\boldsymbol{Z}_r & \textit{if } s=2,\ a=1 \textit{ and } p=2.\end{cases}$$

PROOF. Assume that $(p, q)=(q, r)=1$. Then by Theorem (2.12), ω is contained in $Z(G(p\,;q\,;r))$. Therefore

$$\begin{aligned}G(p\,;q\,;r)&=\langle\omega,\ a_i\ (i\in\boldsymbol{Z})\,;\ \omega=a_{p-1}a_{p-2}\cdots a_1,\ a_{i+p}=a_{i+q}=a_i\\ &\qquad\qquad\text{for } i\in\boldsymbol{Z},\ \omega^r=e\rangle\\ &=\langle\omega,\ a_0\,;\ \omega=a_0^p,\ \omega^r=e\rangle\\ &\cong\boldsymbol{Z}_{pr}.\end{aligned}$$

Assume that $p=1$. Then $\omega=a_0$ and clearly we have

$$G(1\,;q\,;r)\cong\boldsymbol{Z}_r.$$

Assume that $s=2$, $a=1$ and $p=2$. Then we can write $q=2q_1$ and $(q_1, r)=1$. ω is contained in $Z(G(2, 2q_1, r))$ by Theorem (2.12).

$$\begin{aligned}G(2\,;2q_1\,;r)&\cong\langle\omega,\ a_i\ (i\in\boldsymbol{Z})\,;\ \omega^r=e,\ \omega=a_1a_0,\ a_i=a_{i+2}\\ &\qquad\text{for } i\in\boldsymbol{Z},\ [\omega, a_i]=\omega a_i\omega^{-1}a_i^{-1}=e\quad\text{for } i\in\boldsymbol{Z}\rangle\\ &\cong\langle\omega,\ a_0\,;\ \omega^r=e,\ [a_0,\omega]=e\rangle\\ &\cong\boldsymbol{Z}\oplus\boldsymbol{Z}_r.\end{aligned}$$

(The last case corresponds geometrically to the case that C has two irreducible components.)

COROLLARY (2.17). *Assume that p and q are coprime. Then $D(G(p\,;q\,;r))$ is isomorphic to $F((p-1)(a-1))$.*

PROOF. By Theorem (2.12), we have the isomorphism

$$D(G(p\,;q\,;r))\cong D(\boldsymbol{Z}_p*\boldsymbol{Z}_a)$$

because $s=1$. It is well-known that the latter is isomorphic to $F((p-1)(a-1))$. In [**6**], we gave a geometric proof of this isomorphism.

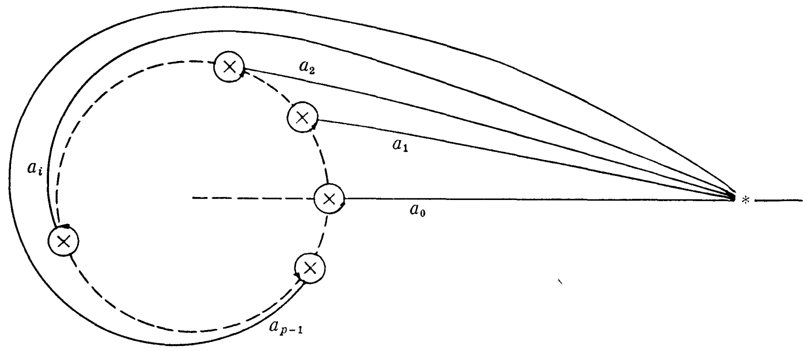
(Group theoretically, this is derived from the next fact: Let G_1 and G_2 be abelian groups. Then $D(G_1*G_2)$ is a free group and generated by these elements $[g_1, g_2]=g_1g_2g_1^{-1}g_2^{-1}$ for $g_1\in G_1$, $g_2\in G_2$. See for example [**3**], problem 34 (p. 197).)

§ 3. Model of the local monodromy relation.

We consider an affine curve

$$V: y^p = x^q \tag{3.1}$$

in $\boldsymbol{C}^2$. Let $\pi: \boldsymbol{C}^2 - V \to \boldsymbol{C}$ be the first projection map i.e. $\pi(x, y) = x$. Then π gives a locally trivial fibration over $\boldsymbol{C} - \{0\}$. Take generators $a_0, a_1, \cdots, a_{p-1}$ of $\pi_1(\pi^{-1}(1), *)$ as in Figure (3.2). (The base point $*$ is chosen so that the absolute value of its y-coordinate is large enough.)



$\pi^{-1}(1)$
Figure (3.2)

We consider $\pi^{-1}(\eta)$ ($\eta \in \boldsymbol{C}$) as a subset of $\boldsymbol{C}$ by the projection into the y-coordinate. Let D be the unit disk $\{z, |z| \leq 1\}$ in the x-coordinate plane $\boldsymbol{C}$. Then $\pi^{-1}(D)$ is a deformation retract of $\boldsymbol{C}^2 - V$. Let D^+ or D^- be the upper or lower closed half disk respectively.

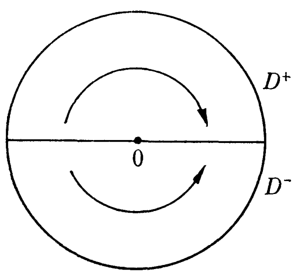
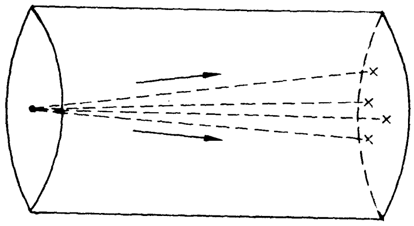


Figure (3.3)

Then $\pi^{-1}(D^+)$ or $\pi^{-1}(D^-)$ can be deformed into $\pi^{-1}([0, 1])$ by the rotation of the argument. Again $\pi^{-1}([0, 1])$ can be deformed into $\pi^{-1}(1)$.



$\pi^{-1}([0, 1])\cap\{|y|\leqq 2\}$

Figure (3.4)

Therefore we have isomorphisms $\pi_1(\pi^{-1}(1), *)\cong\pi_1(\pi^{-1}(D^+), *)$ and $\pi_1(\pi^{-1}(1), *)\cong \pi_1(\pi^{-1}(D^-), *)$. Applying the theorem of Van Kampen to the division $\pi^{-1}(D)= \pi^{-1}(D^+)\cup\pi^{-1}(D^-)$; one obtains this:
$\pi_1(\boldsymbol{C}^2-V, *)$ is generated by the image of $\pi_1(\pi^{-1}(1), *)$ and the generating relations are derived from the monodromy relations i.e. the relations which are obtained by the deformation of the fiber $\pi^{-1}(1)$ along the circle $|x|=1$. (This is exactly the situation which is considered in [2].) More precisely, we get:

$$\text{(3.5)}\qquad \begin{cases} a_0=\omega^m a_r\omega^{-m} \\ a_1=\omega^m a_{r+1}\omega^{-m} \\ \vdots \\ a_{p-r-1}=\omega^m a_{p-1}\omega^{-m} \\ a_{p-r}=\omega^{m+1}a_0\omega^{-(m+1)} \\ \vdots \\ a_{p-1}=\omega^{m+1}a_{r-1}\omega^{-(m+1)} \end{cases}$$

where the integers m and r are defined by the equation: $q=mp+r,\ 0\leqq r\leqq p-1$ and

$$\text{(3.6)}\qquad \omega=a_{p-1}a_{p-2}\cdots a_0\,.$$

To understand these relations more systematically, we introduce infinite elements a_i $(i\in\boldsymbol{Z})$ by

$$\text{(3.7)}\qquad a_{kp+j}=\omega^k a_j\omega^{-k}\qquad \text{for } k\in\boldsymbol{Z} \text{ and } 0\leqq j\leqq p-1\,.$$

Then it is easy to see that (3.7) is equivalent to

$$\text{(3.8)}\qquad a_{j+p}=\omega a_j\omega^{-1}\qquad \text{for any } j\in\boldsymbol{Z}\,.$$

Now (3.5) can be written in the following simple form

$$\text{(3.9)}\qquad a_j=a_{j+q}\qquad 0\leqq j\leqq p-1\,.$$

By (3.8), this implies

$$\text{(3.10)}\qquad a_j=a_{j+q}\qquad \text{for any } j\in\boldsymbol{Z}\,.$$

Thus we obtain

PROPOSITION (3.11). *$\pi_1(\boldsymbol{C}^2-V, *)$ is isomorphic to $G(p; q)$.*

The next corollary is important.

COROLLARY (3.12). *$\pi_1(\boldsymbol{C}^2-V)$ is abelian if and only if $q=1$ (or $p=1$) or $p=q=2$.*

(i) *In the case of $q=1$ or $p=1$, $\pi_1(\boldsymbol{C}^2-V)\cong\boldsymbol{Z}$.*

(ii) *In the case of $p=q=2$, $\pi_1(\boldsymbol{C}^2-V)\cong\boldsymbol{Z}\oplus\boldsymbol{Z}$.*

PROOF. Let $r=(p, q)$ and let $q_1=q/r$. Then by Proposition (2.5), ω^{q_1} is contained in the center. Let N be the (infinite) cyclic group generated by ω^{q_1}. Then the quotient group is isomorphic to $G(p; q; q_1)$. By Theorem (2.12), $G(p; q; q_1)$ is isomorphic to $\boldsymbol{Z}_{p/r} * \boldsymbol{Z}_{q_1} * F(r-1)$. Thus for $\pi_1(\boldsymbol{C}^2-V, *)$ to be abelian, it is necessary that $p=1$ or $q=1$ or $p=q=2$. The other direction is immediate by the definition of $G(p; q)$. Geometrically (ii) corresponds to the case that V has an ordinary double point at the origin.

§ 4. Representation of the fundamental group $\pi_1(\boldsymbol{P}^2-C)$.

We return to the situation of Theorem (1.3) in § 1. Let

$$C: \prod_{j=1}^{l}(Y-\beta_j Z)^{\nu_j}-\prod_{i=1}^{m}(X-\alpha_i Z)^{\lambda_i}=0\,. \tag{4.1}$$

Consider the set $U=\{(\alpha_1, \alpha_2, \cdots, \alpha_m, \beta_1, \cdots, \beta_l)\in\boldsymbol{C}^{l+m}$; the singular points of C defined by (4.1) are contained in the intersection of lines: $\prod_{j=1}^{l}(Y-\beta_j Z)^{\nu_j}=\prod_{i=1}^{m}(X-\alpha_i Z)^{\lambda_i}=0\}$. Here $\nu_1, \nu_2, \cdots, \nu_l, \lambda_1, \cdots, \lambda_m$ are fixed. It is easy to see that U is a Zariski open set. Therefore for a given C satisfying the assumption in Theorem (1.3), we can arrange $\alpha_1, \alpha_2, \cdots, \alpha_m, \beta_1, \beta_2, \cdots, \beta_l$ at a suitable position using the deformation of the following type:

$$C_t: \prod_{j=1}^{l}(Y-\beta_j(t)Z)^{\nu_j}-\prod_{i=1}^{m}(X-\alpha_i(t)Z)^{\lambda_i}=0$$

where $(\alpha_1(t), \alpha_2(t), \cdots, \alpha_m(t), \beta_1(t), \cdots, \beta_l(t))\in U$ for each t. The topological type of C_t or $\boldsymbol{P}^2-C_t$ is constant under the deformation. We arrange $\alpha_1, \alpha_2, \cdots, \alpha_m, \beta_1, \beta_2, \cdots, \beta_l$ on the real line so that $\alpha_1<\alpha_2<\cdots<\alpha_m$ and $\beta_1<\beta_2<\cdots<\beta_l$. U contains such a point by the next argument. For the calculation, we use the method of a pencil section (Zariski, [7]).

Namely we consider the pencil

$$L_\eta: X=\eta Z\,, \qquad \eta\in\boldsymbol{C}\,. \tag{4.2}$$

The center of the pencil (4.2) is the point $\infty\equiv[0; 1; 0]$. We take ∞ as a (fixed) base point of $\boldsymbol{P}^2-C$. Let $x=X/Z$ and $y=Y/Z$ be the affine coordinates of the chart $\{Z\neq 0\}$. (Note that the line: $Z=0$ meets C at n distinct points.) In this affine space $\boldsymbol{C}^2$, C is defined by

$$C:\ \prod_{j=1}^{l}(y-\beta_j)^{\nu_j}-\prod_{i=1}^{m}(x-\alpha_i)^{\lambda_i}=0\,. \tag{4.3}$$

The singular points of C are these:

$$P_{ij}=(\alpha_i,\ \beta_j),\quad 1\leqq i\leqq m;\quad 1\leqq j\leqq l\quad \text{such that } \lambda_i,\ \nu_j\geqq 2\,. \tag{4.4}$$

In a sufficiently small neighborhood of P_{ij}, C is topologically described by

$$(y-\beta_j)^{\nu_j}=c(x-\alpha_i)^{\lambda_i}\,,\qquad (c\neq 0,\ \text{constant})\,. \tag{4.5}$$

If a pencil line L_η: $x=\eta$ meets C at $(\tilde{y},\ \eta)$ with the intersection multiplicity $\geqq 2$, $\tilde{y}$ is a root of the following equations.

$$\prod_{j=1}^{l}(y-\beta_j)^{\nu_j}=\prod_{i=1}^{m}(\eta-\alpha_i)^{\lambda_i} \tag{4.6}$$

$$\sum_{j=1}^{l}\nu_j(y-\beta_j)^{\nu_j-1}\prod_{i\neq j}(y-\beta_i)^{\nu_i}=0\,. \tag{4.7}$$

Considering the real function

$$f(y)=\prod_{j=1}^{l}(y-\beta_j)^{\nu_j}\,, \tag{4.8}$$

one finds that there is at least a real root γ_j of (4.7) in the open interval $(\beta_j,\ \beta_{j+1})$ for each $j=1,\ 2,\ \cdots,\ l-1$. Because the degree of $f'(y)/\prod_{j=1}^{l}(y-\beta_j)^{\nu_j-1}$ is $l-1$, $\gamma_1, \gamma_2, \cdots, \gamma_{l-1}$ and β_j such that $\nu_j\geqq 2$, $1\leqq j\leqq l$, are the roots of (4.7). See Figure (4.9).

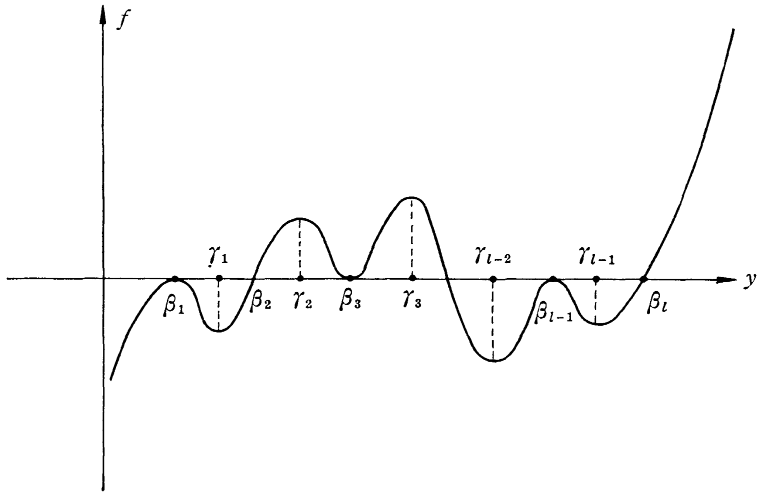


Figure (4.9) (n: odd)

By a slight perturbation of β_j if necessary, we can assume that $\{f(\gamma_1), f(\gamma_2), \cdots, f(\gamma_{l-1})\}$ are mutually distinct. (This is not essential.) Let $g(x)=\prod_{i-1}^{m}(x-\alpha_i)^{\lambda_i}$. By taking $|\alpha_j|$ small enough, we can assume

(4.10) $$|g(x)|<\text{minimum}\{|f(\gamma_1)|, |f(\gamma_2)|, \cdots, |f(\gamma_{l-1})|\}$$

for $\alpha_1-\varepsilon_0\leqq x\leqq\alpha_m+\varepsilon_0$ (ε_0: small enough). Then applying the same argument as above to $g(x)$, we can see that the roots of

(4.11) $$g(x)=f(\gamma_j) \quad \text{for} \quad 1\leqq j\leqq l-1$$

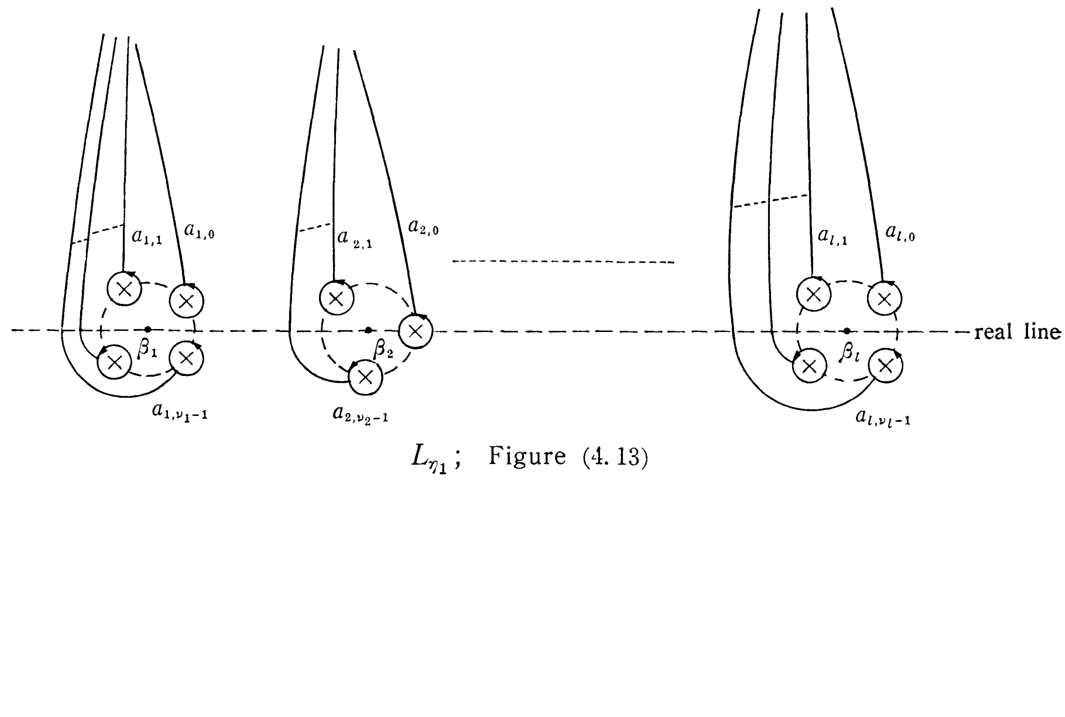
are mutually distinct for each j. In particular, this implies that $(\alpha_1, \alpha_2, \cdots, \alpha_m; \beta_1, \cdots, \beta_l)$ is contained in U. Let $\delta_{j,1}, \delta_{j,2}, \cdots, \delta_{j,n}$ be the roots of (4.11). At each point $(\delta_{j,k}, \gamma_j)$, C is topologically equivalent to

(4.12) $$C: (y-\gamma_j)^2=c\cdot(x-\delta_{j,k}) \qquad (c\neq 0, \text{ constant}).$$

This says that the line: $x=\delta_{j,k}$ is tangent to C with the multiplicity 2. (Note that γ_j is a simple root of (4.7).)

Let $\pi: \boldsymbol{C}^2-C\rightarrow\boldsymbol{C}$ be the projection into the x-coordinate. The fiber $\pi^{-1}(\eta)$ is $\boldsymbol{C}^2\cap L_\eta-C\cap\boldsymbol{C}^2\cap L_\eta$. By the above consideration, π is a locally trivial fibration over $\boldsymbol{C}-\{\alpha_1, \alpha_2, \cdots, \alpha_m; \delta_{j,k}\ (1\leqq j\leqq l-1, 1\leqq k\leqq n)\}$. By the theorem of Van Kampen [2] (see also §3), $\pi_1(\boldsymbol{P}^2-C, \infty)$ is generated by the image of $\pi_1(L_\eta-L_\eta\cap C, \infty)$ for any fixed $\eta\in\boldsymbol{C}-\Sigma$ ($\Sigma=\{\alpha_1, \alpha_2, \cdots, \alpha_m; \delta_{j,k}\ (1\leqq j\leqq l-1, 1\leqq k\leqq n)\}$) and the generating relations between fixed generators of $\pi_1(L_\eta-L_\eta\cap C, \infty)$ are derived from (i) a torsion relation (see below) and (ii) the local monodromy relations at (α_i, β_j) or $(\delta_{j,k}, \gamma_j)$.

Take $\varepsilon>0$ small enough so that we can find ν_j points of $f^{-1}(\varepsilon)$ on a small circle centered at β_j ($j=1, 2, \cdots, l$) and similarly λ_i points of $g^{-1}(\varepsilon)$ on a small circle centered at α_i ($i=1, 2, \cdots, m$). We take $\eta_1\in g^{-1}(\varepsilon)$ on the circle with center α_1 as a base point of $\boldsymbol{C}-\Sigma$ and we take generators a_{jk}, $1\leqq j\leqq l$, $0\leqq k\leqq\nu_i-1$, of $\pi_1(L_{\eta_1}-L_{\eta_1}\cap C, \infty)$ as follows.



L_{η_1}; Figure (4.13)

$\{a_{jk}\}$ are oriented in the counterclockwise direction and are joined to ∞ along the half line: $\{y;\ \text{argument}\ (y)=\pi/2\}$.

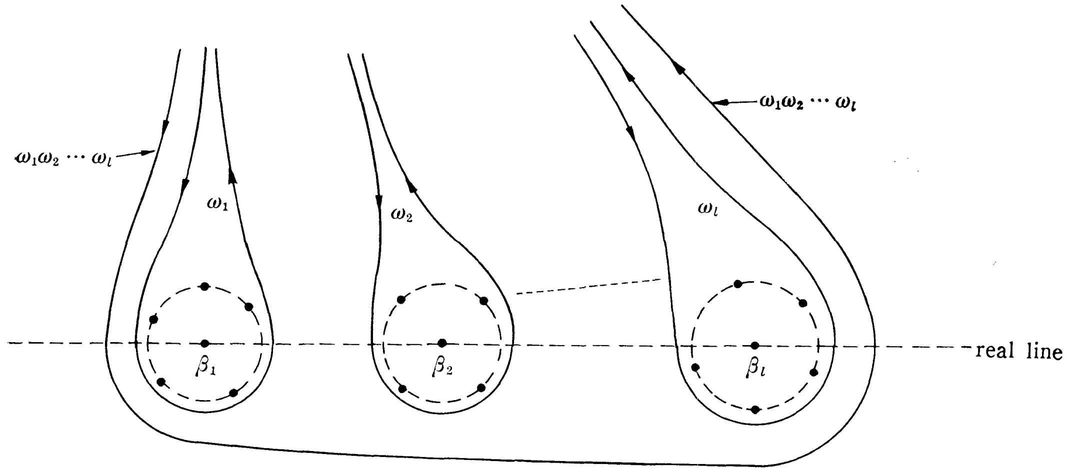
Let us define ω_j ($j=1, 2, \cdots, l$) by

(4.14) $$\omega_j=a_{j,\nu_j-1}a_{j,\nu_j-2}\cdots a_{j,0} \quad \text{for } 1\leqq j\leqq l\,.$$

Then the torsion relation is this:

(4.15) $$\omega_1\omega_2\cdots\omega_l=e\,.$$

See the following picture.



L_{η_1}; Figure (4.16)

To avoid the complexity of the monodromy relations, we introduce $a_{j,k}$ ($1\leqq j\leqq l$, $k\in\boldsymbol{Z}$) by

(4.17) $$a_{j,k+t\nu_j}=\omega_j^t a_{j,k}\omega_j^{-t} \quad \text{for } 1\leqq j\leqq l \text{ and } 0\leqq k\leqq\nu_j-1 \text{ and } t\in\boldsymbol{Z}\,.$$

Once we define $a_{j,k}$ by (4.17), they satisfy

(4.18) $$a_{j,k+\nu_j}=\omega_j a_{j,k}\omega_j^{-1} \quad \text{for } 1\leqq j\leqq l \text{ and } k\in\boldsymbol{Z}\,.$$

First we consider the monodromy relation at $x=\alpha_1$. When x moves around a small circle centered at α_1, each small circle with center β_j in Figure (4.13) is rotated by the angle $2\lambda_1\pi/\nu_j$. Namely by the local argument in § 3, we get

(4.19) $$a_{j,k}=a_{j,k+\lambda_1} \quad \text{for } 1\leqq j\leqq l \text{ and } k\in\boldsymbol{Z}\,.$$

Now let $x=\alpha_i$. Take a point η_i of $g^{-1}(\varepsilon)$ on a small circle with center α_i. Note that the picture of $\pi^{-1}(\eta_i)$ is completely the same as in Figure (4.13). Therefore let $a_{j,k}(\eta_i)$ ($1\leqq j\leqq l$; $k\in\boldsymbol{Z}$) be the elements of $\pi_1(\boldsymbol{P}^2-C, \infty)$ represented by the loops in $\pi^{-1}(\eta_i)\cup\{\infty\}$ corresponding to $a_{j,k}$. Then the same argument as above gives the relation:

(4.20) $$a_{j,k}(\eta_i)=a_{j,k+\lambda_i}(\eta_i) \quad \text{for} \quad k\in \boldsymbol{Z}.$$

To translate (4.20) into the words in $a_{j,k}$, consider the following path P_i. (The circle centered at α_i is mapped to the circle $|z|=\varepsilon$ by g.)

path P_i: Figure (4.21)

The deformation along the arc S_k is nothing but the rotation of the small circle with center β_j of the angle θ_k/ν_j for $j=1, 2, \cdots, l$ for some θ_k where θ_k does not depend on j but only on k and i. The deformation along the line segment $\tilde{l}_k$ is trivial by (4.10). (Consider the points of $f(y)=t$, t: real.) Thus the deformation along P_i from η_i to η_1 is the rotation of the small circles with center β_j by θ/ν_j; $\theta=\sum_{k=1}^{i}\theta_k$ for $j=1, 2, \cdots, l$. Note that $\theta=2\pi\cdot a$ for some $a\in \boldsymbol{Z}$ and the rotation of the above circles with center β_j by the angle $2\pi/\nu_i$ for $i=1, 2, \cdots, l$ corresponds to the transformation

$$a_{j,k} \longrightarrow a_{j,k+1} \quad \text{for} \quad 1\leqq j\leqq l,\ k\in \boldsymbol{Z}.$$

Therefore the relation (4.20) is translated into

(4.22) $$a_{j,k}=a_{j,k+\lambda_i} \quad \text{for} \quad k\in \boldsymbol{Z} \text{ and } 1\leqq j\leqq l.$$

It is not necessary to calculate θ_k or θ explicitly by virtue of the periodicity of (4.20). Thus gathering the monodromy relations at $x=\alpha_i$ ($i=1, 2, \cdots, m$), one obtains

(4.23) $$a_{j,k}=a_{j,k+\lambda_i} \quad \text{for} \quad 1\leqq j\leqq l,\ 1\leqq i\leqq m \text{ and } k\in \boldsymbol{Z}.$$

Now we must read the monodromy relations at $x=\delta_{j,k}$ for $j=1, 2, \cdots, l-1$ and $k=1, 2, \cdots, n$. Let $\tau_1, \tau_2, \cdots, \tau_{r_1}$ ($0<\tau_1<\cdots<\tau_{r_1}$) be the positive numbers of $\{f(\gamma_1), f(\gamma_2), \cdots, f(\gamma_{l-1})\}$ and let $\xi_1, \xi_2, \cdots, \xi_{r_2}$ ($0>\xi_1>\xi_2>\cdots>\xi_{r_2}$) be the negative elements of them ($r_1+r_2=l-1$). We consider the following loops l_s ($s=1, 2, \cdots, r_1$) and m_p ($p=1, 2, \cdots, r_2$) in the complex plane (=the f-value plane).

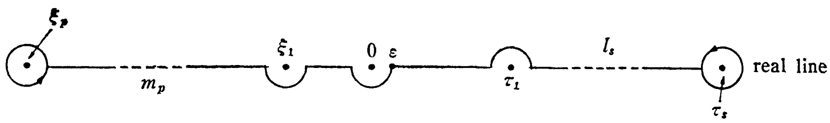
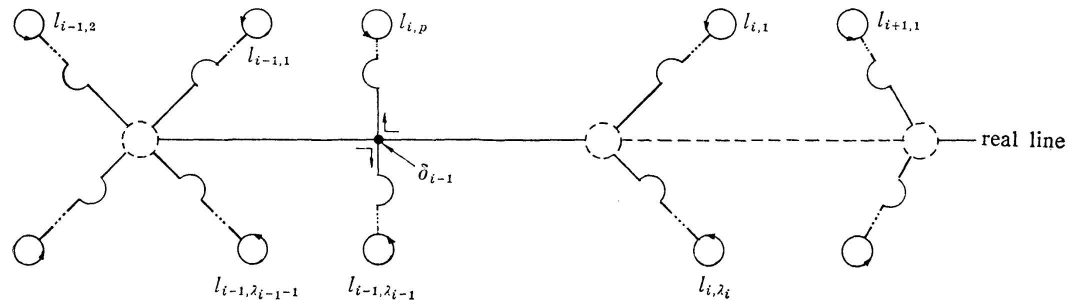


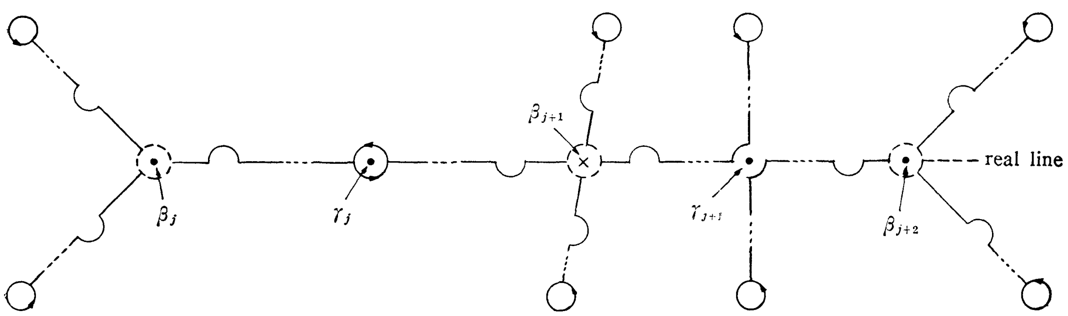
Figure (4.24)

Each loop is based at ε and the half circles are of radius ε. Take γ_j and assume that $f(\gamma_j)=\tau_s$ for example. The inverse image $g^{-1}(l_s)$ consists of n loops which are not necessarily disjoint but meet only at δ_i such that $g(\delta_i)>0$ where δ_i is a root of $g'(x)=0$ such that $\alpha_i<\delta_i<\alpha_{i+1}$, $i=1, \cdots, m-1$. Let $\alpha_{i,1}, \alpha_{i,2}, \cdots, \alpha_{i,\lambda_i}$ be the suitably ordered points of $g^{-1}(\varepsilon)$ on the small circle with the center α_i. Let $l_{i,k}$ $(k=1, \cdots, \lambda_i)$ be the corresponding loop which is based at $\alpha_{i,k}$. At a δ_{i-1} as above, $l_{i-1,p}$ and $l_{i,p}$, turns to the right. They are sketched as follows.



(In the case of $g(\delta_{i-1})>0$, $g(\delta_i)<0$),
$g^{-1}(l_s)$ in x-plane. Figure (4.25)

The inverse image $f^{-1}(l_s)$ consists of $(n-2)$ disjoint loops and two paths on the interval (β_j, β_{j+1}) overlapping each other except the small circle part centered at γ_j.

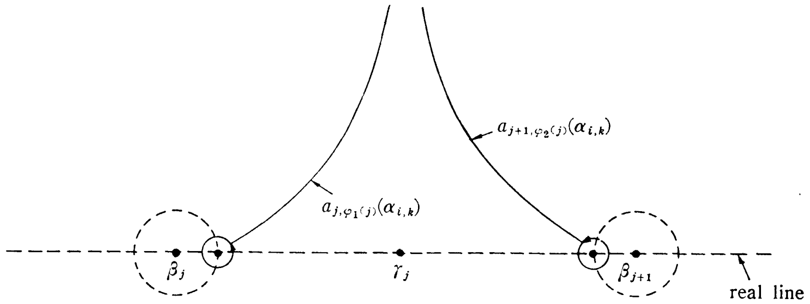


(In the case of $0<f(\gamma_{j+1})<f(\gamma_j)$)
y-plane. Figure (4.26)

When $x=\eta$ moves along $l_{i,k}$ starting at $\alpha_{i,k}$, each point of $L_\eta \cap C$ are deformed along $f^{-1}(l_s)$. Therefore by Proposition (3.11) for $(p, q)=(2, 1)$ and (4.12), we get

$$a_{j,\varphi_1(j)}(\alpha_{i,k})=a_{j+1,\varphi_2(j)}(\alpha_{i,k}), \quad 1\leqq k\leqq\lambda_i \tag{4.27}$$

where $\varphi_1(j)$ and $\varphi_2(j)$ are integers depending only on the first ordering of $a_{j,k}$.



L_{γ}, $\eta=\alpha_{i,k}$;
Figure (4.8)

Using the deformation along the small circle centered at α_i, we can transform the relation (4.27) into the fiber $x=\alpha_{i,1}$ where we may assume that $\alpha_{i,1}$ is equal to η_i defined in (4.20).

(4.29) $$a_{j,\varphi_1(j)+h}(\alpha_{i,1})=a_{j+1,\varphi_2(j)+h}(\alpha_{i,1}), \qquad 0\leqq h\leqq \lambda_i-1 .$$

Now applying the deformation along P_i in Figure (4.21) considered in the argument at $x=\alpha_i$ and using the periodicity (4.23), we obtain

(4.30) $$a_{j,\varphi_1(j)+h}=a_{j+1,\varphi_2(j)+h} \qquad \text{for any} \quad h\in \boldsymbol{Z} .$$

Because this relation is independent of i ($i=1, 2, \cdots, m$), the monodromy relations at $x=\delta_{j,k}$ ($k=1, 2, \cdots, n$) are (4.30) in the existence of (4.23).

Applying the same argument for every $f(\gamma_j)$; $1\leqq j\leqq l-1$, we obtain

(4.31) $$a_{j,\varphi_1(j)+k}=a_{j+1,\varphi_2(j)+k} \qquad \text{for} \quad 1\leqq j\leqq l-1 \text{ and } k\in \boldsymbol{Z}$$

where $\varphi_1(j)$ and $\varphi_2(j)$ are integers depending on j. (If $f(\gamma_j)$ is negative, we use the corresponding loop m_p.) Thus the generating relations are gotten.

§ 5. Decision of the group structure.

Let $G=\pi_1(\boldsymbol{P}^2-C, \infty)$. By the above argument, G is generated by $\omega_1, \omega_2, \cdots, \omega_l$ and $a_{j,k}$ ($1\leqq j\leqq l$, $k\in \boldsymbol{Z}$) and their complete generating relations are these :

(5.1) $$\omega_j=a_{j,\nu_j-1}a_{j,\nu_j-2}\cdots a_{j,0} \qquad \text{for} \quad 1\leqq j\leqq l$$

(5.2) $$\omega_1\omega_2\cdots\omega_l=e$$

(5.3) $$a_{j,k+\nu_j}=\omega_j a_{j,k}\omega_j^{-1} \qquad \text{for} \quad 1\leqq j\leqq l,\ k\in \boldsymbol{Z}$$

(5.4) $$a_{j,k+\lambda_i}=a_{j,k} \qquad \text{for} \quad 1\leqq j\leqq l,\ 1\leqq i\leqq m \text{ and } k\in \boldsymbol{Z}$$

(5.5) $$a_{j,k}=a_{j+1,k+d_j} \qquad \text{for} \quad 1\leqq j\leqq l-1 \text{ and } k\in \boldsymbol{Z} .$$

(Here we put $d_j=\varphi_2(j)-\varphi_1(j)$.)

By Proposition (2.9), (5.4) is equivalent to

(5.6) $$a_{j,k+\lambda}=a_{j,k} \quad \text{for} \quad 1\leqq j\leqq l,\ k\in \boldsymbol{Z},$$

where $\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_m)$. One can see that (5.3)+(5.5) is equivalent to (5.3)$'$+(5.5) where

(5.3)$'$ $$a_{j,k+\nu_p}=\omega_p a_{j,k}\omega_p^{-1} \quad \text{for} \quad 1\leqq j,\ p\leqq l \text{ and } k\in \boldsymbol{Z}.$$

Let $\nu=(\nu_1, \nu_2, \cdots, \nu_l)$ and write $\nu=k_1\nu_1+k_2\nu_2+\cdots+k_l\nu_l$. Then

$$a_{j,k+\nu}=a_{j,k+k_1\nu_1+\cdots+k_l\nu_l}=\omega_l^{k_l}\omega_{l-1}^{k_{l-1}}\cdots\omega_1^{k_1}a_{j,k}\omega_1^{-k_1}\omega_2^{-k_2}\cdots\omega_l^{-k_l} \qquad \text{by (5.3)}'.$$

Expressing $\omega_l^{k_l}\omega_{l-1}^{k_{l-1}}\cdots\omega_1^{k_1}$ as a continuous product of $a_{j,k}$ (j : fixed) by (5.5) and Proposition (2.6), we have

(5.7) $$\omega_l^{k_l}\omega_{l-1}^{k_{l-1}}\cdots\omega_1^{k_1}=a_{j,\nu-1}a_{j,\nu-2}\cdots a_{j,0} \quad \text{for} \quad j=1, 2, \cdots, l.$$

We put

(5.8) $$\omega=a_{j,\nu-1}a_{j,\nu-2}\cdots a_{j,0} \quad \text{for} \quad 1\leqq j\leqq l.$$

Then by the above equation, one gets:

(5.9) $$a_{j,k+\nu}=\omega a_{j,k}\omega^{-1} \quad \text{for} \quad 1\leqq j\leqq l \text{ and } k\in \boldsymbol{Z}.$$

By (5.8), (5.9) and Proposition (2.6), we can write (5.1) and (5.2) as

(5.1)$'$ $$\omega_j=\omega^{\nu_j/\nu}$$

(5.2)$'$ $$\omega^{n/\nu}=e.$$

Thus we get the representation

$$G=\langle \omega, \omega_j, a_{j,k}\ (1\leqq j\leqq l;\ k\in\boldsymbol{Z});\ (5.1)', (5.2)', (5.3)', (5.6), (5.5), (5.8), (5.9)\rangle.$$

It is clear that (5.3)$'$ is contained in (5.9)+(5.1)$'$. Using (5.5), (5.6), (5.8) and (5.9) are recovered from (5.6) ($j=1$) and (5.9) ($j=1$). Thus

$$G=\langle \omega, \omega_j, a_{j,k}\ (1\leqq j\leqq l,\ k\in\boldsymbol{Z});\ (5.1)', (5.2)', (5.5), (5.6) \text{ for } j=1, (5.8) \text{ for } j=1, (5.9) \text{ for } j=1\rangle.$$

Now (5.1)$'$ and (5.5) say that we can eliminate the generators $\omega_1, \cdots, \omega_l$ and $a_{j,k}$ ($2\leqq j\leqq l$). Namely one obtains finally

$$G=\langle \omega, a_{1,k}\ (k\in\boldsymbol{Z});\ \omega=a_{1,\nu-1}a_{1,\nu-2}\cdots a_{1,0}, (5.6) \text{ for } j=1, (5.9) \text{ for } j=1 \text{ and } \omega^{n/\nu}=e\rangle$$
$$\cong G(\nu;\lambda;n/\nu) \text{ by the definition of } G(\nu;\lambda;n/\nu).$$

This completes the proof of Theorem (1.3). Now Corollary (1.4) is obtained from Theorem (2.12), because $(\lambda/s, n/\nu)=\lambda/s$ by (1.2) where $s=(\nu, \lambda)$.

§ 6. Examples.

In this section, we give some typical examples. By Theorem (1.3) and Theorem (2.12), we have the criterion

(i) $\pi_1(\boldsymbol{P}^2-C)$ is abelian

$\Leftrightarrow \lambda=1$ or $\nu=1$ (C: irreducible)

or

$\nu=2$, $\lambda=2$ (C: 2 components).

(ii) $Z(\pi_1(\boldsymbol{P}^2-C))$ is non-trivial and $\pi_1(\boldsymbol{P}^2-C)$ is not abelian

$\Leftrightarrow n>\lambda\nu$, $(\lambda, \nu)=1$ and $\lambda\neq 1$, $\nu\neq 1$ (C: irreducible)

or

$s=(\lambda, \nu)>1$, $ns>\lambda\nu$ except $\nu=2$ and $\lambda=2$ (C: not irreducible).

(iii) $Z(\pi_1(\boldsymbol{P}^2-C))$ is trivial i.e. $\pi_1(\boldsymbol{P}^2-C)$ is centerless

$\Leftrightarrow (\lambda, \nu)=1$, $n=\lambda\nu$ except $\lambda=1$ or $\nu=1$

or

$s=(\lambda, \nu)>1$, $ns=\lambda\nu$ and $n>2$.

(A) Abelian.

EXAMPLE (6.1). Let $C: X^n+Y^n+Z^n=0$. Then C is non-singular and $\pi_1(\boldsymbol{P}^2-C)\cong \boldsymbol{Z}_n$.

EXAMPLE (6.2). Let $C: (Y^r-Z^r)(Y^l-2Z^l)^2-\varepsilon\cdot(X^s-Z^s)(X^m-2Z^m)^2=0$ where $n=r+2l=s+2m$ and ε is a positive small number. Then C has $(n-r)(n-s)/4$ ordinary double points and is irreducible if $r\geqq 1$. $\pi_1(\boldsymbol{P}^2-C)\cong \boldsymbol{Z}_n$.

EXAMPLE (6.3). Let C be an irreducible curve of type (1.1) satisfying the assumption in Theorem (1.3) and assume that n (=the degree of C) is prime. Then $\pi_1(\boldsymbol{P}^2-C)\cong \boldsymbol{Z}_n$ because $\lambda=1$ or $\nu=1$.

(B) Non abelian with a non-trivial center.

EXAMPLE (6.4). Let C: $(X^{pr}+Z^{pr})^q+(Y^{qr}+Z^{qr})^p=0$ and assume that $(p, q)=1$ and $p\geqq 2$, $q\geqq 2$, $r\geqq 2$. Then C is irreducible and

(i) $\pi_1(\boldsymbol{P}^2-C)\cong G(p\,;q\,;qr)$

(ii) $Z(\pi_1(\boldsymbol{P}^2-C))\cong \boldsymbol{Z}_r$ and $\pi_1(\boldsymbol{P}^2-C)/Z(\pi_1(\boldsymbol{P}^2-C))\cong \boldsymbol{Z}_p * \boldsymbol{Z}_q$

(iii) $D(\pi_1(\boldsymbol{P}^2-C))\cong F((p-1)(q-1))$.

C has nr $(=pqr^2)$ singular points and each of them is topologically described by $y^p+x^q=0$. For instance, take $p=r=2$ and $q=3$. Then

$$\begin{aligned}\pi_1(\boldsymbol{P}^2-C)&\cong G(2\,;3\,;6)\\ &\cong\langle a,\,b\,;\,a^6=e,\ b^2=a^3\rangle\\ &\cong SL(2,\,\boldsymbol{Z})\,.\end{aligned}$$

EXAMPLE (6.5). Let $C:(X^{pr}+Z^{pr})^{qs}+(Y^{qr}+Z^{qr})^{ps}=0$ and assume that $(p,\,q)=1$ and $p,\,q,\,r,\,s\geqq 2$. Then C has s irreducible components.

(i) $\pi_1(\boldsymbol{P}^2-C)\cong G(ps\,;qs\,;qr)$

(ii) $Z(\pi_1(\boldsymbol{P}^2-C))\cong\boldsymbol{Z}_r$ and $\pi_1(\boldsymbol{P}^2-C)/Z(\pi_1(\boldsymbol{P}^2-C))\cong\boldsymbol{Z}_p*\boldsymbol{Z}_q*F(s-1)$

(iii) $D(\pi_1(\boldsymbol{P}^2-C))\cong D(\boldsymbol{Z}_p*\boldsymbol{Z}_q*F(s-1))$ (=a free group of infinite rank).

(C) Centerless.

EXAMPLE (6.6). Let $C:(X^p+Z^p)^q+(Y^q+Z^q)^p=0$ and assume that $(p,\,q)=1$ and $p,\,q\geqq 2$. Then C is irreducible and C has pq cusp singularities. We have

$$\pi_1(\boldsymbol{P}^2-C)\cong\boldsymbol{Z}_p*\boldsymbol{Z}_q\qquad(\cong G(p\,;q\,;q))$$

and

$$D(\pi_1(\boldsymbol{P}^2-C))\cong F((p-1)(q-1))\,.$$

This example was first studied by Zariski [8], in the case of $p=2$ and $q=3$. (Then $\pi_1(\boldsymbol{P}^2-C)\cong\boldsymbol{Z}_2*\boldsymbol{Z}_3\cong PSL(2,\,\boldsymbol{Z})$). In our previous paper [6], we studied this example in general case.

EXAMPLE (6.7). Let $C:(X^p+Z^p)^{qr}+(Y^q+Z^q)^{pr}=0$ and assume that $(p,\,q)=1$ and $p,\,q,\,r\geqq 2$. Then C has r irreducible components and

$$\pi_1(\boldsymbol{P}^2-C)\cong\boldsymbol{Z}_p*\boldsymbol{Z}_q*F(r-1)\,.$$

REMARK (6.8). Theorem (1.3) says that C is irreducible if $(\nu,\,\lambda)=1$. Note that this is not necessarily true if we omit the assumption on the singular points of C. For example, consider the curve $C:Y(Y-Z)^2-X(X-Z)^2=0$. C has 2 non-singular irreducible components.

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