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A characterization of the Rudvalis group

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The purpose of this paper is to prove the following:

THEOREM A. *Simple groups with Sylow 2-subgroups of type Rd are isomorphic to Rd, where Rd is the Rudvalis simple group of order* $2^{14}\cdot 3^{3}\cdot 5^{3}\cdot 7\cdot 13\cdot 29$**.

This is a Corollary of the following result:

THEOREM B. *Let G be a finite group with Sylow 2-subgroup T satisfying the following condition:*

(a) *T is of order at least 2^{14};*

(b) *$Z_4(T)$ is of order 16;*

(c) *$\Phi(W)$ is of order at most 8, where $W=C_T(Z_3(T))$;*

(d) *all subgroups of W of index at most 16 contain $\Phi(W)$.*

Assume further that G has no subgroup of index 2. Then one of the following holds:

(1) *$G/O(G)$ is isomorphic to the Rudvalis group.*

(2) *$O(G)W$ is normal in G and $G/O(G)W$ is isomorphic to $GL(3, 2)$.*

Our notation is standard and taken from [**5**].

1. Transfer.

Except for Lemma 1.4, we assume that G is a finite group with Sylow 2-subgroup T satisfying the conditions (a) to (d) in Theorem B and we set $W=C_T(Z_3(T))$.

LEMMA 1.1. *The following hold:*

(1) $Z(W)\geqq W'=\Phi(W)=Z_3(T)\cong Z_2^3$;

(2) $|T|=2^{14}$, $T/W\cong D_8$, $W/W'\cong Z_2^8$, $T/W'\cong Z_2 \wr D_8$;

(3) *$N_G(T)$ has a normal 2-complement;*

(4) *If $|W:W\cap T^g|\leqq 2$ for $g\in G$, then $g\in N_G(W')$;*

(5) *W is weakly closed in T with respect to G.*

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**) This theorem is proved by Assa, too.

PROOF. Clearly, $W' \leqq \Phi(W) \leqq Z_3(T) \leqq C_T(W)$ and $Z_3(T)$ is of order 8. Thus $\Phi(W) = Z_i(T)$ for some $i \leqq 3$. Since $Z_4(T)$ is of order 16, $Z(T/\Phi(W))$ is of order 2. Thus we have that $|W : \Phi(W)| \leqq 2^{|T:W|}$. Since $|T : W| \leqq |\mathrm{Aut}(Z_3(T))|_2 \leqq 8$ and $|T| \geqq 2^{14}$, we have that $|T : W| = |\Phi(W)| = 8$ and $|W : \Phi(W)| = 2^8$. In particular, $\Phi(W) = Z_3(T)$ and T is of order 2^{14}. Since all subgroups of W of index 4 contain $\Phi(W)$ by the condition (d) of Theorem B, we see that $\Phi(W) = W' \leqq Z(W)$. This implies that W' is elementary abelian. From $|Z(T/\Phi(W))| = 2$, we have that T/W' is isomorphic to $Z_2 \wr D_8$. Hence (1) and (2) are proved. It follows from an easy calculation that T/W' has no nontrivial automorphism of odd order and so has T. Thus (3) holds. We shall next show (4). Assume that $g \in G$ and $|W : W \cap T^g| \leqq 2$. We need to prove that $g \in N_G(W')$. Set $V = W \cap T^g$ and $U = gVg^{-1}$, so that U is a subgroup of T. Since V is of index at most 2 in W and all subgroups of W of index 16 contain W', we have that U and V have no quotient groups isomorphic to a dihedral group of order 8. This implies that $|WU : W| = |U : U \cap W| \leqq 4$, so $U \cap W$ is of index at most 8 in W. By the condition (d) in Theorem B for W, $\Phi(W) = \Phi(U \cap W) \leqq \Phi(U) = g\Phi(W)g^{-1}$. Thus $g \in N_G(W')$, proving (4). Suppose $g \in G$ and $W^g \leqq T$, so that $g \in N_G(W')$ by (4). Thus (5) follows from $C_T(W') = W$. The lemma is proved.

LEMMA 1.2. $N_G(W') \cap O^2(G) = O^2(N_G(W'))$.

LEMMA 1.3. *Assume that G has no subgroup of index 2. Then the following hold:*

(1) *$N_G(W)$ covers $N_G(W')/O(N_G(W'))$;*

(2) *$N_G(W)/O(N_G(W))W$ is isomorphic to $GL(3, 2)$;*

(3) *If $T \leqq H \leqq G$, then $N_H(W) \cap O^2(H) = O^2(N_H(W))$.*

We shall first show that Lemma 1.2 implies Lemma 1.3.

PROOF OF LEMMA 1.3. Lemma 1.2 yields that $N_G(W')$ has no subgroup of index 2. Since $N_G(W')/C_G(W')$ is isomorphic to a subgroup of $GL(3, 2) \cong \mathrm{Aut}(W')$ which has no subgroup of index 2, we have that $N_G(W')/C_G(W')$ is isomorphic to $GL(3, 2)$. Set $N = N_G(W)$ and $N_1 = N_G(W')$. Since W is a Sylow 2-subgroup of $C_G(W')$, it follows by Frattini argument that $N_1 = C_G(W')N$. Thus $N/C_N(W') \cong N_1/C_G(W') \cong GL(3, 2)$. To prove (1) and (2), it will suffice to show that $C_G(W')$ has a normal 2-complement. By the well-known Burnside theorem, we see that it will suffice to show that $C_G(W')/W' \cap N_G(W)/W' = C_N(W')/W'$ has a normal 2-complement. Set $C = C_N(W')$ and $\bar{N} = N/O(N)W$. Then $\bar{C} = O(\bar{N})$, $\bar{N}/\bar{C} \cong GL(3, 2)$ and $\bar{N}$ acts faithfully on $W/W' \cong Z_2^8$. Since $|\bar{N}|$ divides $|\mathrm{Aut}(W/W')| = 2^{28} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 17 \cdot 31 \cdot 127$, we have that the automorphism group of each Sylow subgroup of $\bar{C}$ does not involve $GL(3, 2)$. Thus $\bar{C} \leqq Z(\bar{N})$ and so $\bar{C}$ normalizes $\bar{T}$. By Lemma 1.1(3), we have that $\bar{C} = 1$. This implies that C has a normal 2-complement, as required. (1) and (2) are proved.

Finally, let H be a subgroup of G containing T. Then Lemma 1.2 implies that $N_H(W')\cap O^2(H)=O^2(N_H(W'))$. Thus $N_H(W)\cap O^2(H)=N_H(W)\cap O^2(N_H(W'))$. To prove (3), we may assume that W' is normal in H. By (1), $O(H)W$ is normal in H. (3) follows directly from this. Hence we see that Lemma 1.2 implies Lemma 1.3.

Now we shall prove Lemma 1.2. We shall introduce some notation. For subsets A and B of a finite group, we set $A\backslash B=\{Ab|b\in B\}$. Permutation representations are always right permutation representations. Let (G, Ω) and (G, Ω') be two permutation representations of a finite group G. Then the notation $O(\Omega)$ denotes the set of all elements of G which acts on Ω as an odd permutations. When (G, Ω) and (G, Ω') are equivalent as permutation representations, we shall write $\Omega\overset{G}{\sim}\Omega'$ or simply $\Omega\sim\Omega'$.

Set $N_1=N_G(W')$. We will show that $T\cap G'=T\cap N_1'$. Then by the well-known Tate's theorem Lemma 1.2 follows. Suppose by the way of contradiction that $T\cap G'\neq T\cap N_1'$ and take an element x of $T\cap G'-N_1'$ of minimal order. Since $T'=\Phi(T)$, there is a normal subgroup N_2 of N_1 of index 2 which does not contain x. Since $x\in G'$, we have that x acts on the set $N_2\backslash G$ as an even permutation. Clearly, x acts on $N_2\backslash N_1=\{N_2, N_2x\}$ as an odd permutation, and thus there is an element g of $G-N$ such that $x\in O(N_2\backslash N_1gT)$. Since $N_2\lhd N_1=N_2+N_2x$, we have that $N_2\backslash N_2gT\overset{T}{\sim}N_2\backslash N_2xgT$. Thus if $N_2gT\neq N_2xgT$, then x acts on $N_2\backslash N_1gT=N_2\backslash N_2gT+N_2\backslash N_2xgT$ as an even permutation, a contradiction, and so we see that $N_2gT=N_2xgT=N_1gT$. Set $S=T\cap N_1^g$ and $K=T\cap N_2^g$. Then K is of index 2 in S and $N_2\backslash N_2gT\overset{T}{\sim}K\backslash T$, so $x\in O(K\backslash T)$. By the minimality of the order of x, we have that $\langle x^2\rangle^u\cap S\subseteqq K$ for each $u\in T$. As $g\in G-N_1$, Lemma 1.1(4) yields that $|W:W\cap S|\geqq 4$. Thus to derive a contradiction, it will suffice to show the following lemma:

LEMMA 1.4. *Let T be a 2-group and W a normal subgroup of T such that any subgroup of W of index at most 8 contains $\Phi(W)$. Let S and K be subgroups of T and let x be an element of T. Assume the following conditions:*

(a) *K is a maximal subgroup of S;*

(b) *$x\in O(K\backslash T)$;*

(c) *$\langle x^2\rangle^u\cap S\subseteqq K$ for each $u\in T$.*

Then $|W:W\cap S|\leqq 2$.

PROOF. For any 2-group T, $Sing(T)$ will denote the set of all triplets (S, K, x) which satisfy the assumptions (a), (b) and (c) of this lemma. Note that for any $t\in T$ and any $u\in T$, $(S^t, K^t, x^u)\in Sing(T)$. Now suppose by the way of contradiction that the lemma is false and let T be a counterexample of minimal order to the lemma. Choose the subgroup K of maximal order which violates the lemma. Then K contains no nontrivial normal subgroup of T. We set $X=\langle x\rangle$ and $Y=\langle x^2\rangle$.

We shall first show that if $x\in H\leqq T$, then there is $t\in T$ such that $(S^t\cap H, K^t\cap H, x)\in Sing(H)$. As $x\in O(K\backslash T)$, there is $t\in T$ such that $x\in O(K\backslash StH)$. Let s be an element of $S-K$. Then $K\backslash KtH\overset{H}{\sim}K\backslash KstH$. Thus if $KtH\neq KstH$, then x acts on $K\backslash StH=K\backslash KtH+K\backslash KstH$ as an even permutation, a contradiction, so $KtH=KstH=StH$, whence $|S^t\cap H:K^t\cap H|=2$ and $K\backslash StH\sim K\backslash KtH\sim K^t\cap H\backslash H$. In particular, $x\in O(K^t\cap H\backslash H)$. By the condition (c), we have that $Y^u\cap(S^t\cap H)\subseteqq K^t\cap H$ for each $u\in H$. Hence $(S^t\cap H, K^t\cap H, x)\in Sing(H)$, as required. Take $H=WX$ and suppose $H<T$, so that $|W:W\cap S^t\cap H|=|W:W\cap S|\leqq 2$, a contradiction, and hence

(1) $$T=WX.$$

Next take $C=C_T(X)$. Then there is $t\in T$ such that $(S^t\cap C, K^t\cap C, x)\in Sing(C)$. Set $L=K^t\cap C$. Since $x\in Z(C)$, it follows that x is represented as the product of cyclic permutations of the same length on $L\backslash C$. As $x\in O(L\backslash C)$, x acts transitively on $L\backslash C$, and so $C=LX\trianglerighteq L$. Suppose $|C:L|\geqq 4$, then there is an even i such that $x^i\in S^t\cap C-L\subseteqq S^t-K^t$, contrary to the condition (c). Hence we have that

(2) S contains a conjugate of $C_T(x)$.

We shall next show that $N_T(K)=S$. Set $N=N_T(K)$. As $x\in O(K\backslash T)$, there is $t\in T$ such that $x\in O(K\backslash NtX)$. Since K is normal in N, we have that $K\backslash KtX\overset{X}{\sim}K\backslash KntX$ for each $n\in N$. Thus $KtX=NtX$, as $x\in O(K\backslash NtX)$, and so $|N^t\cap X:K^t\cap X|=|N:K|$. If $N\neq S$, then for some even j, $x^j\in S^t-K^t$. This is a contradiction by the condition (c). Hence $N_T(K)=S$.

Now let R be a subgroup of T which contains S as a maximal subgroup. Let r be an element of $R-S$ and set $N=K\cap K^r$. Since $N_T(K)=S$, we have that R/N is dihedral of order 8 and S/N is a four-group. Let L be a subgroup of R such that $N<L\neq S$ and L/N is a four-group. We may assume that r is in L. We shall show that $(R, L, x)\in Sing(T)$. Let $y\in Y^u\cap R$. $u\in T$. Then $y^2\in Y^u\cap S\subseteqq K$ by the assumption (c). Similarly, $(y^2)^r\in K$. Thus $y^2\in K\cap K^r=N$, and so $y\in S\cup L$. If y is in S, then it follows from (c) that $y\in K\cap K^r=N\leqq L$, and thus $y\in L$. Hence $Y^u\cap R\subseteqq L$ for each $u\in T$. We must next show that $x\in O(L\backslash T)$. Since $x\in O(K\backslash T)$, it will suffice to show that for any $u\in T$, the following conditions are equivalent:

(3) $$x\in O(L\backslash RuX);$$

(4) $$x\in O(K\backslash RuX).$$

If necessary replacing x with uxu^{-1}, we may assume that $u=1$. Let k be an element of $K-N$ and set $s=[r, k]\in L\cap S-K$. Suppose first that $Rx\neq R$. If $Lx^i=Lk$ for some i, then $Rx^i=R$, and so i is even, as $x\notin R$. Thus $x^i\in Y\cap R\subsetneqq L$. This is a contradiction. Hence $LX\neq LkX$. In particular, x is represented on $L\backslash RX=L\backslash LX+L\backslash LkX$ as the product of two nontrivial cyclic permutations, and so x acts on $L\backslash RX$ as an even permutation. If $Kx^j=Ks$ for some j, then $Rx^j=R$, so j is even. Thus $x^j\in K$ by (c), a contradiction. Thus $KX\neq KsX$. Since $rK=Krs$, we have that X is represented on $K\backslash RX$ as the product of two or four nontrivial cyclic permutations, so x acts on $K\backslash RX$ as an even permutation. Hence in case of $Rx\neq R$, neither (3) nor (4) holds. Suppose next that $Rx=R$. Then x is in R. Since $x^2\in S$, we have that $x^2\in N$ by (c), and so x is in $S\cup L$. Thus (3) is equivalent to

$$(3)' \qquad x\in S-L=N\langle s\rangle k.$$

It follows easily that any element of $S-L$ acts on $K\backslash R$ as an odd permutation. Thus if (3) holds, then (4) also holds. Assume conversely that (4) holds. Then x fixes an element of $K\backslash R$, and so $x\in K\cup K^r-N=S-N$. Thus (3)′ and also (3) hold. Hence (3) and (4) are equivalent in this case. By the definition of $Sing(T)$, we conclude that $(R, L, x)\in Sing(T)$. By the choice of K, we have that $|W: W\cap R|\leqq 2$. Thus we have that

$$(5) \qquad |W: W\cap S|=4.$$

We can now complete the proof of this lemma. Since $W\cap K$ is of index at most 8 in W, it follows from the properties of W that $W\cap K$ contains $\Phi(W)$, and thus W is elementary abelian. By (2), we may assume that S contains $C_T(x)$ and also x. Since $T=WX$, we have that S is of index 4 in T. Thus x^2 acts trivially on the set $S\backslash T$, so that x^2 is contained in all conjugates of S. By (c), x^2 is also contained in all conjugates of K. Since K contains no nontrivial normal subgroup of T, we have that x is of order 2. Since S contains $Z(T)$ and $Z(T)\cap K=1$, $Z(T)$ is of order 2. Hence W is of order at most 4. This is a contradiction. The lemma is proved.

REMARK. The rewriting of the above proof by the use of the transfer mapping is left as an exercise for the readers. The similar way as the above yields the following transfer theorem:

If a Sylow 2-subgroup T of a finite group G has no quotient group isomorphic to D_8, then $T\cap G^2G'=T\cap N^2N'$, where $N=N_G(T)$. See [**7**].

2. The subgroups W and $N_G(W)$.

In the remainder of this paper, we assume that G, T and W satisfy the hypothesis of Theorem B. In this section, we consider the structures of W and $N_G(W)$. We set $N=N_G(W)$ and $\bar{N}=N/O(N)$. We shall use the bar convention. Then $O_2(\bar{N})=\bar{W}$. By Lemma 1.2 and 1.3, $\bar{N}/\bar{W}$ is isomorphic to $GL(3, 2)$ and acts faithfully on $W/W'\cong Z_2^8$ and $W'\cong Z_2^3$. In the proof of the following two lemmas, we shall use the modular representation theory of finite groups and refer to [**2**] for the notation and the terminology. Note that there is a subgroup $\bar{L}$ of $\bar{N}$ such that $\bar{L}\cap\bar{W}=\bar{W}'$ and $\bar{L}/\bar{W}'\cong GL(3, 2)$ by the well-known Gaschütz's theorem.

Lemma 2.1. *$\bar{N}/\bar{W}$ acts irreducibly on W/W' and W'.*

Proof. Set $\tilde{N}=\bar{N}/\bar{W}$. By Lemma 1.3, the irreducibility of $\tilde{N}$ on W' is clear. Considering the group W/W' as a $GF(2)[\tilde{N}]$-module, W/W' is indecomposable and projective. The group $GL(3, 2)$ has four irreducible representations over $GF(2)$, that is, 1-representation, two of degree 3 and one of degree 8. These are all absolutely irreducible and the principal indecomposable $GF(2)[GL(3, 2)]$-modules corresponding to them have degree 8, 16, 16, 8, respectively. Therefore if W/W' is not irreducible, then W/W' is the principal indecomposable $GF(2)[\tilde{N}]$-module corresponding to 1-representation and its factor module by a maximal submodule is the trivial one. (See [**2**, p. 70]). But this contradicts the fact that N has no subgroup of index 2. The lemma is proved.

Lemma 2.2. *The structure of W is uniquely determined.*

Proof. As in Lemma 2.1, we regard W/W' and W' as $GF(2)[\tilde{N}]$-modules, where $\tilde{N}=\bar{N}/\bar{W}$. We define the mapping of $W/W'\otimes W/W'$ to W' by the rule: $x\otimes y\longmapsto[x, y]$ for $x, y\in W/W'$. Since W is special 2-group, this mapping is a $GF(2)[\tilde{N}]$-epimorphism. It is easily shown that the principal indecomposable $GF(2)[\tilde{N}]$-module which corresponds to the irreducible $GF(2)[\tilde{N}]$-module W' has the multiplicity 1 in a decomposition of $W/W'\otimes W/W'$ into principal indecomposable modules. Therefore $W/W'\otimes W/W'$ has a unique maximal submodule by which factor module is isomorphic to W' and the kernel of the above mapping is uniquely determined. So $[x, y]$ is also uniquely determined for $x, y\in W$. Next we consider the square mapping $g: W/W'\longrightarrow W'$. The mapping g must satisfy the relation: $g(x+y)=g(x)+g(y)+[x, y]$ for each $x, y\in W/W'$. If there exists another mapping h of W/W' to W' which satisfies the above relation, then $g+h$ is a $GF(2)[\tilde{N}]$-homomorphism from W/W' to W' and so a 0-mapping by Lemma 2.1. Therefore we have $g=h$. This means that the square mapping of W/W' to W' is also uniquely determined, and hence the uniqueness of the structure of W is proved.

REMARK. We can now prove that a Sylow 2-subgroup T of the Rudvalis group satisfies really the conditions of Theorem B. It follows from [**1**, Section 2, Lemma 2.1, etc.] that (a), (b), (c) in Theorem B and Lemma 2.1 hold also for the Rudvalis group. So N acts transitively on the set of the four-subgroups of W'. Since $W/Z_2(T)$ is extraspecial of order 2^9, all subgroups of $W/Z_2(T)$ of index at most 16 contain $\Phi(W/Z_2(T))$. From this, the condition (d) follows.

LEMMA 2.3. *$m(W)=5$ and $W-W'$ has exactly two N-conjugate classes of involutions of which representatives are u and e such that $C_{\bar{N}}(\bar{u})$ is of order 2^{11} and $C_{\bar{N}}(\bar{e})$ is isomorphic to the direct product of a four-group and the Sylow 2-normalizer of $Sz(8)$. Furthermore, an element of $\bar{N}$ of order 3 acts faithfully on $Z(C_{\bar{N}}(\bar{e}))\cong Z_2^3$.*

PROOF. By [**1**, Lemma 2.8 and 2.9], $m(W)=5$ and $W-W'$ contains exactly 360 involutions. Let u be an involution in $Z_4(T)-W'$, then $C_T(u)$ is of order 2^{11}. By the irreducibility of $\bar{N}/\overline{W}$ on W/W', we have that $|(uW')^N|\geqq \dim(W/W')=8$. This yields that $C_{\bar{N}}(\bar{u})$ is of order 2^{11}, and so $|u^N|=168$. Let P be a Sylow 7-subgroup of N and let Q be a Sylow 3-subgroup of $N_N(P)$. By [**1**, Lemma 2.1 and 2.14], $C_W(P)$ is a four-group and $C_W(Q)$ is quaternion. Take an involution e in $C_W(P)$. Then we have that $C_{\bar{N}}(\bar{e})$ is of order $2^8\cdot 7$, so $|e^N|=192$. Hence involutions of $W-W'$ are conjugate to u or e in N. By the uniqueness of the structure of W, $C_T(e)$ is isomorphic to a direct product of a four-group and a Sylow 2-subgroup of $Sz(8)$. The proof of the remainder of this lemma is easy. The lemma is proved.

LEMMA 2.4. *Let $\bar{F}$ be a subgroup of $\bar{N}$ of order 21. Then the structure of W as an $\bar{F}$-admissible group is uniquely determined.*

PROOF. Set $A=\mathrm{Aut}(W)$. It will suffice to show that any subgroup of A isomorphic to $\bar{F}$ are conjugate to each other in A. Set $B=C_A(W')$. Then A/B is isomorphic to $GL(3, 2)$. Set $C=O_2(A)$. By Lemma 2.1, we have that C stabilizes the chain: $1<W'<W$, and so $C=C_B(W/W')$. We can regard $\bar{N}/\overline{W}'$ as a subgroup of A. Let u and e be involutions given in Lemma 2.3. Set $Z=Z(C_W(e))\cong Z_2^5$. Since u and e are not conjugate in A, we have that $|(eW')^A|=|(eW')^N|=24$. By Lemma 2.3, we have that there exist exactly eight A-conjugates of Z. Thus A/C is isomorphic to a subgroup of S_8, the symmetric group of degree 8, since $(eW')^A$ generate W/W'. From $O_2(B/C)=1$ and $A/B\cong GL(3, 2)$, it follows that $B=C$, and so $A/O_2(A)$ is isomorphic to $GL(3, 2)$. The lemma follows directly from this.

LEMMA 2.5. *The following hold:*

(1) *There is a subgroup L of N containing $O(N)W'$ such that $L/O(N)W'$ is isomorphic to $GL(3, 2)$ and $T\cap L$ is a Sylow 2-subgroup of L.*

(2) *Set $V=T\cap O_{2'2}(C_L(z))$, where $Z(T)=\langle z\rangle$. Then V is an extraspecial maximal subgroup of $T\cap L$. Furthermore, $O_2(C_{\bar{N}}(\bar{z}))=\overline{WV}$, $V\cap W=W'$ and $WV/W\cong Z_2^2$.*

(3) *Let s be an element of $N_L(V)$ such that $|\bar{s}|=3$. Then $C_{\bar{N}}(\bar{z})$ is generated by $\bar{T}$ and $\bar{s}$. $C_W(s)$ is isomorphic to a quaternion group.*

(4) *Let X be a subgroup of W such that $X>W'$ and $X/W'=C_{W/W'}(V/W')$. Then X is elementary of order 32 and s-invariant.*

(5) *Set $Y=C_T(X)$. Then $Y\cong Z_2^4\times Q_8$, $Y=XC_Y(s)$, $C_Y(s)=C_W(s)$, and Y is normal in $C_N(z)$.*

PROOF. (1) follows from Gaschütz's Theorem. (2) and the first statement of (3) follows from the consideration of the structure of $\bar{L}$. By [**1**, Lemma 2.14], $C_W(s)$ is quaternion. By Lemma 1.1(2), WV/W' is isomorphic to $Z_2^2 \wr Z_2^2$. Thus X is of order 32. Clearly, X is normal in $C_N(z)$. So $X-W'$ contains an involution $u\in Z_4(T)-W'$. As X is s-invariant, X is elementary. By Lemma 2.3 and [**1**, Lemma 2.14], (5) holds. The lemma is proved.

LEMMA 2.6. *N has exactly four conjugate classes of involutions represented by z, u, e, t, which satisfy the following conditions:*

(1) *z is in $Z(T)$ and $C_{\bar{N}}(\bar{z})$ is of order $2^{14}\cdot 3$.*

(2) *u is in $Z_4(T)-W'$ and $C_{\bar{N}}(\bar{u})$ is of order 2^{11}.*

(3) *e is in $W-W'$ and $C_{\bar{N}}(\bar{e})$ is of order $2^8\cdot 7$.*

Furthermore, $C_{\bar{N}}(\bar{e})$ is isomorphic to the direct product of a four-group and a Sylow 2-normalizer in $Sz(8)$ and $C_W(e)$ is a Sylow 2-subgroup of $C_N(e)$.

(4) *t is in $T'W-W$ and $C_{\bar{N}}(\bar{t})$ is of order 2^8.*

$C_T(t)$ is a Sylow 2-subgroup of $C_N(t)$, t is commutative with u and $C_T(t)$ covers T/W.

PROOF. We shall use the notation given by Lemma 2.5. Since V is extraspecial, there is an involution t in $V-W'$. Since $C_{\bar{L}}(\bar{z})/\overline{W}'$ is isomorphic to S_4, we may assume that t is in $(T\cap L)'$. Since t normalizes the elementary abelian subgroup X and t does not centralize W', we have that t centralizes an involution u in $Z_4(T)-W'$. We shall show that $C_T(t)$ is of order 2^8. Since $|C_{W/W'}(t)|=16$ and $C_{W'}(t)=Z_2(T)$, it will suffice to show that an involution of $t(Z_2(T)-Z(T))$ is conjugate to t in T. By the well-known Baer's theorem and the structure of $\bar{L}$, $\bar{t}$ inverts an element $\bar{r}$ of $\bar{L}$ of order 3. Set $W^*=W/[W', \bar{r}]$. Then W^* is extraspecial of order 2^9. $[W', r]$ does not contain $Z_2(T)$. By Lemma 2.5 (5), we have that $C_{W^*}(r)$ is quaternion, and so W^* is the central product of the copies of four quaternion groups, whence $[W^*, r]$ is isomorphic to $Q_8*Q_8*Q_8$. Since the centralizer of any elementary abelian subgroup of order 8 in $[W^*, r]$ is isomorphic to $Z_2^2 \times Q_8$, there is a quaternion subgroup of $[W^*, r]$ which is normalized by r and t. Thus t inverts an element of W^* of order 4. Hence t is conjugate to an element of $t(Z_2(T)-Z(T))$, as required. From this, we have that $C_T(t)$ covers T/W and all involutions in tW are conjugate. (4) is proved. The remainder of this lemma follows from Lemma 2.3.

3. The proof of Theorem B.

In this section, we shall prove Theorem B. When W' is strongly closed in T with repect to G, it follows from Goldschmidt **[3]** that $O(G)W'$ is normal in G, and thus by Lemma 1.3, we conclude that $O(G)W$ is normal in G and that $G/O(G)W$ is isomorphic to $GL(3, 2)$. Hence the theorem holds in this case. So we assume that W' *is not strongly closed in* T *with respect to* G in the remainder of this section. We set $N=N_G(W)$. Furthermore, we use some notation defined in Section 2. The elements z, u, e and t are the involutions given in Lemma 2.6. The notation L, V, X, Y and s denotes subgroups and elements given by Lemma 2.5. $\bar{L}$ is a subgroup of $\bar{N}=N/O(N)$ such that $\bar{L}/\overline{W}'$ is isomorphic to $GL(3,2)$ and $T\cap L$ is a Sylow 2-subgroup of L. $V=T\cap O_{2'2}(C_L(z))$, s is an element of $N_L(V)$ such that $\bar{s}$ is of order 3, and X is an elementary abelian normal subgroup of $C_N(z)$ of order 32. Furthermore, $O_{2'2}(C_N(z))\cap T=WV$ and $Y=C_T(X)$.

LEMMA 3.1. $z\sim u \not\sim e \not\sim t$.

PROOF. We shall first show that e is not conjugate to z or u. Suppose false, then there is an element g in G such that $e^g=z$ or u and $C_T(e)^g \leqq T$. Set $C=C_T(e)$ and $D=C^g$. C is isomorphic to the direct product of a four-group and a Sylow 2-subgroup of $Sz(8)$. Thus $C'=W'\leqq Z(C)=\Omega_1(C)\cong Z_2^5$ and C has no dihedral quotient group of order 8. As T/W is dihedral, we have that $\Phi(D)\leqq W$ and $|DW:W|\leqq 4$. Suppose $\Omega_1(D)\leqq W$. Since $m(W)=5$ by Lemma 2.4, $\Omega_1(D)=Z(D)\geqq W'=Z(W)$. Thus $D\leqq C_T(W')=W$, and so $W'=C'=D'$. This means that $g\in N_G(W')$, contrary to Lemma 1.2 (2) and Lemma 2.6. Hence there is an involution d in $D-W$ and a subgroup D_0 of D such that $D=\langle d\rangle\times D_0$ and $|D_0W:W|\leqq 2$. Since D_0 has no dihedral quotient group of order 8, we have that $\Phi(D)=\Phi(D_0)=\Phi(D_0\cap W)\leqq W'$. Thus $D\leqq C_T(W')=W$, a contradiction. Hence it is proved that e is not conjugate to z or u. Suppose next e is conjugate to t. By Lemma 2.6, $C_T(e)$ and $C_T(t)$ are of same order, so these are conjugate. But by Lemma 2.6, $C_T(t)$ has a dihedral quotient group of order 8 and $C_T(e)$ has not. This is a contradiction. Hence $e\not\sim z$, u, t. We shall finally show that $z\sim u$. Suppose false, so that $z\sim t$, since W' is not strongly closed in T. Take an element g in G such that $t^g=z$ and

$C_T(t)^g \leqq T$. By Lemma 2.6, we have that $u \in C_T(t)$, $tu \sim t$ and $u^G \cap T = u^N \subseteqq W - W'$. Thus $t \sim (tu)^g = zu^g \in W - W'$, so $t \sim u$ or e, a contradiction. The lemma is proved.

LEMMA 3.2. *Assume that $g \in G$ and $Y^g \leqq T$. Then Y^g is contained in W and is conjugate to Y by an element of N. In particular, T has exactly seven conjugates of Y. Furthermore, Y is weakly closed in T with respect to $C_G(z)$.*

PROOF. Suppose first Y^g does not contain W'. As $m(W) = m(Y^g) = 5$, we have that $Y^g - W$ has an involution y. Then $Y^g \leqq C_T(y)$. By Lemma 2.6, $C_W(y)$ is of order 32. Since Y has no dihedral quotient group of order 8, we see that $|Y^g : Y^g \cap W| \leqq 4$, and so $|Y^g \cap W| \geqq 32$. Thus $C_W(y) = Y^g \cap W$. By $m(W) = 5$, $C_W(y)$ is not elementary. Thus $Y^g W'/W'$ is elementary of order 32. But T/W' is isomorphic to $Z_2 \wr D_8$ and so for any elementary abelian subgroup U of T/W' of order 32, $|UW : W| \leqq 2$. Thus $|Y^g : Y^g \cap W| \leqq 2$ which is a contradiction. Hence we have that Y^g contains W'. So $Y^g \trianglelefteq C_T(W') = W$. It follows from the weak closure of W that Y^g is conjugate to Y by an element of N, as required. Since Y is normal in $C_N(z)$, we have that $|N : N_N(Y)| = 7$. So T has seven conjugates of Y. The final statement follows from the fact that Y is the unique conjugate of Y of which commutator subgroup is $Y' = Z(T)$. The lemma is proved.

LEMMA 3.3. *Set $J = C_T(X \bmod Z(T))$. Then $|J| = 2^{11}$, $J' = X$, $[X, J] = Z(T)$, $C_G(X \bmod Z(T)) = O(N_G(X))J \trianglelefteq N_G(X)$, and $N_G(X)/O(N_G(X))J$ is isomorphic to S_5, the symmetric group of degree five.*

PROOF. As z and u are conjugate in G, there is g in G such that $u^g = z$ and $C_T(u)^g \leqq T$. So $Y^g \leqq C_T(u)^g \leqq T$, and thus $g \in N_G(Y)N$ by Lemma 3.2. This implies that u is conjugate to an involution of W' by an element of $N_G(Y)$. Thus all involutions of $X - Z(T)$ are conjugate to each other by elements of $N_G(Y) \leqq C_G(z)$. Set $M = N_G(X)$. By Lemma 1.2, $C_M(X \bmod Z(T))T$ has a normal 2-complement. Similarly, $C_G(X)$ and $C_G(Z_2(T))$ have normal 2-complements. Since Y is a Sylow 2-subgroup of $C_G(X)$ and $Y' = Z(T)$, we have that $M \leqq C_G(z)$. Thus $C_G(X \bmod Z(T)) = O(N_G(X))J \triangleleft M$. Set $\bar{M} = M/C_G(X)$. Then $C_{\bar{M}}(Z_2(T))$ is of order 2^6, and so $\bar{M}$ is of order 30×2^6. Since $C_{VW}(s) = C_Y(s) \cong Q_8$, we have that s acts on $VX/Z(T)$ as a fixed-point-free automorphism of order 3. Furthermore $VX/Z(T)$ is of order 2^6 and $t \in V - W'$ is commute with $u \in X - W'$. Thus an easy calculation derives that $VX/Z(T)$ is abelian, and so V is a subgroup of J. Since WV/W' is isomorphic to $Z_2^2 \wr Z_2^2$, we have that $V[V, W]$ is a subgroup of J of order 2^{11}. J is s-invariant and $|W : W \cap J| \geqq 4$, and hence $J = V[V, W]$ is of order 2^{11}. Thus we see that $M/C_M(X/Z(T))$ is of order 120 and isomorphic to S_5. Hence $M/O(M)J$ is also isomorphic to S_5. Since $[W, V]$ is of index 4 in W, $[W, V]' = W'$ by the assumption of Theorem B. Thus $J' \geqq W'$, so J' contains X since $M/O(M)J \cong S_5$ acts irreducibly on $X/Z(T)$. As WV/W' is isomorphic to $Z_2^2 \wr Z_2^2$, $(J/W')' = [W, V, V]W'/W'$ is of order 4. Hence $J' = X$. $[J, J, J] = [X, J] \neq 1$, so $[X, J] = Z(T)$. The lemma is proved.

LEMMA 3.4. *X is strongly closed in T with respect to $C_G(z)$.*

PROOF. Set $H = C_G(z)$. By Lemma 1.2, H has a normal subgroup of index 2. By Lemma 3.3, JW is a Sylow 2-subgroup of $O^2(H)$. Since $J/Y \cong W/Y \cong Z_2^4$ and s acts fixed-point-free on WJ/Y, we have that all involutions of JW is contained in $J \cup W$. Since $N_G(J)$ acts irreducibly on $J/Y \cong Z_2^4$, any involution of J is conjugate to one of W in $N_G(J)$. Suppose X is not strongly closed in T with respect to H. Then there is $w \in W - Y$ such that w is conjugate to an involution of X in H. Take an element g of H such that $w^g \in X$ and $C_T(w)^g \leqq T$. As w is conjugate to u in G, $C_W(w)$ contains a conjugate Y_1 of Y. If $Y_1' = Z(T)$, then by Lemma 3.2, $Y_1 = Y$. So $w \in C_T(Y) = X$, a contradiction. Thus $Y_1' \neq Z(T)$. Since $Y_1{}^g$ is conjugate to Y_1 by an element of $H \cap N$, we may assume that $g \in N_H(Y_1) \leqq N_H(Y_1')$. By Lemma 1.2, $N_G(Z_2(T))$ has a normal 2-complement and so has $N_H(Y_1')$. This implies a contradiction. The lemma is proved.

PROOF OF THEOREM B. We can now prove Theorem B. By Lemma 3.4 and [**3**], we have that $H = C_G(z) = O(H)N_G(J)$. In particular, H is 2-constrained. By the structure of $N_G(X)$, we have that $t \sim z$. $C_T(e)$ has a strongly closed abelian subgroup with respect to $C_G(e)$. Again by [**3**], we see that $C_G(e)$ is solvable or $C_G(e)/O(C_G(e))$ is isomorphic to $Z_2^2 \times Sz(8)$. By Gorenstein-Walter's theorem [**5**], we have that $O(C_G(z)) \leqq O(G)$ and $O(C_G(e)) \leqq O(G)$. By [**6**] (or [**11**]), we conclude that $G/O(G)$ is isomorphic to the Rudvalis group. Theorem B is proved.

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