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On the hypoellipticity for infinitely degenerate semi-elliptic operators

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§ 0. Introduction.

In this paper we shall study hypoellipticity for a partial differential operator of the form

$$P=a(x, y, D_x)+g(x)b(x, y, D_y) \quad \text{in} \quad R^n=R_x^{n_1}\times R_y^{n_2}, \tag{0.1}$$

where $a(x, y, D_x)$ and $b(x, y, D_y)$ are strongly elliptic operators of order $2l$ and $2m$ with respect to x and y, respectively, and $g(x)$ is a smooth non-negative function with a zero point of infinite order at $x=0$ in $R_x^{n_1}$. The operator of the form $(-\Delta_x)^l+g(x)(-\Delta_y)^m$ is a typical example.

Our main theorem is roughly stated as follows: Assume that $b(x, y, D_y)$ is of second order (, but $a(x, y, D_x)$ is not necessarily of second order). Then we have the statement:

$$(*) \qquad \text{“}u\in\mathcal{D}'(\Omega),\quad Pu\in H_s^{loc}(\Omega)\Rightarrow u\in H_s^{loc}(\Omega)\text{”} \qquad \text{for any } \Omega\subset R^n,$$

and therefore P is hypoelliptic in R^n. When $b(x, y, D_y)$ is of higher order $\geqq 4$, we set the following condition on $g(x)$.

Condition (G).

$$|\partial_x^\beta g(x)|\leqq C_\beta g(x)^{1-\sigma|\beta|} \qquad \text{in a neighborhood of} \quad x=0$$

for a fixed σ $(0<\sigma<\{2(m+l(m-1))\}^{-1})$. Then, we have the statement (*) in this case, too. (Such a σ is determined from Propositions 5.1 and 5.2. See Remark of Proposition 5.2.)

When $g(x)$ has a zero point of finite order, fairly complete results have been obtained by Hörmander [8], [9], Grushin [6], [7], Beals [1], Y. Kato [11], Kumano-go-Taniguchi [15], Taniguchi [17], Tsutsumi [19], etc. In such case except [17] we have the stronger result than (*), that is, the statement

$$(**) \qquad \text{“}u\in\mathcal{D}'(\Omega),\quad Pu\in H_s^{loc}(\Omega)\Rightarrow u\in H_{s+\sigma_0}^{loc}(\Omega)\text{”} \qquad \text{for any} \quad \Omega\subset R^n$$

holds for some positive number σ_0. It should be noted that we can no longer expect the statement (**) for the operator of the form (0.1) when $g(x)$ has a zero point of infinite order (see Theorem 1.2).

As an example which does not satisfy the sufficient condition for hypoellipticity given by Hörmander [9], V. S. Fedii [5] considered an operator $P=\partial_x^2+\phi(x)^2\partial_y^2$ where $\phi(x)$ satisfies $\phi(x)>0$ for $x\neq 0$ and $\phi^{(j)}(0)=0$ for any j, and he proved the statement (*) for that operator, improving the criterion given by Treves [18]. Remark that his result can be obtained by setting $g(x)=\phi(x)^2$.

The plan of this paper is as follows. In Section 1 we state our main result (Theorem 1.1). In Section 2 we introduce a notion "weakly-elliptic" for an operator P concerning sufficient conditions for hypoellipticity given in [5], [16], [18], and we prove that the statement (*) holds for the weakly-elliptic operators (, so P is necessarily hypoelliptic). In Sections 3 and 4 we prove that P of the form (0.1) is weakly-elliptic. Section 5 is devoted to the proof of Lemma 4.1, which plays an important role in Section 4.

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§ 1. Notations and main result.

For a multi-index $\alpha=(\alpha_1, \cdots, \alpha_n)$ of non-negative integers α_j $(j=1, \cdots, n)$ and a point $x=(x_1, \cdots, x_n)\in R^n$ we use the notations:

$$|\alpha|=\alpha_1+\cdots+\alpha_n, \quad \alpha!=\alpha_1!\cdots\alpha_n!, \quad x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n},$$

$$\partial_x^\alpha=\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}, \quad \partial_{x_j}=\frac{\partial}{\partial x_j}, \quad D_x^\alpha=D_{x_1}^{\alpha_1}\cdots D_{x_n}^{\alpha_n},$$

$$D_{x_j}=-i\frac{\partial}{\partial x_j}, \quad \langle x\rangle=\{1+|x|^2\}^{1/2}.$$

Let $C^\infty(\Omega)$ denote the set of infinitely differentiable functions in Ω and let $C_0^\infty(\Omega)$ denote the set of $C^\infty(\Omega)$-functions with compact support, where Ω is an open set in R^n. For a compact set K of Ω, $C_0^\infty(K)$ denotes the set of $C^\infty(\Omega)$-functions whose supports are contained in K. Let $\mathscr{B}(\Omega)$ denote the set of $C^\infty(\Omega)$-functions whose derivatives of any order are all bounded in Ω, and let $\mathscr{D}'(\Omega)$ denote the set of distributions in Ω. $\mathscr{E}'(K)$ is the set of distributions with support in K. $C^\infty(R^n)$, $C_0^\infty(R^n)$, $\mathscr{B}(R^n)$, $\mathscr{D}'(R^n)$ are often denoted simply by C^∞, C_0^∞, $\mathscr{B}$, $\mathscr{D}'$, respectively. $\mathscr{S}=\mathscr{S}(R^n)$ denotes the Schwartz space of rapidly decreasing functions and $\mathscr{S}'$ denotes its dual space. For $u\in\mathscr{S}_x$ the Fourier transform $\hat{u}(\xi)=\mathscr{F}[u](\xi)$ is defined by

$$\mathscr{F}[u](\xi)=\int e^{-ix\cdot\xi}u(x)\,dx, \quad x\cdot\xi=x_1\xi_1+\cdots+x_n\xi_n,$$

and then, for $\hat{u}(\xi)\in\mathcal{S}_\xi$, the inverse Fourier transform $\bar{\mathcal{F}}[\hat{u}]$ is defined by

$$\bar{\mathcal{F}}[\hat{u}](x)=\int e^{ix\cdot\xi}\hat{u}(\xi)\,d\!\!{}^-\xi, \qquad d\!\!{}^-\xi=(2\pi)^{-n}d\xi.$$

For real s we define the Sobolev space H_s as the completion of $\mathcal{S}$ in the norm

$$\|u\|_s=\left\{\int\langle\xi\rangle^{2s}|\hat{u}(\xi)|^2 d\!\!{}^-\xi\right\}^{1/2}.$$

$H_s^{loc}(\Omega)$ denotes the space $\{u\in\mathcal{D}'(\Omega)\,;\,\phi u\in H_s \text{ for } \phi\in C_0^\infty(\Omega)\}$. For $\phi, \psi\in C_0^\infty(\Omega)$ we write $\phi\subset\subset\psi$ when $\psi(x)\equiv 1$ in a neighborhood of $\operatorname{supp}\phi$.

Now we shall study the hypoellipticity of a differential operator P of the form

$$P=a(x, y, D_x)+g(x)b(x, y, D_y) \qquad \text{in} \quad R^n=R_x^{n_1}\times R_y^{n_2} \tag{0.1}$$

under the following conditions:

1°) $a(x, y, D_x)$ is a differential operator of order $2l$ with C^∞ coefficients and satisfies for large $|\xi|$

$$\operatorname{Re} a(x, y, \xi)\geqq C_1|\xi|^{2l} \qquad (C_1>0). \tag{1.1}$$

2°) $b(x, y, D_y)$ is a differential operator of order $2m$ with C^∞ coefficients and satisfies for large $|\eta|$

$$\operatorname{Re} b(x, y, \eta)\geqq C_2|\eta|^{2m} \qquad (C_2>0). \tag{1.2}$$

3°) $g(x)$ belongs to $C^\infty(R_x^{n_1})$, $g(x)>0$ for $x\neq 0$, and $\partial_x^\beta g(0)=0$ for any β.

THEOREM 1.1. *Let P satisfy Conditions* 1°), 2°) *and* 3°). *Then, we obtain the following two assertions:*

i) *If $m=1$ (, i.e., $b(x, y, D_y)$ is of second order), then the statement*

(*) "$u\in\mathcal{D}'(\Omega)$, $Pu\in H_s^{loc}(\Omega)\Rightarrow u\in H_s^{loc}(\Omega)$" *for ony* $\Omega\subset R^n$

holds. Therefore, P is hypoelliptic.

ii) *In case $m\geqq 2$, we add the following condition* (G) *for $g(x)$. Then we have the statement* (*).

Condition (G). *For any β there exists a constant C_β such that*

$$|\partial_x^\beta g(x)|\leqq C_\beta g(x)^{1-\sigma|\beta|} \qquad \textit{in a neighborhood of } x=0. \tag{1.3}$$

Here σ is a fixed positive number such that

$$0<\sigma<\{2(m+l(m-1))\}^{-1}. \tag{1.4}$$

REMARK 1. It is easy to see that in general any non-negative function $g(x)$ always satisfies the inequality (1.3) with $\sigma=1/2$. In fact, for $|x|\leqq 1$, $-1\leqq t\leqq 1$ and $M=\max_{j,|x|\leqq 2}|\partial_{x_j}^2 g(x)|$ the inequality

$$0\leqq g(x+te_j)\leqq g(x)+t\partial_{x_j}g(x)+\frac{t^2}{2}M$$

holds, and so, setting

$$t=-\mathrm{sign}(\partial_{x_j}g(x))\sqrt{g(x)}\,/\,\sqrt{g(x)^2+(M/2)^2}$$

we have

$$|\partial_{x_j}g(x)|\leqq C\sqrt{g(x)} \quad \text{on} \quad \{|x|\leqq 1\}.$$

REMARK 2. We see that the function $e^{-(1/|x|)}$ satisfies Condition 3°) and the inequality (1.3) with arbitrary positive σ. The function

$$f(x)=e^{-(1/|x|)}\sin^2\frac{1}{|x|}+e^{-(1/|x|)^2}$$

does not satisfy the inequality (1.3) for any σ ($<1/2$), though it satisfies Condition 3°). Because we have for integer k

$$\partial_{x_j}^2 f(x)=0(e^{-\pi k}), \quad f(x)=0(e^{-\pi^2 k^2}), \quad |x|=(\pi k)^{-1} \qquad \text{as} \quad k\to\infty.$$

We note that f is not expressed in the form $f(x)=\phi(x)^2$ for any non-negative C^∞-function ϕ.

As stated in Introduction, if we replace the condition "$\partial_x^\beta g(0)=0$ for any β" in Condition 3°) by "$\partial_x^{\beta_0}g(0)\neq 0$ for some β_0", we can expect a stronger statement than (*):

(**) "$u\in\mathcal{D}'(\Omega)$, $Pu\in H_s^{loc}(\Omega)\Rightarrow u\in H_{s+\sigma_0}^{loc}(\Omega)$" for any $\Omega\subset R^n$,

for some $\sigma_0>0$. It is impossible to replace (*) by (**) in Theorem 1.1, in general. To explain this fact we prove the following

THEOREM 1.2. *Let $P=p(x, y, D_x, D_y)$ be a differential operator of the form* (0.1). *We assume that there exists some $\sigma_0>0$ such that*

$$\|u\|_{\sigma_0}^2\leqq C(\|Pu\|_0^2+\|u\|_0^2), \qquad u\in C_0^\infty(\Omega) \tag{1.5}$$

for some constant C when Ω is an open set (containing $\{0\}$) of $R_{x,y}^n$. Then, there exists some β_0 such that $\partial_x^{\beta_0}g(0)\neq 0$.

To prove this theorem we use the following theorem (see Hörmander [8]).

THEOREM 1.3. *Let $P=p(x, D_x)$ be a differential operator of order m with coefficients in $C^\infty(R_x^n)$. Suppose that, for a fixed $\sigma_0>0$ and an open set Ω of R_x^n, we have the estimate*

$$\|u\|_{\sigma_0}^2 \leqq C_1(\|Pu\|_0^2+\|u\|_0^2), \qquad u\in C_0^\infty(\Omega)$$

for some constant C_1. Then, for any integer $N>0$, any θ $(0<\theta\leqq 1/2)$ and any compact set K of Ω we have the estimate

$$(1.6) \qquad \int\langle\xi\rangle^{2\sigma_0}|\phi(y)|^2dy$$

$$\leqq C_2\Big\{\int\Big|\sum_{|\alpha+\beta|<N} p_{(\beta)}^{(\alpha)}(x, \xi)\langle\xi\rangle^{\theta(|\alpha|-|\beta|)}(iy)^\beta D_y^\alpha\phi(y)/(\alpha!\,\beta!)\Big|^2dy$$

$$+\langle\xi\rangle^{2m-2\theta N}\sum_{|\alpha+\beta|\leqq N+m}\int|y^\beta D_y^\alpha\phi(y)|^2dy\Big\},$$

$$(x, \xi)\in K\times R_\xi^n, \quad \phi(y)\in C_0^\infty(R_y^n),$$

for a constant $C_2=C_2(K)$. Here $p_{(\beta)}^{(\alpha)}(x, \xi)=\partial_\xi^\alpha D_x^\beta p(x, \xi)$.

PROOF OF THEOREM 1.2. Assume that $\partial_x^\beta g(0)=0$ for any β. Then, applying Theorem 1.3 to (1.5), that is, setting $(x, y)=(0, 0)$ and $(\xi, \eta)=(0, \eta)$ for $p(x, y, D_x, D_y)$, we have from (1.6)

$$\langle\eta\rangle^{2\sigma_0}\|\phi(z)\|^2$$

$$\leqq C'\{\|\sum_{|\alpha+\beta|<N} p_{(\beta)}^{(\alpha)}(0, 0, 0, \eta)\langle\eta\rangle^{\theta(|\alpha|-|\beta|)}(iz)^\beta$$

$$\times D_z^\alpha\phi(z)/(\alpha!\,\beta!)\|^2+\langle\eta\rangle^{2M-2\theta N}\sum_{|\alpha+\beta|\leqq N+M}\|z^\beta D_z^\alpha\phi(z)\|^2\},$$

$$\phi(z)\in C_0^\infty(R_z^n), \quad R_z^n=R_x^{n_1}\times R_y^{n_2},$$

where $M=\max(2l, 2m)$. Fix $\phi(z)\not\equiv 0$ and θ so that $2\theta l<\sigma_0$. Then, taking N so large that $M-\theta N<0$ and noting that when $|\alpha|>2l$ (=the order of $a(x, y, \xi)$ in ξ) $p_{(\beta)}^{(\alpha)}(0, 0, 0, \eta)=0$ for any β, we have

$$\langle\eta\rangle^{2\sigma_0}\leqq C''\langle\eta\rangle^{4\theta l} \qquad \text{for any } \eta,$$

which is a contradiction. Q.E.D.

§ 2. Weakly-elliptic operator and hypoellipticity.

DEFINITION 2.1. A differential operator $P=p(x, D_x)$ of order m with coefficients in $C^\infty(\Omega)$ is called 'weakly-elliptic' (or simply 'w-elliptic') in Ω if P satisfies the following conditions:

I) For any compact set K of Ω and any $N>0$ there exists a constant $C_1=C_1(K, N)$ such that

$$\|u\|_0 \leqq C_1(\|Pu\|_0+\|u\|_{-N}), \quad u\in C_0^\infty(K). \tag{2.1}$$

II) For any compact set K of Ω, and β ($|\beta|\neq 0$), any $\mu>0$ and any $N>0$ there exists a constant $C_2=C_2(K, \beta, \mu, N)$ such that

$$\|P_{(\beta)}u\|_{-|\beta|} \leqq \mu\|Pu\|_0+C_2\|u\|_{-N}, \quad u\in C_0^\infty(K), \quad (|\beta|\neq 0) \tag{2.2}$$

where $p_{(\beta)}(x, \xi)=D_x^\beta p(x, \xi)$.

III) For any $x_0\in\Omega$ and any neighborhood U of x_0 there exists $\phi(x)\in C_0^\infty(U)$ such that

$$\phi(x)\equiv 1 \quad \text{in some neighborhood of } x_0$$

and the estimate

$$\|[P, \phi]u\|_s \leqq C_3(K, s, N, \phi)(\|Pu\|_{s-\kappa}+\|u\|_{-N}), \quad u\in C_0^\infty(K) \tag{2.3}$$

holds for any compact set K, any real s and any $N>0$, where $\kappa>0$ may depend on x_0, and $[P, \phi]=P\phi-\phi P$.

REMARK. We see that many hypoelliptic operators are w-elliptic. For instance semi-elliptic operators, subelliptic operators in Egorov [**3**], [**4**] and hypoelliptic operators treated in Hörmander [**10**] are w-elliptic. But, in general, hyperbolic operators are not w-elliptic, since the estimates II) and III) do not hold.

THEOREM 2.2. *Let a differential operator P be w-elliptic in Ω. Then, for any open set $\Omega'\subset\Omega$ we have*

$$v\in\mathcal{D}'(\Omega'), \quad Pv\in H_s^{\mathrm{loc}}(\Omega')\Rightarrow v\in H_s^{\mathrm{loc}}(\Omega').$$

REMARK 1. This theorem is a re-formation of theorems given by Fedii [**5**], Oleinik-Radkevich [**16**] and Treves [**18**]. It seems that the assumptions of this theorem does not include the following estimate (, which is derived from the hypotheses of their theorems): "For any real s, any $\mu>0$ and any $N>0$ there exists a constant C such that

$$\|P^{(j)}u\|_s \leqq \mu\|Pu\|_s+C\|u\|_{-N}, \quad u\in C_0^\infty(K),$$

where $p^{(j)}(x, \xi)=\partial_{\xi_j}p(x, \xi)$." We do not know whether the above estimate holds for the operators P in Theorem 1.1, when the order of $a(x, y, D_x)$ is not less than 4.

REMARK 2. Theorem 2.2 will be proved by showing that, for any fixed point $x_0 \in \Omega'$, there exist ϕ, $\psi \in C_0^\infty(\Omega')$ such that $\phi(x) \equiv 1$ in a neighborhood of x_0 and $\phi \subset\subset \psi$, and for some large $N>0$ and a constant C the estimate

$$\|\phi v\|_s \leqq C(\|\psi P v\|_s + \|\psi v\|_{-N}) \tag{2.4}$$

holds. Therefore, by modifying the coefficients of P outside of $\operatorname{supp}\psi$ we may assume without loss of generality that all the coefficients of P belong to $\mathscr{B}(R^n)$.

Before the proof of Theorem 2.2 we define pseudodifferential operators and state several propositions without proofs (see [2], [8], [12], [13] and [14]). Let $\lambda(\xi)$ be a basic weight function in R^n with the following conditions:

$$A_0^{-1}\langle\xi\rangle^{\sigma'} \leqq \lambda(\xi) \leqq A_0\langle\xi\rangle \qquad (A_0>0,\ \sigma'>0), \tag{2.5}$$

$$|\lambda^{(\alpha)}(\xi)| \leqq A_\alpha \lambda(\xi)^{1-|\alpha|}. \tag{2.6}$$

By using this fixed function $\lambda(\xi)$ we define a class of pseudodifferential operators.

DEFINITION 2.3. We say that the C^∞-function $p(x, \xi)$ in $R^{2n}_{x,\xi}$ belongs to $S^m_{\lambda,\rho,\delta}$ ($-\infty<m<\infty$, $\delta=(\delta_1, \cdots, \delta_n)$, $\rho=(\rho_1, \cdots, \rho_n)$, $0 \leqq \delta_j < \rho_j \leqq 1$, $j=1, \cdots, n$), when for any multi-indices $\alpha=(\alpha_1, \cdots, \alpha_n)$, $\beta=(\beta_1, \cdots, \beta_n)$ we have

$$|p^{(\alpha)}_{(\beta)}(x, \xi)| \leqq C_{\alpha,\beta}\, \lambda(\xi)^{m-\rho\cdot\alpha+\delta\cdot\beta} \qquad \text{on} \quad R^{2n}, \tag{2.7}$$

where

$$p^{(\alpha)}_{(\beta)}(x, \xi) = \partial_\xi^\alpha D_x^\beta p(x, \xi), \quad \rho\cdot\alpha = \sum_{j=1}^n \rho_j \alpha_j, \quad \delta\cdot\beta = \sum_{j=1}^n \delta_j \beta_j.$$

Then, the pseudodifferential operator $P=p(x, D_x)$ (denoted by $P \in S^m_{\lambda,\rho,\delta}$) with the symbol $p(x, \xi)$ is defined by

$$Pu = \int e^{ix\cdot\xi} p(x, \xi)\, \hat{u}(\xi)\, d\xi, \quad u \in \mathcal{S}. \tag{2.8}$$

We often denote the symbol of $P=p(x, D_x)$ by $\sigma(P)$.

REMARK 1. When $\rho=(\rho_0, \cdots, \rho_0)$, $\delta=(\delta_0, \cdots, \delta_0)$, we denote $S^m_{\lambda,\rho,\delta}$, $\mathcal{S}^m_{\lambda,\rho,\delta}$ by $S^m_{\lambda,\rho_0,\delta_0}$, $\mathcal{S}^m_{\lambda,\rho_0,\delta_0}$ respectively. Furthermore when $\lambda=\langle\xi\rangle$, $\rho_0=1$, $\delta_0=0$, we often write $S^m_{\lambda,\rho_0,\delta_0}=S^m$, $\mathcal{S}^m_{\lambda,\rho_0,\delta_0}=\mathcal{S}^m$. A differential operator of order m with coefficients in $\mathscr{B}(R^n)$ is clearly the element of $\mathcal{S}^m$.

2. $S^m_{\lambda,\rho,\delta}$ is Fréchet space provided with semi-norms

$$|p|_l^{(m)} = \max_{|\alpha+\beta| \leqq l} \sup_{R^{2n}} \{|p^{(\alpha)}_{(\beta)}(x, \xi)|\, \lambda(\xi)^{-m+\rho\cdot\alpha-\delta\cdot\beta}\}.$$

We say that the subset B of $S^m_{\lambda,\rho,\delta}$ is bounded when $\sup_{p\in B}\{|p|_l^{(m)}\} < \infty$ for any l.

3. We set $S^{-\infty}_{\lambda,\rho,\delta}=\bigcap_{-\infty<m<\infty} S^m_{\lambda,\rho,\delta}$. It is easy to see that $S^{-\infty}_{\lambda,\rho,\delta}=S^{-\infty}$.

PROPOSITION 2.4. *Let $P_j=p_j(x, D_x)\in S^{m_j}_{\lambda,\rho,\delta}$. Then, we have $P_1P_2\in S^{m_1+m_2}_{\lambda,\rho,\delta}$ and*

$$\sigma(P_1P_2)-\sum_{|\alpha|<N}\frac{1}{\alpha!}p_1^{(\alpha)}(x, \xi)p_{2(\alpha)}(x, \xi)\in S^{m_1+m_2-\tau_0 N}_{\lambda,\rho,\delta}, \tag{2.9}$$

where $\tau_0=\min_j(\rho_j-\delta_j)$. Furthermore, we have

$$[P_1, P_2]-\sum_{0<|\alpha+\beta|<N}\frac{(-1)^{|\beta|}}{\alpha!\beta!}P_{2(\beta)}^{(\alpha)}P_{1(\beta)}^{(\alpha)}\in S^{m_1+m_2-\tau_0 N}_{\lambda,\rho,\delta}. \tag{2.10}$$

We define $\|u\|_{s,\lambda}$ by

$$\|u\|_{s,\lambda}=\left(\int\lambda(\xi)^{2s}|\hat{u}(\xi)|^2 d\xi\right)^{1/2} \quad \text{for} \quad u\in\mathcal{S}.$$

When $\lambda=\langle\xi\rangle$, we denote $\|u\|_{s,\lambda}=\|u\|_s$.

PROPOSITION 2.5. *Let $P=p(x, D_x)\in S^m_{\lambda,\rho,\delta}$. Then for any real s there exist C and l depending on m and s such that*

$$\|Pu\|_{s,\lambda}\leqq C|p|_l^{(m)}\|u\|_{s+m,\lambda} \quad \textit{for} \quad u\in\mathcal{S}. \tag{2.11}$$

PROPOSITION 2.6. *Let $P=p(x, D_x)\in S^m_{\lambda,\rho,\delta}$ and let ϕ and $\psi\in\mathcal{B}$ satisfy* $\mathrm{dis}(\mathrm{supp}\,\phi, \mathrm{supp}\,\psi)\geqq C_0>0$. *Then, $\phi P\psi\in S^{-\infty}$.*

PROPOSITION 2.7. *Let $p(x, \xi)\in S^m_{\lambda,\rho,\delta}$ satisfy the following conditions (which is called (H)-condition):*

i) *There exist $C_0>0$ and some real m' such that*

$$|p(x, \xi)|\geqq C_0\lambda(\xi)^{m'} \quad \textit{for large} \quad |\xi|. \tag{2.12}$$

ii) *For any α, β there exists a constant $C_{\alpha,\beta}$ such that*

$$|p_{(\beta)}^{(\alpha)}(x, \xi)/p(x, \xi)|\leqq C_{\alpha,\beta}\lambda(\xi)^{\delta\cdot\beta-\rho\cdot\alpha} \quad \textit{for large} \quad |\xi|. \tag{2.13}$$

Then, there exists a parametrix $Q=q(x, D_x)\in S^{-m'}_{\lambda,\rho,\delta}$ such that $I\equiv QP\equiv PQ$ (mod $S^{-\infty}$), *that is,*

$$I=QP+K=PQ+K' \quad \textit{for some } K \textit{ and } K'\in S^{-\infty}. \tag{2.14}$$

Furthermore, we can write

$$\begin{cases} Q=Q_0Q_1 \\ \textit{for} \quad Q_0\in S^{-m'}_{\lambda,\rho,\delta} \quad \textit{defined by} \quad \sigma(Q_0)=q_0(x, \xi)=p(x, \xi)^{-1} \quad \textit{for large} \quad |\xi|, \\ \textit{and some} \quad Q_1\in S^0_{\lambda,\rho,\delta}. \end{cases} \tag{2.15}$$

PROOF. Note that $q_{0(\beta)}^{(\alpha)}(x, \xi)$ consists of the linear combination of symbols of the form

$$(p^{-1}p_{(\beta^1)}^{(\alpha^1)})\cdots(p^{-1}p_{(\beta^k)}^{(\alpha^k)})p^{-1} \qquad \text{for large } |\xi|$$

$$(\alpha^1+\cdots+\alpha^k=\alpha,\ \beta^1+\cdots+\beta^k=\beta,\ k\leqq|\alpha|+|\beta|).$$

Then, from (2.12) and (2.13) we have $q_0(x, \xi)\in S_{\lambda,\rho,\delta}^{-m'}$, so $Q_0\in S_{\lambda,\rho,\delta}^{-m'}$. Similarly, using (2.9) in Proposition 2.4, from (2.12) and (2.13) it is easy to check $\sigma(I-PQ)\in S_{\lambda,\rho,\delta}^{-\tau_0}$, where $\tau_0=\min_j(\rho_j-\delta_j)$. Set $R=I-PQ_0$. Then, since $R^j\in S_{\lambda,\rho,\delta}^{-\tau_0 j}$, there exists $Q_1\in S_{\lambda,\rho,\delta}^{0}$ such that $Q_1\sim\sum_{j=0}^{\infty}R^j$ in the sense

$$Q_1-\sum_{j=0}^{N-1}R^j\in S_{\lambda,\rho,\delta}^{-\tau_0 N}$$

holds for any N (see [8]). Asymptotically, we have

$$(I-R)Q_1\sim(I-R)\sum_{j=1}^{\infty}R^j\sim I.$$

Consequently we obtain

$$PQ_0Q_1=(I-R)Q_1=I+K' \qquad \text{for some } K'\in S^{-\infty}.$$

Hence, we see that $Q=Q_0Q_1\in S_{\lambda,\rho,\delta}^{-m'}$ is a right parametrix. Similarly, checking $\sigma(I-Q_0P)$, we have a left parametrix Q' such that $Q'P\equiv I \pmod{S^{-\infty}}$. By means of the equality

$$Q'\equiv Q'(PQ)\equiv(Q'P)Q\equiv Q \qquad (\mathrm{mod}\, S^{-\infty}),$$

we have $QP\equiv I \pmod{S^{-\infty}}$, that is, $QP=I+K$, $K\in S^{-\infty}$. Q. E. D.

REMARK. Let $p(x, \xi)\in S_{\lambda,\rho,\delta}^{m}$ satisfy

$$|p(x, \xi)|\geqq C_0\lambda(\xi)^m \quad (C_0>0) \qquad \text{for large } |\xi|.$$

Then, using $|p_{(\beta)}^{(\alpha)}(x, \xi)|\leqq C_{\alpha,\beta}\lambda(\xi)^{m+\delta\cdot\beta-\rho\cdot\alpha}$, we get (2.13). Therefore, $p(x, D_x)\in S_{\lambda,\rho,\delta}^{m}$ has a parametrix $Q\in S_{\lambda,\rho,\delta}^{-m}$.

LEMMA 2.8. *Let $\Lambda_{s,k,\varepsilon}$ denote a pseudodifferential operator with the symbol $\langle\xi\rangle^s(1+\varepsilon\langle\xi\rangle)^{-k}$ for real s, $\varepsilon>0$ and $k\geqq 0$. Then, $\Lambda_{s,k,\varepsilon}\in S^{s-k}$, and for any $\varepsilon>0$ there exists a constant C_ε such that*

(2.16) $$\|\Lambda_{s,k,\varepsilon}u\|_{s'}\leqq C_\varepsilon\|u\|_{s+s'-k}, \quad u\in\mathcal{S}.$$

Furthermore, $\{\Lambda_{s,k,\varepsilon}\}$ is a bounded set in S^s, so the estimate

(2.17) $$\|\Lambda_{s,k,\varepsilon}u\|_{s'}\leqq C_0\|u\|_{s'+s}, \quad u\in\mathcal{S}$$

holds for a constant C_0 *independent of* ε. *Moreover, for any* α *there exists a constant* C_α *such that*

$$\|\Lambda_{s,k,\varepsilon}^{(\alpha)}u\|_0 \leqq C_\alpha \|\Lambda_{s-|\alpha|,k,\varepsilon}u\|_0, \quad u\in\mathcal{S}. \tag{2.18}$$

PROOF. (2.16) and (2.17) are clear. Using the inequality

$$|\partial_\xi^\alpha(\langle\xi\rangle^s(1+\varepsilon\langle\xi\rangle)^{-k})| \leqq C'_\alpha \langle\xi\rangle^{s-|\alpha|}(1+\varepsilon\langle\xi\rangle)^{-k}$$

we obtain (2.18). Q.E.D.

LEMMA 2.9. *Let* $\Xi=\{\boldsymbol{a}=(a_0, a_1, \cdots, a_l)\}$ *be a set of real vectors with* $a_j>0$ $(1\leqq j\leqq l)$ *satisfying the following: For any* $\mu>0$ *there exists some constant* C_μ *such that*

$$a_j \leqq \mu \sum_{k=0}^{l-1} a_k + C_\mu \sum_{k=j+1}^{l} a_k \quad \text{for any} \quad \boldsymbol{a}\in\Xi$$

$$(j=1, \cdots, l-1).$$

Then, for any $\mu'>0$, *we can find a constant* $C'_{\mu'}$ *such that*

$$a_j \leqq \mu' a_0 + C'_{\mu'} a_l \quad \text{for any} \quad \boldsymbol{a}\in\Xi \quad (j=1, \cdots, l-1).$$

PROOF. For any $\mu_j>0$ $(1\leqq j\leqq l-1)$ we have

$$a_j \leqq \mu_j \sum_{k=0}^{l-1} a_k + C_{\mu_j} \sum_{k=j+1}^{l} a_k \quad \text{for any} \quad \boldsymbol{a}\in\Xi$$

for some C_{μ_j}. Multiplying both sides by some $d_j>0$ and taking the sum over $1\leqq j\leqq l-1$, we obtain

$$\sum_{j=1}^{l-1} d_j a_j \leqq \Big(\sum_{j=1}^{l-1} \mu_j d_j\Big) \sum_{k=0}^{l-1} a_k + \sum_{j=0}^{l-1} d_j C_{\mu_j} \sum_{k=j+1}^{l} a_k.$$

Then we have

$$\sum_{j=1}^{l-1} h_j a_j \leqq \Big(\sum_{j=1}^{l-1} \mu_j d_j\Big) a_0 + \Big(\sum_{j=1}^{l-1} d_j C_{\mu_j}\Big) a_l,$$

where

$$h_1 = d_1 - \sum_{k=1}^{l-1} \mu_k d_k, \; h_j = d_j - \sum_{k=1}^{l-1} \mu_k d_k - \sum_{k=1}^{j-1} d_k C_{\mu_k}$$

$$(j=2, \cdots, l-1).$$

For any $\mu'>0$ we can find $d_j>0$, $\mu_j>0$ such that

$$h_j > 1-\mu', \quad \sum_{j=1}^{l-1} \mu_j d_j < \mu' \quad (j=1, \cdots, l-1).$$

(For instance, we can determine $\{d_j, \mu_j\}$ by

$$d_1=1,\ \mu_1=\mu'/2,\ d_2=1+C_{\mu_1}d_1,\ \mu_2 d_2=\mu'/4,\ \cdots,$$

$$d_j=1+\sum_{k=1}^{j-1} C_{\mu_k}\mu_k,\ \mu_j d_j=\mu'/2^j,\ \cdots,\ (j=1,\ \cdots,\ l-1).)$$

Therefore, we have for any $0<\mu'<1$

$$(1-\mu')\sum_{j=1}^{l-1} a_j \leqq \mu' a_0+\Big(\sum_{j=1}^{l-1} d_j C_{\mu_j}\Big)a_l, \qquad \boldsymbol{a}\in\Xi. \qquad \text{Q. E. D.}$$

From now on let P denote a differential operator of order m with coefficients $\in\mathcal{B}(R^n)$.

LEMMA 2.10. *Let P satisfy Condition* II) *of Definition* 2.1. *Then, for any compact set $K\subset\Omega$, any β ($|\beta|\neq 0$), any real s, any $\mu>0$, $N>0$, $\varepsilon>0$ and $k\geqq 0$ there exists a constant $C=C(K, \beta, s, \mu, N, k)$ independent of ε such that*

$$\|\Lambda_{s-|\beta|,k,\varepsilon}P_{(\beta)}u\|_0\leqq\mu\|\Lambda_{s,k,\varepsilon}Pu\|_0+C\|u\|_{-N},\quad u\in C_0^\infty(K). \tag{2.19}$$

PROOF. Take ϕ, $\psi\in C_0^\infty(\Omega)$ such that $\phi\subset\subset\psi$ and $\phi(x)\equiv 1$ on K. Then, for $u\in C_0^\infty(K)$ we have $\psi\Lambda_{s,k,\varepsilon}u\in C_0^\infty(K')$, where $K'=\operatorname{supp}\psi$. Replacing u by $\psi\Lambda_{s,k,\varepsilon}u$ in (2.2) we obtain

$$\begin{aligned}\|P_{(\beta)}\psi\Lambda_{s,k,\varepsilon}u\|_{-|\beta|}&\leqq\mu\|P\psi\Lambda_{s,k,\varepsilon}u\|_0+C_1\|\psi\Lambda_{s,k,\varepsilon}u\|_{-N}\\&\leqq\mu\|P\psi\Lambda_{s,k,\varepsilon}u\|_0+C_1'\|u\|_{-N+s},\quad u\in C_0^\infty(K).\end{aligned}$$

We used the estimate (2.17) in the last step. Since N is arbitrary, we have

$$\begin{aligned}&\|P_{(\beta)}\psi\Lambda_{s,k,\varepsilon}u\|_{-|\beta|}\\&\qquad\leqq\mu\|P\psi\Lambda_{s,k,\varepsilon}u\|_0+C_1'\|u\|_{-N},\quad u\in C_0^\infty(K).\end{aligned}$$

Noting that $\Lambda_{s,k,\varepsilon}u=\psi\Lambda_{s,k,\varepsilon}u+(1-\psi)\Lambda_{s,k,\varepsilon}\phi u$ for $u\in C_0^\infty(K)$ and $P_{(\beta)}(1-\psi)\cdot\Lambda_{s,k,\varepsilon}\phi\in S^{-\infty}$ from Proposition 2.6, we obtain

$$\|P_{(\beta)}\Lambda_{s,k,\varepsilon}u\|_{-|\beta|}\leqq\|P_{(\beta)}\psi\Lambda_{s,k,\varepsilon}u\|_{-|\beta|}+C_2\|u\|_{-N},\quad u\in C_0^\infty(K).$$

In the same way we have

$$\|P\psi\Lambda_{s,k,\varepsilon}u\|_0\leqq\|P\Lambda_{s,k,\varepsilon}u\|_0+C_3\|u\|_{-N},\quad u\in C_0^\infty(K).$$

From the above three estimates it follows that the estimate

$$\|P_{(\beta)}\Lambda_{s,k,\varepsilon}u\|_{-|\beta|}\leqq\mu\|P\Lambda_{s,k,\varepsilon}u\|_0+C_4\|u\|_{-N},\quad u\in C_0^\infty(K)$$

holds. Using the expansion formula

$$[P, \Lambda_{s,k,\varepsilon}]-\sum_{0<|\alpha|<s+m+N}\frac{(-1)^{|\alpha|}}{\alpha\,!}\Lambda^{(\alpha)}_{s,k,\varepsilon}P_{(\alpha)}\in S^{-N}$$

in Proposition 2.4, we see that the estimate

$$\begin{aligned}\|P\Lambda_{s,k,\varepsilon}u\|_0&\leqq\|\Lambda_{s,k,\varepsilon}Pu\|_0+\|[P, \Lambda_{s,k,\varepsilon}]u\|_0\\&\leqq\|\Lambda_{s,k,\varepsilon}Pu\|_0+C_5\Big(\sum_{0<|\alpha|<s+m+N}\|\Lambda^{(\alpha)}_{s,k,\varepsilon}P_{(\alpha)}u\|_0+\|u\|_{-N}\Big)\\&\leqq\|\Lambda_{s,k,\varepsilon}Pu\|_0+C_5'\Big(\sum_{0<|\alpha|<s+m+N}\|\Lambda_{s-|\alpha|,k,\varepsilon}P_{(\alpha)}u\|_0+\|u\|_{-N}\Big),\\&\qquad u\in C_0^\infty(K)\end{aligned}$$

holds for a constant C_5' (independent of ε). We used the estimate (2.18) in the last step. Similarly we have

$$\begin{aligned}\|\Lambda_{s-|\beta|,k,\varepsilon}P_{(\beta)}u\|_0&\leqq\|P_{(\beta)}\Lambda_{s,k,\varepsilon}u\|_{-|\beta|}\\&+C_6\Big(\sum_{|\beta|<|\alpha|<s+m+N}\|\Lambda_{s-|\alpha|,k,\varepsilon}P_{(\alpha)}u\|_0+\|u\|_{-N}\Big),\quad u\in C_0^\infty(K).\end{aligned}$$

From the above three estimates it follows that for any $0<|\beta|<s+m+N$ the estimate

$$\begin{aligned}\|\Lambda_{s-|\beta|,k,\varepsilon}P_{(\beta)}u\|_0&\leqq\mu\Big(\|\Lambda_{s,k,\varepsilon}Pu\|_0+\sum_{0<|\alpha|<s+m+N}\|\Lambda_{s-|\alpha|,k,\varepsilon}P_{(\alpha)}u\|_0\Big)\\&+C_7\Big(\sum_{|\beta|<|\alpha|<s+m+N}\|\Lambda_{s-|\alpha|,k,\varepsilon}P_{(\alpha)}u\|_0+\|u\|_{-N}\Big),\quad u\in C_0^\infty(K)\end{aligned}$$

holds for a constant C_7 (independent of ε). Applying Lemma 2.9 to the above estimate for $0<|\beta|<s+m+N$, that is, setting $\|\Lambda_{s,k,\varepsilon}Pu\|_0$, $\|\Lambda_{s-|\beta|,k,\varepsilon}P_{(\beta)}u\|_0$ $(0<|\beta|<s+m+N)$ and $\|u\|_{-N}$ for a_0, a_j $(j=1,\cdots,l-1)$ and a_l, respectively, it is easy to see that (2.19) holds. Q.E.D.

COROLLARY. *Let P satisfy Condition* II) *of Definition* 2.1. *Then, for any $K\subset\Omega$, any real s, s', any $N>0$, $\varepsilon>0$ and $k\geqq0$ there exists a constant $C=C(K, s, s', N, k)$ independent of ε such that*

$$\|[P, \Lambda_{s,k,\varepsilon}]u\|_{s'}\leqq C(\|\Lambda_{s+s',k,\varepsilon}Pu\|_0+\|u\|_{-N}),\quad u\in C_0^\infty(K). \tag{2.20}$$

PROOF. From the expansion

$$[P, \Lambda_{s,k,\varepsilon}]-\sum_{0<|\alpha|<s+s'+m+N}C_\alpha\Lambda^{(\alpha)}_{s,k,\varepsilon}P_{(\alpha)}\in S^{-N-s'}$$

and (2.19) it is easy to see that (2.20) holds. Q.E.D.

LEMMA 2.11. *Let P satisfy Conditions* I) *and* II) *of Definition* 2.1. *Then, any K, s, $N>0$, $\varepsilon>0$ and $k\geqq 0$ there exists a constant $C=C(K, s, N, k)$ independent of ε such that*

$$\|\Lambda_{s,k,\varepsilon}u\|_0\leqq C(\|\Lambda_{s,k,\varepsilon}Pu\|_0+\|u\|_{-N}),\quad u\in C_0^\infty(K). \tag{2.21}$$

PROOF. Take $\psi\in C_0^\infty(\Omega)$ so that $\psi(x)\equiv 1$ on K, and replace $u\in C_0^\infty(K)$ by $\psi\Lambda_{s,k,\varepsilon}u$ in (2.1). Then, by the same way as in the first half of the proof of Lemma 2.10, we have

$$\|\Lambda_{s,k,\varepsilon}u\|_0\leqq C_1(\|P\Lambda_{s,k,\varepsilon}u\|_0+\|u\|_{-N}),\quad u\in C_0^\infty(K).$$

Using $P\Lambda_{s,k,\varepsilon}=\Lambda_{s,k,\varepsilon}P+[P, \Lambda_{s,k,\varepsilon}]$ and (2.20) we obtain the estimate (2.21).
Q.E.D.

REMARK. Set $k=s_0+m+N$ for $s_0>0$. Then, for any $v\in H_{-N}\cap\mathcal{E}'(K)$ the estimate

$$\|\Lambda_{s,k,\varepsilon}v\|_0\leqq C(\|\Lambda_{s,k,\varepsilon}Pv\|_0+\|v\|_{-N}) \tag{2.22}$$

holds, where $s\leqq s_0$ and C is a constant independent of ε. Indeed, taking the sequence $\{u_j\}_{j=1}^\infty$ such that

$$u_j\in C_0^\infty(K')\quad (K\subset K'\subset\Omega),\quad u_j\to v\quad\text{in}\quad H_{-N},$$

and noting that $\Lambda_{s,k,\varepsilon}$, $\Lambda_{s,k,\varepsilon}P\in S^{-N}$ for fixed $\varepsilon>0$ and that the estimate (2.21) holds for $u_j\in C_0^\infty(K')$, we have (2.22) by $j\to\infty$.

LEMMA 2.12. *Let P satisfy Condition* II) *of Definition* 2.1. *Suppose that Q is a differential operator of order m' with coefficients in $\mathcal{B}(R^n)$ and that for any $K\subset\Omega$, real s and $N>0$ there exists a constant $C_1=C_1(K, s, N)$ such that*

$$\|Qu\|_s\leqq C_1(\|Pu\|_{s-\kappa}+\|u\|_{-N}),\quad u\in C_0^\infty(K), \tag{2.23}$$

where κ is some real number. Then, for any β, any $K\subset\Omega$, real s and $N>0$ we can find some constant $C_2=C_2(\beta, K, s, N)$ such that

$$\|Q_{(\beta)}u\|_s\leqq C_2(\|Pu\|_{s-\kappa+|\beta|}+\|u\|_{-N}),\quad u\in C_0^\infty(K). \tag{2.24}$$

Furthermore for any $\varepsilon>0$ and $k\geqq 0$ we can find a constant $C_3=C_3(\beta, K, s, N, k)$ independent of ε such that

$$\|\Lambda_{s-|\beta|,k,\varepsilon}Q_{(\beta)}u\|_0\leqq C_3(\|\Lambda_{s-\kappa,k,\varepsilon}Pu\|_0+\|u\|_{-N}),\quad u\in C_0^\infty(K). \tag{2.25}$$

PROOF. By induction on $|\beta|$ we show that (2.24) holds. When $|\beta|=0$, it is trivial that (2.24) holds. Assume that the estimate (2.24) holds for $|\beta|\leqq n$.

It is easy to see that if $|\beta|=n+1$, then $Q_{(\beta)}=[D_{x_j}, Q_{(\beta')}]$ for $|\beta'|=n$. Using this, for $|\beta|=n+1$ we have

$$\begin{aligned}\|Q_{(\beta)}u\|_s&=\|[D_{x_j}, Q_{(\beta')}]u\|_s\\&\leqq\|D_{x_j}Q_{(\beta')}u\|_s+\|Q_{(\beta')}D_{x_j}u\|_s\\&\leqq C(\|Q_{(\beta')}u\|_{s+1}+\|Q_{(\beta')}D_{x_j}u\|_s),\quad u\in C_0^\infty(K),\end{aligned}$$

where C is some constant (we often denote different constants by same notation C in what follows). By the inductive hypothesis we have

$$\begin{aligned}&\|Q_{(\beta')}u\|_{s+1}+\|Q_{(\beta')}D_{x_j}u\|_s\\&\quad\leqq C(\|Pu\|_{s+1+|\beta'|-\kappa}+\|PD_{x_j}u\|_{s+|\beta'|-\kappa}+\|u\|_{-N}),\\&\qquad\qquad u\in C_0^\infty(K).\end{aligned}$$

From (2.19) of Lemma 2.10 the estimate

$$\begin{aligned}\|PD_{x_j}u\|_s&\leqq\|[P, D_{x_j}]u\|_s+\|D_{x_j}Pu\|_s\\&\leqq C(\|P_{(j)}u\|_s+\|Pu\|_{s+1})\\&\leqq C(\|Pu\|_{s+1}+\|u\|_{-N}),\quad u\in C_0^\infty(K)\end{aligned}$$

holds for any real s. Hence, from the above three estimates we obtain (2.24) with $|\beta|=n+1$. Next we prove (2.25). Replacing u by $\Lambda_{-|\beta|,k,\varepsilon}u$ in (2.24) as in the first half of the proof of Lemma 2.10, we have

$$\begin{aligned}\|Q_{(\beta)}\Lambda_{-|\beta|,k,\varepsilon}u\|_s&\leqq C(\|P\Lambda_{-|\beta|,k,\varepsilon}u\|_{s-\kappa+|\beta|}+\|u\|_{-N})\\&\leqq C(\|\Lambda_{s-\kappa,k,\varepsilon}Pu\|_0+\|[P, \Lambda_{-|\beta|,k,\varepsilon}]u\|_{s-\kappa+|\beta|}+\|u\|_{-N}),\\&\qquad\qquad u\in C_0^\infty(K).\end{aligned}$$

Applying (2.20) in Corollary to Lemma 2.10 to the second term on the right side, we obtain

$$\|Q_{(\beta)}\Lambda_{-|\beta|,k,\varepsilon}u\|_s\leqq C(\|\Lambda_{s-\kappa,k,\varepsilon}Pu\|_0+\|u\|_{-N}),\quad u\in C_0^\infty(K).$$

Using the expansion

$$[Q_{(\beta)}, \Lambda_{-|\beta|,k,\varepsilon}]-\sum_{0<|\alpha|<s+m'+N-|\beta|}\frac{(-1)^{|\alpha|}}{\alpha!}\Lambda^{(\alpha)}_{-|\beta|,k,\varepsilon}Q_{(\alpha+\beta)}\in S^{-N-s},$$

we have

$$\begin{aligned}\|\Lambda_{s-|\beta|,k,\varepsilon}Q_{(\beta)}u\|_0&\leqq\|Q_{(\beta)}\Lambda_{-|\beta|,k,\varepsilon}u\|_s+\|[Q_{(\beta)}, \Lambda_{-|\beta|,k,\varepsilon}]u\|_s\\&\leqq\|Q_{(\beta)}\Lambda_{-|\beta|,k,\varepsilon}u\|_s+C\Big(\sum_{0<|\alpha|<s+m'+N-|\beta|}\|\Lambda^{(\alpha)}_{-|\beta|,k,\varepsilon}Q_{(\alpha+\beta)}u\|_s+\|u\|_{-N}\Big)\end{aligned}$$

$$\leqq \|Q_{(\beta)}\Lambda_{-|\beta|,k,\varepsilon}u\|_s+C\Big(\sum_{|\beta|<|\alpha|<s+m'+N}\|\Lambda_{s-|\alpha|,k,\varepsilon}Q_{(\alpha)}u\|_0+\|u\|_{-N}\Big),$$

$$u\in C_0^\infty(K).$$

We used (2.18) in the last step. From the above two estimates we obtain for $0\leqq|\beta|<s+m'+N$

$$\|\Lambda_{s-|\beta|,k,\varepsilon}Q_{(\beta)}u\|_0$$

$$\leqq C\Big(\|\Lambda_{s-\kappa,k,\varepsilon}Pu\|_0+\sum_{|\beta|<|\alpha|<s+m'+N}\|\Lambda_{s-|\alpha|,k,\varepsilon}Q_{(\alpha)}u\|_0+\|u\|_{-N}\Big),$$

$$u\in C_0^\infty(K).$$

Applying Lemma 2.9 to the above estimate for $0\leqq|\beta|<s+m'+N$, it is easy to see that (2.25) holds. Hence, the lemma is proved. Q.E.D.

REMARK. Set $k=\max(s_0+m+N,\ s_0+m'+N)$ for $s_0>0$, where m and m' are the orders of P and Q, respectively. Then, for any $v\in H_{-N}\cap\mathcal{E}'(K)$ the estimate

$$\|\Lambda_{s,k,\varepsilon}Qv\|_0\leqq C(\|\Lambda_{s-\kappa,k,\varepsilon}Pv\|_0+\|v\|_{-N}) \tag{2.26}$$

holds, where $s\leqq s_0$ and C is a constant independent of ε. This fact is obtained from (2.25) by the same way as in Remark of Lemma 2.10.

LEMMA 2.13. *Let P satisfy Condition* II) *of Definition* 2.1 *and let κ be some real number. Let Q be a differential operator of order m' with coefficients in $\mathcal{B}(R^n)$ such that for any $K\subset\Omega$, any β, some real s_β and any $N>0$ the estimate*

$$\|Q_{(\beta)}u\|_{s_\beta-|\beta|}\leqq C(K,\beta,N)(\|Pu\|_{s_\beta-\kappa}+\|u\|_{-N}),\quad u\in C_0^\infty(K) \tag{2.27}$$

holds for some constant $C(K,\beta,N)$. Then, for any real s there exists a constant $C(K,\beta,N,s)$ such that the estimate, replaced s_β by s in (2.27), *holds.*

PROOF. Replacing u by $\Lambda_{s-s_\beta}u$ in (2.27) as in the first half of the proof of Lemma 2.10 (where Λ_{s-s_β} denotes the pseudodifferential operator with symbol $\langle\xi\rangle^{s-s_\beta}$), we have

$$\begin{aligned}\|Q_{(\beta)}\Lambda_{s-s_\beta}u\|_{s_\beta-|\beta|}&\leqq C(\|P\Lambda_{s-s_\beta}u\|_{s_\beta-\kappa}+\|u\|_{-N})\\&\leqq C(\|Pu\|_{s-\kappa}+\|[P,\Lambda_{s-s_\beta}]u\|_{s_\beta-\kappa}+\|u\|_{-N})\\&\leqq C(\|Pu\|_{s-\kappa}+\|u\|_{-N}),\quad u\in C_0^\infty(K).\end{aligned}$$

Here we used (2.20) with $k=0$ in the last step. Furthermore, we have

$$\|Q_{(\beta)}u\|_{s-|\beta|}\leqq\|Q_{(\beta)}\Lambda_{s-s_\beta}u\|_{s_\beta-|\beta|}+\|[Q_{(\beta)},\Lambda_{s-s_\beta}]u\|_{s_\beta-|\beta|}$$

$$\leqq C\Big(\|Pu\|_{s-\kappa}+\sum_{|\beta|<|\alpha|<s+m'+N}\|Q_{(\alpha)}u\|_{s-|\alpha|}+\|u\|_{-N}\Big),\qquad u\in C_0^\infty(K).$$

Therefore, by Lemma 2.9 we obtain

$$\|Q_{(\beta)}u\|_{s-|\beta|}\leqq C(\|Pu\|_{s-\kappa}+\|u\|_{-N}),\quad u\in C_0^\infty(K).$$

Q.E.D.

We end this section by completing the proof of Theorem 2.2, that is, we prove that (2.4) holds.

THE PROOF OF THEOREM 2.2. Let x_0 be any fixed point in Ω' and let $\psi(x)\in C_0^\infty(\Omega')$ such that $\psi(x)\equiv1$ in a neighborhood $U(x_0)$ of x_0. Then, for any natural number l we can find a finite sequence $\{\psi_j(x)\}_{j=1}^l\subset C_0^\infty(\Omega')$ such that

$$\psi_1\subset\subset\psi_2\subset\subset\cdots\subset\subset\psi_l\subset\subset\psi,$$
$$\psi_1(x)\equiv1\qquad\text{in a neighborhood of}\quad x_0,$$

and we have

$$\|[P,\psi_j]u\|_s\leqq C(K,s,N,\psi_j)(\|Pu\|_{s-\kappa}+\|u\|_{-N}),\qquad (j=1,\cdots,l),\quad u\in C_0^\infty(K), \tag{2.28}$$

for any $K\subset\Omega$, real s and $N>0$, where κ is some positive number. Indeed, from Condition III) of Definition 2.1 we can take $\phi_1(x)\in C_0^\infty(U(x_0))$ such that $\phi_1(x)\equiv1$ in some neighborhood $V(x_0)$ of x_0 and satisfies (2.3). It is clear that $\phi_1\subset\subset\psi$. For x_0 and $V(x_0)$ we can take again $\phi_2(x)\in C_0^\infty(V(x_0))$ such that $\phi_2(x)\equiv1$ in some neighborhood of x_0 and satisfies (2.3). Repeating these steps l times, we have $\{\phi_j(x)\}_{j=1}^l\subset C_0^\infty(\Omega')$. Set $\psi_j=\phi_{l-j+1}$ $(j=1,\cdots,l)$. Then, $\{\psi_j\}_{j=1}^l$ is a desired sequence. As well-known, for $\psi v\in\mathcal{E}'$ there exists $N>0$ such that $\psi v\in H_{-N}$. Let us choose l bigger than $(s+m+N)/\kappa$. By Remark of Lemma 2.11, for $\psi_j v\in H_{-N}\cap\mathcal{E}'(K)$ (, where $K=\operatorname{supp}\psi$) the estimate

$$\|\Lambda_{s,k,\varepsilon}\psi_1v\|_0\leqq C(\|\Lambda_{s,k,\varepsilon}P\psi_1v\|_0+\|\psi_1v\|_{-N}) \tag{2.29}$$

holds for C independent of ε and $k=s+m+N$. (Let the same notation C denote different constants independent of ε in what follows.) Furthermore, setting $Q=[P,\psi_j]$ in Lemma 2.12, from (2.28) and Remark of Lemma 2.12 it is easy to see that if $k=s+m+N$, then for $s'\leqq s$ the estimate

$$(2.30)\qquad \|\Lambda_{s',k,\varepsilon}[P, \phi_j]\phi_{j+1}v\|_0 \leqq C(\|\Lambda_{s'-\kappa,k,\varepsilon}P\phi_{j+1}v\|_0+\|\phi_{j+1}v\|_{-N})$$

holds. Noting $[P, \phi_j]\phi_{j+1}=P\phi_j-\phi_jP$, we have from (2.29)

$$\|\Lambda_{s,k,\varepsilon}\phi_1 v\|_0\leqq C(\|\Lambda_{s,k,\varepsilon}\phi_1 Pv\|_0+\|\Lambda_{s,k,\varepsilon}[P, \phi_1]\phi_2 v\|_0+\|\phi_1 v\|_{-N}).$$

Applying (2.30) to the second term on the right side, and noting

$$\|\phi_j v\|_{-N}=\|\phi_j\phi v\|_{-N}\leqq C\|\phi v\|_{-N},$$

we obtain

$$\begin{aligned}\|\Lambda_{s,k,\varepsilon}\phi_1 v\|_0&\leqq C(\|\Lambda_{s,k,\varepsilon}\phi_1 Pv\|_0+\|\Lambda_{s-\kappa,k,\varepsilon}P\phi_2 v\|_0+\|\phi v\|_{-N})\\&\leqq C(\|\Lambda_{s,k,\varepsilon}\phi_1 Pv\|_0+\|\Lambda_{s-\kappa,k,\varepsilon}\phi_2 Pv\|_0\\&\quad+\|\Lambda_{s-\kappa,k,\varepsilon}[P, \phi_2]\phi_3 v\|_0+\|\phi v\|_{-N}).\end{aligned}$$

Applying (2.30) to third term on the right side, and repeating the same procedure, we have

$$\|\Lambda_{s,k,\varepsilon}\phi_1 v\|_0\leqq C\Big(\sum_{j=1}^{l}\|\Lambda_{s-\kappa(j-1),k,\varepsilon}\phi_j Pv\|_0+\|\Lambda_{s-\kappa l,k,\varepsilon}P\phi v\|_0+\|\phi v\|_{-N}\Big).$$

Since $\phi_j Pv\in H_s$ from the hypothesis of theorem, and since $\Lambda_{s-\kappa l,k,\varepsilon}P\in S^{-N}$ for any ε, we obtain from (2.17)

$$\begin{aligned}\|\Lambda_{s,k,\varepsilon}\phi_1 v\|_0&\leqq C\Big(\sum_{j=1}^{l}\|\phi_j Pv\|_{s-\kappa(j-1)}+\|\phi v\|_{-N}\Big)\\&\leqq C(\|\phi Pv\|_s+\|\phi v\|_{-N}).\end{aligned}$$

Letting $\varepsilon\to+0$, we finally get (2.4) with $\phi=\phi_1$. The theorem is proved.

Q. E. D.

§ 3. Proof of Theorem 1.1 (Part I).

Let $P=p(x, y, D_x, D_y)=A+gB=a(x, y, D_x)+g(x)b(x, y, D_y)$ denote a differential operator in Theorem 1.1. We shall show that P is w-elliptic in R^n. Since the conditions of Definition 2.1 are stated for some compact set K of R^n, we may assume, without loss of generality, that $g(x)$ and coefficients of A and B belong to $\mathscr{B}$, and $g(x)$ satisfies for any $\varepsilon>0$

$$(3.1)\qquad g(x)\geqq C_\varepsilon>0 \quad \text{on} \quad \{|x|\geqq\varepsilon\}.$$

LEMMA 3.1. *Set $\Omega_\varepsilon=\{(x, y)\in R^n;\ |x|<\varepsilon\}$. Then, for any $\varepsilon>0$, any α, any real s and any $N>0$ there exist constants $C(\varepsilon, s, N)$ and $C(\varepsilon, \alpha, s, N)$ such that*

(3.2) $$\|u\|_s \leqq C(\varepsilon, s, N)(\|Pu\|_{s-\kappa_0} + \|u\|_{-N}),$$

(3.3) $$\|P^{(\alpha)}u\|_s \leqq C(\varepsilon, \alpha, s, N)(\|Pu\|_{s-\rho_0|\alpha|} + \|u\|_{-N}),$$
$$u \in C_0^\infty(R^n - \Omega_\varepsilon),$$

where $\kappa_0 = \min(2l, 2m)$ *and* $\rho_0 = \min(m/l, l/m)$, *respectively.*

PROOF. Modifying $g(x)$ in $\{|x| < \varepsilon\}$, we may assume that (3.1) holds for any $x \in R_x^{n_1}$. As far as we consider the estimates for $u \in C_0^\infty(R^n - \Omega_\varepsilon)$, such modification is permitted. Set $k = \max(2l, 2m)$ and set $\lambda(\xi, \eta) = (|\xi|^{2l} + |\eta|^{2m} + 1)^{1/k}$. Then $\lambda = \lambda(\xi, \eta)$ is a basic weight function and $p(x, y, \xi, \eta)$ belongs to $S_{\lambda,1,0}^k$. From (1.1), (1.2) and (3.1) we obtain

$$|p(x, y, \xi, \eta)| \geqq C_\varepsilon \lambda^k(\xi, \eta) \qquad \text{for large} \quad |\xi| + |\eta|,$$

where $C_\varepsilon > 0$ depends on ε. By Proposition 2.7 and its Remark we have a parametrix $Q \in S_{\lambda,1,0}^{-k}$ such that $I = QP + K$, $K \in S^{-\infty}$. It is easy to check that $Q \in S_{\rho_0,0}^{-\kappa_0}$ and $P^{(\alpha)}Q \in S_{\rho_0,0}^{-\rho_0|\alpha|}$. Therefore, noting that the semi-norms of $\sigma(Q)$ depend on ε, by means of Proposition 2.5 we obtain (3.2) and (3.3).

Q.E.D.

LEMMA 3.2. *Let* $\phi(x) \in C^\infty(R_x^{n_1})$ *such that for any* $\alpha \neq 0$, $\phi_{(\alpha)} = 0$ *on* $\{|x| \leqq \varepsilon\}$, *where* $\phi_{(\alpha)} = D_x^\alpha \phi(x)$. *Then, for any* $\varepsilon > 0$, *any real* s *and any* $N > 0$ *there exists a constant* $C(\varepsilon, s, N)$ *such that*

(3.4) $$\|[P, \phi]u\|_s \leqq C(\varepsilon, s, N)(\|Pu\|_{s-\rho_0} + \|u\|_{-N}), \quad u \in C_0^\infty(R^n),$$

where $\rho_0 = \min(l/m, m/l)$.

PROOF. Let $u \in C_0^\infty(R^n)$. Then, $\phi_{(\alpha)}u \in C_0^\infty(R^n - \Omega_\varepsilon)$ for any $\alpha \neq 0$. Hence, by Lemma 3.1 we have for any $\alpha \neq 0$ and $\beta \neq 0$

(3.5) $$\|P^{(\beta)}\phi_{(\alpha)}u\|_s \leqq C(\|P\phi_{(\alpha)}u\|_{s-\rho_0} + \|u\|_{-N}), \quad u \in C_0^\infty(R^n).$$

Using this we obtain

$$\begin{aligned}
\|[P, \phi]u\|_s &\leqq C \sum_{1 \leq |\alpha| \leq 2l} \|P^{(\alpha)}\phi_{(\alpha)}u\|_s \\
&\leqq C\Big(\sum_{1 \leq |\alpha| \leq 2l} \|P\phi_{(\alpha)}u\|_{s-\rho_0} + \|u\|_{-N}\Big) \\
&\leqq C\Big(\|Pu\|_{s-\rho_0} + \sum_{1 \leq |\alpha| \leq 2l} \|[P, \phi_{(\alpha)}]u\|_{s-\rho_0} + \|u\|_{-N}\Big) \\
&\leqq C\Big(\|Pu\|_{s-\rho_0} + \sum_{\substack{1 \leq |\alpha| \leq 2l \\ 1 \leq |\beta| \leq 2l}} \|P^{(\beta)}\phi_{(\alpha+\beta)}u\|_{s-\rho_0} + \|u\|_{-N}\Big),
\end{aligned}$$
$$u \in C_0^\infty(R^n).$$

Applying (3.5) to the second term on the right side, we have

$$\|[P,\ \phi]u\|_s \leqq C(\|Pu\|_{s-\rho_0}+\sum_{2\leqq|\alpha|\leqq 4l}\|P\phi_{(\alpha)}u\|_{s-2\rho_0}+\|u\|_{-N}),$$

$$u\in C_0^\infty(R^n).$$

Repeating this procedure j times, we obtain

$$\|[P,\ \phi]u\|_s \leqq C(\|Pu\|_{s-\rho_0}+\sum_{j\leqq|\alpha|\leqq 2jl}\|P\phi_{(\alpha)}u\|_{s-j\rho_0}+\|u\|_{-N}),$$

$$u\in C_0^\infty(R^n).$$

If we choose j so that $s-\rho_0 j+2l<-N$, we obtain (3.4). Q.E.D.

LEMMA 3.3. *For any* $\mu>0$ *and any* $N>0$ *there exists a constant* $C(\mu,\ N)$ *such that*

$$\|u\|_0 \leqq \mu\|Pu\|_0+C(\mu,\ N)\|u\|_{-N},\quad u\in C_0^\infty(R^n). \tag{3.6}$$

PROOF. From Conditions 1°), 2°) and 3°) for P it is easy to see that

$$\begin{aligned}\|\langle D_x\rangle^l u\|_0^2 &\leqq C(\mathrm{Re}(Pu,\ u)+\|u\|_0^2)\\ &\leqq C(\|Pu\|_0^2+\|u\|_0^2),\quad u\in C_0^\infty(R^n).\end{aligned} \tag{3.7}$$

Here $\langle D_x\rangle^l$ is an operator defined by

$$\int e^{i(x\cdot\xi+y\cdot\eta)}\langle\xi\rangle^l\hat{u}(\xi,\ \eta)d\xi\, d\eta,\quad u\in\mathcal{S}.$$

Let $\phi(x)\in C_0^\infty(R^n)$ such that $\operatorname{supp}\phi\subset\{|x|<\varepsilon\}$, $\phi(x)\equiv 1$ on $\{|x|\leqq\varepsilon/2\}$. Then, on account of Poincaré's inequality we have

$$\|\phi u\|_0 \leqq \delta(\varepsilon)\|\langle D_x\rangle^l\phi u\|_0,\quad u\in C_0^\infty(R^n),$$

where $\delta(\varepsilon)\to 0$ $(\varepsilon\to 0)$. From this and the estimate obtained by setting $u=\phi u$ in (3.7) we have

$$\|\phi u\|_0 \leqq C\delta(\varepsilon)(\|P\phi u\|_0+\|\phi u\|_0),\quad u\in C_0^\infty(R^n).$$

Hence, taking a sufficiently small $\varepsilon>0$ for any fixed $\mu>0$, we obtain

$$\|\phi u\|_0 \leqq \mu\|P\phi u\|_0,\quad u\in C_0^\infty(R^n).$$

Since $(1-\phi)u\in C_0^\infty(R^n-\Omega_\varepsilon)$, from (3.2) we get

$$\|(1-\phi)u\|_0 \leqq C_\varepsilon(\|P(1-\phi)u\|_{-\kappa_0}+\|u\|_{-N}),\quad u\in C_0^\infty(R^n)$$

for a constant C_ε depending on ε. From the above two estimates we have

$$\|u\|_0 \leqq \mu\|P\phi u\|_0 + C_\mu(\|P(1-\phi)u\|_{-\kappa_0} + \|u\|_{-N})$$
$$\leqq \mu C\|Pu\|_0 + C'_\mu(\|Pu\|_{-\kappa_0} + \|[P, \phi]u\|_0 + \|u\|_{-N}), \quad u \in C_0^\infty(R^n).$$

Since $\phi(x)$ satisfies the hypothesis of Lemma 3.2, by applying the estimate (3.4) to the third term on the right side we obtain

$$\|u\|_0 \leqq \mu C\|Pu\|_0 + C''_\mu(\|Pu\|_{-\rho_0} + \|u\|_{-N}), \quad u \in C_0^\infty(R^n).$$

Using the interpolation inequality to the second term on the right side we finally obtain (3.6). Thus, the lemma is proved. Q.E.D.

By Lemma 3.3, immediately we see that P satisfies Condition I) of Definition 2.1.

§4. Proof of Theorem 1.1 (Part II).

To show that P satisfies Conditions II) and III) of Definition 2.1, we prepare

LEMMA 4.1. *For some* $0<\kappa<1$ *there exists a constant* C_0 *such that*

(4.1) $$\|g(x)\langle D_y\rangle^{2m-1+\kappa}u\|_0 \leqq C_0(\|Pu\|_0 + \|u\|_0), \quad u \in C_0^\infty(R^n).$$

Furthermore, for any β $(0<|\beta|<2m)$ *and any* $\mu>0$ *there exists a constant* $C(\mu, \beta)$ *such that*

(4.2) $$\|g_{(\beta)}(x)\langle D_y\rangle^{2m}u\|_{-|\beta|} \leqq \mu\|Pu\|_0 + C(\mu, \beta)\|u\|_0, \quad (|\beta|\neq 0) \quad u \in C_0^\infty(R^n).$$

Here $\langle D_y\rangle^s u$ *is an operator defined by*

$$\int e^{i(x\cdot\xi + y\cdot\eta)}\langle\eta\rangle^s \hat{u}(\xi, \eta)\, d\xi\, d\eta \quad \textit{for} \quad u \in \mathcal{S}.$$

Since the proof of this lemma is so long, it will be given in the next section.

Using Lemmas 3.3 and 4.1 we obtain the following two lemmas.

LEMMA 4.2. *For any* $|\beta|\neq 0$, *any* $\mu>0$ *and any* $N>0$ *there exists a constant* $C=C(\beta, \mu, N)$ *such that*

(4.3) $$\|P_{(\beta)}u\|_{-|\beta|} \leqq \mu\|Pu\|_0 + C\|u\|_{-N}, \quad (|\beta|\neq 0) \quad u \in C_0^\infty(R^n).$$

PROOF. We note

$$P_{(\beta)} = A_{(\beta)} + \sum_{\beta'+\beta''=\beta} C_{\beta'\beta''} g_{(\beta')} B_{(\beta'')}.$$

First we observe that

$$\|A_{(\beta)}u\|_{-|\beta|}\leqq C_\beta(\|Au\|_{-1}+\|u\|_0),\quad (|\beta|\neq 0)\quad u\in C_0^\infty(R^n). \tag{4.4}$$

If $|\beta|\geqq 2l$, then (4.4) is obvious. For $|\beta|<2l$, we have

$$\begin{aligned}\|A_{(\beta)}u\|_{-|\beta|}&=\|\Lambda_{-|\beta|}A_{(\beta)}u\|_0\\ &\leqq C(\|A_{(\beta)}\Lambda_{-|\beta|}u\|_0+\sum_{0<|\alpha|<2l-|\beta|}\|\Lambda_{-|\beta+\alpha|}A_{(\beta+\alpha)}u\|_0+\|u\|_0),\\ &\qquad u\in C_0^\infty(R^n),\end{aligned} \tag{4.5}$$

where Λ_s is a pseudodifferential operator with the symbol $(|\xi|^2+|\eta|^2+1)^{s/2}$. From Condition 1°) for A it follows that

$$\|A_{(\beta)}u\|_0\leqq C(\|Au\|_0+\|u\|_0),\quad u\in C_0^\infty(R^n).$$

Applying this estimate to the first term on the right side of (4.5), we obtain

$$\begin{aligned}\|A_{(\beta)}u\|_{-|\beta|}&\leqq C(\|A\Lambda_{-|\beta|}u\|_0+\sum_{0<|\alpha|<2l-|\beta|}\|\Lambda_{-|\beta+\alpha|}A_{(\beta+\alpha)}u\|_0+\|u\|_0)\\ &\leqq C(\|Au\|_{-|\beta|}+\sum_{0<|\alpha|<2l-|\beta|}\|\Lambda_{-|\beta+\alpha|}A_{(\alpha)}u\|_0\\ &\quad+\sum_{0<|\alpha|<2l-|\beta|}\|A_{(\beta+\alpha)}u\|_{-|\beta+\alpha|}+\|u\|_0),\quad u\in C_0^\infty(R^n).\end{aligned}$$

Using the interpolation inequality, we have for $\mu>0$ and $0<|\beta|<2l$

$$\begin{aligned}\|A_{(\beta)}u\|_{-|\beta|}&\leqq C(\|Au\|_{-1}+\sum_{|\beta|<|\alpha|<l}\|A_{(\alpha)}u\|_{-|\alpha|}+\|u\|_0)\\ &\quad+\mu\sum_{0<|\alpha|<2l}\|A_{(\alpha)}u\|_{-|\alpha|},\quad u\in C_0^\infty(R^n).\end{aligned}$$

By means of Lemma 2.9, we obtain (4.4). On the other hand

$$\begin{aligned}\|Au\|_{-1}&\leqq\|Pu\|_{-1}+\|g(x)Bu\|_{-1}\\ &\leqq\mu\|Pu\|_0+C(\|g\langle D_y\rangle^{2m-1}u\|_0+\|u\|_0),\quad u\in C_0^\infty(R^n).\end{aligned}$$

Combining this with (4.4), we have

$$\|A_{(\beta)}u\|_{-|\beta|}\leqq\mu\|Pu\|_0+C(\|g\langle D_y\rangle^{2m-1}u\|_0+\|u\|_0),\quad (|\beta|\neq 0)\quad u\in C_0^\infty(R^n).$$

Noting that

$$\|g(x)\langle D_y\rangle^{2m-1}u\|_0\leqq\mu\|g(x)\langle D_y\rangle^{2m-1+\kappa}u\|_0+C_\mu\|u\|_0,\quad u\in C_0^\infty(R^n),$$

we obtain from (4.1)

$$\|A_{(\beta)}u\|_{-|\beta|}\leqq\mu\|Pu\|_0+C\|u\|_0,\quad (|\beta|\neq 0)\quad u\in C_0^\infty(R^n). \tag{4.6}$$

Using (4.1) and (4.2) we get for $0<|\beta|<2m$

$$\sum_{\beta'+\beta''=\beta}\|g_{(\beta')}B_{(\beta'')}u\|_{-|\beta|}\leqq C\sum_{|\beta'|\leqq|\beta|}\|g_{(\beta')}\langle D_y\rangle^{2m}u\|_{-|\beta|} \tag{4.7}$$

$$\leqq C\Big(\sum_{0<|\beta'|<2m}\|g_{(\beta')}\langle D_y\rangle^{2m}u\|_{-|\beta'|}+\|g\langle D_y\rangle^{2m-1}u\|_0\Big)$$

$$\leqq\mu\|Pu\|_0+C\|u\|_0, \qquad u\in C_0^\infty(R^n).$$

If $|\beta|\geqq 2m$, then the left side of (4.7) is bounded only by $C\|u\|_0$. Therefore, from (4.6) and (4.7) we have

$$\|P_{(\beta)}u\|_{-|\beta|}\leqq\mu\|Pu\|_0+C\|u\|_0, \quad (|\beta|\neq 0) \quad u\in C_0^\infty(R^n).$$

By means of (3.6) we obtain (4.3). Q.E.D.

LEMMA 4.3. *Let* $\psi(x, y)=\phi_1(x)\phi_2(y)$ $(\in C_0^\infty(R^n))$ *where* $\phi_1\equiv 1$ *on* $\{|x|<\varepsilon\}$, $\phi_1(x)\in C_0^\infty(R_x^{n_1})$, $\phi_2(y)\in C_0^\infty(R_y^{n_2})$. *Then, for any* $\varepsilon>0$, *real* s *and* $N>0$ *there exists a constant* $C=C(\varepsilon, s, N)$ *such that*

$$\|[P, \psi]u\|_s\leqq C(\|Pu\|_{s-\kappa'}+\|u\|_{-N}), \quad u\in C_0^\infty(R^n), \tag{4.8}$$

where $\kappa'=\min(l\kappa/(2m-1+\kappa), \kappa)$ *and* κ *is the positive number in Lemma* 4.1.

PROOF. We devide the proof into three steps.

i) Assume that for any $\alpha\neq 0$, β, real s and $N>0$ there exists a constant C such that

$$\|P^{(\alpha)}\psi_{(\alpha+\beta)}u\|_s\leqq C(\|P\psi_{(\alpha+\beta)}u\|_{s-\kappa'}+\|u\|_{-N}), \quad u\in C_0^\infty(R^n). \tag{4.9}$$

Then, (4.8) is valid. Indeed, using (4.9) we have

$$\|[P, \psi]u\|_s\leqq C\sum_{|\alpha|>0}\|P^{(\alpha)}\psi_{(\alpha)}u\|_s$$

$$\leqq C\Big(\sum_{|\alpha|>0}\|P\psi_{(\alpha)}u\|_{s-\kappa'}+\|u\|_{-N}\Big)$$

$$\leqq C\Big(\|Pu\|_{s-\kappa'}+\sum_{|\alpha|\neq 0}\|[P, \psi_{(\alpha)}]u\|_{s-\kappa'}+\|u\|_{-N}\Big)$$

$$\leqq C\Big(\|Pu\|_{s-\kappa'}+\sum_{|\alpha|\neq 0}\sum_{|\beta|\neq 0}\|P^{(\beta)}\psi_{(\alpha+\beta)}u\|_{s-\kappa'}+\|u\|_{-N}\Big),$$

$$u\in C_0^\infty(R^n).$$

Applying (4.9) to the second term on the right side, again, we get

$$\|[P, \psi]u\|_s \leqq C\Big(\|Pu\|_{s-\kappa'} + \sum_{|\alpha|\neq 0}\sum_{|\beta|\neq 0}\|P\psi_{(\alpha+\beta)}u\|_{s-2\kappa'} + \|u\|_{-N}\Big)$$

$$\leqq C\Big(\|Pu\|_{s-\kappa'} + \sum_{2\leqq|\alpha|\leqq 2k}\|[P, \psi_{(\alpha)}]u\|_{s-2\kappa'} + \|u\|_{-N}\Big),$$

$$u \in C_0^\infty(R^n),$$

where $k=\max(2l, 2m)$. Repeating this procedure, as in the proof of Lemma 3.2, we finally obtain (4.8).

ii) We prove (4.9). Let $\alpha=(\alpha_1, \alpha_2)$ where α_1 and α_2 are multi-indices with respect to x and y, respectively. If $\alpha_1\neq 0$, then for any β, $\psi_{(\alpha+\beta)}=0$ on $\Omega_\varepsilon=\{(x, y); |x|\leqq\varepsilon\}$. Hence, by Lemma 3.1, the estimate (4.9) clearly holds. When $\alpha_1=0$, that is, $\alpha=(0, \alpha_2)$, $\alpha_2\neq 0$, we prove

(4.10) $$\|P^{(\alpha)}u\|_s \leqq C(\|Pu\|_{s-\kappa'} + \|u\|_{-N}), \quad u\in C_0^\infty(R^n).$$

Then, replacing u by $\psi_{(\alpha+\beta)}u$, we get easily (4.9). Since we have

$$\|P^{(\alpha)}u\|_s = \|B^{(\alpha_2)}(x, y, D_y)\langle D_y\rangle^{1-2m}g(x)\langle D_y\rangle^{2m-1}u\|_s$$
$$\leqq C\|g(x)\langle D_y\rangle^{2m-1}u\|_s, \quad u\in C_0^\infty(R^n),$$

the estimate (4.10) is obtained if we derive the estimate

(4.11) $$\|g\langle D_y\rangle^{2m-1}u\|_s \leqq C(\|Pu\|_{s-\kappa'} + \|u\|_{-N}), \quad u\in C_0^\infty(R^n).$$

iii) On account of Lemma 2.13, the estimate (4.11) holds if we prove that the estimate

(4.12) $$\|g_{(\beta)}\langle D_y\rangle^{2m-1}u\|_{\kappa'-|\beta|} \leqq C(\|Pu\|_0 + \|u\|_{-N}), \quad u\in C_0^\infty(R^n)$$

holds for any β and $N>0$. (Though $g(x)\langle D_y\rangle^{2m-1}$ is not a differential operator, we can apply Lemma 2.13 to this operator since $\|g(x)\langle D_y\rangle^{2m-1}u\|_{\kappa'-|\beta|}$ is equivalent to $\sum_{|\alpha|\leqq 2m-1}\|g(x)D_y^\alpha u\|_{\kappa'-|\beta|}$.) By means of Minkovski's inequality, it is easy to see

$$(|\xi|+|\eta|+1)^{\kappa'}\langle\eta\rangle^{2m-1} \leqq \langle\xi\rangle^{\kappa'}\langle\eta\rangle^{2m-1} + \langle\eta\rangle^{2m-1+\kappa'}$$
$$\leqq C(\langle\xi\rangle^l + \langle\eta\rangle^{2m-1+\kappa'(2m-1)/(l-\kappa')} + \langle\eta\rangle^{2m+\kappa'-1}).$$

If we take κ' so that $\max(\kappa'(2m-1)/(l-\kappa'), \kappa')\leqq\kappa$, where κ is the positive number in Lemma 4.1, we have for any β

$$\|g_{(\beta)}\langle D_y\rangle^{2m-1}u\|_{\kappa'-|\beta|}$$
$$\leqq C(\|\langle D_x\rangle^l g_{(\beta)}u\|_{-|\beta|} + \|g_{(\beta)}\langle D_y\rangle^{2m-1+\kappa}u\|_{-|\beta|}$$
$$\leqq C(\|\langle D_x\rangle^l u\|_0 + \|g_{(\beta)}\langle D_y\rangle^{2m-1+\kappa}u\|_{-|\beta|}, \quad u\in C_0^\infty(R^n).$$

Using (3.7), the first term on the right side is bounded by $C(\|Pu\|_0+\|u\|_0)$. By Lemma 4.1 the second term is also bounded by $C(\|Pu\|_0+\|u\|_0)$ since $\kappa<1$. Therefore, using (3.6) we have (4.12). Q.E.D.

The completion of the proof of Theorem 1.1. By Lemmas 3.3 and 4.2 we see that P satisfies I) and II) of Definition 2.1, respectively. Lemma 4.3 shows that Condition III) is satisfied for points $(0, y)$ $(y\in R^{n_2})$. For points (x, y) with $x\neq 0$, take $\phi(x, y)\in C_0^\infty(R^n)$ such that $\operatorname{supp}\phi\subset\{|x|\geqq\varepsilon\}$. Then, we can get the estimate (4.8) with $\kappa'=\min(l/m, m/l)$, by the same way as in the proof of Lemma 3.2. Therefore Condition III) is satisfied for points (x, y) with $x\neq 0$. Consequently, P is w-elliptic in R^n. Q.E.D.

§5. Proof of Lemma 4.1.

From Condition (G) of Section 1 or Remark of Theorem 1.1 we have for any β

(5.1) $$|g_{(\beta)}(x)|\leqq Cg(x)^{1-\sigma|\beta|} \quad \text{in a neighborhood of } x=0,$$

where

(5.2) $$\begin{cases} \sigma=1/2 & \text{when } m=1 \\ 0<\sigma<\{2(m+l(m-1))\}^{-1} & \text{when } m\geqq 2. \end{cases}$$

Since P is semi-elliptic in $R^n-\{x=0\}$, it is easy to see that (4.1) and (4.2) hold for $u\in C_0^\infty(R^n-\{x=0\})$, by the same way as in the proof of Lemma 3.1. We may only prove Lemma 4.1 in a neighborhood of $x=0$. Therefore we may assume, without loss of generality, that (5.1) holds for all x, by modifying $g(x)$ outside of $x=0$.

Let $\phi_0(t)$, $\phi_1(t)$ and $\phi_2(t)$ be C^∞-functions in $[0, \infty)$ such that

$$\begin{aligned} &\operatorname{supp}\phi_0(t)\subset[0, 1), \quad \phi_0(t)\equiv 1 \quad \text{on } [0, 1/2], \\ &\operatorname{supp}\phi_1(t)\subset[0, 2), \quad \phi_1(t)\equiv 1 \quad \text{on } [0, 1], \\ &\operatorname{supp}\phi_2(t)\subset(1, \infty), \quad \phi_2(t)\equiv 1 \quad \text{in } [2, \infty), \end{aligned}$$

and

(5.3) $$\phi_1+\phi_2\equiv 1 \quad \text{in } [0, \infty).$$

Set $\lambda(\xi, \eta)=(|\xi|^{2l(l+1)(l+2)}+\langle\eta\rangle^{2m\tau(l+1)(l+2)})^{1/\{2l(l+1)(l+2)\}}$

(5.4) $$\begin{cases} \tau=(l+1)/(l+2) & \text{for } m=1 \\ \tau=l/(m(l+1)) & \text{for } m\geqq 2. \end{cases}$$

Then, since $2m\tau(l+1)(l+2)$ is an integer, λ satisfies (2.5) and (2.6), so it is a basic weight function.

PROPOSITION 5.1. *Set* $h(\xi, \eta)=\lambda(\xi, \eta)^{2l(1-\tau)/\tau}$ *and set* $\chi_j(x, \xi, \eta)=\phi_j(g(x)h(\xi, \eta))$ $(j=0, 1, 2)$. *Then,* $\chi_j(x, D_x, D_y)\in S^0_{\lambda,\mathbf{1},\delta}$, *where* $\mathbf{1}=(1, \cdots, 1)$ *and* $\delta=(\delta_1, \cdots, \delta_{n_1}, 0, \cdots, 0)$, $\delta_k=\delta_0=2l\sigma(1-\tau)/\tau$ $(k=1, \cdots, n_1)$. *Furthermore*

$$\chi_1+\chi_2=I. \tag{5.5}$$

PROOF. The equality (5.5) is obvious from (5.3). We check that $\chi(x, \xi, \eta)$ $(=\chi_j(x, \xi, \eta), j=0, 1, 2)$ satisfies for any α and β

$$|(\partial_\xi \partial_\eta)^\alpha D_x^\beta \chi(x, \xi, \eta)|\leqq C_{\alpha\beta}\lambda^{\delta\cdot\beta-|\alpha|} \quad \text{in} \quad R^n_{x,y}\times R^n_{\xi,\eta}. \tag{5.6}$$

From the Leibnitz formula we have for $|\alpha+\beta|\neq 0$

$$\chi^{(\alpha)}_{(\beta)}=\sum_{0<k\leqq|\alpha+\beta|} C_k\phi^{(k)}(g(x)h(\xi, \eta))$$

$$\times \sum_{\substack{\alpha^1+\alpha^2+\cdots+\alpha^k=\alpha \\ \beta^1+\beta^2+\cdots+\beta^k=\beta}} C_{\alpha^1,\cdots,\alpha^k,\beta^1,\cdots,\beta^k}\, g_{(\beta^1)}\cdots g_{(\beta^k)} h^{(\alpha^1)}\cdots h^{(\alpha^k)}.$$

Using (5.1) and $|h^{(\tilde{\alpha})}|\leqq C_{\tilde{\alpha}} h\lambda^{-|\tilde{\alpha}|}$, we obtain

$$|\chi^{(\alpha)}_{(\beta)}|\leqq C\sum_{0<k\leqq|\alpha+\beta|} \phi^{(k)}(g(x)h(\xi, \eta))\,g(x)^{k-\sigma|\beta|}h^k\lambda^{-|\alpha|}.$$

Noting that for $k\neq 0$,

$$1/2\leqq g(x)h(\xi, \eta)\leqq 2 \qquad \text{on} \quad \operatorname{supp}\phi^{(k)}(g(x)h(\xi, \eta)),$$

we obtain (5.6). Q.E.D.

REMARK. We have proved that $\chi_j\in S^0_{\lambda,\mathbf{1},\delta}$. Recall that by Definition 2.3 $\delta_0<1$, that is

$$2l\sigma(1-\tau)/\tau<1. \tag{5.7}$$

On account of (5.2) and (5.4) it is easy to see that (5.7) is valid.

We shall first see that the estimates with replaced u by $\chi_j u$ $(j=1, 2)$ in (4.1) and (4.2), hold, and prove that the estimate

$$\|[P, \chi_j]u\|_0\leqq C(\|Pu\|_0+\|u\|_0), \quad u\in C_0^\infty(R^n) \tag{5.8}$$

holds.

PROPOSITION 5.2. *Set* $v_1=\chi_1(x, D_x, D_y)u$ *for* $u\in C_0^\infty(R^n)$. *Then, we can find a constant* C_0 *such that*

$$\|g(x)\langle D_y\rangle^{2m-1+\kappa}v_1\|_0\leqq C_0(\|Pv_1\|_0+\|v_1\|_0), \tag{5.9}$$

where $\kappa=2\sigma m(1-\tau)$, *and for any* $0<|\beta|<2m$ *and any* $\mu>0$ *we can find* $C(\mu, \beta)$ *such that*

$$\|g_{(\beta)}(x)\langle D_y\rangle^{2m}v_1\|_{-|\beta|}\leqq\mu\|Pv_1\|_0+C(\mu,\beta)\|v_1\|_0. \tag{5.10}$$

PROOF. From (5.2) and (5.4) it follows that

$$m\tau+2\sigma m(1-\tau)-1\leqq 0. \tag{5.11}$$

For the brevity we write $v=v_1$. Let $\phi_3(t)$ be a C^∞-function in $[0,\infty)$ such that

$$\operatorname{supp}\phi_3(t)\subset[0,3),\quad \phi_3(t)\equiv 1 \quad \text{on} \quad [0,2].$$

Set $\chi_3(x,\eta)=\phi_3(g(x)h(0,\eta))$. Then we see $\chi_3(x,D_y)v=v$, since

$$\chi_3(x,D_y)v(x,y)=\int e^{iy\cdot\eta}\chi_3(x,\eta)\hat{v}(x,\eta)d\eta$$

$$=\int e^{i(x\cdot\xi+y\cdot\eta)}\chi_3(x,\eta)\chi_1(x,\xi,\eta)ud\xi d\eta$$

and $\chi_3(x,\eta)\equiv 1$ on $\operatorname{supp}\chi_1(x,\xi,\eta)$. Using (5.11) and the fact that $g(x)\leqq 3\langle\eta\rangle^{-2m(1-\tau)}$ on $\operatorname{supp}\chi_3(x,\eta)$, we have

$$\begin{aligned}|g\langle\eta\rangle^{2m-1+2\sigma m(1-\tau)}\chi_3|&\leqq\sqrt{3}\,\langle\eta\rangle^{m\tau-1+2\sigma m(1-\tau)}|g^{1/2}\langle\eta\rangle^m|\\&\leqq\sqrt{3}\,g^{1/2}\langle\eta\rangle^m.\end{aligned}$$

Similarly, from (5.1) and (5.11) it follows that for $0<|\beta|<2m$

$$\begin{aligned}|g_{(\beta)}\langle\eta\rangle^{2m-|\beta|}\chi_3|&\leqq C_\beta|g^{1/2-\sigma|\beta|}\langle\eta\rangle^{m-|\beta|}\chi_3 g^{1/2}\langle\eta\rangle^m|\\&\leqq C'_\beta\langle\eta\rangle^{m\tau-|\beta|(1-2\sigma m(1-\tau))}|g^{1/2}\langle\eta\rangle^m|\\&\leqq C'_\beta|g^{1/2}\langle\eta\rangle^m|.\end{aligned}$$

In the second inequality we used the fact that $1/2-\sigma|\beta|\geqq 0$ for $|\beta|<2m$ since σ satisfies (5.2). Hence, using $v=\chi_3 v$, we obtain

$$\begin{aligned}&\|g(x)\langle D_y\rangle^{2m-1+2\sigma m(1-\tau)}v\|_0+\sum_{0<|\beta|<2m}\|g_{(\beta)}\langle D_y\rangle^{2m}v\|_{-|\beta|}\\&\leqq C\|g^{1/2}\langle D_y\rangle^m v\|_0.\end{aligned}$$

From Conditions 1°), 2°) and 3°), clearly we have

$$\begin{aligned}\|g^{1/2}\langle D_y\rangle^m v\|_0^2&=(g(x)\langle D_y\rangle^{2m}v,v)\\&\leqq C(\operatorname{Re}(Pv,v)+\|v\|_0^2)\\&\leqq\mu\|Pv\|_0^2+C_\mu\|v\|_0^2.\end{aligned}$$

Consequently we have (5.9) and (5.10). Q.E.D.

REMARK. By the condition (5.2), i.e.

$$\begin{cases} \sigma=1/2 & \text{when } m=1 \\ 0<\sigma<\{2(m+l(m-1))\}^{-1} & \text{when } m\geqq 2, \end{cases}$$

we can choose τ in (5.4) to satisfy (5.7) and (5.11) simultaneously.

To get (5.8) and the estimates with replaced u by $\chi_2 u$ in (4.1) and (4.2), we consider an operator $\tilde{p}(x, y, D_x, D_y)$ which is obtained by modifying $p(x, y, D_x, D_y)$ in "a neighborhood of $x=0$" as follows: Set

$$\begin{aligned} \tilde{p}(x, y, \xi, \eta) &= a(x, y, \xi)+(g(x)h(\xi, \eta)+\chi_0(x, \xi, \eta))h(\xi, \eta)^{-1}b(x, y, \eta) \\ &\equiv a(x, y, \xi)+\tilde{\chi}(x, \xi, \eta)h(\xi, \eta)^{-1}b(x, y, \eta). \end{aligned}$$

Then, we have

PROPOSITION 5.3. *$\tilde{P}=\tilde{p}(x, y, D_x, D_y)\in S^{2l/\tau}_{\lambda,\rho,\delta}$ and $\tilde{p}(x, y, \xi, \eta)$ satisfies* (H)-*condition, in the following sense:*

i) *There exists a constant $C_0>0$ such that*

$$|\tilde{p}(x, y, \xi, \eta)|\geqq C_0\lambda(\xi, \eta)^{2l} \quad \textit{for large } |\xi|+|\eta|. \tag{5.12}$$

ii) *For any α and β there exists a constant $C_{\alpha\beta}$ such that*

$$|\tilde{p}^{(\alpha)}_{(\beta)}(x, y, \xi, \eta)/\tilde{p}(x, y, \xi, \eta)|\leqq C_{\alpha\beta}\lambda(\xi, \eta)^{\delta\cdot\beta-\rho\cdot\alpha} \quad \textit{for large } |\xi|+|\eta|, \tag{5.13}$$

where $\rho=(1, \cdots, 1, \rho_{n_1+1}, \cdots, \rho_{n_1+n_2})$, $\rho_k=\rho_0=\min(1, l/m)$

and $\delta=(\delta_1, \cdots, \delta_{n_1}, 0, \cdots, 0)$, $\delta_k=\delta_0=2l\sigma(1-\tau)/\tau$.

PROOF. Using $|h^{(\alpha)}|\leqq C_\alpha h\lambda^{-|\alpha|}$ and $\chi_0\in S^0_{\lambda,1,\delta}$, it is easy to check $\tilde{p}(x, y, \xi, \eta)\in S^{2l/\tau}_{\lambda,\rho,\delta}$. When $|\xi|\geqq|\eta|^{m\tau/l}$ the inequality (5.12) is obvious. Indeed, from Conditions 1°), 2°) and 3°)

$$|\tilde{p}(x, y, \xi, \eta)|\geqq \mathrm{Re}\,\tilde{p}\geqq \mathrm{Re}\,a\geqq C|\xi|^{2l}\geqq \frac{C}{3}\lambda^{2l}.$$

On the other hand, if $|\xi|\leqq|\eta|^{m\tau/l}$, then we have

$$\mathrm{Re}\,h(\xi, \eta)^{-1}b(x, y, \eta)\geqq\langle\eta\rangle^{2m(\tau-1)}\,\mathrm{Re}\,b\geqq C\langle\eta\rangle^{2m\tau}\geqq\frac{C}{2}\lambda^{2l}.$$

From the definition of χ_0 it follows that

$$\tilde{\chi}(x, \xi, \eta)=g(x)h(\xi, \eta)+\chi_0(x, \xi, \eta)\geqq 1/2 \quad \text{in } R^n_{x,y}\times R^n_{\xi,\eta}.$$

Hence (5.12) is valid in this case, too. We check (5.13) in dividing it into two cases.

1) The case $x\in\Omega_1=\{x; g(x)h(\xi, \eta)\leqq 2\}$: We have for any α and β

$$|\tilde{p}_{(\beta)}^{(\alpha)}/\tilde{p}|\leqq|a_{(\beta)}^{(\alpha)}/\tilde{p}|+\sum_{\substack{\alpha'+\alpha''=\alpha\\ \beta'+\beta''=\beta}} C_{\alpha'\alpha''\beta'\beta''}|(\tilde{\chi}h^{-1})_{(\beta')}^{(\alpha')}b_{(\beta'')}^{(\alpha'')}/\tilde{p}|. \tag{5.14}$$

Let $\alpha=(\alpha_1, \alpha_2)$ where α_1 and α_2 are multi-indices with respect to x and y, respectively. Using (5.12) and noting that $a_{(\beta)}^{(\alpha)}=0$ if $|\alpha_2|\neq 0$ or $|\alpha_1|>2l$, the first term on the right side is bounded by $C\lambda^{-|\alpha|}$. By means of (5.1) and $g(x)\leqq 2h(\xi, \eta)^{-1}$ on Ω_1, we have

$$|(\tilde{\chi}h^{-1})_{(\beta')}^{(\alpha')}|\leqq C\lambda^{-2l(1-\tau)/\tau+\delta\cdot\beta'-|\alpha'|}.$$

If $\alpha''=(\alpha_1'', \alpha_2'')$ with $|\alpha_2''|>2m$ or $|\alpha_1''|>0$, then $b_{(\beta'')}^{(\alpha'')}=0$. When $\alpha''=(0, \alpha_2'')$ such that $|\alpha_2''|\leqq 2m$, we obtain

$$\begin{aligned}|(\tilde{\chi}h^{-1})_{(\beta')}^{(\alpha')}b_{(\beta'')}^{(\alpha'')}/\tilde{p}|&\leqq C\lambda^{-2l(1-\tau)/\tau+\delta\cdot\beta'-|\alpha'|}|\eta|^{2m-|\alpha''|}\lambda^{-2l}\\ &\leqq C\lambda^{\delta\cdot\beta-|\alpha'|-l|\alpha''|/(m\tau)}.\end{aligned}$$

From (5.4) it follows that $l/(m\tau)>1$. Hence the second term on the right side of (5.14) is bounded by $C\lambda^{\delta\cdot\beta-|\alpha|}$.

2) The case $x\in\Omega_2=\{x; g(x)h(\xi, \eta)>2\}$: Clearly we have $\tilde{p}=p$, so we have for any α and β

$$|\tilde{p}_{(\beta)}^{(\alpha)}/\tilde{p}|\leqq|a_{(\beta)}^{(\alpha)}/p|+\sum_{\beta'+\beta''=\beta} C_{\beta'\beta''}|g_{(\beta')}b_{(\beta'')}^{(\alpha)}/p|.$$

In the same way as in 1) the first term is bounded by $C\lambda^{-|\alpha|}$. Since $g(x)\in\mathscr{B}$, we may assume that $|g(x)|\leqq M$. For $\alpha=(0, \alpha_2)$ such that $|\alpha_2|\leqq 2m$, we have from (5.1)

$$\begin{aligned}|g_{(\beta')}b_{(\beta'')}^{(\alpha)}/p|&\leqq Cg^{1-\sigma|\beta'|}|\eta|^{2m-|\alpha_2|}\Big/\left(\frac{g}{M}|\xi|^{2l}+g\langle\eta\rangle^{2m}\right)\\ &\leqq C(M)g^{-\sigma|\beta'|}((|\xi|^{2l}+\langle\eta\rangle^{2m})^{1/2l})^{(-l/m)|\alpha_2|}\\ &\leqq C(M)g^{-\sigma|\beta'|}\lambda^{-\rho\cdot\alpha}.\end{aligned}$$

Using that $g(x)^{-1}<\dfrac{1}{2}h(\xi, \eta)$ in Ω_2, the left side is bounded by $C\lambda^{\delta\cdot\beta'-\rho\cdot\alpha}$. Thus the proposition is proved. Q. E. D.

In the consequence of Propositions 2.7 and 5.3 we have a parametrix $Q\in S_{\lambda,\rho,\delta}^{-2l}$ such that for $\tilde{P}\in S_{\lambda,\rho,\delta}^{2l/\tau}$

$$I=Q\tilde{P}+K, \quad K\in S^{-\infty}, \tag{5.15}$$

furthermore

$$(5.16)\qquad \begin{cases} Q=Q_0Q_1, \quad Q_0\in S^{-2l}_{\lambda,\tilde\rho,\delta}, \quad Q_1\in S^{0}_{\lambda,\rho,\delta}, \\ \sigma(Q_0)=\tilde p(x, y, \xi, \eta)^{-1} \qquad \text{for large } |\xi|+|\eta|. \end{cases}$$

PROPOSITION 5.4. *For any* β *and* $N>0$ *there exists a constant* $C=C(\beta, N)$ *such that*

$$(5.17)\qquad \|g_{(\beta)}(x)\langle D_y\rangle^{2m} u\|_{-|\beta|,\lambda}\leqq C(\|\tilde P u\|_{\delta\cdot\beta-|\beta|,\lambda}+\|u\|_{-N,\lambda}), \quad u\in C_0^\infty(R^n),$$

where

$$\|u\|_{s,\lambda}=\left(\int\lambda(\xi, \eta)^{2s}|\hat u(\xi, \eta)|^2 d\xi d\eta\right)^{1/2}.$$

PROOF. In the same way as in the proof of Proposition 5.3, by checking the symbol of $g_{(\beta)}(x)\langle D_y\rangle^{2m}Q_0$ we have $g_{(\beta)}(x)\langle D_y\rangle^{2m}Q_0\in S^{\delta\cdot\beta}_{\lambda,\tilde\rho,\delta}$. From (5.16) it follows that $g_{(\beta)}(x)\langle D_y\rangle^{2m}Q\in S^{\delta\cdot\beta}_{\lambda,\tilde\rho,\delta}$. By means of (5.15) we have

$$g_{(\beta)}\langle D_y\rangle^{2m}=g_{(\beta)}\langle D_y\rangle^{2m}QP+K', \quad K'\in S^{-\infty}.$$

Therefore, by Proposition 2.5 we obtain (5.17). Q.E.D.

COROLLARY. *Set* $v_2=\chi_2(x, D_x, D_y)u$ *for* $u\in C_0^\infty(R^n)$. *Then, we can find* C_0 *such that*

$$(5.18)\qquad \|g(x)\langle D_y\rangle^{2m} v_2\|_0\leqq C_0(\|Pv_2\|_0+\|v_2\|_0),$$

and for any $\mu>0$ *and any* β $(\neq 0)$ *we can find* $C(\mu, \beta)$ *such that*

$$(5.19)\qquad \|g_{(\beta)}(x)\langle D_y\rangle^{2m} v_2\|_{-|\beta|}\leqq\mu\|Pv_2\|_0+C(\mu, \beta)\|v_2\|_0, \quad (|\beta|\neq 0).$$

PROOF. Noting that $\|\cdot\|_{0,\lambda}=\|\cdot\|_0$ and that

$$\tilde P\chi_2=P\chi_2+K_1\chi_2 \qquad \text{for some } K_1\in S^{-\infty},$$

from (5.17) with $\beta=0$ we have (5.18). Using that $\|\cdot\|_{-|\beta|}\leqq\|\cdot\|_{-|\beta|,\lambda}$ and interpolation inequality $\|\cdot\|_{\delta\cdot\beta-|\beta|,\lambda}\leqq\mu\|\cdot\|_{0,\lambda}+C_\mu\|\cdot\|_{-N,\lambda}$ $(|\beta|\neq 0)$, we have (5.19) from (5.17). Q.E.D.

PROPOSITION 5.5. *For any real* s *and* $N>0$ *there exists a constant* $C=C(s, N)$ *such that*

$$(5.20)\qquad \|[P, \chi_j]u\|_{s,\lambda}\leqq C(\|Pu\|_{s-\tau_0,\lambda}+\|u\|_{-N,\lambda}), \quad u\in C_0^\infty(R^n),$$

where

$$\tau_0=\min_{1\leq j\leq n}(\rho_j-\delta_j)=\min(1-2l\sigma(1-\tau)/\tau, l/m).$$

PROOF. In the same way as in the proof of (5.13), by expanding the symbol of $P_{(\beta)}^{(\alpha)}Q$ we get $P_{(\beta)}^{(\alpha)}Q\in S_{\lambda,\rho,\delta}^{\delta\cdot\beta-\rho\cdot\alpha}$. It follows from (5.15) that

$$\|P_{(\beta)}^{(\alpha)}u\|_{s,\lambda}\leqq C(\|\tilde{P}u\|_{s+\delta\cdot\beta-\rho\cdot\alpha,\lambda}+\|u\|_{-N,\lambda}),\quad u\in C_0^\infty(R^n).$$

If $\tilde{\alpha}+\tilde{\beta}\neq 0$, then $\tilde{P}\chi_{j}{}_{(\tilde{\alpha})}^{(\tilde{\beta})}\equiv P\chi_{j}{}_{(\tilde{\alpha})}^{(\tilde{\beta})}$ $(j=1, 2)$ (mod $S^{-\infty}$). Consequently, for any α, β, $\tilde{\alpha}$ and $\tilde{\beta}$ $(\tilde{\alpha}+\tilde{\beta}\neq 0)$ we obtain

$$\|P_{(\beta)}^{(\alpha)}\chi_{j}{}_{(\tilde{\alpha})}^{(\tilde{\beta})}u\|_{s,\lambda}\leqq C(\|P\chi_{j}{}_{(\tilde{\alpha})}^{(\tilde{\beta})}u\|_{s+\delta\cdot\beta-\rho\cdot\alpha,\lambda}+\|u\|_{-N,\lambda}),\quad u\in C_0^\infty(R^n). \tag{5.21}$$

For the brevity we write $\chi=\chi_j$. Using the expansion formula (2.10) in Proposition 2.4, we have from (5.21)

$$\begin{aligned}
\|[P, \chi]u\|_{s,\lambda} &\leqq C\Big(\sum_{0<|\alpha+\beta|<N_0}\|P_{(\beta)}^{(\alpha)}\chi_{(\alpha)}^{(\beta)}u\|_{s,\lambda}+\|u\|_{-N,\lambda}\Big)\\
&\leqq C\Big(\sum_{0<|\alpha+\beta|<N_0}\|P\chi_{(\alpha)}^{(\beta)}u\|_{s+\delta\cdot\beta-\rho\cdot\alpha,\lambda}+\|u\|_{-N,\lambda}\Big)\\
&\leqq C\Big(\sum_{0<|\alpha+\beta|<N_0}(\|\chi_{(\alpha)}^{(\beta)}Pu\|_{s+\delta\cdot\beta-\rho\cdot\alpha,\lambda}\\
&\qquad +\|[P, \chi_{(\alpha)}^{(\beta)}]u\|_{s+\delta\cdot\beta-\rho\cdot\alpha,\lambda})+\|u\|_{-N,\lambda}\Big)\\
&\leqq C\Big(\|Pu\|_{s-\tau_0,\lambda}+\sum_{\substack{1\leq|\alpha+\beta|<N_0\\ 1\leq|\tilde{\alpha}+\tilde{\beta}|<N_0}}\|P_{(\tilde{\beta})}^{(\tilde{\alpha})}\chi_{(\alpha+\tilde{\alpha})}^{(\beta+\tilde{\beta})}u\|_{s+\delta\cdot\beta-\rho\cdot\alpha,\lambda}\\
&\qquad +\|u\|_{-N,\lambda}\Big),\quad u\in C_0^\infty(R^n).
\end{aligned}$$

Here $N_0=(2l/\tau+s+N)/\tau_0$. We apply (5.21) to the second term on the last side. Repeating this procedure M times, we finally obtain

$$\begin{aligned}
\|[P, \chi]u\|_{s,\lambda}\leqq C\Big(&\|Pu\|_{s-\tau_0,\lambda}+\sum_{M\leq|\alpha+\beta|<MN_0}\|P\chi_{(\alpha)}^{(\beta)}u\|_{s+\delta\cdot\beta-\rho\cdot\alpha,\lambda}\\
&+\|u\|_{-N,\lambda}\Big),\quad u\in C_0^\infty(R^n).
\end{aligned}$$

If we take M so that $2l/\tau+s-\tau_0M<-N$, then we get (5.20). Q.E.D.

REMARK. Clearly we obtain (5.8) from (5.20).

THE COMPLETION OF THE PROOF OF LEMMA 4.1. By Proposition 5.2 and Corollary to Proposition 5.4 we have for $\kappa=2\sigma m(1-\tau)>0$

$$\begin{aligned}
&\|g(x)\langle D_y\rangle^{2m-1+\kappa}u\|_0 \\
&\quad\leqq\|g(x)\langle D_y\rangle^{2m-1+\kappa}v_1\|_0+\|g(x)\langle D_y\rangle^{2m}v_2\|_0 \\
&\quad\leqq C\sum_{j=1}^{2}(\|Pv_j\|_0+\|v_j\|_0) \\
&\quad\leqq C\Big(\|Pu\|_0+\sum_{j=1}^{2}\|[P,\chi_j]u\|_0+\|v_j\|_0\Big).
\end{aligned}$$

Using (5.8) and $\|v_j\|_0\leqq C\|u\|_0$ we obtain (4.1). Similarly, we have (4.2), too.

Q.E.D.

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