J. Math. Soc. Japan
Vol. 30, No. 2, 1978

A variation of Lyndon-Keisler's homomorphism theorem and its applications to interpolation theorems

By Tsuyoshi FUJIWARA

(Received Aug. 30, 1976)
(Revised May 24, 1977)

As is well known, R. C. Lyndon [9] proved the preservation theorem for homomorphisms and some related theorems by using his interpolation theorem (cf. [8]). H. J. Keisler [4] gave a simple proof of a generalization of the essential part of the above theorems of Lyndon on homomorphisms by using his theory of generalized atomic sets of formulas. On the other hand, L. Henkin [3] proved an extended form of the Craig-Lyndon interpolation theorem which includes the completeness theorem. Improving Henkin's proof, A. Oberschelp [12] proved an interpolation theorem of Lyndon type whose interpolant has some information about the equality symbol. Before this, the Craig-Lyndon interpolation theorem for the infinitary language $L_{\omega_1\omega}$ was given by E.G.K. Lopez-Escobar [7].

The main purpose of this paper is to prove some interpolation theorems whose interpolants have more information about the equality symbol and non-logical symbols than those of the above well-known interpolation theorems, by using a new notion of morphisms.

In Section 1, we shall introduce the notion of a morphism which can be naturally obtained from the notion of a homomorphism by using a many-to-many correspondence instead of a mapping. As preparation for the next section, we shall state a theorem on morphisms which is an immediate variation of Lyndon-Keisler's homomorphism theorem (cf. Lyndon [9; p. 151, lines 3-6], Keisler [4; Theorem 3]).

In Section 2, we shall prove the following interpolation theorem, which may be regarded as a strengthened version of Craig's, of Lyndon's, and of Oberschelp's (cf. Craig [2; Theorem 5], Lyndon [8; p. 140], Oberschelp [12; Theorem 2], and also Chang and Keisler [1; Theorems 2.2.20 and 2.2.24], Robinson [14; Theorem 5.1.8], Shoenfield [15; p. 80]).

Let Φ and Ψ be sentences of a first order language with the sentential constants $\mathfrak{t}$ *and* $\mathfrak{f}$ *and with or without equality such that* $\Phi\models\Psi$. *Then there exists a sentence* Θ *such that* (1) $\Phi\models\Theta$ *and* $\Theta\models\Psi$, (2) *all relation symbols occurring positively* (*resp. negatively*) *in* Θ *occur positively* (*resp. negatively*) *in* *both* Φ *and* Ψ, (3) *all operation symbols (including constant symbols) occurring in* Θ *occur in both* Φ *and* Ψ, (4) *if the equality symbol occurs positively (resp. negatively) in* Θ, *then it occurs positively in* Φ *(resp. negatively in* Ψ*)*.

This interpolation theorem (strictly speaking, a more general form which can be immediately obtained from this interpolation theorem by using the completeness theorem) was announced without proof by Oberschelp [**11**]. However, according to his paper [**12**], he found later that his proof covered only some special cases of this theorem. The theorem on morphisms is the cornerstone of our proof of this interpolation theorem.

In Section 3, by using Motohashi's interpolation and characterization theorems on primitive sets (cf. Motohashi [**10**; p. 116, Theorems 3.3 and 3.4]), we shall show that our interpolation theorem can be extended to an analogous theorem for the infinitary language $L_{\omega_1\omega}$, which is a stronger form of the interpolation theorem due to Lopez-Escobar [**7**; Theorem 4.1].

The author would like to express his hearty thanks to the referee and Dr. N. Motohashi whose valuable advices led him to obtain the result of Section 3.

§ 1. Terminologies, notations, and an immediate variation of Lyndon-Keisler's homomorphism theorem.

In this paper, every (object) language is assumed to be a first order language with the sentential constants $\mathfrak{t}$ and $\mathfrak{f}$. We shall be concerned only with finitary languages in Sections 1 and 2, and with infinitary languages in Section 3.

Let L be a language with or without equality, and let $\mathfrak{A}$ be a structure for L. The domain of $\mathfrak{A}$ is denoted by $|\mathfrak{A}|$, the relation of $\mathfrak{A}$ corresponding to a relation symbol r of L is denoted by $(r)_{\mathfrak{A}}$, and the operation of $\mathfrak{A}$ corresponding to an operation symbol f of L is denoted by $(f)_{\mathfrak{A}}$. A constant symbol is regarded as a nullary operation symbol.

Let t be a term of L which contains at most some of the distinct variables $x_1, \cdots, x_n$. Then t is denoted by $t(x_1, \cdots, x_n)$ if the variables $x_1, \cdots, x_n$ need to be indicated. Let $\mathfrak{A}$ be a structure for L, and let $a_1, \cdots, a_n$ be elements in $|\mathfrak{A}|$. We denote by $t[a_1, \cdots, a_n]$ the value of $t(x_1, \cdots, x_n)$ when $x_1, \cdots, x_n$ are assigned the values $a_1, \cdots, a_n$ respectively.

Let Θ be a formula of L which contains at most some of the distinct variables $x_1, \cdots, x_n$ as free variables. Then Θ is denoted by $\Theta(x_1, \cdots, x_n)$ if the variables $x_1, \cdots, x_n$ need to be indicated. Let $t_1, \cdots, t_n$ be terms of L. We denote by $\Theta[t_1, \cdots, t_n]$ the formula obtained from $\Theta(x_1, \cdots, x_n)$ by replacing all free occurrences of $x_1, \cdots, x_n$ by the terms $t_1, \cdots, t_n$ respectively. Let $\mathfrak{A}$ be a structure for L, and let $a_1, \cdots, a_n$ be elements in $|\mathfrak{A}|$. We write $\mathfrak{A} \models \Theta[a_1/x_1, \cdots, a_n/x_n]$ or simply $\mathfrak{A}\models\Theta[a_1,\cdots,a_n]$, if $a_1,\cdots,a_n$ satisfy $\Theta(x_1,\cdots,x_n)$ in $\mathfrak{A}$ when the free variables $x_1,\cdots,x_n$ are assigned the values $a_1,\cdots,a_n$ respectively. If $\mathfrak{A}\models\Theta[a_1,\cdots,a_n]$ holds for any elements $a_1,\cdots,a_n$ in $|\mathfrak{A}|$, we say that Θ *holds in* $\mathfrak{A}$ and write $\mathfrak{A}\models\Theta$. If $\mathfrak{A}\models\Theta$ holds for every structure $\mathfrak{A}$ for L, we write $\models\Theta$.

Let Φ and Ψ be formulas of L. If $\models\Phi\leftrightarrow\Psi$ holds, we say that Φ and Ψ are *logically equivalent*. Let Γ and Δ be sets of formulas of L. If any formula in Γ (resp. Δ) is logically equivalent to some formula in Δ (resp. Γ), we say that Γ and Δ are logically equivalent and write $\Gamma\equiv_l\Delta$.

Let Σ be a sentence or a set of sentences of L. A structure $\mathfrak{A}$ for L is called a *model* of Σ if $\mathfrak{A}\models\Sigma$ or $\mathfrak{A}\models\Theta$ for every Θ in Σ. We denote by $\boldsymbol{M}(\Sigma)$ the class of all models of Σ. We use $\boldsymbol{M}_L(\Sigma)$ in place of $\boldsymbol{M}(\Sigma)$ if the language L or the similarity type of models needs to be indicated. Σ is said to be *consistent* if $\boldsymbol{M}(\Sigma)$ is non-empty. Furthermore, let Φ be a sentence of L. If $\boldsymbol{M}_L(\Sigma)\subseteqq\boldsymbol{M}_L(\Phi)$, we say that Φ is a *consequence* of Σ and write $\Sigma\models\Phi$.

Let L' be a language included in L, and let $\mathfrak{A}$ be a structure for L. The reduct of $\mathfrak{A}$ to L' is denoted by $\mathfrak{A}\upharpoonright L'$. Let Σ be a set of sentences. We denote by $L(\Sigma)$ the language with equality which is determined by all non-logical symbols occurring in sentences of Σ.

Two formulas are said to be *congruent*, if they differ only in their bound occurrences of variables, and corresponding bound occurrences are bound by corresponding quantifiers. (For the precise definition of congruence of formulas, see Kleene [**6**; p. 153]).

A set F of formulas of L is called a *generalized atomic set* (or briefly, *GA set*) of L, if the following two conditions hold:

(1) If $\Theta(x_1,\cdots,x_n)\in F$ and y is a variable of L whose new occurrences in $\Theta[y,x_2,\cdots,x_n]$ are all free, then $\Theta[y,x_2,\cdots,x_n]\in F$;

(2) If Θ is a formula of L which is congruent to some formula in F, then $\Theta\in F$.

Let F be a GA set of L. We denote by $\mathscr{P}(F)$ or simply $\mathscr{P}F$ the set of all formulas formed from formulas of F by using only the connectives $\wedge$, $\vee$ and the quantifiers $\forall$, $\exists$. Here, and throughout this paper, the conjunction and the disjunction of the empty set of formulas are allowed, and regarded as the identically true sentential constant $\mathfrak{t}$ and the identically false sentential constant $\mathfrak{f}$ respectively.

Let Θ be a formula. Then the formula $\tilde{\Theta}$ is defined by

$$\tilde{\Theta}=\begin{cases}\Phi & \text{if } \Theta \text{ is of the form } \neg\Phi,\\ \neg\Theta & \text{otherwise.}\end{cases}$$

Let F be a set of formulas. The set $\{\tilde{\Theta}\mid\Theta\in F\}$ is denoted by $\tilde{F}$.

It is obvious that if F is a GA set of L, then $\mathscr{P}F$ and $\tilde{F}$ are also GA sets of L. If F and G are GA sets of L, then $F\cup G$ and $F\cap G$ are also GA sets of L.

Let L_1 and L_2 be languages with equality. We denote by $L_1 \cap L_2$ the language of all symbols that are contained in both L_1 and L_2. Now let F be a set of formulas of $L_1 \cap L_2$, and let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for L_1 and L_2 respectively. Let M be a subset of the Cartesian product $|\mathfrak{A}| \times |\mathfrak{B}|$. We say that M is an *F-morphism* of $\mathfrak{A}$ onto $\mathfrak{B}$, if M satisfies the following three conditions:

(1) For any element a in $|\mathfrak{A}|$, there exists an element b in $|\mathfrak{B}|$ such that $\langle a, b\rangle \in M$;

(2) For any element b in $|\mathfrak{B}|$, there exists an element a in $|\mathfrak{A}|$ such that $\langle a, b\rangle \in M$;

(3) For any formula $\Theta(x_1, \cdots, x_n)$ in F and any elements $\langle a_1, b_1\rangle, \cdots, \langle a_n, b_n\rangle$ in M,

$$\mathfrak{A} \models \Theta[a_1, \cdots, a_n] \text{ implies } \mathfrak{B} \models \Theta[b_1, \cdots, b_n].$$

An F-morphism M is called an *F-homomorphism*, if $\langle a, b\rangle, \langle a, c\rangle \in M$ implies $b=c$. We say that $\mathfrak{B}$ is an F-morphic (resp. F-homomorphic) image of $\mathfrak{A}$, if there exists an F-morphism (resp. F-homomorphism) of $\mathfrak{A}$ onto $\mathfrak{B}$. If M is an F-morphism of $\mathfrak{A}$ onto $\mathfrak{B}$, then the set $\{\langle b, a\rangle | \langle a, b\rangle \in M\}$ is denoted by M^{-1}, and obviously M^{-1} is an $\tilde{F}$-morphism of $\mathfrak{B}$ onto $\mathfrak{A}$. If M is an F-homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$ and $\langle a, b\rangle \in M$, then b is denoted by $M(a)$ or simply Ma.

In the remainder part of this section, we want to state some properties about morphisms, which will be used in the next section.

LEMMA 1.1. *Let L be a language with equality. Let F be a GA set of L, and let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for L. If M is an F-morphism of $\mathfrak{A}$ onto $\mathfrak{B}$, then M is a $\mathcal{P}F$-morphism of $\mathfrak{A}$ onto $\mathfrak{B}$.*

PROOF. This lemma can be easily proved by induction on formation of formulas of $\mathcal{P}F$.

The following theorem is an immediate variation of Lyndon-Keisler's homomorphism theorem (cf. Lyndon [**9**; p. 151, lines 3-6], Keisler [**4**; Theorem 3]).

THEOREM 1.2. *Let L be a language with equality. Let F be a GA set of L, and let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for L. Then the following two conditions are equivalent:*

(i) *Every sentence in $\mathcal{P}F$ that holds in $\mathfrak{A}$ also holds in $\mathfrak{B}$;*

(ii) *There exist an elementary extension $\mathfrak{A}^*$ of $\mathfrak{A}$ and an elementary extension $\mathfrak{B}^*$ of $\mathfrak{B}$ such that $\mathfrak{B}^*$ is an F-morphic image of $\mathfrak{A}^*$.*

PROOF. (ii)⇒(i) follows immediately from Lemma 1.1.
(i)⇒(ii) can be proved in essentially the same manner as the well-known proof of Lyndon-Keisler's homomorphism theorem (cf. Keisler [**4**; Proofs of Theorems 1, 2 and 3], Chang and Keisler [**1**; The main part of Proof of Theorem 3.2.4],

[**1**; Proofs of Lemma 5.2.9-Proposition 5.2.12]). We omit the detail of the proof.

The following proposition is a modification of Theorem 1.2.

PROPOSITION 1.3. *Let L be a language with equality. Let F be a GA set of L, and let Σ_1 and Σ_2 be sets of sentences of L. Then the following two conditions are equivalent:*

(i) *There exists no sentence Θ in $\mathscr{P}F$ such that $\Sigma_1 \models \Theta$ and $\Sigma_2 \models \neg \Theta$;*

(ii) *There exist a structure $\mathfrak{A}$ in $\boldsymbol{M}_L(\Sigma_1)$ and a structure $\mathfrak{B}$ in $\boldsymbol{M}_L(\Sigma_2)$ such that $\mathfrak{B}$ is an F-morphic image of $\mathfrak{A}$.*

PROOF. Since (ii)⇒(i) is obvious from Lemma 1.1, we shall prove (i)⇒(ii).

Proof of (i)⇒(ii). Suppose that (i) holds. First we put

$$\Delta_1 = \{\neg \Theta \mid \Theta \text{ is a sentence in } \mathscr{P}F,\ \Sigma_2 \models \neg \Theta\}.$$

Now assume, by way of contradiction, that $\Sigma_1 \cup \Delta_1$ is not consistent. Then by the compactness theorem, there exists a finite subset $\{\neg \Theta_1, \cdots, \neg \Theta_n\}$ of Δ_1 such that

$$\Sigma_1 \cup \{\neg \Theta_1, \cdots, \neg \Theta_n\} \text{ is inconsistent.}$$

Hence we have

$$\Sigma_1 \models \Theta_1 \vee \cdots \vee \Theta_n \text{ and } \Sigma_2 \models \neg(\Theta_1 \vee \cdots \vee \Theta_n).$$

This contradicts the assumption (i), because $\Theta_1 \vee \cdots \vee \Theta_n$ is a sentence in $\mathscr{P}F$. Therefore $\Sigma_1 \cup \Delta_1$ is consistent, and therefore there exists a structure $\mathfrak{C}$ in $\boldsymbol{M}_L(\Sigma_1 \cup \Delta_1)$.

Next we put

$$\Delta_2 = \{\Theta \mid \Theta \text{ is a sentence in } \mathscr{P}F,\ \mathfrak{C} \models \Theta\}.$$

Now assume, by way of contradiction, that $\Sigma_2 \cup \Delta_2$ is not consistent. Then by the compactness theorem, there exists a finite subset $\{\Theta'_1, \cdots, \Theta'_m\}$ of Δ_2 such that

$$\Sigma_2 \cup \{\Theta'_1, \cdots, \Theta'_m\} \text{ is inconsistent.}$$

Therefore we have

$$\Sigma_2 \models \neg(\Theta'_1 \wedge \cdots \wedge \Theta'_m) \text{ and } \mathfrak{C} \models \Theta'_1 \wedge \cdots \wedge \Theta'_m.$$

Hence we have

$$\neg(\Theta'_1 \wedge \cdots \wedge \Theta'_m) \in \Delta_1 \text{ and } \mathfrak{C} \models \Theta'_1 \wedge \cdots \wedge \Theta'_m,$$

because $\Theta'_1 \wedge \cdots \wedge \Theta'_m$ is a sentence in $\mathscr{P}F$. This contradicts the fact that $\mathfrak{C}$ is a model of Δ_1. Therefore $\Sigma_2 \cup \Delta_2$ is consistent, and therefore there exists a structure $\mathfrak{D}$ in $\boldsymbol{M}_L(\Sigma_2 \cup \Delta_2)$.

Since $\mathfrak{D}$ is a model of Δ_2, every sentence in $\mathscr{P}F$ that holds in $\mathfrak{C}$ also holds in $\mathfrak{D}$. Hence by (i)$\Rightarrow$(ii) of Theorem 1.2, there exist an elementary extension $\mathfrak{A}$ of $\mathfrak{C}$ and an elementary extension $\mathfrak{B}$ of $\mathfrak{D}$ such that $\mathfrak{B}$ is an F-morphic image of $\mathfrak{A}$. From this, it follows immediately that (ii) holds.

§ 2. Applications to interpolation theorems.

In this section, we shall consider only formulas which are built up using the connectives $\wedge$, $\vee$, $\neg$, and the quantifiers $\forall$, $\exists$.

Let L be a language with or without equality. Let s be a symbol of L, and let Φ be a sentence of L. Then s is said to occur *positively* (resp. *negatively*) in Φ, if s has an occurrence in Φ which is within the scope of an even (resp. odd) number of negation symbols. Let Σ be a set of sentences of L. Then s is said to occur positively (resp. negatively) in Σ, if s occurs positively (resp. negatively) in some sentence in Σ.

LEMMA 2.1. *Let L be a language with equality, and let Φ be a sentence of L. Let G be the smallest GA set of L that satisfies the following two conditions:*

(1) *If r is an n-ary relation symbol which occurs positively in Φ, and $x_1, \cdots, x_n$ are variables, then $r(x_1, \cdots, x_n) \in G$;*

(2) *If r is an n-ary relation symbol which occurs negatively in Φ, and $x_1, \cdots, x_n$ are variables, then $\neg r(x_1, \cdots, x_n) \in G$.*

Let O be the smallest GA set of L that satisfies the following two conditions:

(1) *If f is an n-ary operation symbol which occurs in Φ, and $x, x_1, \cdots, x_n$ are variables, then $f(x_1, \cdots, x_n)=x \in O$;*

(2) *If x and y are variables, then $x=y \in O$ and $\neg x=y \in O$,*

Let O' be the smallest GA set of L that satisfies the following two conditions:

(1) *If f is an n-ary operation symbol which occurs in Φ, and $x, x_1, \cdots, x_n$ are variables, then $\neg f(x_1, \cdots, x_n)=x \in O'$;*

(2) *If x and y are variables, then $\neg x=y \in O'$.*

Then the following three assertions hold:

(i) *Φ is logically equivalent to some sentence in $\mathscr{P}(G \cup O)$;*

(ii) *Φ is logically equivalent to some sentence in $\mathscr{P}(G \cup \tilde{O})$;*

(iii) *If the equality symbol does not occur positively in Φ, then Φ is logically equivalent to some sentence in $\mathscr{P}(G \cup O')$.*

PROOF. This lemma can be easily proved by using De Morgan laws with respect to $\wedge$, $\vee$ and $\forall$, $\exists$, the law of double negation, and the following property (S) concerning substitution of terms.

(S) Let $\Theta(x_1, \cdots, x_m)$ be a formula of L, and let $t(y_1, \cdots, y_n)$ be a term of L. If all new occurrences of $y_1, \cdots, y_n$ in $\Theta[t(y_1, \cdots, y_n), x_2, \cdots, x_m]$ are free, and each of $y_1, \cdots, y_n$ differs from x_1, then the following three formulas are logically equivalent:

(1) $\Theta[t(y_1, \cdots, y_n), x_2, \cdots, x_m]$,

(2) $\exists x_1(t(y_1, \cdots, y_n)=x_1 \wedge \Theta(x_1, \cdots, x_m))$,

(3) $\forall x_1(\neg t(y_1, \cdots, y_n)=x_1 \vee \Theta(x_1, \cdots, x_m))$.

Now we shall prove the following lemma, which may be regarded as a stronger form of the Robinson joint consistency theorem for languages with equality (cf. Chang and Keisler [**1**; Theorem 2.2.23], Shoenfield [**15**; p. 79]).

LEMMA 2.2. *Let L be a language with equality, and let Σ_1 and Σ_2 be sets of sentences of L. Let F_1 be the smallest GA set of L that satisfies the following four conditions:*

(1) *If r is an n-ary relation symbol which occurs positively in Σ_1, and $x_1, \cdots, x_n$ are variables, then $r(x_1, \cdots, x_n) \in F_1$;*

(2) *If r is an n-ary relation symbol which occurs negatively in Σ_1, and $x_1, \cdots, x_n$ are variables, then $\neg r(x_1, \cdots, x_n) \in F_1$;*

(3) *If f is an n-ary operation symbol which occurs in Σ_1, and $x, x_1, \cdots, x_n$ are variables, then $f(x_1, \cdots, x_n)=x \in F_1$;*

(4) If x and y are variables, then $x=y \in F_1$ and $\neg x=y \in F_1$.

Let F_2 be the smallest GA set of L that satisfies the following four conditions:

(1) *If r is an n-ary relation symbol which occurs negatively in Σ_2, and $x_1, \cdots, x_n$ are variables, then $r(x_1, \cdots, x_n) \in F_2$;*

(2) *If r is an n-ary relation symbol which occurs positively in Σ_2, and $x_1, \cdots, x_n$ are variables, then $\neg r(x_1, \cdots, x_n) \in F_2$;*

(3) *If f is an n-ary operation symbol which occurs in Σ_2, and $x, x_1, \cdots, x_n$ are variables, then $f(x_1, \cdots, x_n)=x \in F_2$;*

(4) *If x and y are variables, then $x=y \in F_2$; and if the equality symbol occurs positively in Σ_2, and x and y are variables, then $\neg x=y \in F_2$.*

Let the GA set $F_1 \cap F_2$ be denoted by F. Then the following three conditions are equivalent:

(i) *There exists no sentence Θ in $\mathscr{P}F$ such that $\Sigma_1 \models \Theta$ and $\Sigma_2 \models \neg \Theta$;*

(ii) *There exist a structure $\mathfrak{A}$ in $\boldsymbol{M}_L(\Sigma_1)$ and a structure $\mathfrak{B}$ in $\boldsymbol{M}_L(\Sigma_2)$ such that $\mathfrak{B}$ is an F-homomorphic image of $\mathfrak{A}$;*

(iii) *$\Sigma_1 \cup \Sigma_2$ is consistent.*

PROOF. (iii)$\Rightarrow$(i) is obvious. Since F contains all formulas of the form $x=y$, every F-morphism is an F-homomorphism. Hence (i)$\Rightarrow$(ii) follows immediately from (i)$\Rightarrow$(ii) of Proposition 1.3. It remains only to prove (ii)$\Rightarrow$(iii).

Proof of (ii)$\Rightarrow$(iii). Assume that (ii) holds. And let P be an F-homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$, where $\mathfrak{A} \in \boldsymbol{M}_L(\Sigma_1)$ and $\mathfrak{B} \in \boldsymbol{M}_L(\Sigma_2)$.

First we construct a structure $\mathfrak{A}^*$ for L as follows:

(1) $|\mathfrak{A}^*|=|\mathfrak{A}|$;

(2) If r is a relation symbol such that $r(x_1, \cdots, x_n) \in F_2-F$ and $\neg r(x_1, \cdots, x_n) \in F_2-F$, then

$$(r)_{\mathfrak{A}^*}=\{\langle a_1, \cdots, a_n\rangle | \langle Pa_1, \cdots, Pa_n\rangle \in (r)_{\mathfrak{B}}\} ;$$

(3) If r is a relation symbol such that $r(x_1, \cdots, x_n)\in F_2-F$ and $\neg r(x_1, \cdots, x_n)\notin F_2-F$, then

$$(r)_{\mathfrak{A}^*}=(r)_{\mathfrak{A}}\cap\{\langle a_1, \cdots, a_n\rangle | \langle Pa_1, \cdots, Pa_n\rangle \in (r)_{\mathfrak{B}}\} ;$$

(4) If r is a relation symbol such that $\neg r(x_1, \cdots, x_n)\in F_2-F$ and $r(x_1, \cdots, x_n)\notin F_2-F$, then

$$(r)_{\mathfrak{A}^*}=(r)_{\mathfrak{A}}\cup\{\langle a_1, \cdots, a_n\rangle | \langle Pa_1, \cdots, Pa_n\rangle \in (r)_{\mathfrak{B}}\} ;$$

(5) If f is an operation symbol such that $f(x_1, \cdots, x_n)=x\in F_2-F$, then $(f)_{\mathfrak{A}^*}$ is defined so that

$$P((f)_{\mathfrak{A}^*}(a_1, \cdots, a_n))=(f)_{\mathfrak{B}}(Pa_1, \cdots, Pa_n)$$

holds for any elements $a_1, \cdots, a_n$ in $|\mathfrak{A}^*|$;

(6) The other relations and operations are the same as those of $\mathfrak{A}$.

Now it can be easily verified that the mapping P is an F_2-homomorphism of $\mathfrak{A}^*$ onto $\mathfrak{B}$. Hence P^{-1} is an $\tilde{F}_2$-morphism of $\mathfrak{B}$ onto $\mathfrak{A}^*$, and hence by Lemma 1.1, P^{-1} is a $\mathscr{P}\tilde{F}_2$-morphism of $\mathfrak{B}$ onto $\mathfrak{A}^*$. By (ii) and (iii) of Lemma 2.1, each sentence in Σ_2 is logically equivalent to some sentence in $\mathscr{P}\tilde{F}_2$. Therefore, since $\mathfrak{B}\in \boldsymbol{M}_L(\Sigma_2)$, we have $\mathfrak{A}^*\in \boldsymbol{M}_L(\Sigma_2)$.

On the other hand, it is obvious that

$$r(x_1, \cdots, x_n)\in F_1 \text{ implies } r(x_1, \cdots, x_n)\notin F_2-F,$$
$$\neg r(x_1, \cdots, x_n)\in F_1 \text{ implies } \neg r(x_1, \cdots, x_n)\notin F_2-F,$$
$$f(x_1, \cdots, x_n)=x\in F_1 \text{ implies } f(x_1, \cdots, x_n)=x\notin F_2-F.$$

Hence it can be easily verified that the identity mapping I of $|\mathfrak{A}|$ onto $|\mathfrak{A}^*|$ is an F_1-homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}^*$. Hence by Lemma 1.1, I is a $\mathscr{P}F_1$-homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}^*$. By (i) of Lemma 2.1, each sentence in Σ_1 is logically equivalent to some sentence in $\mathscr{P}F_1$. Therefore, since $\mathfrak{A}\in \boldsymbol{M}_L(\Sigma_1)$, we have $\mathfrak{A}^*\in \boldsymbol{M}_L(\Sigma_1)$. Moreover, since $\mathfrak{A}^*\in \boldsymbol{M}_L(\Sigma_2)$, we have $\mathfrak{A}^*\in \boldsymbol{M}_L(\Sigma_1\cup\Sigma_2)$. Hence $\Sigma_1\cup\Sigma_2$ is consistent. This completes the proof.

Let Φ and Ψ be sentences of L, and put $\Sigma_1=\{\Phi\}$ and $\Sigma_2=\{\neg\Psi\}$ in Lemma 2.2. Then the following interpolation theorem follows immediately from [not (iii)]$\Rightarrow$[not (i)].

THEOREM 2.3. *Let L be a language with equality, and let Φ and Ψ be sentences of L such that $\Phi\models\Psi$. Then there exists a sentence Θ of L which satisfies the following four conditions:*

(1) *$\Phi\models\Theta$ and $\Theta\models\Psi$;*

(2) *All relation symbols occurring positively (resp. negatively) in Θ occur positively (resp. negatively) in both Φ and Ψ ;*

(3) *All operation symbols occurring in Θ occur in both Φ and Ψ ;*

(4) *If the equality symbol occurs negatively in Θ, then it occurs negatively in Ψ.*

REMARK 2.4. Applying this theorem to the sentences $\neg\Psi$, $\neg\Phi$ with $\neg\Psi\models\neg\Phi$ in place of Φ, Ψ with $\Phi\models\Psi$, the above condition (4) can be replaced by the following:

(4)′ *If the equality symbol occurs positively in Θ, then it occurs positively in Φ.*

Theorem 2.3 can be improved so that Θ satisfies not only the conditions (1)-(4) but also (4)′. To show this, we shall first prove the following lemma, which can be regarded as a generalization of the Robinson joint consistency theorem for languages without equality (cf. Robinson [**13**; Theorem 2.9], [**14**; Theorem 5.1.6]).

LEMMA 2.5. *Let L be a language with equality, and let Σ_i $(i=1, 2)$ be a set of sentences of L in which the equality symbol does not occur positively. Let F_i $(i=1, 2)$ be the smallest GA set of L that satisfies the following two conditions:*

(1_i) *If r is an n-ary relation symbol occurring positively in Σ_i, and $t_1, \cdots, t_n$ are terms of $L(\Sigma_i)$, then $r(t_1, \cdots, t_n)\in F_i$;*

(2_i) *If r is an n-ary relation symbol occurring negatively in Σ_i, and $t_1, \cdots, t_n$ are terms of $L(\Sigma_i)$, then $\neg r(t_1, \cdots, t_n)\in F_i$.*

Let F denote the GA set $F_1\cap\tilde{F}_2$, and let O be the smallest GA set of L that satisfies the following two conditions:

(1) *If f is an n-ary operation symbol of $L(\Sigma_1)\cap L(\Sigma_2)$, and $x, x_1, \cdots, x_n$ are variables, then $f(x_1, \cdots, x_n)=x\in O$;*

(2) *If x and y are variables, then $x=y\in O$.*

Then the following four conditions are equivalent:

(i) *There exists no sentence Θ in $\mathscr{P}F$ such that $\Sigma_1\models\Theta$ and $\Sigma_2\models\neg\Theta$;*

(ii) *There exist a structure $\mathfrak{A}$ in $\boldsymbol{M}_L(\Sigma_1)$ and a structure $\mathfrak{B}$ in $\boldsymbol{M}_L(\Sigma_2)$ such that $\mathfrak{B}$ is an F-morphic image of $\mathfrak{A}$;*

(iii) *There exist a structure $\mathfrak{A}^*$ in $\boldsymbol{M}_L(\Sigma_1)$ and a structure $\mathfrak{B}^*$ in $\boldsymbol{M}_L(\Sigma_2)$ such that $\mathfrak{B}^*$ is an $(F\cup O)$-homomorphic image of $\mathfrak{A}^*$;*

(iv) *$\Sigma_1\cup\Sigma_2$ is consistent.*

PROOF. (iv)$\Rightarrow$(i) is obvious. (i)$\Rightarrow$(ii) follows immediately from (i)$\Rightarrow$(ii) of Proposition 1.3. (iii)$\Rightarrow$(iv) follows immediately from (ii)$\Rightarrow$(iii) of Lemma 2.2, because under the assumption of Lemma 2.5 that the equality symbol does not occur positively in Σ_2, $F\cup O$ in Lemma 2.5 includes F in Lemma 2.2, i.e. an $(F\cup O)$-homomorphism in Lemma 2.5 is an F-homomorphism in Lemma 2.2. What remains is to prove the implication (ii)$\Rightarrow$(iii).

Before we describe the proof of (ii)$\Rightarrow$(iii), we define some GA sets as follows:

G is the GA set of L which consists of all formulas in F not containing any operation symbol.

G_1 is the GA set of L which consists of all formulas in F_1 not containing any operation symbol.

O_1 is the smallest GA set of L that satisfies the following two conditions:

(1) If f is an n-ary operation symbol of $L(\Sigma_1)$, and $x, x_1, \cdots, x_n$ are variables, then $f(x_1, \cdots, x_n)=x\in O_1$;

(2) If x and y are variables, then $x=y\in O_1$.

Proof of (ii)$\Rightarrow$(iii). We assume that (ii) holds, that is, we assume that there exist a structure $\mathfrak{A}$ in $\boldsymbol{M}_L(\Sigma_1)$ and a structure $\mathfrak{B}$ in $\boldsymbol{M}_L(\Sigma_2)$ such that an F-morphism M of $\mathfrak{A}$ onto $\mathfrak{B}$ exists. Let L' be the language formed from L by omitting all operation symbols not belonging to $L(\Sigma_1)\cap L(\Sigma_2)$, and let $\mathfrak{M}$ be the substructure of $(\mathfrak{A}\times\mathfrak{B})\upharpoonright L'$ generated by M, where $\mathfrak{A}\times\mathfrak{B}$ denotes the direct product of $\mathfrak{A}$ and $\mathfrak{B}$.

First we shall prove that the mapping P defined by

$$P=\{\langle\langle a, b\rangle, a\rangle \mid \langle a, b\rangle\in|\mathfrak{M}|\}$$

is a $(\tilde{G}\cup O)$-homomorphism of $\mathfrak{M}$ onto $\mathfrak{A}$. (Note that $\tilde{G}\cup O$ is a GA set of L'.)

Since it is obvious that P is an O-homomorphism of $\mathfrak{M}$ onto $\mathfrak{A}$, we shall prove that P is a $\tilde{G}$-homomorphism. To prove this, it suffices to prove that P^{-1} is a G-morphism of $\mathfrak{A}$ onto $\mathfrak{M}$.

Let $\Theta(x_1, \cdots, x_n)$ be an atomic formula such that $\Theta(x_1, \cdots, x_n)\in G$, and let $\langle a_1, \langle a_1, b_1\rangle\rangle, \cdots, \langle a_n, \langle a_n, b_n\rangle\rangle$ be members of P^{-1}. Now suppose that

$$(*)\qquad \mathfrak{A}\models\Theta[a_1, \cdots, a_n].$$

Since $\langle a_1, b_1\rangle, \cdots, \langle a_n, b_n\rangle$ are in $|\mathfrak{M}|$, there exist terms $t_1(y_1, \cdots, y_m), \cdots, t_n(y_1, \cdots, y_m)$ of $L(\Sigma_1)\cap L(\Sigma_2)$ and elements $\langle c_1, d_1\rangle, \cdots, \langle c_m, d_m\rangle$ in M such that

$$a_i=t_i[c_1, \cdots, c_m], \qquad i=1, 2, \cdots, n;$$

$$b_i=t_i[d_1, \cdots, d_m], \qquad i=1, 2, \cdots, n.$$

Hence from (*), we have

$$\mathfrak{A}\models\Theta[t_1, \cdots, t_n][c_1/y_1, \cdots, c_m/y_m].$$

This implies that

$$\mathfrak{B}\models\Theta[t_1, \cdots, t_n][d_1/y_1, \cdots, d_m/y_m],$$

because $\Theta[t_1, \cdots, t_n]\in F$ and M is an F-morphism of $\mathfrak{A}$ onto $\mathfrak{B}$. Therefore we have

$$\mathfrak{B}\models\Theta[b_1, \cdots, b_n].$$

Hence from (*) and the definition of $\mathfrak{M}$, we obtain

$$\mathfrak{M}\models\Theta[\langle a_1, b_1\rangle, \cdots, \langle a_n, b_n\rangle].$$

Next let $\Theta(x_1, \cdots, x_n)$ be an atomic formula such that $\neg\Theta(x_1, \cdots, x_n)\in G$, and let $\langle a_1, \langle a_1, b_1\rangle\rangle, \cdots, \langle a_n, \langle a_n, b_n\rangle\rangle$ be members of P^{-1}. Now suppose that

$$\mathfrak{A}\models\neg\Theta[a_1, \cdots, a_n].$$

Then

$$\mathfrak{M}\models\neg\Theta[\langle a_1, b_1\rangle, \cdots, \langle a_n, b_n\rangle]$$

follows immediately from the definition of $\mathfrak{M}$.

By the above arguments, we have that P^{-1} is a G-morphism. Hence P is a $(\tilde{G}\cup O)$-homomorphism of $\mathfrak{M}$ onto $\mathfrak{A}$.

Now we construct a structure $\mathfrak{A}^*$ for L as follows:

(1) $|\mathfrak{A}^*|=|\mathfrak{M}|$;

(2) If r is a relation symbol such that $r(x_1, \cdots, x_n)\in\tilde{G}_1-\tilde{G}$ and $\neg r(x_1, \cdots, x_n)\in\tilde{G}_1-\tilde{G}$, then

$$(r)_{\mathfrak{A}^*}=\{\langle\langle a_1, b_1\rangle, \cdots, \langle a_n, b_n\rangle\rangle\,|\,\langle a_1, \cdots, a_n\rangle\in(r)_{\mathfrak{A}}\};$$

(3) If r is a relation symbol such that $r(x_1, \cdots, x_n)\in\tilde{G}_1-\tilde{G}$ and $\neg r(x_1, \cdots, x_n)\notin\tilde{G}_1-\tilde{G}$, then

$$(r)_{\mathfrak{A}^*}=(r)_{\mathfrak{M}}\cap\{\langle\langle a_1, b_1\rangle, \cdots, \langle a_n, b_n\rangle\rangle\,|\,\langle a_1, \cdots, a_n\rangle\in(r)_{\mathfrak{A}}\};$$

(4) If r is a relation symbol such that $\neg r(x_1, \cdots, x_n)\in\tilde{G}_1-\tilde{G}$ and $r(x_1, \cdots, x_n)\notin\tilde{G}_1-\tilde{G}$, then

$$(r)_{\mathfrak{A}^*}=(r)_{\mathfrak{M}}\cup\{\langle\langle a_1, b_1\rangle, \cdots, \langle a_n, b_n\rangle\rangle\,|\,\langle a_1, \cdots, a_n\rangle\in(r)_{\mathfrak{A}}\};$$

(5) If f is an n-ary operation symbol of L such that $f(x_1, \cdots, x_n)=x\notin O$, then $(f)_{\mathfrak{A}^*}$ is defined so that

$$P((f)_{\mathfrak{A}^*}(\langle a_1, b_1\rangle, \cdots, \langle a_n, b_n\rangle))=(f)_{\mathfrak{A}}(a_1, \cdots, a_n)$$

holds for any elements $\langle a_1, b_1\rangle, \cdots, \langle a_n, b_n\rangle$ in $|\mathfrak{A}^*|$;

(6) The other relations and operations are the same as those of $\mathfrak{M}$.

Then by using the fact that P is a $(\tilde{G}\cup O)$-homomorphism of $\mathfrak{M}$ onto $\mathfrak{A}$, it can be easily verified that P is a $(\tilde{G}_1\cup O_1)$-homomorphism of $\mathfrak{A}^*$ onto $\mathfrak{A}$. Hence P^{-1} is a $(G_1\cup\tilde{O}_1)$-morphism of $\mathfrak{A}$ onto $\mathfrak{A}^*$, and hence by Lemma 1.1, P^{-1} is a $\mathcal{P}(G_1\cup\tilde{O}_1)$-morphism of $\mathfrak{A}$ onto $\mathfrak{A}^*$. By (iii) of Lemma 2.1, each sentence in Σ_1 is logically equivalent to some sentence in $\mathcal{P}(G_1\cup\tilde{O}_1)$. Therefore, since $\mathfrak{A}\in\boldsymbol{M}_L(\Sigma_1)$, we have $\mathfrak{A}^*\in\boldsymbol{M}_L(\Sigma_1)$.

Hereafter, we shall prove that the mapping Q defined by

$$Q=\{\langle\langle a, b\rangle, b\rangle\,|\,\langle a, b\rangle\in|\mathfrak{A}^*|\}$$

is an $(F\cup O)$-homomorphism of $\mathfrak{A}^*$ onto $\mathfrak{B}$.

First we have that Q is an O-homomorphism of $\mathfrak{A}^*$ onto $\mathfrak{B}$, because $(f)_{\mathfrak{A}^*}=(f)_{\mathfrak{M}}$ for all operation symbols f occurring in formulas in O and Q is obviously an O-homomorphism of $\mathfrak{M}$ onto $\mathfrak{B}$.

Next we shall show that Q is a G-morphism (G-homomorphism) of $\mathfrak{A}^*$ onto $\mathfrak{B}$.

Let $\Theta(x_1, \cdots, x_n)$ be an atomic formula formed from a relation symbol r such that $\Theta(x_1, \cdots, x_n) \in G$, and let $\langle\langle a_1, b_1\rangle, b_1\rangle, \cdots, \langle\langle a_n, b_n\rangle, b_n\rangle$ be members of Q. Now suppose that

$$\mathfrak{A}^* \models \Theta[\langle a_1, b_1\rangle, \cdots, \langle a_n, b_n\rangle].$$

Since $\Theta(x_1, \cdots, x_n) \in G$ i.e. $\neg\Theta(x_1, \cdots, x_n) \in \tilde{G}$, $(r)_{\mathfrak{A}^*} \subseteq (r)_{\mathfrak{M}}$ follows from the definition of $\mathfrak{A}^*$. Hence we have

$$\mathfrak{M} \models \Theta[\langle a_1, b_1\rangle, \cdots, \langle a_n, b_n\rangle].$$

Therefore by the definition of $\mathfrak{M}$, we have

$$\mathfrak{B} \models \Theta[b_1, \cdots, b_n].$$

Next let $\Theta(x_1, \cdots, x_n)$ be an atomic formula formed from a relation symbol r such that $\neg\Theta(x_1, \cdots, x_n) \in G$, and let $\langle\langle a_1, b_1\rangle, b_1\rangle, \cdots, \langle\langle a_n, b_n\rangle, b_n\rangle$ be members of Q. Now suppose that

$$\mathfrak{A}^* \models \neg\Theta[\langle a_1, b_1\rangle, \cdots, \langle a_n, b_n\rangle].$$

Since $\neg\Theta(x_1, \cdots, x_n) \in G$ i.e. $\Theta(x_1, \cdots, x_n) \in \tilde{G}$, $(r)_{\mathfrak{A}^*} \supseteq (r)_{\mathfrak{M}}$ follows from the definition of $\mathfrak{A}^*$. Hence we have

$$\mathfrak{M} \models \neg\Theta[\langle a_1, b_1\rangle, \cdots, \langle a_n, b_n\rangle].$$

Therefore by the definition of $\mathfrak{M}$, we have

$$\mathfrak{A} \models \neg\Theta[a_1, \cdots, a_n] \quad \text{or} \quad \mathfrak{B} \models \neg\Theta[b_1, \cdots, b_n].$$

Since $\langle a_1, b_1\rangle, \cdots, \langle a_n, b_n\rangle$ are in $|\mathfrak{A}^*|$ i.e. in $|\mathfrak{M}|$, there exist terms $t_1(y_1, \cdots, y_m), \cdots, t_n(y_1, \cdots, y_m)$ of $L(\Sigma_1) \cap L(\Sigma_2)$ and elements $\langle c_1, d_1\rangle, \cdots, \langle c_m, d_m\rangle$ in M such that

$$a_i = t_i[c_1, \cdots, c_m], \qquad i = 1, 2, \cdots, n;$$

$$b_i = t_i[d_1, \cdots, d_m], \qquad i = 1, 2, \cdots, n.$$

Hence, in the case where

$$\mathfrak{A} \models \neg\Theta[a_1, \cdots, a_n],$$

we have

$$\mathfrak{A} \models \neg\Theta[t_1, \cdots, t_n][c_1/y_1, \cdots, c_m/y_m].$$

This implies that

$$\mathfrak{B} \models \neg\Theta[t_1, \cdots, t_n][d_1/y_1, \cdots, d_m/y_m],$$

because $\neg\Theta[t_1, \cdots, t_n] \in F$ and M is an F-morphism of $\mathfrak{A}$ onto $\mathfrak{B}$. Therefore we have

$$\mathfrak{B} \models \neg \Theta[b_1, \cdots, b_n].$$

Hence in any case, we have

$$\mathfrak{B} \models \neg \Theta[b_1, \cdots, b_n].$$

By the above arguments, we have that Q is a G-morphism of $\mathfrak{A}^*$ onto $\mathfrak{B}$. Therefore Q is a $(G \cup O)$-homomorphism of $\mathfrak{A}^*$ onto $\mathfrak{B}$, because Q is an O-homomorphism of $\mathfrak{A}^*$ onto $\mathfrak{B}$. Hence by Lemma 1.1, Q is a $\mathscr{P}(G \cup O)$-homomorphism of $\mathfrak{A}^*$ onto $\mathfrak{B}$. Therefore we have that Q is an $(F \cup O)$-homomorphism of $\mathfrak{A}^*$ onto $\mathfrak{B}$, because $\mathscr{P}(F \cup O) \equiv_l \mathscr{P}(G \cup O)$ is obvious from (S) in the proof of Lemma 2.1. Hence, if we put $\mathfrak{B}^* = \mathfrak{B}$, then we have (iii). This completes the proof of Lemma 2.5.

Let Φ be a sentence of L in which the equality symbol does not occur positively, and let Ψ be a sentence of L in which the equality symbol does not occur negatively. Put $\Sigma_1 = \{\Phi\}$ and $\Sigma_2 = \{\neg \Psi\}$ in Lemma 2.5. Then the following interpolation theorem follows immediately from [not (iv)]$\Rightarrow$[not (i)].

THEOREM 2.6. *Let L be a language with equality, and let Φ and Ψ be sentences of L such that $\Phi \models \Psi$. Assume that the equality symbol does not occur positively in Φ and does not occur negatively in Ψ. Then there exists a sentence Θ of L which satisfies the following four conditions:*

(1) $\Phi \models \Theta$ *and* $\Theta \models \Psi$;
(2) *All relation symbols occurring positively* (*resp. negatively*) *in Θ occur positively* (*resp. negatively*) *in both Φ and Ψ*;
(3) *All operation symbols occurring in Θ occur in both Φ and Ψ*;
(4) *The equality symbol does not occur in Θ.*

The following interpolation theorem which is the aim of this section follows immediately from the collection of Theorem 2.3, Remark 2.4, and Theorem 2.6.

THEOREM 2.7. *Let L be a language with or without equality, and let Φ and Ψ be sentences of L such that $\Phi \models \Psi$. Then there exists a sentence Θ of L which satisfies the following four conditions:*

(1) $\Phi \models \Theta$ *and* $\Theta \models \Psi$;
(2) *All relation symbols that occur positively* (*resp. negatively*) *in Θ occur positively* (*resp. negatively*) *in both Φ and Ψ*;
(3) *All operation symbols that occur in Θ occur in both Φ and Ψ*;
(4) *If the equality symbol occurs positively* (*resp. negatively*) *in Θ, then it occurs positively in Φ* (*resp. negatively in Ψ*).

§ 3. An extension of the interpolation theorem to $L_{\omega_1 \omega}$.

In this section, we shall consider only languages with a set of ω_1 variables.

Let L be a usual first order language with or without equality. We denote by $L_{\omega_1\omega}$ the infinitary language which has the same symbols as L and in which the conjunction symbol $\wedge$ and disjunction symbol $\vee$ may be applied to countable sets of formulas. For the infinitary language $L_{\omega_1\omega}$, we use the same terminologies and notations as those for a finitary language, except that we use $\mathscr{P}^*$ instead of $\mathscr{P}$. However, in our discussion, we shall be concerned only with formulas having finitely many free variables. To make sure, we want to state some of the terminologies and notations.

A set F of formulas of $L_{\omega_1\omega}$ is called a *GA set* of $L_{\omega_1\omega}$, if the following three conditions hold:

(1) Each formula of F has only finitely many free variables;

(2) If $\theta(x_1, \cdots, x_n)\in F$ and y is a variable of L whose new occurrences in $\Theta[y, x_2, \cdots, x_n]$ are all free, then $\Theta[y, x_2, \cdots, x_n]\in F$;

(3) If Θ is a formula of $L_{\omega_1\omega}$ which is congruent to some formula in F, then $\Theta\in F$.

Let F be a GA set of $L_{\omega_1\omega}$. We denote by $\mathscr{P}^*(F)$ or simply $\mathscr{P}^*F$ the set of all formulas of $L_{\omega_1\omega}$ which are formed from formulas of F by using only the (countable) connectives $\wedge$, $\vee$ and the quantifiers $\forall$, $\exists$ and each of which has only finitely many free variables. Note that $\mathscr{P}^*F$ is also a GA set of $L_{\omega_1\omega}$.

The following lemma, which is analogous to Lemma 1.1, can be easily obtained.

LEMMA 3.1. *Let L be a language with equality. Let F be a GA set of $L_{\omega_1\omega}$, and let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for L. If M is an F-morphism of $\mathfrak{A}$ onto $\mathfrak{B}$, then M is a $\mathscr{P}^*F$-morphism of $\mathfrak{A}$ onto $\mathfrak{B}$.*

Let F be a GA set of $L_{\omega_1\omega}$, and let G be a subset of F. If the smallest GA set of $L_{\omega_1\omega}$ including G coincides with F, then we say that F is *generated* by G. If F is generated by a countable set, then we say that F is *countably generated*. Note that if F is generated by G, then every G-morphism is an F-morphism, (and obviously, every F-morphism is a G-morphism).

The following proposition, which is analogous to Proposition 1.3, can be immediately obtained from Motohashi's interpolation and characterization theorems on primitive sets (cf. Motohashi [**10**; p. 116, Theorems 3.3 and 3.4]).

PROPOSITION 3.2. *Let L be a language with equality. Let F be a countably generated GA set of $L_{\omega_1\omega}$, and let Φ_1 and Φ_2 be sentences of $L_{\omega_1\omega}$. Then the following two conditions are equivalent:*

(i) *There exists no sentence Θ in $\mathscr{P}^*F$ such that $\Phi_1\models\Theta$ and $\Phi_2\models\neg\Theta$;*

(ii) *There exist a structure $\mathfrak{A}$ in $\boldsymbol{M}_L(\Phi_1)$ and a structure $\mathfrak{B}$ in $\boldsymbol{M}_L(\Phi_2)$ such that $\mathfrak{B}$ is an F-morphic image of $\mathfrak{A}$.*

We are now in a position to show the following theorem, which is an extension of Theorem 2.7 and which is a strengthening of Lopez-Escobar's interpolation theorem (cf. Lopez-Escobar [**7**; Theorem 4.1], or also Keisler [**5**; Theorems 4, 6 and 6A]).

THEOREM 3.3. *Let L be a language with or without equality, and let Φ and Ψ be sentences of $L_{\omega_1\omega}$ such that $\Phi \models \Psi$. Then there exists a sentence Θ of $L_{\omega_1\omega}$ which satisfies the following four conditions:*

(1) $\Phi \models \Theta$ *and* $\Theta \models \Psi$;

(2) *All relation symbols that occur positively (resp. negatively) in Θ occur positively (resp. negatively) in both Φ and Ψ;*

(3) *All operation symbols that occur in Θ occur in both Φ and Ψ;*

(4) *If the equality symbol occurs positively (resp. negatively) in Θ, then it occurs positively in Φ (resp. negatively in Ψ).*

PROOF. Change the description of Section 2 as follows:

(1) Sentences of L are replaced by sentences of $L_{\omega_1\omega}$;

(2) GA sets of L are replaced by GA sets of $L_{\omega_1\omega}$;

(3) $\mathscr{P}$ is replaced by $\mathscr{P}^*$;

(4) Σ_1 and Σ_2 in Lemma 2.2 and in Lemma 2.5 are assumed to be unit sets;

(5) Lemma 1.1 and Proposition 1.3 are replaced by Lemma 3.1 and Proposition 3.2 respectively.

Then, it can be easily seen that all the above modified arguments hold. Hence we have Theorem 3.3.

References

[1] C.C. Chang and H.J. Keisler, Model Theory, North-Holland, Amsterdam, 1973.

[2] W. Craig, Linear reasoning. A new form of the Herbrand-Gentzen theorem, J. Symbolic Logic, **22** (1957), 250-268.

[3] L. Henkin, An extension of the Craig-Lyndon interpolation theorem, J. Symbolic Logic, **28** (1963), 201-216.

[4] H.J. Keisler, Theory of models with generalized atomic formulas, J. Symbolic Logic, **25** (1960), 1-26.

[5] H.J. Keisler, Model Theory for Infinitary Logic, North-Holland, Amsterdam, 1971.

[6] S.C. Kleene, Introduction to Metamathematics, D. Van Nostrand, Princeton, 1952.

[7] E.G.K. Lopez-Escobar, An interpolation theorem for denumerably long formulas, Fund. Math., LVII (1965), 253-272.

[8] R.C. Lyndon, An interpolation theorem in the predicate calculus, Pacific J. Math., **9** (1959), 129-142.

[9] R.C. Lyndon, Properties preserved under homomorphism, Pacific J. Math., **9** (1959), 143-154.

[10] N. Motohashi, Interpolation theorem and characterization theorem, Ann. Japan Assoc. Philos. Sci., **4** (1972), 85-150.

[11] A. Oberschelp, On the Craig-Lyndon interpolation theorem, Notices of Amer. Math. Soc., **14** (1967), 142.

[12] A. Oberschelp, On the Craig-Lyndon interpolation theorem, J. Symbolic Logic, **33** (1968), 271-274.

[13] A. Robinson, A result on consistency and its application to the theory of definition, Koninkl. Ned. Akad. Wetensch. Proc. Ser. A, **59** (=Indag. Math., **18**) (1956), 47-58.
[14] A. Robinson, Introduction to Model Theory and to the Metamathematics of Algebra, 2nd, revised printing, North-Holland, Amsterdam, 1974.
[15] J.R. Shoenfield, Mathematical Logic, Addison-Wesley, Reading, Massachusetts, 1967.

Tsuyoshi FUJIWARA
Department of Mathematics
Faculty of General Education
University of Osaka Prefecture
Sakai, Osaka, Japan