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Fine movability

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1. Introduction.

The notion of shape for compacta was introduced by K. Borsuk [4]. In a series of papers [5], [6] and [7] Borsuk has defined the concepts of fundamental absolute neighborhood retracts (FANR's), movability and n-movability and has proved that all of these properties are shape invariants.

In this paper we shall introduce the concepts of fine movability and n-fine movability, $n=0, 1, 2, \cdots$, which are shape invariants and define the n-fine movability pro-group $\boldsymbol{m}_n(X, x_0)$ for a pointed compactum (X, x_0). For a compactum X, we shall prove that

(1) X is a pointed FANR, i. e. an FANR having the shape of a CW-complex if and only if X is fine movable,

(2) X is n-fine movable if and only if $\boldsymbol{m}_k(X, x_0)=0$ for $x_0 \in X$ and $k=0, 1, 2, \cdots, n$,

(3) an n-dimensional compactum X is fine movable if and only if $\boldsymbol{m}_k(X, x_0)=0$ for $x_0 \in X$ and $k=0, 1, 2, \cdots, n+1$,

(4) an LC^{n-1} compactum X is n-fine movable,

(5) if X is n-fine movable X is n-movable,

(6) if X_1, X_2 and $X_1 \cap X_2$ are n-fine movable compacta $X_1 \cup X_2$ is n-fine movable.

From (1), (2) and (5) we have the following implications: a pointed FANR $\rightarrow$ an n-fine movable compactum $\rightarrow$ an n-movable compactum. It is known that each of converse implications does not generally hold. S. Mardešić [18] has proved that an n-dimensional LC^{n-1} compactum is movable. The assertions (4) and (5) extend this result. For pointed FANR's or equivalently for fine movable compacta (6) has proved by Dydak, Nowak and Strok [12].

Throughout this paper all spaces are metrizable and maps are continuous. AR and ANR mean those for metric spaces. By $\dim X$ we mean the covering dimension of X.

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2. Fine movability.

Let M and N be spaces and X and Y closed sets of M and N, respectively. A map $f: M-X\to N$ is said to be a *fine map* from $M-X$ into N *rel.* X, Y if for every neighborhood V of Y in N there exists a neighborhood U of X in M such that $f(U-X)\subset V$. (Sometimes Y is not contained in the range of f. If $f(M-X)$ is disjoint from Y, then we often write $f: M-X\to N-Y$ rel. X, Y.) Two fine maps $f, g: M-X\to N$ rel. X, Y are said to be *fine homotopic* (notation: $f\underset{f}{\simeq}g$) if there exists a homotopy $H:(M-X)\times I\to N$ satisfying the following conditions; $H(x, 0)=f(x)$, $H(x, 1)=g(x)$ for $x\in M-X$ and for every neighborhood V of Y in N there exists a neighborhood U of X in M such that $H((U-X)\times I)\subset V$. H is called a *fine homotopy rel.* X, Y connecting f and g.

A closed subset X of a space M is said to be *unstable* in M (Sher [**23**, p. 346]) if there exists a homotopy $H: M\times I\to M$ such that $H(x, 0)=x$ for $x\in M$ and $H(x, t)\in M-X$ for $x\in M$ and $0<t\leqq 1$. It is known by Anderson [**1**] and Chapman [**9**] that every compactum of the Hilbert space l_2 is unstable in l_2 and every Z-set of the Hilbert cube Q is unstable in Q. Also, it is known by [**15**, Theorem 1] that every metric space X is imbedded as an unstable subset into an AR $M(X)$ with $\dim M(X)=\dim X+1$ and $w(M(X))=w(X)$.

LEMMA 1. *A closed set X of a space M is unstable in M if and only if for every open neighborhood V of X in M there exists a homotopy $H: M\times I\to M$ such that*

$$(2.1)\qquad H(V\times I)\subset V,\ H(M\times(0, 1])\subset M-X \quad \textit{and} \quad H(x, t)=x$$
$$\textit{for}\quad (x, t)\in M\times\{0\}\cup(M-V)\times I, \quad \textit{where } (0, 1]=\{t: 0<t\leqq 1\}.$$

PROOF. Since the if part is obvious, we only prove the only if part. Let X be an unstable set of M. There is a homotopy $H': M\times I\to M$ such that $H'(x, 0)=x$ for $x\in M$ and $H'(M\times(0, 1])\subset M-X$. Let V be any open neighborhood of X in M. By the paracompactness of M there exists a map $\alpha: M\to I$ such that $H'(\bigcup_{x\in V}\{x\}\times[0, \alpha(x)])\subset V$ and $M-V=\alpha^{-1}(0)$. Define $r: M\times I\to M\times I$ and $H: M\times I\to M$ by

$$\begin{aligned} r(x, t)&=(x, \alpha(x)) && \text{for } \alpha(x)\leqq t, \\ &=(x, t) && \text{for } \alpha(x)\geqq t, \\ H(x, t)&=H'r(x, t) && \text{for } (x, t)\in M\times I. \end{aligned}$$

It is easy to see that H satisfies (2.1).

LEMMA 2. *Let X be an unstable compact set of M and V a neighborhood of X in M. For every fine map $f: Z-B\to M$ from $Z-B$ to M rel. B, X, where*

Z is a space and B is a closed set of Z, there is a fine homotopy $H:(Z-B)\times I \to M$ rel. B, X such that $H(x, 0)=f(x)$, $H(x, 1)\in M-X$ for $x\in Z-B$ and $H(x, t) =f(x)$ for $x\in(Z-B)\cap f^{-1}(M-V)$ and $t\in I$.

PROOF. By Lemma 1 there exists a homotopy $H':M\times I\to M$ such that $H'(V\times I)\subset V$, $H'(M\times(0, 1])\subset M-X$ and $H'(x, t)=x$ for $(x, t)\in M\times\{0\}\cup(M-V)\times I$. For each $x\in Z$, put $\alpha(x)=\min(\rho(x, B), 1)$, where ρ is a metric in Z. Define a homotopy $H:(Z-B)\times I\to M$ by

$$H(x, t)=H'(f(x), t\cdot\alpha(x)) \qquad \text{for} \quad (x, t)\in(Z-B)\times I.$$

Obviously $H(x, 0)=f(x)$, $H(x, 1)\in M-X$ for $x\in Z-B$ and $H(x, t)=f(x)$ for $x\in (Z-B)\cap f^{-1}(M-V)$ and $t\in I$. To prove that H is a fine homotopy rel. B, X, let U be any neighborhood of X in M. Since X is compact, we can find a $\varepsilon>0$ and a neighborhood U' of X in M such that $H'(U'\times[0, \varepsilon])\subset U$. Since f is a fine map rel. B, X, there exists a neighborhood V' of B in Z such that $f(V'-B)\subset U'$. Put $V''=V'\cap\{x:\alpha(x)<\varepsilon, x\in Z-B\}$. Then we have $H(V''\times I)\subset H'(f(V'')\times[0, \varepsilon])\subset U$. Therefore H is a fine homotopy rel. B, X.

Let X be a compactum and M an AR containing X. Let $n=0, 1, 2, \cdots$. We say that X is *fine movable* (resp. *n-fine movable*) in M if for every neighborhood V of X in M there exists a neighborhood V' of X satisfying the following condition:

(2.2) Let V'' be any neighborhood of X in M, and let Z be a space and B a closed set of Z (resp. with $\dim(Z-B)\leqq n$). For every fine map $f:Z-B\to V'$ rel. B, X there exists a fine homotopy $H:(Z-B)\times I\to V$ rel. B, X such that $H(x, 0)=f(x)$ and $H(x, 1)\in V''$ for $x\in Z-B$.

LEMMA 3. *Let M and M' be* AR's *containing a compactum X. If X is fine movable (resp. n-fine movable) in M, then X is fine movable (resp. n-fine movable) in M'.*

PROOF. We shall prove the lemma only for fine movability. Since M and M' are AR's, there exist maps $\varphi:M\to M'$ and $\psi:M'\to M$ such that $\varphi|X=\psi|X =1_X$, where 1_X is the identity map of X. Let U be a neighborhood of X in M'. By the fine movability of X in M we can find a neighborhood V' of X in M satisfying (2.2) for $V=\varphi^{-1}(U)$. Since M' is an ANR and $\varphi\psi(x)=x$ for $x\in X$, there exist a neighborhood U' of X in M' and a homotopy $H':U'\times I\to U$ such that $U'\subset\psi^{-1}(V')\cap U$, $H'(x, t)=x$ for $(x, t)\in U'\times\{0\}\cup X\times I$ and $H'(x, 1)=\varphi\psi(x)$ for $x\in U'$. Let us prove that U' satisfies (2.2). Let $f:Z-B\to U'$ be a fine map rel. B, X. Take any neighborhood U'' of X in M'. By the definition of V' there is a fine homotopy $H'':(Z-B)\times I\to\varphi^{-1}(U)$ rel. B, X such that $H''(x, 0)= \psi f(x)$ for $x\in Z-B$ and $H''((Z-B)\times\{1\})\subset\varphi^{-1}(U'')$. Define $H:(Z-B)\times I\to U$ by

$$H(x,t)=\begin{cases} H'(f(x),2t), & 0\leqq t\leqq 1/2 \\ =H''(x,2t-1), & 1/2\leqq t\leqq 1 \end{cases}, \quad x\in Z-B.$$

Then $H(x,0)=f(x)$ and $H(x,1)\in U''$ for $x\in Z-B$. Obviously H is a fine homotopy rel. B, X.

By Lemma 3 we can simply define: A compactum X is *fine movable* (resp. *n-fine movable*) if there exists an AR M containing X such that X is fine movable (resp. n-fine movable) in M.

THEOREM 1. *The fine movability and the n-fine movability are shape invariants.*

PROOF. We only prove the theorem for fine movability. Let X and Y be compacta with the same shape. Suppose that X is fine movable. To show that Y is fine movable, let M and N be compact AR's containing X and Y unstably, respectively. Since $\mathrm{Sh}(X)=\mathrm{Sh}(Y)$ there exist proper maps $\xi: M-X\to N-Y$ and $\eta: N-Y\to M-X$ such that $\eta\xi\underset{p}{\simeq}1_{M-X}$ and $\xi\eta\underset{p}{\simeq}1_{N-Y}$, where $\underset{p}{\simeq}$ means properly homotopic. (Cf. [**9**, Theorem 2] or [**17**, Theorem 1 and Corollary 2].) Let U be any neighborhood of Y in N. Take a neighborhood V of X in M such that $\xi(V-X)\subset U$. For the neighborhood V, by the fine movability of X there exists a neighborhood V' of X in M satisfying the condition (2.2). Since $\xi\eta\underset{p}{\simeq}1_{N-Y}$, we can find a neighborhood U' of Y in N such that $\eta(U'-Y)\subset V'$ and $\xi\eta|(U-Y)\underset{p}{\simeq}i_{U'-Y}$ in $U-Y$, where $i_{U'-Y}$ is the inclusion: $U'-Y\to U-Y$. To prove that U' satisfies (2.2) for the neighborhood U, let U'' be any neighborhood of Y and let $f: Z-B\to U'$ rel. B, Y be a fine map. By Lemma 2 we can assume that $f(Z-B)\subset U'-Y$. Choose a neighborhood V'' of X in M such that $\xi(V''-X)\subset U''$. By the definition of V', (2.2) and Lemma 2, there exists a fine homotopy $H': (V'-X)\times I\to V-X$ rel. X, X such that $H'(x,0)=x$ and $H'(x,1)\in V''-X$ for every $x\in V'-X$. Define $H: (Z-B)\times I\to U$ by

$$H(x,t)=\xi H'(\eta f(x),t), \qquad (x,t)\in(Z-B)\times I.$$

Then $H(x,0)=\xi\eta f(x)$ and $H(x,1)\in\xi H'((V'-X)\times\{1\})\subset\xi(V''-X)\subset U''$ for each $x\in Z-B$. Obviously H is a fine homotopy rel. B, Y. Since $\xi\eta f\underset{p}{\simeq}f$ in $U-X$, this completes the proof.

In [**17**] we have defined the fine shape $\mathrm{Sh}_f(X)$ for a compactum X. Let us remind its definition. Let X, Y be compacta and M, N AR's containing X, Y respectively. Consider the subspaces $Z=M\times[0,1)\cup X\times\{1\}$ and $Z'=N\times[0,1)\cup Y\times\{1\}$ of $M\times I$ and $N\times I$, respectively. Suppose that there exist fine maps $F: Z-X\times\{1\}\to N$ rel. $X\times\{1\}, Y$ and $G: Z'-Y\times\{1\}\to M$ rel. $Y\times\{1\}, X$ such that

(2.3) $\qquad G*F\underset{f}{\simeq}p_M$ rel. $X\times\{1\}, Y$ and $F*G\underset{f}{\simeq}p_N$ rel. $Y\times\{1\}, X$, where $G*F: Z-X\times\{1\}\to M$ and $F*G: Z'-Y\times\{1\}\to N$ are defined by $G*F(x,t)=G(F(x,t),t)$ and $F*G(y,t)=F(G(y,t),t)$, and $p_M: Z-X\times\{1\}=M\times[0,1)\to M$ and $p_N: Z'-Y\times\{1\}=N\times[0,1)\to N$ are the projections. Then we say that X and Y has the *same fine shape* and we write $\mathrm{Sh}_f(X)=\mathrm{Sh}_f(Y)$. If only the first relation in (2.3) is true, then we say that Y is *fine dominate* X and we write $\mathrm{Sh}_f(Y)\geqq \mathrm{Sh}_f(X)$. The following has proved in [**17**, Theorem 1].

(2.4) $\mathrm{Sh}_f(X)\geqq\mathrm{Sh}_f(Y)$ *if and only if, for any compact* AR's M *and* N *containing unstably* X *and* Y *respectively, there exist proper maps* $f: M-X\to N-Y$ *and* $g: N-Y\to M-X$ *such that* $fg\underset{p}{\simeq}1_{N-Y}$.

By (2.4) and the proof of Theorem 1, we have

COROLLARY 1. *The fine movability and the n-fine movability are hereditarily fine shape properties* (cf. Borsuk [8, p. 348]).

LEMMA 4. *Every compactum is* 0-*fine movable.*

PROOF. Let X be a compactum. By Theorem 1 of [**15**] there exist a compact AR $M(X)$ such that X is an unstable set of $M(X)$ and $P=M(X)-X$ is an infinite simplicial polytope. Let V be a neighborhood of X in $M(X)$. Since $M(X)$ is an ANR, we can find a neighborhood V' of X and a fine homotopy $H: (P^0\cap V')\times I\to V$ rel. X, X such that $H(x,0)=x$ and $H(x,1)\in X$ for $x\in P^0\cap V'$, where P^0 is the set of the vertices of P. Obviously the neighborhood V' satisfies the condition (2.2) of the 0-fine movability for V.

Following Borsuk [8, p. 204] and Dydak [**13**], a compactum X is said to be a *pointed* FANR if for every point $x\in X$ (X,x) is an FANR in the pointed shape category whose objects are pointed compacta and whose morphisms are pointed F-sequences. By [8, Chap. VIII, (1.5)], [**14**, Theorem 1.1] and [**24**, Théorème 5.8] X is a pointed FANR if and only if X is an FANR and X has the shape of a CW-complex (cf. Dydak [**13**]).

THEOREM 2. *A compactum* X *is a pointed* FANR *if and only if* X *is fine movable.*

PROOF. The only if part follows immediately from [**24**, Théorème 5.8] and Lemma 3. Conversely, suppose X is fine movable. Imbed X into the Hilbert space l_2. Let V be a neighborhood of X in l_2. We shall prove that there exists a neighborhood V' of X in l_2 satisfying the following condition:

(2.5) For every neighborhood V'' of X there exists a homotopy $H: V'\times I\to V$ such that $H(x,t)=x$ for $(x,t)\in V'\times\{0\}\cup X\times I$ and $H(x,1)\in V''$ for $x\in V'$.

Obviously, since the existence of V' implies that X is a pointed FANR, it is enough to find V' satisfying (2.5). By the fine movability of X there exists a neighborhood V' of X satisfying (2.2) for the neighborhood V. We claim that

V' satisfies (2.5), too. To prove it, let V'' be any neighborhood of X. By (2.2) there exists a fine homotopy $H' : (V'-X)\times I \to V$ rel. X, X such that $H'(x, 0) = x$ and $H'(x, 1)\in V''$ for every $x\in V'-X$. By Lemma 2 we can assume that $H'((V'-X)\times I)\subset V-X$. Since H' is a fine homotopy rel. X, X, we can find an open neighborhood W of X such that $W\subset V''$ and $H'((W-X)\times I)\subset V''-X$. Since W is an l_2-manifold, by the negligibility of X in W [2, Theorem 5], there exists an onto homeomorphism $\varphi : l_2\to l_2-X$ such that $\varphi | l_2-W=1_{l_2-W}$ and $\varphi(W) = W-X$. Define $H'' : V'\times I\to V$ by

$$H''(x, t)=\varphi^{-1}H'(\varphi(x), t) \qquad \text{for} \quad (x, t)\in V'\times I .$$

Then

$$H''(x, 0)=x \quad \text{and} \quad H''(x, 1)\in V'' , \qquad x\in V' , \tag{2.6}$$

$$H''(X\times I)\subset W .$$

Now, consider the product space $V'\times I\times J$, where $J=\{s : 0\leqq s\leqq 1\}$, and its subset $K=V'\times I\times \{0\}\cup V'\times(\{0\}\cup\{1\})\times J\cup X\times I\times J$. We shall define a map $\xi : K\to V$ such that

$$\xi(x, t, 0)=H''(x, t) , \qquad (x, t)\in V'\times I , \tag{2.7}$$

$$\xi(x, 0, s)=x \quad \text{and} \quad \xi(x, 1, s)\in W , \qquad (x, s)\in V'\times J ,$$

$$\xi(x, t, 1)=x , \qquad (x, t)\in X\times I .$$

To construct a homotopy H satisfying (2.5), by the homotopy extension theorem, extend ξ to a map $\xi' : V'\times I\times J\to V$. If we define $H : V'\times I\to V$ by $H(x, t)= \xi'(x, t, 1)$, $(x, t)\in V'\times I$, then the homotopy H is a required one. Thus it remains to construct a map ξ satisfying (2.7). Define η on the set $V'\times I\times \{0\}\cup X\times \{1\} \times J\cup V'\times \{0\}\times J$ by

$$\eta(x, t, 0)=H''(x, t) , \qquad (x, t)\in V'\times J ,$$

$$\eta(x, 1, s)=H''(x, 1-s) , \qquad (x, t)\in X\times J ,$$

$$\eta(x, 0, s)=x , \qquad (x, s)\in V'\times J .$$

Since $\eta(V'\times \{1\}\times \{0\}\cup X\times \{1\}\times J)\subset W$, by the homotopy extension theorem, $\eta | V'\times \{1\}\times \{0\}\cup X\times \{1\}\times J$ is extended to a map $\eta' : V'\times \{1\}\times J\to W$. By the definition of η on the set $X\times I\times \{0\}\cup X\times \{1\}\times J$, there is a map $\eta'' : X\times I\times J\to X$ such that $\eta''(x, t, 0)=\eta(x, t, 0)$, $\eta''(x, 1, s)=\eta(x, 1, s)$ and $\eta''(x, 0, s)=\eta''(x, t, 1)=x$. Define $\xi : V'\times I\times J\to V$ by $\xi=\eta$ on $V'\times I\times \{0\}\cup V'\times \{0\}\times J$, $\xi=\eta'$ on $V'\times \{1\}\times J$ and $\xi=\eta''$ on $X\times I\times J$. Obviously ξ satisfies (2.7). This completes the proof of the theorem.

THEOREM 3. *An n-dimensional compactum X is a pointed* FANR *if and only* *if X is $(n+1)$-fine movable.*

PROOF. Let $\dim X=n$. By [**15**, Theorem 1] there exists a compact AR $M(X)$ such that X is an unstable subset of $M(X)$ and $\dim M(X)=n+1$. If X is $(n+1)$-fine movable in $M(X)$, it is obvious that X is fine movable in $M(X)$, because $\dim(M(X)-X)\leqq n+1$. By Theorems 1 and 2 X is a pointed FANR.

3. Pro-groups $\boldsymbol{m}_k(X, x_0)$.

In this section we shall define progroups $\boldsymbol{m}_k(X, x_0)$ satisfying (2) and (3) in Introduction. The following notations are used throughout the section.

(3.1)
$$I^k=\{(x_1, \cdots, x_k): 0\leqq x_i\leqq 1,\ i=1, \cdots, k\};$$
$$J^{k-1}=\{(x_1, \cdots, x_k)\in I^k: x_k=1 \text{ or } \prod_{i=1}^{k-1} x_i(1-x_i)=0\};$$
$$I_0^k=I^k-J^{k-1};$$
$$J_0^{k-1}=J^{k-1}\cap I_0^k; \qquad k=1, 2, 3, \cdots.$$

Let (M, X) be a pair of spaces and $x_0\in X$. Consider the set $F_k(M, X, x_0)$ of all fine maps $f: I^k-I^{k-1}=I_0^k\to M$ rel. I^{k-1}, X such that $f(J_0^{k-1})=x_0$. In the set $F_k(M, X, x_0)$ we define the relation "$\equiv$" as follows:

(3.2) For $f, g\in F_k(M, X, x_0)$, $f\equiv g$ if and only if there exists a fine homotopy $H: I_0^k\times I\to M$ rel. I^{k-1}, X such that $H(x, 0)=f(x)$, $H(x, 1)=g(x)$ for $x\in I_0^k$ and $H(J_0^{k-1}\times I)=x_0$.

Obviously the relation $\equiv$ is an equivalence relation in $F_k(M, X, x_0)$. By $\mu_k(M, X, x_0)$ we denote the set of the equivalence classes. For $k\geqq 2$, a usual group structure is introduced in $\mu_k(M, X, x_0)$; namely, for $f, g\in F_k(M, X, x_0)$, define

(3.3)
$$f\cdot g(x_1, \cdots, x_k)=f(2x_1, x_2, \cdots, x_k), \qquad 0\leqq x_1\leqq 1/2,$$
$$=g(2x_1-1, x_2, \cdots, x_k), \qquad 1/2\leqq x_1\leqq 1.$$

The operation in (3.3) induces a group structure in $\mu_k(M, X, x_0)$. For $k\geqq 2$, by $\mu_k(M, X, x_0)$ we mean the group with this structure. For $k=1$ $\mu_1(M, X, x_0)$ is a set. We call $\mu_k(M, X, x_0)$, $k=1, 2, \cdots$, the k-dimensional *fine group* of (M, X, x_0). Obviously

(3.4) if $k\geqq 3$ then $\mu_k(M, X, x_0)$ is abelian.

Let X be an unstable compact subset of M. For a pair (N, Y) of spaces and $y_0\in Y$, suppose that there is a map $f: (M-X)\cup\{x_0\}\to N$ such that $f|M-X$

is a fine map from $M-X$ to N rel. X, Y and $f(x_0)=y_0$. Then a homomorphism $f_*: \mu_k(M, X, x_0)\to\mu_k(N, Y, y_0)$ is defined as follows. For any $g\in F_k(M, X, x^0)$, by the proof of Lemma 2 there exists a $g'\in F_k(M, X, x_0)$ such that $g\equiv g'$ and $g'(I_0^k)\subset(M-X)\cup\{x_0\}$. Define $f_*: \mu_k(M, X, x_0)\to\mu_k(N, Y, y_0)$ by

$$f_*([g])=[fg'],$$

where $[h]$ means the equivalence class containing h. Obviously f_* is a homomorphism for $k\geqq 2$. We call f_* the homomorphism *induced by f.*

Let X be a compactum and M an AR containing X unstably. Choose a neighborhood basis $\{M_i: i=1, 2, \cdots\}$ of X in M such that $M_{i+1}\subset M_i$, $i=1, 2, \cdots$. For each i, the inclusion map $\psi^i: M_{i+1}\to M_i$ induces the homomorphism $\psi_*^i: \mu_k(M_{i+1}, X, x_0)\to\mu_k(M_i, X, x_0)$, $k=1, 2, \cdots$, where $x_0\in X$.

Let pro $(\mathcal{G})$ and pro $(\mathcal{E}_0)$ be the pro-categories of groups and pointed-sets indexed by the directed set of positive integers, respectively. (See Mardešić [**19**; 2, 3, 4] and Moszyńska [**21**] for the definition of the pro-category.) Denote $\{\mu_k(M_i, X, x_0), \psi_*^i: i\in J\}$ by $\boldsymbol{m}_k(X, x_0: M, \{M_i\})$. Then $\boldsymbol{m}_k(X, x_0: M, \{M_i\})$ is an object of pro $(\mathcal{G})$ for $k\geqq 2$ and an object of pro $(\mathcal{E}_0)$ for $k=1$. If M' is another AR containing X unstably and $\{M_i'\}$ is a neighborhood basis of X in M' such that $M_{i+1}'\subset M_i'$, then it is obvious that $\boldsymbol{m}_k(X, x_0: M, \{M_i\})$ and $\boldsymbol{m}_k(X, x_0: M', \{M_i'\})$ are isomorphic. Thus we can define the pro-group $\boldsymbol{m}_k(X, x_0)$ up to isomorphism ($\boldsymbol{m}_1(X, x_0)$ is the pro-set). For a convenience we set $\boldsymbol{m}_0(X, x_0)=0$. Here 0 is a zero object in pro $(\mathcal{G})$ or pro $(\mathcal{E}_0)$, We call $\boldsymbol{m}_k(X, x_0)$ the *k-fine movability pro-group* of (X, x_0).

Finally, we give an alternate description of $\mu_k(M, X, x_0)$ which is convenient for use. Let us use the following notations.

(3.5)
$$E^k=\{(x_1, \cdots, x_k): \sum_{i=1}^k x_i^2\leqq 1,\ -1\leqq x_i\leqq 1,\ i=1, 2, \cdots, k\};$$
$$E_0^k=\{(x_1, \cdots, x_k)\in E^k: x_k>0\};$$
$$S^k=\{(x_1, \cdots, x_{k+1}): \sum_{i=1}^{k+1} x_i^2=1\};$$
$$S_0^k=\{(x_1, \cdots, x_{k+1})\in S^k: x_{k+1}>0\};$$
$$S_1^k=\{(x_1, \cdots, x_{k+1})\in S^k: x_{k+1}\geqq 0\};$$
$$l_k=\{(x_1, 0, \cdots, 0, x_{k+1})\in S_0^k: x_1\geqq 0\};\ k=1, 2, \cdots.$$

For each k, there exists a fine map $h: I^k-I^{k-1} (=I_0^k)\to S_1^k-S^{k-1} (=S_0^k)$ rel. I^{k-1}, S^{k-1} such that $h(J_0^{k-1})=l_k$ and $h|I_0^k-J_0^{k-1}: I_0^k-J_0^{k-1}\to S_0^{k-1}-l_k$ is an onto homeomorphism. (See (3.1) for notations.) Thus every element of the group $\mu_k(M, X, x_0)$ is represented by some fine map $f: S_0^k\to M$ rel. S^{k-1}, X such that $f(l_k)=x_0$.

LEMMA 5. *An element* $\alpha \in \mu_k(M, X, x_0)$ *is zero if and only if a fine map* $f: S_0^k \to M$ *rel.* S^{k-1}, X *representing* α *is extended to a fine map* $F: E_0^{k+1} \to M$ *rel.* E^k, X.

The proof is obvious.

4. m-fine movability and m-movability.

LEMMA 6. *Let* X *be a compactum. If* $\boldsymbol{m}_1(X, x_0)=0$ *for each* $x_0 \in X$, *then* X *has a finite number of components.*

PROOF. Let M be an AR containing X. Suppose that X has an infinite number of components. There is a component C_0 of X such that

(4.1) every neighborhood of C_0 in M contains a component C of X different from C_0.

Let $x_0 \in C_0$. We shall prove that $\boldsymbol{m}_1(X, x_0) \neq 0$. Let V be any neighborhood of X and V_0 the component of V containing C_0. By (4.1) there exists a component C of X such that $C \neq C_0$ and $C_0 \cup C \subset V_0$. Let $f: (0, 1] \to V_0$ be a map such that $f(1)=x_0$ and $f((0, 1/2])$ is a point of C. Then f is a fine map rel. $\{0\}$, X and represents a non zero element of $\mu_1(V, X, x_0)$. Suppose that $\phi_*([f])=0$, where $\phi_*: \mu_1(V, X, x_0) \to \mu_1(M, X, x_0)$ is induced by the inclusion $\phi: V \subset M$. This means that there exists a fine homotopy $H: (0, 1] \times I \to M$ rel. $\{0\}$, X such that $H(x, 0)=f(x)$ for $x \in (0, 1]$ and $H(\{1\} \times I \cup (0, 1] \times \{1\})=x_0$. Since there is a neighborhood V' of X in M such that V' is a union of disjoint open sets V_1 and V_2, $V_1 \supset C_0$ and $V_2 \supset C$, this contradicts the connectedness of $(0, 1]$. Thus $\boldsymbol{m}_1(X, x_0) \neq 0$. This completes the proof.

THEOREM 4. *A compactum* X *is* n-*fine movable if and only if* $\boldsymbol{m}_k(X, x_0)=0$ *for* $k=0, 1, 2, \cdots, n$ *and* $x_0 \in X$.

PROOF. For $n=0$, the theorem follows from Lemma 4 and the definition of $\boldsymbol{m}_0(X, x_0)$. Let $n>0$ and let M be an AR containing M unstably. For neighborhoods V and V' of X in M, we write $V' \underset{n}{\subset} V$ if V' satisfy the condition (2.2) of the n-fine movability for V.

Suppose that X is n-fine movable. Let M_0 be any neighborhood of X in M. Choose a neighborhood basis $\{M_i : i=1, 2, \cdots\}$ of X such that $M_i \underset{n}{\subset} M_{i-1}$, $i=1, 2, \cdots$. We shall prove that the homomorphism $\phi_*: \mu_k(M_1, X, x_0) \to \mu_k(M_0, X, x_0)$ is zero for each $k \leqq n$ and $x_0 \in X$, where ϕ is the inclusion of M_1 into M_0. Obviously this shows that $\boldsymbol{m}_k(X, x_0)=0$ for $k \leqq n$. Let $f: S_0^k \to M_1$ be a fine map rel. S^{k-1}, X such that $f(l_k)=x_0$. Since $M_i \underset{n}{\subset} M_{i-1}$, $i \geqq 1$, and $n \geqq k$, there exist homotopies $\xi_i: S_0^k \times I \to M_i$ rel. S^{k-1}, X such that $\xi_0(x, 0)=f(x)$, $\xi_i(x, 0)=\xi_{i-1}(x, 1) \in M_{i+1}$ for $x \in S_0^k$ and $i=1, 2, \cdots$. Let R_0 be the space of non negative reals. Define a map $\xi: S_0^k \times R_0 \to M_0$ by

$$\xi(x, t)=\xi_i(x, i+1-t) \qquad \text{for } x \in S_0^k \text{ and } i \leqq t \leqq i+1,\ i=0, 1, 2, \cdots.$$

By the definition of ξ, for any neighborhood V of X in M there exist an $r\in R_0$ and a neighborhood W of S^{k-1} in S_1^k such that

$$\xi(W\times R_0\cup S_0^k\times(r,\infty))\subset V. \tag{4.2}$$

Since there is a homeomorphism onto $\varphi: E_0^{k+1}\to S_0^k\times R_0$ such that $\varphi(x)=(x,0)$ for $x\in S_0^k$, the existence of the map ξ satisfying (4.2) implies that the map f is extended to a fine map: $E_0^{k+1}\to M_0$ rel. E_0^k, X. By Lemma 5, this means that $\phi_*([f])=0$. Thus ϕ_* is zero.

Conversely, assume that $\boldsymbol{m}_k(X, x_0)=0$ for $k\leqq n$ and $x_0\in X$. Since $n\geqq 1$, by Lemma 6 X has a finite number of components. Hence we can assume that X is connected. Let M be a compact AR containing X unstably. Fix a point $x_0\in X$. For neighborhoods W and W' of X in M we write $W'\underset{n}{\ll} W$ if $W'\subset W$ and $\phi_*: \mu_k(W', X, x_0)\to\mu_k(W, X, x_0)$ is zero for $k=1, 2, \cdots, n$, where ϕ is the inclusion map of W' into W. Let V be any open neighborhood of X in M. Since X is connected, there exists an open neighborhood basis $\{V_i : i=0, 1, \cdots\}$ of X such that $V=V_0$, $V_{i+1}\underset{n}{\ll} V_i$ and each V_i is connected. We shall prove that the condition (2.2) of the n-fine movability is satisfied by V and $V'=V_{n+1}$. To do it, let $f: Z-B\to V_{n+1}$ be a fine map rel. B, X, where Z is a space, B is closed in Z and $\dim(Z-B)\leqq n$. Without loss of generality, by Lemma 2 it can be assume that

$$f(Z-B)\subset V_{n+1}-X. \tag{4.3}$$

Also, by the proof of Theorem 3.1 of Dugundji [**11**], we can assume that $P=Z-B$ is an n-dimensional locally finite simplicial polytope satisfying the following;

(4.4) P is open in $Z=P\cup B$ and there is a sequence $\{\varepsilon_i\}$ of positive numbers such that $\varepsilon_i\to 0$ $(i\to\infty)$ and if for a simplex s of P $\rho(s, B)<1/i$ then diameter of $s<\varepsilon_i$, where ρ is a metric in Z.

We shall construct a map $\xi: P\times[0,1)\to V$ satisfying the condition;

(4.5) $\xi(x,0)=f(x)$ for $x\in P$, and for any neighborhood W of X in M there exist a closed neighborhood U of B in Z and a map α: $P\to[0,1)$ such that $\alpha^{-1}(0)\cup B=U$ and $\xi(x,\alpha(x))\in W$ for $x\in P$.

At first, suppose that a map ξ is constructed. Take any neighborhood V'' of X in M. There exists a neighborhood U of B in Z and a map $\alpha: P\to[0,1)$ satisfying (4.5) for $W=V''$. Define $H: P\times I\to V$ by

$$H(x,t)=\xi(x, t\cdot\alpha(x)) \qquad \text{for } x\in P \text{ and } t\in I.$$

Then $H(x,0)=f(x)$ and $H(x,1)\in V''$ for $x\in X$. It is obvious that H is a fine homotopy rel. B, X. This shows that X is n-fine movable. Thus it remains to construct a map ξ in (4.5).

For each simplex s of P, put $p(s)=\max\{i : f(s)\subset V_i\}$. By (4.3) $p(s)<\infty$ for every simplex s. Denote by P^k the k-skeleton of P. For $x\in P^0$, since $V_{p(x)}$ is connected, there is a map $f_x : \{x\}\times[0, 1)\to V_{p(x)}$ such that $f_x(x, 0)=f(x)$ and $f_x([1/2, 1))=x_0$. Define $\xi_0 : P\times\{0\}\cup P^0\times[0, 1)\to V_n$ by

$$\xi_0(x, 0)=f(x)\,, \qquad x\in P\,,$$

$$\xi_0(x, t)=f_x(x, t)\,, \qquad x\in P^0 \text{ and } t_0\in[0, 1)\,.$$

For some $m<n$, suppose that for each $k\leqq m$, the map $\xi_k : P\times\{0\}\cup P^k\times[0, 1)\to V_{n-k}$ such that

(4.6) $\xi_k|P\times\{0\}\cup P^{k-1}\times[0, 1)=\xi_{k-1}$, where $\xi_{-1}=f$, if s is a k-simplex of P then $\xi_k|s\times[0, 1) : s\times[0, 1)\to V_{p(s)-k}$ is a fine map rel. $s\times\{1\}, S$.

Let s be an $(m+1)$-simplex of P. Consider a map $f'_s : s\times\{0\}\cup\partial s\times[0, 1)\to V_{p(s)-m}$ defined by

$$f'_s(x, 0)=f(x) \qquad \text{for} \quad x\in s\,,$$

$$f'_s(x, t)=\xi_m(x, t) \qquad \text{for} \quad (x, t)\in\partial s\times[0, 1)\,,$$

where ∂s is the boundary of s. Then f'_s is a fine map rel. $\partial s\times\{1\}, X$. Since $\phi_* : \mu_m(V_{p(s)-m}, X, x_0)\to\mu_m(V_{p(s)-m-1}, X, x_0)$ is zero, f'_s is extended to a fine map $f_s : s\times[0, 1)\to V_{p(s)-m-1}$ rel. $s\times\{1\}, X$ by Lemma 5. Define $\xi_{m+1} : P\times\{0\}\cup P^{m+1}\times[0, 1)\to V_{n-m-1}$ by

$$\xi_{m+1}(x, 0)=f(x)\,, \qquad x\in P\,,$$

$$\xi_{m+1}(x, t)=f_s(x, t)\,, \qquad x\in s\in P^{m+1} \text{ and } t\in[0, 1)\,.$$

Obviously (4.6) is satisfied for $k=m+1$. Thus, for each k, ξ_k satisfying (4.6) has been constructed. Put $\xi=\xi_m$. By the construction of ξ, for every neighborhood W of X and for every simplex s of P, there is $t_s\in[0, 1)$ such that $\xi(s\times[t_s, 1))\subset W$. Thus, for a given neighborhood W of X, it follows from the paracompactness of P that there are a map $\alpha : P\to[0, 1)$ and a neighborhood U of X satisfying (4.5). This completes the proof.

From Theorems 2, 3 and 4, and the proof of Theorem 4 the following corollaries are obtained.

COROLLARY 2. *A finite dimensional compactum X is a pointed* FANR *if and only if $\boldsymbol{m}_k(X, x_0)=0$ for $k=0, 1, 2, \cdots$ and $x_0\in X$.*

COROLLARY 3. *A continuum X is n-fine movable if and only if for some* *point* x_0 *of* X $\boldsymbol{m}_k(X, x_0)=0$ *for* $k=0, 1, \cdots, n$.

COROLLARY 4. *An n-dimensional continuum X is a pointed* FANR *if and only if for some point* $x_0 \in X$ $\boldsymbol{m}_k(X, x_0)=0$ *for* $k=0, 1, \cdots, n+1$.

Next, we shall investigate the connection between n-fine movability, LC^{n-1} and n-movability in the sense of Borsuk [**7**].

THEOREM 5. *Every* LC^{n-1} *compactum is n-fine movable.*

PROOF. Let X be an LC^{n-1} compactum. By $M(X)$ denote a compact AR containing X unstably which is constructed in [**15**, Theorem 1]. Then $P=M(X)-X$ is a locally finite simplicial polytope. Let P^n be the n-skeleton of P. Since X is LC^{n-1}, there exist a neighborhood W of X in $M(X)$ and a retraction $r: X \cup (W \cap P^n) \to X$ satisfying the following:

(4.7) there exists a sequence $\{\varepsilon_i\}$ of positive numbers such that $\varepsilon_i \to 0$ $(i \to \infty)$ and if s is a simplex of P^n and $\rho(s, X) < 1/i$ then the diameter $s \cup r(s) < \varepsilon_i$, where ρ is a metric in $M(X)$.

Let V be any neighborhood of X in $M(X)$. Since $M(X)$ is an ANR, by (4.7) there exist a neighborhood V' of X, $V' \subset V \cap W$, and a homotopy $H: ((V' \cap P^n) \cup X) \times I \to V$ satisfying the following:

(4.8) $H(x, 0)=x$, $H(x, 1)=r(x)$ for $x \in (V' \cap P^n) \cup X$, there exists a sequence $\{\delta_i\}$ of positive numbers such that $\delta_i \to 0$ $(i \to \infty)$ and if s is a simplex of P^n and $\rho(s, X) < 1/i$ then diameter $H((s \cap V') \times I) < \delta_i$.

Then it is easy to prove that the condition (2.2) of the n-fine movability is satisfied for V and V'. This completes the proof.

Theorem 5 extends [**18**, Theorem 1] and [**7**, Theorem 4.1].

THEOREM 6. *Every n-fine movable compactum is m-movable.*

PROOF. Let X be an n-fine movable compactum. We use the same notations as in the proof of Theorem 5. For a given neighborhood V of X in $M(X)$, we must find a neighborhood V' of X satisfying the following;

(4.9) for any neighborhood V'' and for a compactum C of V' with $\dim C \leqq n$ there exists a homotopy $H: C \times I \to V$ such that $H(x, 0)=x$ and $H(x, 1) \in V''$ for $x \in C$.

Let us show that a neighborhood V' satisfying the condition (2.2) of the n-fine movability is a required one. By (2.2) there is a fine homotopy $H': (P^n \cap V') \times I \to V$ rel. X, X such that $H'(x, 0)=x$ and $H'(x, 1) \in V''$ for $x \in P^n \cap V'$. From the construction of $M(X)$, we can assume that $V'-X$ is a subpolytope of P. Since every compactum C of V' with $\dim C \leqq n$ is deformable to $(V'-X) \cap P^n$ in V', by composing this deformation and the homotopy H' we find a homotopy satisfying (4.9). This completes the proof.

Finally, we shall prove a sum theorem of n-fine movable compacta.

THEOERM 7. *Let X_1 and X_2 be n-fine movable compacta such that $X_1\cap X_2$ is n-fine movable. Then $X=X_1\cup X_2$ is n-fine movable.*

PROOF. By [**15**, Theorem 1], construct a compact AR $M(X)$ containing X unstably such that $K=M(X)-X$ is an infinite polytope. Since X_1 and X_2 are closed in X, by the construction of $M(X)$, there exist compact AR's $M(X_1)$ and $M(X_2)$ such that $M(X_1)\cup M(X_2)=M(X)$, $M(X_1)\cap M(X_2)$ is an AR, $M(X_i)\cap X=X_i$ and $M(X_i)-X_i$ is a subpolytope of P for $i=1, 2$. Put $K_i=K\cap M(X_i)$, $i=1, 2$. Let V be any neighborhood of X in $M(X)$. By using the same argument as the proof of the if part of Theorem 4 we can show that there exists a neighborhood V' of X in $M(X)$ satisfying the following;

(4.10) if Z is a space, B is a closed set such that $P=Z-B$ is an n-dimensional infinite polytope satisfying (4.4) and if $f: Z-B\to V'-X$ is a fine map rel. B, X, then there is a map $\xi: P\times[0, 1)\to V$ such that ξ satisfies (4.5) and $\xi(f^{-1}(V'\cap K_i)\times[0, 1))\subset V\cap K_i$, $i=1, 2$.

The map ξ is constructed inductively for the skeletons of K_i, $i=1, 2$, and $K_1\cap K_2$ by making use of the n-fine movability of X_i, $i=1, 2$, and $X_1\cap X_2$. Finally, the n-fine movability of X is proved by using the map ξ (cf. the proof of the if part of Theorem 4).

COROLLARY 5. *Let X be an n-dimensional compactum which is a union of compacta X_1 and X_2. If X_1, X_2 and $X_1\cap X_2$ are n-fine movable, then X is movable.*

This is a consequence of Theorems 6 and 7, and [**16**, Theorem 3].

The following theorem was proved by J. Ono in case X is an FANR. Also K. Sakai [**22**, 6-4] proved it in case X is an MANR and $X_1\cap X_2$ is an MAR in the sense of Godlewski. Our proof is similar to them.

THEOREM 8. *Let X be a compactum which is a union of compacta X_1 and X_2. If X is n-fine movable and $X_1\cap X_2$ is an* FAR, *then X_1 and X_2 are n-fine movable.*

PROOF. Let $M(X)$ be a compact AR containing X unstably constructed in [**15**, Theorem 1]. Denote by $M(X_i)$, $i=1, 2$, and $M(X_1\cap X_2)$ the compact AR's in $M(X)$ corresponding to X_i, $i=1, 2$, and $X_1\cap X_2$, respectively, that is, $M(X_1)\cup M(X_2)=M(X)$, $M(X_1)\cap M(X_2)=M(X_1\cap X_2)$ and $M(X_i)\cap X=X_i$ for $i=1, 2$. Put $M=M(X)/M(X_1\cap X_2)$ and $M_i=M(X_i)/M(X_1\cap X_2)$, $i=1, 2$. By $*$ denote the point corresponding to $M(X_1\cap X_2)$. Each of M and M_i, $i=1, 2$, is a compact AR and M is a one point union of M_1 and M_2 and $M_1\cap M_2=\{*\}$. Let $\varphi: M(X)\to M$ be the quotient map. Set $X'=\varphi(X)$ and $X_i'=\varphi(X_i)$, $i=1, 2$. Since $X'=$

$X/X_1 \cap X_2$, $X_i' = X_i/X_1 \cap X_2$, $i=1, 2$, and $X_1 \cap X_2$ is an FAR, $\mathrm{Sh}(X) = \mathrm{Sh}(X')$ and $\mathrm{Sh}(X_i) = \mathrm{Sh}(X_i')$, $i=1, 2$. Thus it is enough to prove that X_i', $i=1, 2$, is n-fine movable (cf. Theorem 1). We give the proof only for X_1'. Let V be any neighborhood of X_1' in M_1'. Since X' is n-fine movable, there exists a neighborhood W of X' in M' satisfying the condition (2.2) of the n-fine movability for the neighborhood $V \cup M_2'$. Put $V' = W \cap M_1'$. Then V' is a neighborhood of X' in M_1'. Since $M_1' \cap M_2' = \{*\}$, there is a retraction $r: M' \to M_1'$ such that $r(M_2') = \{*\}$. By using the retraction r, it is easy to see that the neighborhood V' of X_1' in M_1' satisfies the condition (2.2) for V. Thus X_1' is n-fine movable. This completes the proof.

It is not known whether Theorem 8 holds in case $X_1 \cap X_2$ is n-fine movable:

PROBLEM. Let X be a union of compacta X_1 and X_2. If X and $X_1 \cap X_2$ are n-fine movable (resp. fine movable), then are X_1 and X_2 n-fine movable (resp. fine movable)?

REMARK. Since the n-fine movability is a shape property, all statements for finite dimensionality can be replaced by finite fundamental dimensionality in the sense of Borsuk [8, p. 227]. For example, Theorem 3 is generalized slightly as follows.

THEOREM 3'. *A compactum X with* $\mathrm{Fd}\, X \leqq n$ *is a pointed* FANR *if and only if X is $(n+1)$-fine movable, where* $\mathrm{Fd}\, X$ *is the fundamental dimension of X.*

By Theorems 2, 5 and 6 we have the following implications.

$$\begin{array}{ccc} \text{pointed FANR} & \longleftrightarrow & \text{fine movability} \\ & & \downarrow \\ \mathrm{LC}^{n-1} \text{ compactum} & \longrightarrow & n\text{-fine movability} \\ & & \downarrow \\ & & n\text{-movability} \end{array}$$

Each of the converse implications does not hold. Also, it is known by the following examples that the hypothesis of finite dimensionality in Theorem 3 or Corollaries 2 and 4 is necessary and the n-fine movability in Theorem 7 or Corollary 5 can not be replaced by the movability.

EXAMPLES. (1) An infinite 0-dimensional compactum is movable by [**20**] but not 1-fine movable.

(2) Let X be a dyadic solenoid. Imbed X into a compact AR $M(X)$ constructed in [**15**, Theorem 1]. We use the same notations as in the proof of it. Let $x_0 \in X$ and consider the subset $X_0 = X \cup (\bigcup_{n=1}^{\infty} K_n) \cup J$, where J is an arc in $M(X)$ connecting x_0 and a point of K_1. Obviously X_0 is a movable continuum. However it is easy to see that $\boldsymbol{m}_1(X_0, x_0) = 0$ and $\boldsymbol{m}_2(X_0, x_0) \neq 0$. Thus X_0 is 1-fine movable but not 2-fine movable.

(3) Let $X=\bigvee_{i=1}^{\infty} S_i$ be a continuum which is a one point union of a countable number of copies S_i of a 1-sphere. Then X is LC^0 and 1-dimensional. However, for every point x_0 of X $\boldsymbol{m}_1(X, x_0)=0$ and $\boldsymbol{m}_2(X, x_0)\neq 0$. Thus X is not 2-fine movable.

(4) Let X be the continuum constructed by Borsuk [3, p. 124]. Borsuk proved that X is locally contractible and $\check{H}^n(X)\neq 0$ for each $n\geqq 0$, where $\check{H}^*$ is the integral Čech cohomology. Since X is LC^{∞}, X is n-fine movable for each n. However X is not fine movable.

(5) Let Y be the non movable compactum constructed by Cox [10, Theorem 5] such that Y is a one point union of movable compacta X_1 and X_2. Obviously both X_1 and X_2 are not 1-fine movable.

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