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Perturbations of M-accretive operators and quasi-linear evolution equations

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1. Introduction.

Let X be a complex Banach space and let A be an m-accretive operator with domain $D(A)\subseteqq X$ and range $R(A)\subseteqq X$. Then a rather common problem in nonlinear perturbation theory is the following: given an accretive operator $B: D(B)\to X(D(B)\subseteqq D(A))$, what additional assumptions on B ensure the m-accretiveness of $A+B$? This problem can be rephrased as follows: let $U(v)u=Au+Bv$, $(u, v)\in D(A)\times D(A)$. Assume that U is m-accretive in u and accretive in v. What additional assumptions on U w. r. t. v ensure the m-accretiveness of the operator $U_1: u\to U(u)u$? Our main purpose here is to present such a result for operators $U(v)u$ which are not necessarily equal to the sum of two operators A and B as above. This result will be shown after we establish the existence of solutions to quasi-linear problems of the form

$$\text{(I)} \qquad x'(t)+U(x(t))x(t)=0\,, \qquad x(0)=x_0\,, \qquad t\in[0, \infty)\,.$$

The method here employs the contraction principle on an operator T associated with the equation

$$\text{(II)}_u \qquad x'(t)+U(u(t))x(t)=0\,, \quad x(0)=x_0\,, \quad t\in[0, T]\,,$$

where u is taken from a suitable family of continuous functions. This operator T maps $u(t)$ into the unique solution $x_u(t)$, $t\in[0, T]$ of (II)_u which is assumed to exist by known results. In case $U(v)u$ is linear in u, the problem (II)_u is linear, and this is why problems like (I) are called "quasi-linear".

Quasi-linear problems for ordinary differential equations go at least as far back as Corduneanu [**1**]. The reader is also referred to the papers of Lasota and Opial [**11**], Opial [**15**], Avramescu [**2**], Kartsatos [**4-6**] and Kartsatos and Ward [**7**] for some further results. For quasi-linear problems concerning partial differential equations, the reader is referred to Kato [**10**], Ward [**16**] and the references therein. Ward employed in [**16**] the Schauder-Tychonov theorem for a suitable space of functions associated with the weak topology of X.

The second purpose of this paper is to obtain solutions in a Banach function space of

$$(**)\qquad x'+A(t)x=G(t, x), \qquad t\in[0, +\infty)$$

where $A(t)$ is now linear and m-accretive. This result extends in a certain direction the applicability of Theorem 2.2 of Massera and Schäffer [**13**, p. 292]. These authors assumed that $A(t)$ is a bounded linear operator, excluding thus large classes of linear partial differential operators.

2. Preliminaries.

In what follows, X will be a complex Banach space with uniformly convex dual X^*. By F we denote the duality map of X, i. e., for each $x\in X$, $F(x)$ is the unique functional in X^* with $\langle x, F(x)\rangle=\|x\|^2=\|F(x)\|^2$. Here $\langle x, f\rangle$ denotes the value of $f\in X^*$ at x and $\|\cdot\|$ is the norm in X or X^*. This map F is well defined and uniformly continuous on bounded subsets of X (cf. Kato [8]). An operator $A: D(A)\to X$ with (domain) $D(A)\subseteqq X$ is said to be "accretive" if

$$\mathrm{Re}\langle Ax-Ay, F(x-y)\rangle\geqq 0 \quad \text{for every} \quad x, y\in D(A).$$

An accretive operator A is said to be "m-accretive" if the range $R(I+\lambda A)=X$ for every $\lambda>0$. Here I is the identity operator. Now consider the Cauchy problem

$$(2.1)\qquad x'+A(t)x=0, \qquad x(0)=x_0, \qquad t\in[0, T],$$

where T is a positive constant and $x_0\in D(A(0))=D(A(t))$, $t\in[0, T]$. By a "strong solution" of (2.1) we mean a function $x(t)$, $t\in[0, T]$ which is strongly continuous on $[0, T]$, strongly differentiable a. e., and satisfies (2.1) a. e.

3. Main results.

We shall first establish a theorem concerning the existence of a unique strong solution of the problem

$$(\mathrm{I})\qquad x'(t)+A(t, x(t))x(t)=0, \qquad x(0)=x_0,$$

where $A(t, u)v$ is Lipschitzian in t, u and m-accretive in v.

THEOREM 3.1. *Let* (I) *satisfy the following:*

(i) *the domain of the operator* $U(t,\cdot,\cdot)$ *with* $U(t, u, v)=A(t, u)v$ *is the set* $\bar{D}\times D$, $D\subseteqq X$ *for every* $t\in[0, T)$, *and the range* $R(U(t,\cdot,\cdot))\subseteqq X$. *Moreover,* $x_0\in D$,

(ii) *for every* $(t, u)\in[0, T)\times\bar{D}$, $A(t, u)v$ *is m-accretive in* v,

(iii) $$\|A(t, u_1)v-A(s, u_2)v\| \leqq r(\|u_1\|, \|u_2\|, \|v\|)[|t-s|(1+\|A(s, u_2)v\|)+\|u_1-u_2\|]$$

for every $t, s\in[0, T)$, $u_1, u_2, v\in\bar{D}$. *Here* $r: R_+^3\to R_+=[0, +\infty)$ *is increasing in all three variables. Then there exists* $T_1<T$ *such that* (I) *has a unique strong solution* $x(t)$, $t\in[0, T_1]$ *which is also uniformly Lipschitz continuous on* $[0, T_1]$.

PROOF. Let $M=1+\|A(0, x_0)x_0\|$ and L be a positive constant with $L/M<T$. Let $0<T_1\leqq L/M$. Consider the set $S=\{u:[0, T_1]\to X; u(0)=x_0, u(t)\in\bar{D}, t\in[0, T_1], \|u(t)-u(t')\|\leqq M|t-t'|, t, t'\in[0, T_1]\}$. Then for every $u\in S$ we have $\|u(t)-x_0\|\leqq Mt\leqq MT_1\leqq L$. Moreover, $S\neq\emptyset$ because $u(t)\equiv x_0\in S$. Now let $u\in S$ and consider the problem

$$\text{(I)}_u \qquad x'(t)+A(t, u(t))x(t)=0\,, \quad x(0)=x_0\,, \quad t\in[0, T_1]\,.$$

This problem has a unique strong solution $x_u(t)$ because the operator $B_u(t)v\equiv A(t, u(t))v$ satisfies all the assumptions of Theorem 1 in [8]. Actually, this solution $x_u(t)$ is also weakly continuously differentiable on $[0, T_1]$ and such that $A(t, u(t))x(t)$ is weakly continuous in t. Furthermore, $x(t)$ satisfies (I)_u everywhere if $x'(t)$ denotes now the weak derivative of $x(t)$. We are planning to show that the operator $T: u\to x_u$ is a contraction mapping on S if T_1 is chosen small enough. To this end, fix $u\in S$ and consider the approximating equations

$$(3.1)_n \qquad x_n'+A_n(t)x_n=0\,, \quad x_n(0)=x_0\,.$$

Here $A_n(t)=A_n(t, u(t))\equiv A(t, u(t))[I+(1/n)A(t, u(t))]^{-1}$, $n=1, 2, \cdots$, are defined and Lipschitz-continuous on X with Lipschitz constants not exceeding $2n$. Moreover, the operators $J_n(t)\equiv[I+(1/n)A(t, u(t))]^{-1}: X\to D$ are also Lipschitz-continuous on X with Lipschitz constants not exceeding 1. Each one of the equations $(3.1)_n$ has a unique strongly continuously differentiable solution $x_n(t)$ defined on $[0, T_1]$, and such that $\lim_{n\to\infty} x_n(t)=x_u(t)$ strongly and uniformly w.r.t. t on $[0, T_1]$ (cf. Kato [8]). We are planning to show that the sequence $\{x_n(t)\}$, $n=1, 2, \cdots$, is uniformly bounded on $[0, T_1]$ independently of $u\in S$, and that $\{x_n(t)\}$ is also uniformly Lipschitz-continuous on $[0, T_1]$ independently of $u\in S$. To this end, let us first note that the following inequality holds as in Kato [8, Lemma 4.1]:

$$\begin{aligned}&\|A_n(t, u(t))v-A_n(s, u(s))v\|\\&\quad\leqq r(\|u(t)\|, \|u(s)\|, \|v\|)|t-s|(1+M+\|v\|+2\|A_n(s, u(s))v\|)\end{aligned}$$

for any $t, s\in[0, T_1]$, $v\in D$. Now we have

$$\begin{aligned}(3.1) \qquad (d/dt)\|x_n(t)-x_0\|^2&=2\,\mathrm{Re}\,\langle x_n'(t),\ F(x_n(t)-x_0)\rangle\\&=-2\,\mathrm{Re}\,\langle A_n(t, u(t))x_n(t)-A_n(t, u(t))x_0\,,\ F(x_n(t)-x_0)\rangle\end{aligned}$$

$$-2\operatorname{Re}\langle A_n(t, u(t))x_0,\ F(x_n(t)-x_0)\rangle$$
$$\leqq 2\|A_n(t, u(t))x_0\|\,\|x_n(t)-x_0\|$$
$$\leqq 2[\|A_n(0, x_0)x_0\|+r(\|u(t)\|, \|x_0\|, \|x_0\|)\cdot$$
$$[(1+M+\|x_0\|+2\|A_n(0, x_0)\|)T_1+\|u(t)-x_0\|]]\,\|x_n(t)-x_0\|$$
$$\leqq 2[\|A_n(0, x_0)x_0\|+r(\|x_0\|+L, \|x_0\|, \|x_0\|)\cdot$$
$$[(1+\|x_0\|+M+2\|A_n(0, x_0)x_0\|)(L/M)+L]]\,\|x_n(t)-x_0\|.$$

This inequality holds almost everywhere in $[0, T_1]$. Dividing by $2\|x_n(t)-x_0\|$ and integrating from 0 to $t\leqq T_1$ we obtain

$$\|x_n(t)-x_0\|\leqq[\|A_n(0, x_0)x_0\|+r(\|x_0\|+L, \|x_0\|, \|x_0\|)\cdot (1+M+\|x_0\|+2\|A_n(0, x_0)x_0\|)(L/M)+L)]T_1=K_1T_1 \tag{3.2}$$

where the constant $K_1>0$ is independent of T_1, $u\in S$, but depends on n. In order to find an upper bound for the derivative $x_n'(t)$, consider first the function $z_n(t)\equiv x_n(t+h)-x_n(t)$, $0\leqq t$, $t+h<T_1$. Then we have

$$(1/2)(d/dt)\|z_n(t)\|^2=\operatorname{Re}\langle z_n'(t),\ F(z_n(t))\rangle \tag{3.3}$$
$$=-\operatorname{Re}\langle A_n(t+h, u(t+h))x_n(t+h)$$
$$-A_n(t, u(t))x_n(t), F(z_n(t))\rangle$$
$$=-\operatorname{Re}\langle A_n(t+h, u(t+h))x_n(t+h)$$
$$-A_n(t+h, u(t+h))x_n(t), F(z_n(t))\rangle$$
$$-\operatorname{Re}\langle A_n(t+h, u(t+h))x_n(t)-A_n(t, u(t))x_n(t), F(z_n(t))\rangle$$
$$\leqq r(\|u(t+h)\|, \|u(t)\|, \|x_n(t)\|)[(1+M+\|x_n(t)\|$$
$$+2\|A_n(t, u(t))x_n(t)\|)|h|+\|u(t+h)-u(t)\|]\,\|z_n(t)\|$$
$$\leqq r(\|x_0\|+L, \|x_0\|+L, \|x_0\|+K_1T_1)\cdot$$
$$[(1+\|x_n'(t)\|+2M+\|x_0\|+K_1T_1)]\,\|z_n(t)\|\,|h|.$$

Dividing above by $\|z_n(t)\|\,|h|$ and integrating we obtain, after passage to the limit for $h\to 0$,

$$\|x_n'(t)\|\leqq\|x_n'(0)\| +\int_0^t r(\|x_0\|+L,\ \|x_0\|+L,\ \|x_0\|+K_1T_1)\cdot\|x_n'(s)\|ds +r(\|x_0\|+L, \|x_0\|+L, \|x_0\|+K_1T_1)(1+2M+\|x_0\|+K_1T_1)T_1. \tag{3.4}$$

Thus, by Gronwall's inequality, we have

$$\|x_n'(t)\|\leqq[K_2T_1+\|A_n(0, x_0)x_0\|]e^{K_2T_1}, \tag{3.5}$$

where K_2 is independent of T_1, $u \in S$. Now since $\|A_n(0, x_0)x_0\| \leqq \|A(0, x_0)x_0\|$ (cf. Kato [8]), we obtain from (3.2), (3.5) that $\|x_n(t)\| \leqq \|x_0\| + K_3T_1$, $\|x_n'(t)\| \leqq (K_2T_1 + K_4)e^{K_2T_1}$ with K_2, K_3, K_4 independent of T_1, $u \in S$ and n. Moreover, we also have $\|x_u(t)\| \leqq \|x_0\| + K_3T_1$, $\|x_u(t) - x_u(t')\| \leqq (K_2T_1 + K_4)e^{K_2T_1}|t - t'|$ for every $t, t' \in [0, T_1]$.

New let $u_1, u_2 \in S$ and x_1, x_2 be the corresponding solutions of $(\mathrm{I})_u$. Then we have

$$
\begin{aligned}
(3.6) \qquad & (1/2)(d/dt)\|x_1(t) - x_2(t)\|^2 \\
&= -\mathrm{Re}\langle A(t, u_1(t))x_1(t) - A(t, u_2(t))x_2(t), \\
&\qquad\qquad F(x_1(t) - x_2(t))\rangle \\
&= -\mathrm{Re}\langle A(t, u_1(t))x_1(t) - A(t, u_1(t))x_2(t), F(x_1(t) - x_2(t))\rangle \\
&\quad - \mathrm{Re}\langle A(t, u_1(t))x_2(t) - A(t, u_2(t))x_2(t), F(x_1(t) - x_2(t))\rangle \\
&\leqq r(\|u_1(t)\|, \|u_2(t)\|, \|x_2(t)\|)\cdot \|u_1(t) - u_2(t)\| \|x_1(t) - x_2(t)\|
\end{aligned}
$$

from which, by division by $\|x_1(t) - x_2(t)\|$ and integration, we get

$$
\begin{aligned}
(3.7) \qquad & \sup_{t \in [0, T_1]} \|x_1(t) - x_2(t)\| \\
&\leqq T_1 r(\|x_0\| + L, \|x_0\| + L, \|x_0\| + K_3L/M) \sup_{t \in [0, T_1]} \|u_1(t) - u_2(t)\| \\
&= K_5 \sup_{t \in [0, T_1]} \|u_1(t) - u_2(t)\| .
\end{aligned}
$$

Now we may (and do) choose T_1 small enough so that

$$[K_2T_1 + K_4]e^{K_2T_1} \leqq M$$

and $K_5 < 1$; then the operator $T: u \to x_u$ maps the set S into itself and is a contraction. Since S is a complete metric space under the sup-norm, T has a fixed point $x(t)$, $t \in [0, T_1]$. This is the desired strong solution of (I). Uniqueness follows from 3.6 by replacing u_1, u_2 by x_1, x_2 respectively.

The above result generalizes the existence result in the proof of Theorem 11.2 of Kato [9]. Kato considered the case $A(t, u, v) \equiv Au + Bv$ under a "localized" Lipschitz condition on B and m-accretiveness of a multi-valued A.

It should be noted that if $U(t, u, u) \equiv A(u)u$ (independent of t) and accretive in u, then the solution guaranteed by Theorem 3.1 is extendable to $[0, \infty)$ if we further assume that $A(u)u$ is "demiclosed" (i.e., if $u_n \in D$, $n = 1, 2, \cdots$, and $u_n \to u$ and $A(u_n)u_n \rightharpoonup v \in X$ then $u \in D$ and $A(u)u = v$). In fact, (cf. proof of Theorem 11.2 of [9]) in this case, if $[0, T')$ is the maximal interval of existence of $x(t)$ with $T' < +\infty$, then $\lim_{t \to T'} x(t) = x(T') \in D$ exists.

Now we are ready for the following perturbation result:

THEOREM 3.2. *Let D be a subset of X. Let A: $\bar{D} \times D \to X$ be such that $A(u)v$ is m-accretive in v and $\|A(u_1)v - A(u_2)v\| \leqq r(\|u_1\|, \|u_2\|, \|v\|)\|u_1 - u_2\|$ for* *any* $u_1, u_2 \in \bar{D}, v \in D$. *Then if* $A(u)u$ *is demiclosed and accretive, it is m-accretive.*

PROOF. Taking into consideration the proof of Theorem 11.2 in Kato's paper [**9**] (cf. also Mermin [**14**, Lemma 4.2]), it suffices to show the existence of some $x_0 \in D$ such that for all $p \in X$ the Cauchy problem

$$x'(t)+A(x(t))x(t)+x(t)-p=0,\ x(0)=x_0 \tag{3.8}$$

has a unique strong solution on $[0, \infty)$. To show this, we simply remark that the operator $B(u)v \equiv A(u)v+v-p$ satisfies the assumptions placed on A in Theorem 3.1, and that the local strong solution obtained there is extendable to $[0, \infty)$ by the discussion above.

Theorem 3.1 holds of course if we perturb Equation (I) by a Lipschitzian function. This is the content of the following

COROLLARY 3.1. *Let the operator* $A(u)v$ *be as in Theorem* 3.1, *and let* G: $[0, T)\times\bar{D}\to X$ *satisfy:*

$$\|G(t_1, u_1)-G(t_2, u_2)\| \leqq r_1(\|u_1\|, \|u_2\|)[|t_1-t_2|+\|u_1-u_2\|]$$

for every $t_1, t_2 \in [0, T)$ *and* $u_1, u_2 \in \bar{D}$. *Then the conclusion of Theorem* 1 *is true for the equation*

$$x'(t)+A(t, x(t))x(t)=G(t, x(t))\,. \tag{I$_G$}$$

PROOF. It suffices to consider instead of $A(t, u)v$ the operator $B(t, u)v \equiv A(t, u)v-G(t, u)$.

The above corollary has points of contact with the main result of Gröger [**3**] who considered $A(t, u)v \equiv A(t)v$ and G Lipschitzian and defined on the whole of X w. r. t. the second variable.

4. **Linear** M**-accretive** $A(t)$.

Let C be the space of all X-valued continuous functions on R_+ with the topology of uniform convergence on finite intervals. Then C is a Fréchet space. Now consider the differential equation

$$x'+A(t)x=f(t),\ x(0)=x_0 \in D,\ t\in[0, \infty)\,, \tag{*}$$

where $D(A(t))=D(A(0))=D$, $R(A(t))\subseteqq X$ with A linear, closed and $f\in C$. Let f be Lipschitzian on $[0, +\infty)$ and, moreover, let

$$\|A(t)v-A(s)v\| \leqq L|t-s|\cdot\|A(s)v\| \tag{S}$$

for every $s, t\in R_+, v\in D$, where L is a positive constant. Then the operator $A_1(t): D\to X$ with $A_1(t)x\equiv A(t)x-f(t)$ satisfies all the hypotheses of Theorem 2.1 of Mermin's dissertation [**14**] (cf. also Kato [8, Theorems 1, 2]). Consequently, the equation (*) has a unique solution $x(t), t\in R_+$ which is strongly differentiable a. e., weakly continuously differentiable, and satisfying (*) ($x'(t)$ here is the weak derivative) on R_+. Thus, under the above assumptions on $A(t)$, Equation(*) has always solutions on $[0, \infty)$ if f belongs to a proper Banach space of Lipschitzian functions. The Banach spaces B considered below consist of functions $f: [0, +\infty) \to X$ which are at least Lipschitzian on $[0, +\infty)$.

Let B, E be two complex Banach spaces in C which are stronger than C (convergence in B or E implies convergence in C). Then the pair (B, E) is "admissible" if for every $f \in B$ there exists al least one solution $x \in E$ of(*). We denote by X_{0E} the linear manifold of X consisting of all initial values of E-solutions of the homogeneous Cauchy problem

$$x' + A(t)x = 0 . \tag{4.1}$$

Now let X_1 be any (but fixed) subspace of X supplementary to X_{0E} and let X_{1E} be the linear manifold consisting of all initial values of E-solutions of (*) belonging to X_1 and corresponding to all possible $f \in B$. We have the following theorem which extends a variation of Theorem 2.2 of Massera and Schäffer [**13**] to unbounded operators:

THEOREM 4.1. *Let $D(A(t)) = D(A(0)) = D$ with A linear, closed m-accretive and satisfying* (S). *Moreover, let B, E be two complex Banach spaces such that the pair (B, E) is admissible, X_{0E}, X_{1E} as above and $P_1(D) = \{P_1 u;\ u \in D\} \subseteqq D$, where P_1 is the projection of X_1. Then there exists a constant $K>0$ such that for every $f \in B$ Equation* (*) *has a unique solution $x \in E$ with $x(0) \in X_{1E}$ and satisfying $\|x\|_E \leqq K \|f\|_B$.*

PROOF. We partially follow the steps of Massera and Schäffer in [**12**]. Let Y be the linear manifold consisting of all possible E-solutions of (*) with initial values in X_{1E} while f ranges in B. Now let $x \in Y$. We define

$$\|x\|_Y = \|x\|_E + \|x'(0)\| + \|x' + A(\cdot)x\|_B ,$$

where $x'(0)$ is the weak derivative of the solution $x(t)$.

Then $\|\cdot\|_Y$ is a norm on Y, and we show that under this norm Y is complete. Let $x_n \in Y$, $n = 1, 2, \cdots$ be a Cauchy sequence. Then for every $\varepsilon > 0$ there exists $N(\varepsilon) > 0$ such that

$$\|x_m - x_n\|_Y = \|x_m - x_n\|_E + \|x_m'(0) - x_n'(0)\| + \|x_m' + A(\cdot)x_m - [x_n' + A(\cdot)x_n]\|_B < \varepsilon \tag{4.2}$$

for every m, n with $m, n > N(\varepsilon)$. Thus, in particular, $\{x_n\}$ is a Cauchy sequence in E and, since E is stronger than C (which is complete), there is a continuous function $x(t)$, $t \in R_+$ such that $x_n \to x$ in C. In particular, $x_n(0) \to x(0)$ as $n \to \infty$. On the other hand, since X is a Banach space, there exists a vector $y \in X$ such that the sequence $\{x_n'(0)\}$ converges strongly to y as $n \to \infty$. It is also true that there exists $f \in B$ such that

$$\lim_{n\to\infty} \|f_n - f\|_B = 0 , \qquad C\text{-}\lim_{n\to\infty} f_n = f , \tag{4.3}$$

where $f_n(t)=x_n'(t)+A(t)x_n(t)$. Since $A(0)$ is closed and $x_n'(0)+A(0)x_n(0)=f_n(0)$ with $x_n'(0)\to y\in X$ and $f_n(0)\to f(0)$, we obtain $x(0)\in D$ and $A(0)x(0)=-y+f(0)$. Now let $\bar{x}(t), t\in R_+$ be the solution of (*) with $\bar{x}(0)=x(0)$. This solution exists because $x(0)\in D$. Then we have

$$(4.4)\qquad x_n'(t)+A(t)x_n(t)=f_n(t), \qquad t\in R_+,$$

$$(4.5)\qquad \bar{x}'(t)+A(t)\bar{x}(t)=f(t), \qquad t\in R_+.$$

Subtracting (4.5) from (4.4) and applying the functional $F(x_n(t)-\bar{x}(t))$ on both members of the resulting equation, we easily obtain

$$\begin{aligned}(4.6)\qquad (d/dt)\|x_n(t)-\bar{x}(t)\|^2 &= 2\,\mathrm{Re}\,\langle x_n'(t)-\bar{x}'(t), F(x_n(t)-\bar{x}(t))\rangle \\ &= -2\,\mathrm{Re}\,\langle A(t)x_n(t)-A(t)\bar{x}(t), F(x_n(t)-\bar{x}(t))\rangle \\ &\quad -2\,\mathrm{Re}\,\langle f_n(t)-f(t), F(x_n(t)-\bar{x}(t))\rangle \\ &\leqq \|f_n(t)-f(t)\|\|x_n(t)-\bar{x}(t)\|\end{aligned}$$

almost everywhere in $[0, c]$, where c is a fixed positive constant. From (4.6) we obtain

$$(4.7)\qquad (d/dt)\|x_n(t)-\bar{x}(t)\|\leqq\|f_n(t)-f(t)\|, \qquad \text{a.e. in } [0, c],$$

which implies

$$\begin{aligned}(4.8)\qquad \|x_n(t)-\bar{x}(t)\| &\leqq \|x_n(0)-\bar{x}(0)\|+\int_0^c\|f_n(s)-f(s)\|ds \\ &\leqq \|x_n(0)-\bar{x}(0)\|+c\sup_{t\in[0,c]}\|f_n(t)-f(t)\|.\end{aligned}$$

Consequently, $x_n(t)$ converges strongly and uniformly to $x(t)$ on the interval of $[0, c]$. Since $c>0$ is arbitrary $\bar{x}(t)\equiv x(t)$ and $x'(t)+A(t)x(t)=f(t)$. Consequently, $x'(0)=y$ and

$$\lim_{n\to\infty}\|x_n-x\|_Y=0$$

which proves the completeness of Y. Now consider the operator $T: Y\to B$ with $(Tx)(t)=x'(t)+A(t)x(t)$.

The operator T is linear and bounded. In fact, $\|Tx\|_B\leqq\|x\|_Y$. T is one-to-one. To this end, let $x_1, x_2\in Y$ with $Tx_1=Tx_2$. Then since $T(x_1-x_2)=0$ and $x_1-x_2\in E$, we must have $x_1(0)-x_2(0)\in X_{0E}$. Since $X_{0E}\cap X_{1E}=\{0\}$, $x_1(0)=x_2(0)$ which implies $x_1(t)\equiv x_2(t)$, $t\in R_+$. To show that T is onto, let $f\in B$, and let $x\in E$ with $x'+A(t)x=f$. Then since $P_1(D)\subseteqq D$, $P_1x(0)\in D$. Let $x_1(t)$ be the solution of (*) with $x_1(0)=P_1x(0)$. Then $x(0)-x_1(0)=P_0x(0)\in X_{0E}$. Here P_0 is the projection of X_{0E}. Thus, $x-x_1\in E$ which implies $x_1\in E$. Since $x_1(0)\in X_{1E}$, $Tx_1=f$, which proves the ontoness of T. Now it follows from a well known theorem in Functional Analysis that the operator T^{-1}: $B\to Y$ is bounded and since $\|T\|\leqq 1$, we must have $\|T^{-1}\|\geqq 1$. Let $K=\|T^{-1}\|-1$. Then $\|x\|_E\leqq\|x\|_Y-\|f\|_B\leqq\|T^{-1}f\|_B-\|f\|_B\leqq(\|T^{-1}\|-1)\|f\|_B=K\|f\|_B$. This completes the proof.

As an application of the above considerations, we show the existence of solutions in E of a perturbed linear equation of the form

$$x'+A(t)x=G(t, x), \qquad t\in[0, +\infty). \tag{4.9}$$

COROLLARY 4.1. *Let $A(t), B, E$ satisfy the hypotheses of Theorem 4.1. Let $M=\{u\in E;\ \|u\|\leqq r\}$, where r is a positive constant. Let $G: R_+\times\{v\in X;\ \|v\|\leqq r\}\to X$ satisfy:*

(i) *the operator U defined by $(Ux)(t)=G(t, x(t))$ maps M into B,*

(ii) $\|G(\cdot, u_1(\cdot))-G(\cdot, u_2(\cdot))\|_B\leqq L\|u_1-u_2\|_E$

for every $u_1, u_2\in M$ and $\|G(\cdot, 0)\|_B\leqq\lambda$ with the constants λ, L, r satisfying $(\lambda+Lr)K\leqq r$ and $KL<1$. Here K is the constant of Theorem 4.1. Then (4.9) has at least one solution $x(t), t\in[0, \infty)$ with $x(0)\in X_{1E}$.

PROOF. Consider the operator $T: M\to E$ which maps the function $u\in M$ into the unique solution $x_u\in E$ ($x_u(0)\in X_{1E}$) of the equation

$$x'+A(t)x=G(t, u(t)).$$

The solution $x_u(t)$ is guaranteed by Theorem 4.1. Moreover,

$$\begin{aligned}\|x_u\|_E&\leqq K\|G(\cdot, u(\cdot))\|_B\\ &\leqq K(\|G(\cdot, 0)\|_B+L\|u\|_E)\leqq K(\lambda+Lr)\leqq r.\end{aligned}$$

Thus, $T(M)\subseteq M$. We also have

$$\begin{aligned}\|Tu_1-Tu_2\|&\leqq K\|G(\cdot, u_1(\cdot))-G(\cdot, u_2(\cdot))\|_B\\ &\leqq KL\|u_1-u_2\|.\end{aligned}$$

This proves that T is a contraction on M and completes the proof.

In Theorem 3.1 we assumed that $P_1(D)\subseteq D$ to ensure that $P_1x(0)\in D$, otherwise the existence of $x_1(t)$ cannot be shown. In view of the usual spaces of definition of partial differential operators, this is not really a strong assumption. It is actually true that $A(s)$ generates (for any but fixed $s\in[0, +\infty)$) a linear contraction semigroup $T(t)$ on $\bar{D}$. Thus, we may assume without loss of generality that $X=\bar{D}$ and that A is densely defined in X.

In the results considered in this section we could have restricted ourselves to finite intervals. Systems of the form

$$x'+A(t)x=G(t, x), \qquad Tx=0, t\in[0, T] \tag{4.10}$$

can be considered, where T is a bounded linear operator mapping $C[0, T]$ into X. E now would consist of all $u\in C[0, T]$ with $Tu=0$ and satisfying other suitable conditions.

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References

[1] C. Corduneanu, Sur les systèmes différentiels de la forme $y'=A(x, y)y+b(x, y)$, An. Sti. Univ. "Al I. Cuza", Iasi, (N. S) **4** (1958), 45-52.

[2] C. Avramescu, Sur une équation non linéaire dans les espaces de Banach, Act. Conf. Intern. "Equa-Diff 73", Bruxelles et Louvain-La-Neuve, (1973), 97-107.

[3] K. Gröger, Zur Theorie von Evolutionsgleichungen, in den die "Vorgeschichte" eine Rolle spielt, Math. Nachr., **56** (1973), 63-72.

[4] A. G. Kartsatos, Nonzero solutions to boundary value problems for nonlinear systems, Pacific J. Math., **52** (1974), 425-433.

[5] A. G. Kartsatos, Bounded solutions to perturbed nonlinear systems and asymptotic relationships, J. Reine Angew. Math., **273** (1975), 170-177.

[6] A. G. Kartsatos, Stability via Tychonov's theorem, Internat. J. Systems Sci., **5** (1974), 933-937.

[7] A. G. Kartsatos, and J. R. Ward, Boundedness and existence of periodic solutions of quasi-linear systems, J. Inst. Math. Appl., **15** (1975), 187-194.

[8] T. Kato, Nonlinear semigroups and evolution equations, J. Math. Soc. Japan, 19 (1967), 503-520.

[9] T. Kato, Accretive operators and nonlinear evolution equations in Banach spaces, Proc. Symp. Pure Math., **18** (**1970**), 138-161.

[10] T. Kato, Quasi-linear equations of evolution with applications to partial differential equations, Proc. Symp. Spectral Theory Differential Equations, Lecture Notes No. 448, Springer-Verlag, New York, 1975.

[11] A. Lasota and Z. Opial, Sur les solutions periodiques des équations différentielles ordinaires, Ann. Polon. Math., **16** (1964), 69-94.

[12] J. L. Massera and J. J. Schäffer, Linear differential equations and functional analysis, I., Ann. of Math., **67** (1958), 517-573.

[13] J. L. Massera and J. J. Schäffer, Linear differential equations and functional analysis, IV, Math. Ann., **139** (1960), 287-342.

[14] J. Mermin, Accretive operators and nonlinear semigroups, Doct. Dissert., University of California, Berkeley, 1968.

[15] Z. Opial, Linear problems for systems of nonlinear differential equations, J. Differential Equations, **3** (1967), 580-594.

[16] J. R. Ward, Existence of solutions to quasi-linear differential equations in a Banach space, to appear.

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