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Induced characters of some 2-groups

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Let G be a 2-group and χ a complex, irreducible charactor of G. The Schur index of χ with regard to the rational field $\boldsymbol{Q}$ is denoted by $m_{\boldsymbol{Q}}(\chi)$. It is known that $m_{\boldsymbol{Q}}(\chi)=1$ or 2, and that if $m_{\boldsymbol{Q}}(\chi)=2$, then there exist a subgroup H of G and an irreducible character ϕ of H such that χ is induced from ϕ, i.e., $\chi=\phi^{G}$, $m_{\boldsymbol{Q}}(\phi)=2$, and the factor group H/N, N=kernel of ϕ, is a generalized quaternion group (cf. (11.7) and (14.3) of [2], or [3]).

Now, let H' be a generalized quaternion group. The faithful irreducible characters of H' are algebraically conjugate to each other and their Schur indices are equal to 2, whereas any non-faithful irreducible character of H' has Schur index 1 (cf. [5, § 6]). So we ask a question: Let G be a 2-group, let H, N be subgroups of G such that $H \triangleright N$ and H/N is a generalized quaternion group, and let ϕ be a faithful irreducible character of H/N, which is also regarded as a character of H. Suppose that the induced character ϕ^{G} is irreducible. Is it true that the Schur index $m_{\boldsymbol{Q}}(\phi^{G})=2$?

A simple case for the question is that $N=\{1\}$. Namely, let $G \supset H$ be 2-groups, where H is a generalized quaternion group, and let ϕ be an irreducible charater of H such that $m_{\boldsymbol{Q}}(\phi)=2$ and ϕ^{G} is irreducible. Is it ture that $m_{\boldsymbol{Q}}(\phi^{G})=2$? The purpose of the paper is to show that this is true for a class of induced characters ϕ^{G}, which are associated with *cyclotomic algebras*. Our result yields, as a special case, that if $[G: H]=2$, then the question is affirmative.

Let us briefly explain the contents of the paper. From now on, H_{n} denotes the generalized quaternion group of order 2^{n+1} $(n \geqq 2)$, ϕ_{n} an irreducible character of H_{n} with $m_{\boldsymbol{Q}}(\phi_{n})=2$, and ζ_{s} a primitive s-th root of unity, where s is a natural number. In § 1, we investigate a 2-group G such that $G \supset H_{n}$ and that the induced character ϕ_{n}^{G} is irreducible (Theorem 1). We also determine the values of ϕ_{n}^{G} at elements x of G (Proposition 1).

In § 2, we study a cyclotomic algebra B made with the extension $\boldsymbol{Q}(\zeta_{2^n})/k$, where k is a subfield of the field $\boldsymbol{Q}(\zeta_{2^n})$ of 2^{n}-th roots of unity. It will be shown that the index of B is 1 or 2, and if B has index 2, then there exists a 2-group G, which is a finite subgroup of the multiplicative group $B^{\times}$ such that

$G \supset H_n$ and ϕ_n^G is irreducible. Moreover, B is isomorphic to the simple component of the group algebra $\boldsymbol{Q}[G]$, which corresponds to ϕ_n^G. In particular, $m_{\boldsymbol{Q}}(\phi_n^G)=2$ (Theorem 2). We will call such a group G a *cyclotomic 2-group.*

Let G be a cyclotomic 2-group and let F be a 2-group such that $F \supset G$, $[F: G]=2$, and $(\phi_n^G)^F=\phi_n^F$ is an irreducible character of F. The purpose of § 3 is to prove $m_{\boldsymbol{Q}}(\phi_n^F)=2$ (Theorem 3). As a special case of the result, we have: If G is any 2-group such that $G \supset H_n$, $[G: H_n]=2$, and ϕ_n^G is irreducible, then $m_{\boldsymbol{Q}}(\phi_n^G)=2$ (Corollary 1).

NOTATION. $\boldsymbol{Z}$ is the integers. If K is a Galois extension of k, then $\mathcal{G}(K/k)$ is the Galois group of K over k. For $x \in K$ and $\sigma \in \mathcal{G}(K/k)$, x^σ denotes the image of x by σ. Let χ be an irreducible character of a group G such that $K(\chi)=K$. Then χ^σ is the character of G defined by $\chi^\sigma(g)=(\chi(g))^\sigma$, $g \in G$. Let $N \triangleleft G$ and ψ a character of N. Then ψ^g $(g \in G)$ is the character of N defined by $\psi^g(x)=\psi(gxg^{-1})$, $x \in N$. $\langle a, b, \cdots\rangle$ is the group generated by $a, b, \cdots$.

§ 1. Induced characters.

Let $H=H_n=\langle a, b\rangle$ denote the generalized quaternion group of order 2^{n+1} $(n \geqq 2)$ with relations

$$a^{2^n}=1\,, \qquad bab^{-1}=a\,, \qquad b^2=a^{2^{n-1}}\,. \tag{1}$$

We summarize known results about characters of H (cf. pp. 225-226 of [**5**]).

There are $2^{n-1}-1$ irreducible characters $\phi_\nu (1 \leqq \nu \leqq 2^{n-1}-1)$ of H, which are not one-dimensional:

$$\phi_\nu(a^i)=\zeta_{2^n}^{\nu i}+\zeta_{2^n}^{-\nu i}\,, \qquad \phi_\nu(a^i b)=0\,, \qquad (i=0, 1, \cdots, 2^n-1)\,.$$

Each ϕ_ν is induced from the linear character η_ν of $\langle a\rangle$: $\eta_\nu(a^i)=\zeta_{2^n}^{i\nu}$.

The character ϕ_ν is faithful, if and only if ν is odd. If ν is odd then the Schur index $m_{\boldsymbol{Q}}(\phi_\nu)=2$, and if ν is even then $m_{\boldsymbol{Q}}(\phi_\nu)=1$. The faithful characters ϕ_ν $(1 \leqq \nu \leqq 2^{n-1}-1,\ 2 \nmid \nu)$ are algebraically conjugate to each other and $\boldsymbol{Q}(\phi_\nu)=Q(\zeta_{2^n}+\zeta_{2^n}^{-1})$.

THEOREM 1. *Let G be a 2-group which contains the generalized quaternion group H of order 2^{n+1} with $[G: H]=2^r$. Let ϕ be a faithful, irreducible character of H. Suppose that the induced character $\chi=\phi^G$ is irreducible. Then $r \leqq n-2$, $\boldsymbol{Q}(\chi)=\boldsymbol{Q}(\zeta_{2^{n-r}}+\zeta_{2^{n-r}}^{-1})$, and $[\boldsymbol{Q}(\phi): \boldsymbol{Q}(\chi)]=2^r=[G: H]$.*

PROOF. We may assume that

$$\phi(a^j)=\zeta_{2^n}^j+\zeta_{2^n}^{-j}\,, \qquad \phi(a^j b)=0\,, \qquad (j=0, 1, \cdots, 2^n-1)\,. \tag{2}$$

We can find a sequence of subgroups G_i of G and an element u_i of G_i such that

$$H=G_0 \subset G_1 \subset \cdots \subset G_r=G\,,$$

$$[G_i : G_{i-1}]=2, \quad G_i=G_{i-1}\cup G_{i-1}u_i, \quad u_i^2\in G_{i-1}, \quad (i=1, \cdots, r).$$

First we will prove Theorem 1 for the case $n=2$. Namely, suppose that H is the quaternion group of order 8. We observe that

$$\phi(1)=2, \quad \phi(a^2)=-2, \quad \phi(y)=0 \ \text{ for } \ y\in(H-\langle a^2\rangle). \tag{3}$$

Assume that $r\geqq 1$. Then $H\neq G_1=H\cup Hu_1$. Since $\{1, a^2\}$ is the center of H, it follows that $u_1a^2u_1^{-1}=a^2$, so $u_1(H-\langle a^2\rangle)u_1^{-1}=H-\langle a^2\rangle$. By (3), this yields that $\phi^{u_1}=\phi$. Hence by (45.5) of [1], ϕ^{G_1} is not irreducible, contradiction. Therefore, if n=2, then $r=0$, proving Theorem 1 for the case $n=2$.

Hereafter we assume $n\geqq 3$ and $[G: H]=2^r>1$. It is easy to see that Theorem 1 follows from the following proposition.

PROPOSITION 1. *Let the notation and assumption be as above. Then* $r\leqq n-2$, *and*

$$\phi^{G_i}(a^{2^i z})=2^i\phi(a^{2^i z}), \quad (z\in \boldsymbol{Z}), \quad (0\leqq i\leqq r) \tag{4}$$

$$\phi^{G_i}(y)=0 \ \textit{ for } \ y\in(G_i-\langle a^{2^i}\rangle), \quad (0\leqq i\leqq r) \tag{5}$$

$$\sigma(\langle a^{2^i}\rangle)=\langle a^{2^i}\rangle \ \textit{ for any } \ \sigma\in \mathrm{Aut}\,(G_i), \quad (0\leqq i\leqq r) \tag{6}$$

$$u_i a^{2^{i-1}}u_i^{-1}=a^{2^{i-1}(1+2^{n-i})} \ \textit{ for an appropriate } \ u_i\in G_i, \quad (1\leqq i\leqq r) \tag{7}$$

$$x^{2^i}\in\langle a^{2^i}\rangle \ \textit{ for any } \ x\in G_i, \quad (1\leqq i\leqq r). \tag{8}$$

PROOF. We will use induction on i. The equations (4)-(6) clearly hold for $i=0$.

Suppose that the equations (4)-(8) hold for $i\geqq 0$. We will show that they also hold for $i+1$. (The following argument yields that the equations (4)-(8) for $i=1$ follow from the equations (4)-(6) for $i=0$.) We assume that $i+1\leqq r\leqq n-1$. By (6), $u_{i+1}a^{2^i}u_{i+1}^{-1}=a^{2^i s}$ for some $s\in \boldsymbol{Z}$. Putting $\alpha=u_{i+1}^2\in G_i$, we have $\alpha a^{2^i}\alpha^{-1}=a^{2^i s^2}$. By (7), $u_j a^{2^i}u_j^{-1}=a^{2^i(1+2^{n-j})}=a^{2^i}$ for $j=1, \cdots, i$. So the conjugates of a^{2^i} in G_i are $\{a^{2^i}, a^{-2^i}\}$, because $(a^z b^{z'} u_1^{z_1}\cdots u_i^{z_i})a^{2^i}(a^z b^{z'} u_1^{z_1}\cdots u_i^{z_i})^{-1}=a^{\pm 2^i}$, $(z', z_1, \cdots, z_i=0$ or $1)$. Hence $2^i s^2\equiv\pm 2^i \pmod{2^n}$, so $s^2\equiv\pm 1 \pmod{2^{n-i}}$. But there is no $s\in\boldsymbol{Z}$ such that $s^2\equiv -1 \pmod{2^{n-i}}$, $(n-i\geqq 2)$. Hence $s^2\equiv 1 \pmod{2^{n-i}}$, and so $s\equiv\pm 1, \pm 1+2^{n-i-1} \pmod{2^{n-i}}$. If $s\equiv\pm 1 \pmod{2^{n-i}}$, then $u_{i+1}a^{2^i z}u_{i+1}^{-1}=a^{\pm 2^i z}$ $(z\in\boldsymbol{Z})$, so by (4),

$$(\phi^{G_i})^{u_{i+1}}(a^{2^i z})=\phi^{G_i}(a^{\pm 2^i z})=2^i\phi(a^{\pm 2^i z})=2^i\phi(a^{2^i z})=\phi^{G_i}(a^{2^i z}).$$

Since $u_{i+1}\langle a^{2^i}\rangle u_{i+1}^{-1}=\langle a^{2^i}\rangle$, it follows that for $y\in(G_i-\langle a^{2^i}\rangle)$, $u_{i+1}yu_{i+1}^{-1}\in(G_i-\langle a^{2^i}\rangle)$, so $(\phi^{G_i})^{u_{i+1}}(y)=0=\phi^{G_i}(y)$ by (5). Hence $(\phi^{G_i})^{u_{i+1}}=\phi^{G_i}$, so $\phi^{G_{i+1}}$ is not irreducible, contradiction. Thus $s\equiv\pm 1+2^{n-i-1} \pmod{2^{n-i}}$. In particular, this implies that $r\leqq n-2$. For, if $i+1=n-1\leqq r$, then $2^{n-i}=4$ and $\pm 1+2^{n-i-1}\equiv\pm 1 \pmod{2^{n-i}}$, so $\phi^{G_{i+1}}$ would not be irreducible.

If $s\equiv -1+2^{n-i-1} \pmod{2^{n-i}}$, put $v_{i+1}=bu_{i+1}$. Then

$$v_{i+1}a^{2^i}v_{i+1}^{-1}=a^{-2^i(-1+2^{n-i-1})}=a^{2^i(1+2^{n-i-1})}.$$

So we may assume $u_{i+1}a^{2^i}u_{i+1}^{-1}=a^{2^i(1+2^{n-i-1})}$, proving (7) for $i+1$. We also have

$$\phi^{G_{i+1}}(a^{2^{i+1}z})=\phi^{G_i}(a^{2^{i+1}z})+(\phi^{G_i})^{u_{i+1}}(a^{2^{i+1}z})$$
$$=2^i\phi(a^{2^{i+1}z})+2^i\phi(a^{2^{i+1}z})=2^{i+1}\phi(a^{2^{i+1}z}),$$

proving (4) for $i+1$.

If $y\in(G_{i+1}-G_i)$, then $\phi^{G_{i+1}}(y)=0$, because $G_{i+1}\triangleright G_i$. Since $u_{i+1}\langle a^{2^i}\rangle u_{i+1}^{-1}=\langle a^{2^i}\rangle$, it follows that for $y\in(G_i-\langle a^{2^i}\rangle)$, $u_{i+1}yu_{i+1}^{-1}\in(G_i-\langle a^{2^i}\rangle)$, and so $\phi^{G_{i+1}}(y)=\phi^{G_i}(y)+(\phi^{G_i})^{u_{i+1}}(y)=0+0=0$ by (5). If $2\nmid z$, then by (4),

$$\phi^{G_{i+1}}(a^{2^iz})=2^i\phi(a^{2^iz})+2^i\phi(a^{2^iz(1+2^{n-i-1})})=2^i\phi(a^{2^iz})-2^i\phi(a^{2^iz})=0.$$

Thus the equation (5) holds for $i+1$.

As a special case of the argument we have proved the equations (4), (5), (7) for $i=1$. We will prove the equations (6), (8) for $i=1$. Put $u_1^2=\alpha\in G_0$. We have $\alpha a\alpha^{-1}=u_1^2au_1^{-2}=a^{(1+2^{n-1})2}=$a, so $\alpha=a^\lambda$ for some $\lambda\in\boldsymbol{Z}$. Since $a^\lambda=\alpha=u_1\alpha u_1^{-1}=a^{\lambda(1+2^{n-1})}$, $2|\lambda$. Since $u_1\langle a\rangle u_1^{-1}=\langle a\rangle$, it follows that $u_1bu_1^{-1}=a^\nu b$ for some $\nu\in\boldsymbol{Z}$. Then $a^{2\lambda}b=a^\lambda ba^{-\lambda}=\alpha b\alpha^{-1}=u_1^2bu_1^{-2}=u_1a^\nu bu_1^{-1}=a^{\nu(1+2^{n-1})}a^\nu b$, and consequently $2\lambda\equiv 2\nu(1+2^{n-2}) \pmod{2^n}$, so $2|\nu$, because $2|\lambda$. We have $(a^jbu_1)^2=a^jba^{j(1+2^{n-1})}a^\nu bu_1^2=a^{-j2^{n-1}-\nu+2^{n-1}+\lambda}\in\langle a^2\rangle$, $(a^ju_1)^2=a^{2j(1+2^{n-2})+\lambda}\in\langle a^2\rangle$, $(a^jb)^2=a^{2^{n-1}}\in\langle a^2\rangle$. Thus for any $x\in G_1$, $x^2\in\langle a^2\rangle$, so for every $\sigma\in\mathrm{Aut}(G_1)$, $\sigma(a^2)=(\sigma(a))^2\in\langle a^2\rangle$, proving (6), (8) for $i=1$.

We now proceed to prove the equations (6), (8) for $i+1$, provided that they hold for $i\geqq 1$. If $x\in G_i$, then $x^{2^i}\in\langle a^{2^i}\rangle$ by (8), so $x^{2^{i+1}}\in\langle a^{2^{i+1}}\rangle$. If $x\in(G_{i+1}-G_i)$, we write $x=a^\nu b^{\nu'}u_1^{\nu_1}\cdots u_i^{\nu_i}u_{i+1}$, where $\nu', \nu_1, \cdots, \nu_i=0, 1$. Since $x^2\in G_i$, $x^{2^{i+1}}\in\langle a^{2^i}\rangle$ by (8). Put $x^{2^{i+1}}=a^{2^iz}$. By (7), we have $a^{2^iz}=xa^{2^iz}x^{-1}=a^{\pm 2^iz(1+2^{n-i-1})}=a^{\pm 2^iz+2^{n-1}z}$. Hence if $i<r\leqq n-2$, then $2|z$, so $x^{2^{i+1}}\in\langle a^{2^{i+1}}\rangle$. This proves (8) for $i+1$. For any $\sigma\in\mathrm{Aut}(G_{i+1})$, we have $\sigma(a^{2^{i+1}})=(\sigma(a))^{2^{i+1}}\in\langle a^{2^{i+1}}\rangle$, by what has just been proved. This proves (6) for $i+1$.

The proof of Proposition 1 is completed.

§ 2. Cyclotomic groups and Schur index.

Let ζ_{2^n} ($n\geqq 2$) be a primitive 2^n-th root of unity. Let k be a subfield of $\boldsymbol{Q}(\zeta_{2^n})$. Let B be a *cyclotomic algebra* made with the extension $\boldsymbol{Q}(\zeta_{2^n})/k$, i.e., a crossed product of the form:

$$B=(\beta, \boldsymbol{Q}(\zeta_{2^n})/k)=\sum_{\sigma\in\mathcal{G}}\boldsymbol{Q}(\zeta_b)u_\sigma, \tag{9}$$

$$u_\sigma xu_\sigma^{-1}=x^\sigma \qquad (x\in\boldsymbol{Q}(\zeta_{2^n})), \tag{10}$$

$$u_\sigma u_\tau=\beta(\sigma, \tau)u_{\sigma\tau}, \qquad \beta(\sigma, \tau)\in\langle\zeta_{2^n}\rangle \tag{11}$$

for all $\sigma, \tau\in\mathcal{G}=\mathcal{G}(\boldsymbol{Q}(\zeta_{2^n})/k)$. (See Chapter 2 of [8].)

For a prime $\mathfrak{p}$ of k, $\mathrm{inv}_\mathfrak{p}(B)$ denotes the Hasse invariant of B at $\mathfrak{p}$.

PROPOSITION 2. *Let $B=(\beta, \boldsymbol{Q}(\zeta_{2^n})/k)$ be a cyclotomic algebra defined by*

(9)-(11). *Then the index of* B *equals* 1 *except the case that the automorphism* ι *of the extension* $\boldsymbol{Q}(\zeta_{2^n})/\boldsymbol{Q}$, *defined by* $\zeta_{2^n}^{\iota}=\zeta_{2^n}^{-1}$, *belongs to* $\mathcal{G}(\boldsymbol{Q}(\zeta_{2^n})/k)$ *and* $\beta(\iota,\iota)=-1$. *In this case,* (i) *if* $k\neq\boldsymbol{Q}$, *then* $\mathrm{inv}_{\mathfrak{p}}(B)=0$ *for any finite prime* $\mathfrak{p}$ *of* k, *and* $\mathrm{inv}_{\mathfrak{p}_\infty}(B)=1/2$ *for any infinite prime* $\mathfrak{p}_\infty$ *of* k; (ii) if $k=\boldsymbol{Q}$, *then* $\mathrm{inv}_p(B)=0$ *for any rational prime* $p\neq 2$, ∞, *and* $\mathrm{inv}_2(B)=\mathrm{inv}_\infty(B)=1/2$.

PROOF. Let $\mathfrak{p}$ be a prime of k. If $\mathfrak{p}\nmid 2$, ∞, then $\mathrm{inv}_{\mathfrak{p}}(B)=0$, because $\mathfrak{p}$ is unramified in $\boldsymbol{Q}(\zeta_{2^n})/k$ and the values of the factor set β are roots of unity. If $\mathfrak{p}|2$, then it follows easily from Theorems 3.1 and 4.1 of [7] that $\mathrm{inv}_{\mathfrak{p}}(B)=0$ except the case $k=\boldsymbol{Q}$ and $\beta(\iota,\iota)=-1$, where $\mathrm{inv}_2(B)=1/2$. Let $\mathfrak{p}_\infty$ denote an infinite prime of k. If $\iota\notin\mathcal{G}(\boldsymbol{Q}(\zeta_{2^n})/k)$ then k is not real, so $\mathrm{inv}_{\mathfrak{p}_\infty}(B)=0$. Suppose that $\iota\in\mathcal{G}(\boldsymbol{Q}(\zeta_{2^n})/k)$. We note that $\beta(\iota,\iota)=\pm 1$ (cf. Theorem 4.1 of [7]) and that $B\otimes_k k_{\mathfrak{p}_\infty}\sim(\beta(\iota,\iota),\boldsymbol{C}/\boldsymbol{R},\iota)$, where $\boldsymbol{C}$ and $\boldsymbol{R}$ are the complex numbers and the real numbers, respectively. If $\beta(\iota,\iota)=-1$, then the above cyclic algebra is the ordinary quaternion algebra over $\boldsymbol{R}$ and has index 2. If $\beta(\iota,\iota)=1$, the cyclic algebra has index 1. The assertions of Proposition 2 now follow immediately.

Suppose that in the notation of Proposition 2, $\iota\in\mathcal{G}(\boldsymbol{Q}(\zeta_{2^n})/k)$ and $\beta(\iota,\iota)=-1$. Then the cyclotomic algebra B has index 2, $k=\boldsymbol{Q}(\zeta_{2^{n-r}}+\zeta_{2^{n-r}}^{-1})$ for some r with $0\leqq r\leqq n-2$, and $\mathcal{G}(\boldsymbol{Q}(\zeta_{2^n})/k)=\langle\iota\rangle\times\langle\tau\rangle$, where $(\zeta_{2^n})^{\tau}=\zeta_{2^n}^{1+2^{n-r}}$. We may assume that B is of the form:

$$B=(\beta,\boldsymbol{Q}(\zeta_{2^n})/k)=\sum_{i=0}^{1}\sum_{j=0}^{z-1}\boldsymbol{Q}(\zeta_{2^n})u_\iota^i u_\tau^j\,, \tag{12}$$

$$u_\iota\zeta_{2^n}u_\iota^{-1}=\zeta_{2^n}^{-1}\,,\qquad u_\tau\zeta_{2^n}u_\tau^{-1}=\zeta_{2^n}^{1+2^{n-r}}\,, \tag{13}$$

$$u_\iota^2=\beta(\iota,\iota)=-1\,,\qquad u_\tau^z=1\,,\qquad (z=2^r)\,,\qquad u_\iota u_\tau=u_\tau u_\iota\,. \tag{14}$$

In fact, since $u_\tau u_\tau^z u_\tau^{-1}=u_\tau^z$, we have $u_\tau^z=\zeta_{2^{n-r}}^x$ for some $x\in\boldsymbol{Z}$. Put $c=1+(1+2^{n-r})+\cdots+(1+2^{n-r})^{z-1}$, $(n-r\geqq 2)$. It is easy to see that $c=2^r c'$, $(2,c')=1$. Let y be an integer such that $yc'+x\equiv 0 \pmod{2^{n-r}}$. Then

$$(\zeta_{2^n}^y u_\tau)^z=\zeta_{2^n}^{yc}u_\tau^z=\zeta_{2^{n-r}}^{yc'+x}=1\,,\qquad (\zeta_{2^n}^{2^r}=\zeta_{2^{n-r}})\,.$$

So, from now on we assume $u_\tau^z=1$. Let $u_\iota u_\tau=\zeta_{2^n}^t u_\tau u_\iota$. Then we have

$$1=(u_\tau^z)^{\iota-1}=(\zeta_{2^n}^t)^{1+\tau+\cdots+\tau^{z-1}}=\zeta_{2^n}^{tc}\,.$$

(See the equation (1.11) of [6, p. 582].) So, $2^{n-r}|t$. Putting $j=t/2^{n-r}$ and $v_\iota=\zeta_{2^n}^j u_\iota$, we see easily that $v_\iota u_\tau=u_\tau v_\iota$ and $v_\iota^2=-1$. Hence we may assume $u_\iota u_\tau=u_\tau u_\iota$.

Now put $a=\zeta_{2^n}$, $b=u_\iota$, $u=u_\tau$. Then the cyclotomic algebra B contains the finite group G:

$$G=\langle a,b,u\rangle\,,\quad a^{2^n}=1\,,\quad b^2=a^{2^{n-1}}\,,\quad u^{2^r}=1\,, \tag{15}$$

$$bab^{-1}=a^{-1}, \quad uau^{-1}=a^{1+2^{n-r}}, \quad bu=ub. \tag{16}$$

The cyclotomic algebra B contains the field $\boldsymbol{Q}(\zeta_{2^n})$ as a maximal subfield, and so has an absolutely irreducible, faithful representation U which is realized in $\boldsymbol{Q}(\zeta_{2^n})$. Since the group G spans the algebra B with coefficients in $\boldsymbol{Q}$, the representation U also gives an absolutely irreducible, faithful representation of G, the character χ of which is given by

$$\chi(a^i)=\sum_{\nu=0}^{1}\sum_{\mu=0}^{z-1}(\zeta_{2^n}^i)^{\iota^\nu\tau^\mu}, \qquad (i=0, 1, \cdots, 2^{n-1}) \tag{17}$$

$$\chi(x)=0, \qquad \text{if} \quad x\notin\langle a\rangle. \tag{18}$$

The simple component of the group algebra $\boldsymbol{Q}[G]$ which corresponds to χ is isomorphic to B, so $m_{\boldsymbol{Q}}(\chi)=2$. The group G contains the generalized quaternion group $H_n=\langle a, b\rangle$ of order 2^{n+1}. Let η be the linear character of the cyclic group $\langle a\rangle$, given by $\eta(a^i)=\zeta_{2^n}^i$. Let $\phi=\phi_n$ be the character of H_n given by (2). Then it is easy to see that $\phi=\eta^H$ and $\chi=\eta^G=\phi^G$. Thus we have

THEOREM 2. *Let G be the 2-group defined by* (15)-(16) *and embedded in the cyclotomic algebra B with index 2. Let χ be the faithful irreducible character of G given by* (17)-(18). *Then χ is induced from the faithful irreducible character ϕ_n of the generalized quaternion group $H_n\subset G$, and the Schur index $m_{\boldsymbol{Q}}(\chi)=m_{\boldsymbol{Q}}(\phi_n^G)=2$.*

We will call the 2-group G given by (15)-(16), the *cyclotomic 2-group of type* (n, r) and denote it by $G_{n,r}$ $(0\leqq r\leqq n-2)$.

RRMARK 1. From Satz 12 of [4] we easily conclude that the faithful irreducible characters of $G_{n,r}$ are algebraically conjugate to each other and induced from the faithful irreducible characters of H_n.

REMARK 2. $H_n=G_{n,0}$.

In § 3, we will prove the following.

THEOREM 3. *Let $G=G_{n,r}$ be the cyclotomic 2-group of type (n,r) and χ its faithful irreducible character. Let F be a 2-group such that $[F: G]=2$ and that χ^F is irreducible. Then the Schur index $m_{\boldsymbol{Q}}(\chi^F)=2$.*

COROLLARY 1. *Let H be the generalized quaternion group of order 2^{n+1} and ϕ its faithful irreducible character. Let F be a group such that $[F: H]=2$ and ϕ^F is irreducible. Then $m_{\boldsymbol{Q}}(\phi^F)=2$.*

PROOF. Since $H=G_{n,0}$, the assertion is clear by Theorem 3.

COROLLARY 2. *Let the notation be as in Theorems 2 and 3. Then $H_n\subset G_{n,r}\subset F$. If ϕ_n^F is irreducible, then $m_{\boldsymbol{Q}}(\phi_n^F)=2$.*

PROOF. This follows at once from Theorems 2 and 3.

§ 3. Proof of Theorem 3.

In this section we will use the notation of Theorem 3. Since $[F: G]=2$, there exists an element $v \in F$ such that $v \notin G$, $v^2 \in G$. Since F contains the generalized quaternion group $H=H_n=\langle a, b\rangle$ with $[F: H]=2^{r+1}$, the equation (7) implies that $va^{2^r}v^{-1}=a^{2^r(1+2^{n-r-1})}$. From the equation $uau^{-1}=a^{1+2^{n-r}}$, it follows that $ua^{2^r}u^{-1}=a^{2^r}$, so

$$(vuv^{-1})(va^{2^r}v^{-1})(vuv^{-1})^{-1}=va^{2^r}v^{-1}=a^{2^r(1+2^{n-r-1})}.$$

Writing $vuv^{-1}=a^i b^j u^e$, the left side of the above equation is equal to

$$a^i b^j u^e a^{2^r(1+2^{n-r-1})} u^{-e} b^{-j} a^{-i}=a^{(-1)^j 2^r(1+2^{n-r-1})}.$$

Hence $j \equiv 0 \pmod 2$, so $vuv^{-1}=a^i u^e$. Since $u^{2^r}=1$, we have $1=(a^i u^e)^{2^r}=a^{i2^r l}$ for some $l \in \boldsymbol{Z}$, $2 \nmid l$. Hence $2^{n-r}|i$, so we write $vuv^{-1}=a^{2^{n-r}h}u^e$. It is easy to see that elements $a^\nu u^\mu\ (2|\nu)$ and $a^{\nu'} b u^{\mu'}$ have order less than 2^n, so $vav^{-1}=a^\nu u^\mu$ for some $\nu, \mu \in \boldsymbol{Z}$, $2 \nmid \nu$. Summarizing, we have

$$vav^{-1}=a^\nu u^\mu,\ (2 \nmid \nu), \qquad vuv^{-1}=a^{2^{n-r}h}u^e. \tag{19}$$

LEMMA 1. *Let t be a non-negative integer, and put $\gamma_t=2^{t(n-r)}$, $\delta_t=2^{(t+1)r-tn}$, $M_t=\langle a^{\gamma_t}, u^{\delta_t}\rangle$, $M'_t=\langle a^{\delta_t}, u^{\gamma_t}\rangle$.* (I) *If*

$$\frac{2t-1}{2t}n<r\leqq\frac{2t}{2t+1}n \quad \textit{for some} \quad t>0, \tag{20}$$

then $F \triangleright M_t$ and M_t is abelian. (II) *If*

$$\frac{2t}{2t+1}n<r\leqq\frac{2(t+1)-1}{2(t+1)}n \quad \textit{for some} \quad t\geqq 0, \tag{21}$$

then $F \triangleright M'_t$ and M'_t is abelian.

PROOF. First we will prove $M_t \triangleleft F$ for the case (I). It suffices to prove that the elements $va^{\gamma_t}v^{-1}$, $au^{\delta_t}a^{-1}$, and $vu^{\delta_t}v^{-1}$ belong to M_t. It follows from (20) that $\delta_t \leqq \gamma_t < \delta_{t-1}$. Hence $va^{\gamma_t}v^{-1}=(a^\nu u^\mu)^{\gamma_t}=a^{\gamma_t \nu l}u^{\gamma_t \mu} \in M_t$, $au^{\delta_t}a^{-1}=(aua^{-1})^{\delta_t}=(a^{-2^{n-r}}u)^{\delta_t}=a^{-2^{n-r}\delta_t l'}u^{\delta_t}=a^{-\delta_{t-1}l'}u^{\delta_t} \in M_t$, $vu^{\delta_t}v^{-1}=(a^{2^{n-r}h}u^e)^{\delta_t}=a^{\delta_{t-1}hl''}u^{e\delta_t} \in M_t$, where l, l', l'' are some integers.

The proof of $M'_t \triangleleft F$ for the case (II) is similar. It follows from (21) that $\gamma_t<\delta_t\leqq\gamma_{t+1}$. Hence we have $va^{\delta_t}v^{-1}=(a^\nu u^\mu)^{\delta_t}=a^{\nu\delta_t l}u^{\mu\delta_t} \in M'_t$, $au^{\gamma_t}a^{-1}=(aua^{-1})^{\gamma_t}=(a^{-2^{n-r}}u)^{\gamma_t}=a^{-\gamma_{t+1}l'}u^{\gamma_t} \in M'_t$, $vu^{\gamma_t}v^{-1}=(a^{2^{n-r}h}u^e)^{\gamma_t}=a^{\gamma_{t+1}hl''}u^{e\gamma_t} \in M'_t$, where l, l', l'' are some integers.

Let i and j are non-negative integers with $i+j=r$. Then $u^{2^j}a^{2^i}u^{-2^j}=a^{2^i(1+2^{n-r})2^j}=a^{2^i}$, so the group $\langle a^{2^i}, u^{2^j}\rangle$ is abelian. It is easy to see that $\gamma_t, \delta_t \geqq 1$ by (20), (21) and that $\gamma_t\delta_t=2^r$. Therefore, M_t and M'_t are abelian, completing the proof of Lemma 1.

We observe that there is one and only one integer t which satisfies either (20) or (21). If the integer t satisfies (20), put $M=M_t$. If the integer t satisfies (21), put $M=M'_t$. We write $M=\langle a^{2^i}, u^{2^j}\rangle$, where $2^i=\gamma_t$, $2^j=\delta_t$ for $M=$

M_t, and $2^i=\delta_t$, $2^j=\gamma_t$ for $M=M_t'$. Let $N=\langle a, u^{2^j}\rangle \supset M$. As before, let η be the linear character of $\langle a\rangle$ defined by $\eta(a)=\zeta_{2^n}$. So, $\chi=\eta^G=(\eta^N)^G$.

LEMMA 2. *Let the notation be as above. There exists a linear character* ξ *of* M *such that* $\eta^N=\xi^N$, $\xi(a^{2^i})=\zeta_{2^n}^{2^i}$, $\xi(u^{2^j})=1$.

PROOF. By Lemma 1, $N\rhd M$, N/M is cyclic, and M is abelian. It follows from (9.12) of [2] that $\eta^N|M=\sum_\nu \rho^{n_\nu}$, where ρ is a linear character of M and $\{n_\nu\}$ is a complete system of coset representatives of $I(\rho)$ in N, $I(\rho)$ being the inertial group of ρ. We have $[N: I(\rho)]=\eta^N(1)=[N: \langle a\rangle]=2^{r-j}=2^i=[N: M]$. Hence $I(\rho)=M$. This implies that $\rho_\nu^N=\eta^N$, where $\rho_\nu=\rho^{a^\nu}$ $(\nu=1, \cdots, 2^i)$. Put $w=u^{2^j}$. Then $a^{2^i}w=wa^{2^i}$ and $w^{2^i}=1$, so $\eta^N(a^{2^i})=\eta(a^{2^i})+\eta(wa^{2^i}w^{-1})+\cdots+\eta(w^{2^i-1}a^{2^i}w^{-(2^i-1)})=2^i\eta(a^{2^i})$. On the other hand, we have $\eta^N(a^{2^i})=\rho_\nu(a^{2^i})+\rho_\nu(aa^{2^i}a^{-1})+\cdots+\rho_\nu(a^{2^i-1}a^{2^i}a^{-(2^i-1)})=2^i\rho_\nu(a^{2^i})$. Hence $\rho_\nu(a^{2^i})=\eta(a^{2^i})=\zeta_{2^n}^{2^i}$, $(1\leqq\nu\leqq 2^i)$. If $\rho_\nu(w)=\rho_{\nu'}(w)$, then $\rho_\nu(x)=\rho_{\nu'}(x)$ for all $x\in M$. Therefore, $\rho_\nu(w)\neq\rho_{\nu'}(w)$ for $\nu\neq\nu'$. We have $(\rho_\nu(w))^{2^i}=\rho_\nu(w^{2^i})=\rho_\nu(1)=1$. Hence $\rho_1(w), \cdots, \rho_{2^i}(w)$ are distinct 2^i-th roots of unity, and so for some ν, $\rho_\nu(w)=1$. Then, $\xi=\rho_\nu$ is the linear character of M, as is stated in the lemma.

We now proceed to prove Theorem 3. Put $\chi'=\chi^F$. Recall that $\langle a\rangle\subset N=\langle a, u^{2^j}\rangle\subset G\subset F$, $M=\langle a^{2^i}, u^{2^j}\rangle\subset N$, $\chi=\eta^G=(\eta^N)^G=(\xi^N)^G=\xi^G$. Hence $\chi'=\chi^F=\xi^F$. Put $k=\boldsymbol{Q}(\chi')$. Recall that χ' is induced from the character ϕ of the generalized quaternion group H and that $[F: H]=2^{r+1}$. Hence by Theorem 1, $k=\boldsymbol{Q}(\zeta_{2^{n-r-1}}+\zeta_{2^{n-r-1}}^{-1})$. We also recall that $M\lhd F$ and ξ is a linear character of M. Set $E=\{g\in F\,;\ \xi^g=\xi^{\tau(g)}$ for some $\tau(g)\in\mathcal{G}(k(\xi)/k)\}$. Then by Proposition 3.4 of [8], $k(\xi^E)=k$. Since $(\xi^E)^F=\xi^F=\chi'$, it follows from Corollary 3.9 of [8] that $m_{\boldsymbol{Q}}(\chi')=m_k(\chi')=m_k(\xi^E)$. By Proposition 3.5 of [8], we conclude that $m_k(\xi^E)$ is the index of a cyclotomic algebra of the form: $B=(\beta(\tau, \tau'),\ k(\xi)/k)$. Note that $\boldsymbol{Q}(\xi)=\boldsymbol{Q}(\zeta_{2^{n-i}})\supset\boldsymbol{Q}(\zeta_{2^{n-r}})\supset k$, so $k(\xi)=\boldsymbol{Q}(\xi)$. Let ι denote the automorphism of $\boldsymbol{Q}(\xi)/k$ such that $\zeta_{2^{n-i}}^{\iota}=\zeta_{2^{n-i}}^{-1}$. We have $ba^{2^i}b^{-1}=a^{-2^i}$, $bu^{2^j}b^{-1}=u^{2^j}$, so $\xi^b(a^{2^i})=\xi(a^{-2^i})=\zeta_{2^{n-i}}^{-1}=(\xi(a^{2^i}))^{\iota}$, $\xi^b(u^{2^j})=\xi(u^{2^j})=1=(\xi(u^{2^j}))^{\iota}$. Hence $\xi^b=\xi^{\iota}$, $\tau(b)=\iota$. This implies that $b\in E$. From construction of the cyclotomic algebra B (cf. Proposition 3.5 of [8]) it follows that $\beta(\iota, \iota)=\xi(b^2)=\xi(a^{2^{n-1}})=-1$. Hence by Proposition 2, the index of B equals 2, as was to be shown.

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