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Non-singular bilinear maps which come from some positively filtered rings

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Let K be a commutative ring, and $K[X]$ the polynomial ring over K. Then it is known that $K[X]/(f(X))$ is a free Frobenius extension of K (in the sense of [3]) for a monic polynomial $f(X)$ ([5], [2]). The purpose of this paper is to extend this result to non-commutative rings. To this end we take a "positively filtered ring" satisfying some condition in place of a "polynomial ring" $K[X]$, and an ideal generated by a monic polynomial is replaced by a one sided ideal generated by a monic submodule, which is a generalization of a monic polynomial. Main results are Theorem 9, 11, and 12. In particular, Theorem 12 yields that $K[X]$ is a free Frobenius extension of $K[f(X)]$ for a monic polynomial $f(X)$ over a commutative ring K, and Corollary to Theorem 12 is a generalization of [5; Theorem 2.1].

§ 1.

All rings are associative, but not necessarily commutative. Every ring has 1, which is preserved by homomorphisms, inherited by subrings and acts as the identity operator on modules. Let ${}_AM$, ${}_AN$ be left A-modules over a ring A. By $\mathrm{Hom}_r({}_AM, {}_AN)$ we denote the module of left A-homomorphisms from ${}_AM$ to ${}_AN$ acting on the right side. We denote $\mathrm{Hom}_r({}_AM, {}_AM)$ by $\mathrm{End}_r({}_AM)$. Similarly Hom_l is used for right A-modules and right A-homomorphisms acting on the left side. Let ${}_AM_{A'}$ be a left A, right A'-module. If ${}_AM$ is finitely generated, projective, and generator, and $\mathrm{End}_r({}_AM)\simeq A'$ under the mapping induced by $M_{A'}$, we call ${}_AM_{A'}$ an invertible module. It is well known that this is right-left symmetric.

Let $R\supseteqq K$ be rings, and $R_0=K\subseteqq R_1\subseteqq R_2\subseteqq \cdots$ an ascending sequence of additive subgroups such that $R=\cup R_i$ and $R_i\cdot R_j\subseteqq R_{i+j}$ for all $i, j\geqq 0$. We call $R=\cup R_i$ a positively filtered ring over K. If, further, $R=\cup R_i$ satisfies the following condition we call $R=\cup R_i$ a (*)-positively filtered ring over K:

(*) Each R_n/R_{n-1} $(n\geqq 1)$ is an invertible module as a K-bimodule, and $(R_n/R_{n-1})\otimes_K(R_m/R_{m-1})\simeq R_{n+m}/R_{n+m-1}$ canonically, for all $n, m\geqq 1$.

We denote this by $K[R_1]$, and put $R_i=0$, if $i<0$. For any $i\geqq 0$, we put $gr_iR=R_i/R_{i-1}$. It is easily seen that the latter half of (*) can be replaced by the condition that $R_n=R_1^n$ for all $n\geqq 1$, because both sides are invertible K-bimodules.

For example, let K be a ring, σ an automorphism of K, and D a σ-derivation (i. e. an additive endomorphism of K satisfying $(ab)^D=a^Db^\sigma+ab^D$). Then the skew polynomial ring $R=K[X;\sigma,D]$ defined by $aX=Xa^\sigma+a^D$ $(a\in K)$ is a (*)-positively filtered ring over K, where $R_n=K+XK+\cdots+X^nK$ $(n\geqq 1)$ (cf. [1]). This is the case that $R_1/K\cong K$ as right K-modules (or equivalently, as left K-modules). If $f(X)$ is a monic polynomial of degree n then $R_n=R_{n-1}\oplus Kf(X)=R_{n-1}\oplus f(X)K$. Another example is the tensor K-ring $T(M)$ generated by an invertible K-bimodule M. For any $n\geqq 1$, we put $R_n=K\oplus M\oplus M^2\oplus\cdots\oplus M^n$, where $M^i=M\otimes_K\cdots\otimes_K M$ (i-times). It is evident that $R_{n-1}\oplus M^n=R_n$.

In what follows, $R=K[R_1]$ is always a (*)-positively filtered ring, and unadorned $\otimes$ means $\otimes_K$. Since gr_nR is right K-projective, there is a right K-submodule P_n of R_n such that $R_n=R_{n-1}\oplus P_n$ (direct sum). We call P_n a monic right K-submodule of degree n. Symmetrically we define a monic left K-submodule of degree n. Evidently $P_n\simeq gr_nR$ canonically, as right K-modules. In the sequel P_n and Q_n denote always a monic right K-submodule and a monic left K-submodule, respectively. Then $R_n=K\oplus P_1\oplus\cdots\oplus P_n=K\oplus Q_1\oplus\cdots\oplus Q_n$, and so $R=K\oplus P_1\oplus P_2\oplus\cdots=K\oplus Q_1\oplus Q_2\oplus\cdots$. Therefore, if $n<m$ then $R_n\cap(P_m+P_{m+1}+\cdots)=R_n\cap(Q_m+Q_{m+1}+\cdots)=0$.

PROPOSITION 1. *For all $n, m\geqq 0$, $R_{n+m}=R_{n+m-1}\oplus(P_n\otimes Q_m)$.*

PROOF. There are canonical isomorphisms $P_n\otimes Q_m\simeq gr_nR\otimes gr_mR\simeq gr_{n+m}R$, and therefore $R_{n+m}=R_{n+m-1}\oplus(P_n\otimes Q_m)$.

COROLLARY 1. *For all $n, m\geqq 1$, $R_{n+m}=R_{n-1}\oplus(P_n\otimes R_m)$, and so $R=R_{n-1}\oplus(P_n\otimes R)$. Similar facts hold for monic left K-submodules.*

PROOF. $R_{n+m}=R_{n-1}\oplus P_n\oplus(P_n\otimes Q_1)\oplus\cdots\oplus(P_n\otimes Q_m)=R_{n-1}\oplus(P_n\otimes R_m)$.

COROLLARY 2. *Let Y be a monic K-bisubmodule of degree n. Then $R=R_{n-1}\otimes K[Y]=K[Y]\otimes R_{n-1}$, where $K[Y]=K+Y+Y^2+\cdots$.*

PROOF. $P_i\otimes Y^j$ is a monic right K-submodule of degree $nj+i (i,j\geqq 0)$, and $R_{n-1}\otimes Y^j=Y^j\oplus(P_1\otimes Y^j)\oplus(P_2\otimes Y^j)\oplus\cdots\oplus(P_{n-1}\otimes Y^j)$. Hence $R=\oplus_{j\geqq 0}(R_{n-1}\otimes Y^j)=R_{n-1}\otimes K[Y]$. Similarly we have $R=K[Y]\otimes R_{n-1}$.

Let I be a right ideal of R such that $R=R_{n-1}\oplus I$. Put $P=I\cap R_n$. Then $R_n=R_{n-1}\oplus P$, and so P is a monic right K-submodule of degree n. Therefore $R=R_{n-1}\oplus(P\otimes R)$. Hence $I=P\otimes R$.

PROPOSITION 2. *If $R_1x\subseteqq R_n$ then $x\in R_{n-1}$. Therefore if $R_rx\subseteqq R_n$ for some $r\geqq 1$ then $x\in R_{n-r}$.*

PROOF. We may assume that $n\geqq 0$. Consider a left K-submodule $(Kx+R_{n-1})/R_{n-1}=U$ of gr_nR. Then, under the canonical isomorphism $\mathrm{gr}_1R\otimes gr_nR\simeq\mathrm{gr}_{n+1}R$, the image of $gr_1R\otimes U$ is equal to 0. Hence $U=0$, that is, $x\in R_{n-1}$, because gr_1R is invertible.

PROPOSITION 3. *For any $n, m\geqq 0$, $gr_mR\simeq\mathrm{Hom}_r({}_Kgr_nR,\ {}_Kgr_{n+m}R)$, by right* *multiplication.*

PROOF. This follows easily from that $gr_nR\otimes gr_mR\simeq gr_{n+m}R$ canonically, and that gr_nR is invertible.

PROPOSITION 4. *Let M be a K-bisubmodule of R_n such that $M\supseteqq R_{r-1}$, and such that $gr_rR\oplus(M/R_{r-1})=R_n/R_{r-1}$, where $0\leqq r\leqq n-1$. Let $0\leqq v\leqq r$, and Q_v a monic left K-submodule of degree v, and let x be an element of R_{n-v}. Then $Q_v(x-x_{r-v})\subseteqq M$ for some x_{r-v} in R_{r-v}.*

PROOF. $qx\in R_n=R_r+M$ for all $q\in Q_v$, and so qx is written as $qx=c+d$ $(c\in R_r, d\in M)$. Then the map $q\rightarrow c+R_{r-1}$ is a left K-homomorphism from Q_v to gr_rR. Since $Q_v\simeq gr_vR$ canonically, as left K-modules, Proposition 3 implies that there is an element x_{r-v} in R_{r-v} such that $c+R_{r-1}=qx_{r-v}+R_{r-1}$ for all $q\in Q_v$. Then $Q_v(x-x_{r-v})\subseteqq M$, as desired.

PROPOSITION 5. *Assume that M is as in Proposition 4. Put $K_s=\{x\in R_n\,|\,R_sx\subseteqq M\}$ and $K_s^*=\{x\in R_n\,|\,xR_s\subseteqq M\}$, where $0\leqq s\leqq n$. Then $K_s\cap R_r=K_s^*\cap R_r=R_{r-1-s}$.*

PROOF. Since $R_{r-1}\subseteqq M$, it is evident that $R_{r-1-s}\subseteqq K_s\cap R_r$. Let $x\in K_s\cap R_r$. If $s=0$ then $K_s=M$, and so $K_s\cap R_r=M\cap R_r=R_{r-1}$. Thus we may assume that $s\geqq 1$. Since $M\cap R_r\subseteqq R_{r-1}$, we have $x\in R_{r-1}$, and so $R_1x\subseteqq R_r\cap M\subseteqq R_{r-1}$. Then by Proposition 2, $x\in R_{r-2}$. Therefore $R_2x\subseteqq R_r\cap M\subseteqq R_{r-1}$, and hence $x\in R_{r-3}, \cdots$. Finally $x\in R_{r-1-s}$. Hence $K_s\cap R_r=R_{r-1-s}$. Symmetrically $K_s^*\cap R_r=R_{r-1-s}$.

PROPOSITION 6. *Let M be as in Proposition 4, and $0\leqq s\leqq r$, $r+1+s\leqq n$. Then $R_r+K_s=R_{n-s}$, $R_{r-1}+R_sK_s=M$, and $R_r\cap K_s=R_{r-1-s}$.*

PROOF. Put $K_s=Y$. To prove the first assertion we use Proposition 4. Evidently $R_r+Y\subseteqq R_{n-s}$. If $s=0$ then $Y=M$. Therefore we may assume that $s\geqq 1$. Let $x\in R_{n-s}$ and each Q_v a monic left K-submodule of degree v. Then, since $x\in R_n=R_r+M$, $Q_0(x-x_r)\subseteqq M$ for some $x_r\in R_r$, where $Q_0=K$. Then $Q_1(x-x_r-x_{r-1})\subseteqq M$ for some $x_{r-1}\in R_{r-1}$. If $s\geqq 2$ then $Q_2(x-x_r-x_{r-1}-x_{r-2})\subseteqq M$ for some $x_{r-2}\in R_{r-2}$, and so on. Eventually $Q_s(x-x_r-\cdots-x_{r-s})\subseteqq M$ for some $x_{r-s}\in R_{r-s}$. Then, since $R_{r-1}\subseteqq M$, $Q_v(x-x_r-\cdots-x_{r-s})\subseteqq M$ for all $v=0, \cdots, s$. Therefore $R_s(x-x_r-\cdots-x_{r-s})\subseteqq M$, or equivalently, $x-x_r-\cdots-x_{r-s}\in Y$, and hence $x\in R_r+Y$. Thus $R_{n-s}=R_r+Y$. To prove the second assertion we put $n-s=t$. Then, as $R_r\subseteqq R_{t-1}$, $R_t=R_{t-1}+Y$, and so $R_{t+1}=R_t+R_1Y=R_{t-1}+R_1Y$, because $K\subseteqq R_1$. Then $R_{t+2}=R_t+R_2Y=R_{t-1}+R_2Y$, and so on. Eventually $R_n=R_t+R_sY=R_r+R_sY=R_r+(R_{r-1}+R_sY)$, because $R_t=R_r+Y$. Since $R_{r-1}+R_sY\subseteqq M$, the assumption for M yields that $R_{r-1}+R_sY=M$. The last assertion follows from Proposition 5.

Let $0\leqq s\leqq r, r+1+s\leqq n$, and let Y be a K-bisubmodule of R_{n-s} such that $R_sY\cap R_r\subseteqq R_{r-1}$, $R_r+Y=R_{n-s}$, and $Y\supseteqq R_{r-1-s}$. Then, as in the proof of Proposition 6, we can prove that $R_n=R_r+R_sY$. Therefore if we put $R_{r-1}+R_sY=M$ then M satisfies the conditions in Proposition 4, and so $Y\cap R_r=R_{r-1-s}$ by Proposition 5. Then, by Proposition 5 and Proposition 6, $Y=\{x\in R_n\,|\,R_sx\subseteqq M\}$.

Let U be a left B, right A-module, and V a left A, right B'-module, and let W be an invertible left B, right B'-module. If φ is a bilinear map from $U\times V$ to W such that $\varphi(bu, v)=b\varphi(u, v)$, $\varphi(ua, v)=\varphi(u, av)$, and $\varphi(u, vb')=\varphi(u, v)b'$ ($u\in U, v\in V, b\in B, a\in A, b'\in B'$), φ is called a (B, A, B')-bilinear map. Then φ induces a left A, right B'-homomorphism μ from V to $\mathrm{Hom}_r({}_BU, {}_BW)$. If ${}_BU$ is finitely generated and projective, and μ is an isomorphism, we call φ a non-singular (B, A, B')-bilinear map from $U\times V$ to W. To be easily seen, this is right-left symmetric. In fact $V\simeq\mathrm{Hom}_r({}_BU, {}_BW)$ yields a left B, right A-isomorphism $\mathrm{Hom}_l(V_{B'}, W_{B'})\simeq\mathrm{Hom}_l(\mathrm{Hom}_r({}_BU, {}_BW)_{B'}, W_{B'})$. And, as is well known, the latter is isomorphic to ${}_BU_A$ canonically. Further, since $\mathrm{Hom}_r({}_BU, {}_BW)_{B'}\simeq \mathrm{Hom}_r({}_BU, {}_BB)\otimes_B W_{B'}$, $V_{B'}$ is finitely generated and projective.

LEMMA 7. *Let φ be a non-singular (B, A, B')-bilinear map from $U\times V$ to W, and let I, I' be ideals of B, B' respectively such that $IW=WI'$. Then φ induces a non-singular $(B/I, A, B'/I')$-bilinear map φ_0 from $U/IU\times V/VI'$ to $W/IW=W/WI'$.*

PROOF. To be easily seen U/IU is finitely generated and projective as a left B/I-module, and the homomorphism $V/VI'\to\mathrm{Hom}_r({}_{B/I}U/IU, {}_{B/I}W/IW)$ induced by φ_0 is given by a sequence of isomorphisms $V/VI'\simeq V\otimes_{B'}(B'/I')\simeq\mathrm{Hom}_r({}_BU, {}_BW)\otimes_{B'}(B'/I')\simeq\mathrm{Hom}_r({}_BU, {}_BW\otimes_{B'}(B'/I'))\simeq\mathrm{Hom}_r({}_BU, {}_B(B/I)\otimes_B W)\simeq\mathrm{Hom}_r({}_{B/I}(B/I)\otimes_B U, {}_{B/I}(B/I)\otimes_B W)\simeq\mathrm{Hom}_r({}_{B/I}U/IU, {}_{B/I}W/IW)$.

LEMMA 8. *Let C/B and C'/B' be extension rings over B and B' respectively, and assume that C/B and C'/B' are Morita equivalent by a pair of invertible modules ${}_BW_{B'}\subseteqq {}_CW_{1C'}$, in the sense of* [4; §3]. *Then φ induces a non-singular (C, A, C')-bilinear map φ_1 from $C\otimes_B U\times V\otimes_{B'}C'$ to W_1.*

PROOF. Noting that $W_1=C\otimes_B W=W\otimes_{B'}C'$, the proof proceeds as in the proof of Lemma 7. (In fact, if we take ring homomorphisms in place of ring extensions, Lemma 7 is a special case of Lemma 8. Cf. [4].)

THEOREM 9. *Let M be as in Proposition* 4, *and let $0\leqq s\leqq r$, $0\leqq t\leqq r$,* and *$r\leqq s+t\leqq n$. Then the multiplication in R induces a non-singular (K, K, K)-bilinear map ϕ from $R_s/R_{r-1-t}\times R_t/R_{r-1-s}$ to R_n/M $(\simeq gr_rR)$.*

To prove this we need the following proposition, in which we use the following definition. Let $-1\leqq i<j$, and A a left K-submodule such that $A\supseteqq R_i$, $R_{j-1}/R_i\oplus A/R_i=R_j/R_i$. Then we call A a monic left K-submodule of degree j over R_i. Since $A/R_i\simeq(R_j/R_i)/(R_{j-1}/R_i)\simeq gr_jR$, ${}_KA/R_i$ is projective, and so A is written as $A=R_i\oplus Q_j$ with some monic left K-submodule Q_j of degree j (and conversely). Similarly we define a monic right K-submodule of degree j over R_i.

PROPOSITION 10. *Assume the same assumptions as in Theorem* 9, *and let each A_i be a monic left K-submodule of degree i over R_{r-t-1} $(i=r-t, \cdots, s)$. Then there are monic right K-submodules B_j $(j=r-s, \cdots, t)$ of degree j over R_{r-s-1} such that $A_iB_j\subseteqq M$ provided $i+j\neq r$.*

PROOF. Put $B_{r-j}=\{x\in R_t\mid A_{r-t}x, \cdots, A_{j-1}x, A_{j+1}x, \cdots, A_sx\subseteqq M\}$ for $j=r-t$,

$\cdots, s$. Then $R_{r-s-1} \subseteqq B_{r-j}$ for all j, because $R_{r-1} \subseteqq M$. If $x \in R_{r-j-1} \cap B_{r-j}$ then $A_j x \subseteqq R_{r-1} \subseteqq M$. Further, since $x \in R_t$, $R_{r-t-1} x \subseteqq R_{r-1} \subseteqq M$. Therefore $R_s x \subseteqq M$. Hence $x \in R_{r-s-1}$, by Proposition 5. Thus $R_{r-j-1} \cap B_{r-j} = R_{r-s-1}$. Next we shall show that $R_{r-j-1} + B_{r-j} = R_{r-j}$ $(r-t \leqq j < s)$. Since $R_{j-1} B_{r-j} \subseteqq M$, we have $B_{r-j} \subseteqq K_{j-1} \cap R_r = R_{r-1-(j-1)} = R_{r-j}$ by Proposition 5. Let $A_i = R_{r-t-1} \oplus Q_i$, where Q_i is a monic left K-submodule of degree i $(i=r-t, \cdots, s)$. Let x be any element of R_{r-j} $(r-t \leqq j < s)$. Then $Q_{j+1}(x - x_{r-j-1}) \subseteqq M$ for some $x_{r-j-1} \in R_{r-j-1}$, by Proposition 4. Then $Q_{j+2}(x - x_{r-j-1} - x_{r-j-2}) \subseteqq M$ for some $x_{r-j-2} \in R_{r-j-2}, \cdots$. Finally $Q_s(x - x_{r-j-1} - \cdots - x_{r-s}) \subseteqq M$ for some $x_{r-s} \in R_{r-s}$, because $s+t \leqq n$. On the other hand, if $v \leqq j-1$ then $R_v x \subseteqq R_{r-1} \subseteqq M$. Thus $x - x_{r-j-1} - \cdots - x_{r-s} \in B_{r-j}$, and so $x \in R_{r-j-1} + B_{r-j}$. Hence $R_{r-j-1} + B_{r-j} = R_{r-j}$ for all $j = r-t, \cdots, s$. Finally $B_{r-s} = \{x \in R_t \mid R_{s-1} x \subseteqq M\} \subseteqq R_r \cap K_{s-1} = R_{r-s} \subseteqq B_{r-s}$ by Proposition 5, and so $B_{r-s} = R_{r-s}$. Thus each B_j is a monic right K-bisubmodule of degree j over R_{r-s-1} such that $A_i B_j \subseteqq M$ provided $i+j \neq r$.

PROOF OF THEOREM 9. By the assumption for M, $gr_r R \simeq R_n/M$ canonically, and $gr_{r-t} R \oplus \cdots \oplus gr_s R \simeq Q_{r-t} \oplus \cdots \oplus Q_s \simeq R_s/R_{r-t-1}$ as left K-modules. Therefore $\mathrm{Hom}_r({}_K R_s/R_{r-t-1}, {}_K R_n/M)$ is right K-isomorphic to $\mathrm{Hom}_r({}_K gr_{r-t} R \oplus \cdots \oplus gr_s R, {}_K gr_r R)$. Then, by Proposition 3, the latter is isomorphic to $gr_t R \oplus gr_{t-1} R \oplus \cdots \oplus gr_{r-s} R \simeq R_t/R_{r-s-1}$. Thus we have an isomorphism τ from R_t/R_{r-s-1K} to $\mathrm{Hom}_r({}_K R_s/R_{r-t-1}, {}_K R_n/M)_K$. Take A_i $(i=r-t, \cdots, s)$ and B_j $(j=r-s, \cdots, t)$ as in Proposition 10. Then ${}_K R_s/R_{r-t-1} = A_{r-t}/R_{r-t-1} \oplus \cdots \oplus A_s/R_{r-t-1}$ and $R_t/R_{r-s-1K} = B_{r-s}/R_{r-s-1} \oplus \cdots \oplus B_t/R_{r-s-1}$. Note that A_i and B_i are written as $A_i = R_{r-t-1} \oplus Q_i$ and $B_i = R_{r-s-1} \oplus P_i$ with a monic left K-submodule Q_i and a monic right K-submodule P_i of degree i. Then the isomorphism τ implies that $\mathrm{Hom}_r({}_K R_s/R_{r-t-1}, {}_K R_n/M)$ is equal to the set of all homomorphisms $(a_{r-t} + \cdots + a_s + R_{r-t-1} \to \Sigma_i a_i b_{r-i} + M)$, where $a_i \in A_i$, $b_{r-i} \in B_{r-i}$ $(i=r-t, \cdots, s)$. But $A_i B_j \subseteqq M$ provided $i+j \neq r$, and so $\Sigma_i a_i b_{r-i} + M = (\Sigma_i a_i)(\Sigma_j b_j) + M$. Hence ϕ is non-singular.

THEOREM 11. *Let M be as in Proposition 4, and let $0 \leqq s \leqq r < t < n$, and $r \leqq s+t \leqq n$. Put $Y = \{x \in R_t \mid R_s x \subseteqq M\}$. Then $M = R_{r-1} + R_s K_s$, and the multiplication in R induces a non-singular (K, K, K)-bilinear map φ from $R_s \times R_t/Y$ to R_n/M $(\simeq gr_r R)$.*

PROOF. By Proposition 6, $R_r/R_{r-1-s} \simeq R_{n-s}/K_s$ canonically, $M = R_{r-1} + R_s K_s$, and $R_{n-s} = R_r + K_s$. Therefore $R_t = R_r + (R_t \cap K_s) = R_r + Y$. Then $R_r/R_{r-1-s} \simeq R_t/Y$ canonically, because $R_r \cap Y = R_r \cap R_t \cap K_s = R_r \cap K_s = R_{r-1-s}$. Then, by Theorem 9, φ is non-singular.

Let Y be a monic K-bisubmodule of degree $r+1$ $(r \geqq 0)$. Then $R_r + R_r Y = R_{2r+1}$, and $R_r \cap R_r Y = \{0\} \subseteqq R_{r-1}$ (Corollary 1 to Proposition 1). Therefore, if we put $M = R_{r-1} + R_r Y$ then M satisfies the condition in Proposition 4, where $n = 2r+1$. By Proposition 5 and Proposition 6, there exists uniquely a monic K-bisubmodule Y^* of degree $r+1$ such that $R_{r-1} + R_r Y = R_{r-1} + Y^* R_r$. (In fact, $Y = K_r$ and $Y^* = K_r^*$.) The notation Y^* for a monic submodule Y will be used in the remainder of this paper. Since $R=K[Y^*]R_r=R_rR[Y]$ (Corollary 2 to Proposition 1), we obtain $Y^*R=Y^*R_rK[Y]\subseteqq(R_{r-1}+R_rY)K[Y]\subseteqq R_{r-1}+RY$, and so $R_{r-1}+Y^*R\subseteqq R_{r-1}+RY$. Similarly $R_{r-1}+RY\subseteqq R_{r-1}+Y^*R$, and hence $R_{r-1}+RY=R_{r-1}+Y^*R$. Conversely this equality implies that $R_{r-1}+R_rY=R_{r-1}+Y^*R_r$, because $R_{r-1}+R_rY=R_{2r+1}\cap(R_{r-1}+RY)=R_{2r+1}\cap(R_{r-1}+Y^*R)=R_{r-1}\oplus Y^*R_r$ by Corollary 1 to Proposition 1. Similarly we can see that $R_{r-1}\oplus R_{r-1}Y=R_{r-1}\oplus Y^*R_{r-1}$ and $R_{r-1}\oplus Y=R_{r-1}\oplus Y^*$. And, the former implies that $R_{r-1}K[Y]=K[Y^*]R_{r-1}$, where $K[Y]=K+Y+Y^2+\cdots$. Therefore two ring extensions $K[Y^*]/K$ and $K[Y]/K$ are Morita equivalent by a pair of invertible modules $R_r/R_{r-1}\subseteqq K[Y^*]\otimes(R_r/R_{r-1})=(R_r/R_{r-1})\otimes K[Y]$, in the sense of [4], where $R=K[Y^*]\otimes R_r=R_r\otimes K[Y]$. Apply Lemma 8 and Theorem 9 for $s=t=r$, $n=2r+1$. Then we obtain the following

THEOREM 12. *Let Y be a monic K-bisubmodule of degree $r+1$ $(r\geqq 0)$. Then there exists uniquely a monic K-bisubmodule Y^* of degree $r+1$ such that $R_{r-1}\oplus R_rY=R_{r-1}\oplus Y^*R_r$, and the multiplication in R induces a non-singular $(K[Y^*], R, K[Y])$-bilinear map $R\times R\to R/R_{r-1}K[Y]$.*

COROLLARY. *Assume the same assumptions as in Theorem 12. Then, for any $i\geqq 1$, the multiplication in R induces a non-singular $(K[Y^*]/Y^{*i}K[Y^*], R, K[Y]/K[Y]Y^i)$-bilinear map $R/Y^{*i}R\times R/RY^i\to R/L_i$, where $L_i=Y^{*i}R+K[Y^*]R_{r-1}=RY^i+R_{r-1}K[Y]$.*

PROOF. We have already seen that $R_{r-1}+Y^*R=R_{r-1}+RY$, and $R_{r-1}+Y^*R_{r-1}=R_{r-1}+R_{r-1}Y$. Assume that $Y^{*i}R+K[Y^*]R_{r-1}=RY^i+R_{r-1}K[Y]$. Then $Y^{*i+1}R+K[Y^*]R_{r-1}=Y^*(Y^{*i}R+K[Y^*]R_{r-1})+R_{r-1}=Y^*RY^i+Y^*R_{r-1}K[Y]+R_{r-1}\subseteqq RY^{i+1}+R_{r-1}K[Y]$. Symmetrically $RY^{i+1}+R_{r-1}K[Y]\subseteqq Y^{*i+1}+K[Y^*]R_{r-1}$. Hence $Y^{*i}R+K[Y^*]R_{r-1}=RY^i+R_{r-1}K[Y]$ for all $i=1, 2, 3, \cdots$. Then, by virtue of Lemma 7, this corollary follows from Theorem 12.

PROPOSITION 13. *Let Y and W be monic K-bisubmodules of degree $r+1$ and $s+1$ $(r, s\geqq 0)$ respectively. Then YW is a monic K-bisubmodule of degree $r+s+2$, and $(YW)^*=Y^*W^*$.*

PROOF. The first half is evident from Proposition 1. Now, $R_{r-1}+RY=R_{r-1}+Y^*R$ and $R_{s-1}+RW=R_{s-1}+W^*R$, and these yield $R_{r-1}W+RYW=R_{r-1}W+Y^*RW$ and $Y^*R_{s-1}+Y^*RW=Y^*R_{s-1}+Y^*W^*R$. Then $R_{r+s}+RYW=R_{r+s}Y^*RW=R_{r+s}+Y^*W^*R$, and hence $(YW)^*=Y^*W^*$, as desired.

PROPOSITION 14. *Let Y and W be monic K-bisubmodules of degree $r+1$ $(r\geqq 0)$. Then $R/RY\simeq R/RW$ as left R, right K-modules if and only if $R/Y^*R\simeq R/W^*R$ as left K, right R-modules.*

PROOF. Since $R=R_r\oplus RY$ and $RY+R_{r-1}K[Y]=R_{r-1}\oplus RY$, $R/(RY+R_{r-1}K[Y])$ is isomorphic to gr_rR as K-bimodules, and gr_rR is uniquely determined by r. Hence this proposition follows from Corollary to Theorem 12.

PROPOSITION 15. *Let M be as in Proposition 4.*

(i) *Let $1\leqq s\leqq n$, and $s\leqq n-r-1$, and assume that $R_r+K_s=R_{n-s}$. Then*

$R_r+K_{s-1}=R_{n-s+1}$, and $R_{r-s}+R_1K_s=K_{s-1}$. In particular, if $r<s\leqq n-r-1$, then $R_1K_s=K_{s-1}$.

(ii) *Let $1\leqq s\leqq r-1$ and $s\leqq n-r-1$. Then $R_1K_s=K_{s-1}$.*

(iii) *If $n\geqq 2r+1$ then $K+R_1K_r=K_{r-1}$.*

(iv) *If $n-r\leqq s$ and $r\leqq s$ then $K_s=0$.*

(v) *If $R_r+K_s=R_{n-s}$ for some s such that $r<s\leqq n-r-1$, then $R_r\oplus K_{n-r-1}=R_{r+1}$. Therefore, by* (i), *$R_1K_i=K_{i-1}$ for all $i=r+1,\cdots,n-r-1$.*

PROOF. (i) $R_r+K_s=R_{n-s}$ yields $R_{r+1}+R_1K_s=R_{n-s+1}$. But $r+1\leqq n-s$ by assumption, and so $R_{r+1}\subseteqq R_{n-s}=R_r+K_s$. Hence $R_{n-s+1}=R_r+R_1K_s$. Evidently $R_1K_s\subseteqq K_{s-1}$. Hence $R_r+K_{s-1}=R_{n-s+1}$. Since $R_{r-s}+R_1K_s\subseteqq K_{s-1}$ and $R_r\cap K_{s-1}=R_{r-s}$ (Proposition 6), we obtain $K_{s-1}=R_{r-s}+R_1K_s$. In particular, if $s>r$ then $R_1K_s=K_{s-1}$, because $R_{r-s}=0$. (ii), (iii) Assume $0\leqq s\leqq r$. Then, by Proposition 6, $R_r+K_s=R_{n-s}$. Then, by (i), $R_{r-s}+R_1K_s=K_{s-1}$. But, since $K_s\supseteqq R_{r-1-s}$, we have $R_1K_s\supseteqq R_1R_{r-1-s}$. Thus $R_1K_s=K_{s-1}$, if $s\leqq r-1$. On the other hand, if $s=r$ then $K+R_1K_r=K_{r-1}$. (iv) Since $K_s\subseteqq R_{n-s}\subseteqq R_r$, we have $K_s=K_s\cap R_r=R_{r-1-s}=0$ by Proposition 5. (v) By Proposition 5, $R_r\oplus K_s=R_{n-s}$, and so $R_{r+1}=R_r\oplus(R_{r+1}\cap K_s)$. Hence $R_{r+1}\cap K_s$ is a monic K-bisubmodule of degree $r+1$. Since $K_{s-1}\supseteqq K_s$, $R_{n-s}\cap K_{s-1}=(R_r+K_s)\cap K_{s-1}=(R_r\cap K_{s-1})\oplus K_s$. But, since $r<s$, $R_r\cap K_{s-1}=R_{r-s}=0$ by Proposition 5. Hence $R_{n-s}\cap K_{s-1}=K_s$, that is, $R_1(R_{n-s}\cap K_{s-1})\subseteqq K_{s-1}$. Then $R_{n-r-1-s}(R_{r+1}\cap K_{s-1})\subseteqq R_{n-r-2-s}(R_{r+2}\cap R_{s-1})\subseteqq\cdots\subseteqq R_1(R_{n-s-1}\cap K_{s-1})\subseteqq R_{n-s}\cap K_{s-1}=K_s$, and so $R_{n-r-1}(R_{r+1}\cap K_{s-1})\subseteqq R_sK_s\subseteqq M$, that is, $R_{r+1}\cap K_{s-1}\subseteqq K_{n-r-1}$. Since $R_{r+1}\cap K_{s-1}\supseteqq K_{n-r-1}$ is evident, we obtain $K_{n-r-1}=R_{r+1}\cap K_{s-1}$. Hence $R_{r+1}=R_r\oplus(R_{r+1}\cap K_s)=R_r\oplus K_{n-r-1}$, as desired.

§ 2. Examples.

EXAMPLE 1. Let K be a (commutative) field, σ an automorphism of K such that $\sigma\neq id$ and $\sigma^2=id$. Take an element a of K such that $\sigma(a)\neq a$. We consider the skew polynomial ring $R=K[X;\sigma]$ defined by $Xb=\sigma(b)X$ $(b\in K)$. Put $y=a+X^2$ and $y^*=\sigma(a)+X^2$. Then $Xy^*=yX$, and $Xy=y^*X$. It is easy to see that $Ry\subseteqq K+y^*R$, and similarly $y^*R\subseteqq K+Ry$. Hence $K+Ry=K+y^*R$. But $K+Ry\neq K+yR$, because $\sigma(a)\neq a$.

EXAMPLE 2. Let K be a field of characteristic 2, and D a derivation from K to K such that $D\neq 0$ and $D^2=0$. We consider the skew polynomial ring $R=K[X;D]$ defined by $Xb=bX+D(b)(b\in K)$. Take an element a of K with $D(a)\neq 0$. Put $W=a+X^2$. Then, as $D^2=0$, we have $bW=Wb$ for all $b\in K$, and it is easy to see that $RW\subseteqq K+WR$, and similarly $WR\subseteqq K+RW$. But $XW\notin WR$, because $D(a)\neq 0$. Hence $RW\subsetneqq WR$.

EXAMPLE 3. Let K be a field of characteristic 3, and D_1 a derivation from K to K such that $D_1\neq 0$ and $D_1^3=0$. Then $X^3a=aX^3$ for all $a\in K$, in $K[X;D_1]$. Put $M=R_1\oplus(X^2+X^3)K$, where $R_1=K+XK$. Then M is a K-

bisubmodule of R_3. Assume that $M=R_1\oplus U$ for some K-bisubmodule U. Then, as in easily seen, U is a monic K-bisubmodule of degree 3, and so U is generated by a monic polynomial $f(X)$ of degree 3 such that $f(X)a=af(X)$ for all $a\in K$, because $U\simeq R_3/R_2\simeq K$ as K-bimodules. It is easy to see that $f(X)=a_0+X^3$ for some a_0 in K. Then $M=R_1+f(X)K=R_1\oplus X^3K$. But X^2+X^3 does not belong to $R_1\oplus X^3K$, a contradiction.

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