

Scattering theory for Schrödinger operators with long-range potentials, I, abstract theory

By Hitoshi KITADA

(Received March 8, 1976)

(Revised Dec. 22, 1976)

§ 1. Introduction.

In the recent development of the scattering theory, the Schrödinger operator $-\Delta + V$ in $\mathfrak{H} = L^2(R^N)$ with a real long-range potential V has been one of the main concerns. Roughly speaking, V is said to be long-range when $V(x) = O(|x|^{-\varepsilon_0})$, $|x| \rightarrow \infty$, $\varepsilon_0 > 0$, while short-range when ε_0 can be taken as $\varepsilon_0 > 1$. The study of the long-range scattering was initiated by the work of Dollard [7] for the pure Coulomb potential $V(x) = \text{const.}/|x|$. Among other things he showed in [7] that the ordinary wave operator $W^\pm = \text{s-lim}_{t \rightarrow \pm\infty} e^{itH_2} e^{-itH_1}$, where $H_1 = -\Delta$, $H_2 = -\Delta + V$, does not exist, but alternatively the modified wave operator $W_{\mathcal{D}}^\pm = \text{s-lim}_{t \rightarrow \pm\infty} e^{itH_2} e^{-itH_1 - iX^\pm(t)}$, where $X^\pm(\pm t) = \mathcal{F}^{-1} \left[\int_{\pm 1}^{\pm t} V(s\xi) ds \right] \mathcal{F}$ for $t > 1$, exists and is complete. Here \mathcal{F} denotes the Fourier transformation in $L^2(R^N)$ and by "complete" it is meant that the range of $W_{\mathcal{D}}^\pm$ is equal to the absolutely continuous subspace $\mathfrak{H}_{2,ac}$ of H_2 . Dollard's proof of the completeness is based on the eigenfunction expansions of H_2 .

After Dollard's work many authors have investigated general long-range potentials and proved the existence of modified wave operators (cf. e. g. Buslaev and Matveev [5], Alsholm and Kato [3], Alsholm [2], Hörmander [8]). It may be said that the study of the existence of modified wave operators has now reached a rather satisfactory stage.

As to the completeness of modified wave operators, however, it seems that no results have yet been published except for spherically symmetric long-range potentials (cf. Amrein, Martin and Misra [4])¹⁾. In the case of short-range

1) Professor T. Ikebe also obtained a proof of the completeness of time dependent modified wave operators for the case $\varepsilon_0 > 1/2$, but his method is different from ours mentioned below (a lecture in 1975). Professor S. Agmon also informed the author that he had obtained an eigenfunction expansion theorem and proved the completeness of modified wave operators. His method covers general elliptic operators with general long-range perturbations (private communication). The writer expresses his thanks to Professor Agmon for kindly communicating these results.

scattering, the completeness of wave operators was proved by using the "so-called" stationary method (cf. e.g. Kato and Kuroda [14], Kuroda [17], [18], Saitō [23], [24], Agmon [1]). In these works the so-called limiting absorption principle played an important role among others. Although this principle itself was extended to the long-range case by Ikebe and Saitō [10], it was not enough to assure the completeness of modified wave operators, as stationary methods used in the short-range case could not be applied to the long-range case. Recently Ikebe [9] and Saitō [25]~[28] developed an eigenfunction expansion theory for $H_2 = -\Delta + V$, where V is a long-range potential. Using their eigenfunctions (or more exactly eigenoperators), one can construct a unitary operator $W_{\tilde{I}-S}^\sharp$ from \mathfrak{H} onto $\mathfrak{H}_{2,ac}$ which intertwines H_1 and $H_{2,ac}$, where $H_{2,ac}$ is the restriction of H_2 on $\mathfrak{H}_{2,ac}$. However, the relation between $W_{\tilde{I}-S}^\sharp$ and $W_{\tilde{D}}^\sharp$ has not been clarified and the completeness of time dependent modified wave operators has remained unproved.

The main purpose of the present and the succeeding papers is to prove the completeness of time dependent modified wave operators using a stationary method which gives a connection between the time dependent modified wave operators and the eigenfunction expansions of Ikebe and Saitō mentioned above. Our method has its origin in Kato-Kuroda theory [13], [14], but differs from it in the proof of the completeness of stationary wave operators, inasmuch as our proof is based on the modified Lippmann-Schwinger equation (see Theorem 5.1 and Remark 5.3 below) while the proof of Kato and Kuroda is based on the construction of the inverse of the linkage operator \hat{G} (see Theorem 2.4 of [13]). However the proof of the fact that the time dependent and the stationary wave operators coincide with each other somewhat resembles their proof. To clarify this logical situation, we summarize our abstract framework in Part I, which will be applied to long-range scattering in Part II for a long-range potential $V(x) = O(|x|^{-\varepsilon_0})$, $|x| \rightarrow \infty$, $\varepsilon_0 > 0$. In [16], we have sketched a proof of the completeness for the case $\varepsilon_0 > 1/2$. But the method of proof of the present paper is somewhat different from [16], and the key result, Theorem 3 of [16], which was left unproved, will be proved in Part II as one of the consequences of more general result.

In addition to the results stated above, we prove the invariance principle for modified wave operators²⁾. Moreover we shall prove that our stationary wave operator coincides with those of Pinchuk [22] and Isozaki [11] which are constructed on the basis of the eigenfunction expansions by Ikebe or Saitō

2) In [6] Chandler and Gibson asserted that the invariance principle for long-range scattering holds. However, it seems to the author that the proof is incomplete because the uniform boundedness of $Q_\varphi(t)$ which was tacitly used at the end of the proof of Theorem 1 of § 2, [6], does not seem to follow from the assumptions adopted in [6].

mentioned before.

The plan of Part I is as follows: In § 2, we summarize the main results of this Part in a form which will be used in Part II to prove the completeness. For the purpose of illustration we shall also give some applications to long-range situation with $\epsilon_0 > 1/2$, using the results given by the author in [15]. In § 3, we review some fundamental concepts (especially, that of spectral forms) of Kato-Kuroda theory, and in § 4, we construct a stationary wave operator in essentially the same way as Kato and Kuroda. § 5 is devoted to studying the sufficient conditions for the completeness of stationary wave operators, which are different from those of Kato-Kuroda theory (see Theorem 5.1). Furthermore we shall prove the uniqueness theorem concerning stationary wave operators which will be used in Part II to verify that our stationary wave operator coincides with those of Pinchuk and Isozaki stated above. The results in §§ 3~5 are described in a rather general abstract framework as in [13]. In § 6, we restrict ourselves to considering only self-adjoint operators and investigate the relation between the time dependent and the stationary wave operator, and at the same time prove our main results stated in § 2.

Here the author wishes to express his sincere appreciation to Professor S. T. Kuroda for encouraging and stimulating discussions in the course of the preparation of this paper, and for a comment on the proof of Lemma 6.6 which simplified it.

§ 2. Main results and some applications.

2.1. Main theorems. In the present subsection the main results (Theorems 2.1~2.3) of Part I will be presented in a form which will be used in Part II to prove the completeness of time dependent wave operators for long-range scattering. The theorems are fairly complicated and are combinations of a few theorems, each of which can be stated in a more general context and may have some interests of its own. In subsequent sections, while aiming at proving Theorems 2.1~2.3, we shall state and prove those separate theorems in a general context. This is also a part of our purpose in the present part.

Let H_1 and H_2 be self-adjoint operators in a Hilbert space \mathfrak{H} such that $\mathcal{D}(H_1) = \mathcal{D}(H_2)$ ³⁾. Let E_j be the spectral measure associated with H_j and $E_{j,ac}$ be its absolutely continuous part. Let Γ be a Borel set in R^1 , which we shall fix throughout the paper, and put $\mathfrak{H}_{j,ac}(\Gamma) \equiv E_{j,ac}(\Gamma)\mathfrak{H}$. Let $\{X(t)\}_{t \in R^1}$ be a family of self-adjoint operators in \mathfrak{H} . We shall consider the existence and the completeness of the modified wave operator

$$(2.1) \quad W_D^\pm(\Gamma) = \text{s-lim}_{t \rightarrow \pm\infty} e^{itH_2} e^{-itH_1 - iX(t)} E_{1,ac}(\Gamma).$$

3) $\mathcal{D}(T)$ denotes the domain of an operator T .

The invariance principle will also be discussed. We shall first introduce various conditions concerning H_j , $X(t)$, etc.

We suppose that there are given dense⁴⁾ linear subspaces \mathfrak{X}_j , $j=1, 2$, of \mathfrak{H} having the following properties:

- i) \mathfrak{X}_1 is a separable normed space with its own norm;
- ii) \mathfrak{X}_2 is a separable Hilbert space with its own inner product;
- iii) \mathfrak{X}_j is continuously imbedded into \mathfrak{H} , $j=1, 2$;
- iv) $\mathfrak{X}_1 \subset \mathcal{D}(H_1) = \mathcal{D}(H_2)$, $\mathfrak{X}_1 \subset \mathfrak{X}_2$.

We denote by I_j the imbedding operator $I_j: \mathfrak{X}_j \rightarrow \mathfrak{H}$. Then we can regard \mathfrak{H} as being continuously imbedded into \mathfrak{X}_j^* by I_j^* .

Let $R_j(z) = (H_j - z)^{-1}$ for $z \in \mathbf{C}^\pm = \{z \in \mathbf{C} \mid \text{Im } z \gtrless 0\}$ ⁵⁾. We first suppose that the following condition (L.A.P.) is satisfied.

(L. A. P.) Put $K_{\Gamma}^\pm = \{z \in \mathbf{C}^\pm \mid \text{Re } z \in \Gamma\}$ ⁶⁾. Let $j=1$ or 2 . Then the mapping

$$(2.2) \quad K_{\Gamma}^\pm \times \mathfrak{X}_j \ni (z, x) \longmapsto I_j^* R_j(z) I_j x \in \mathfrak{X}_j^*$$

can be extended uniquely to a continuous mapping

$$(2.3) \quad \bar{K}_{\Gamma}^\pm \times \mathfrak{X}_j \longrightarrow \mathfrak{X}_j^*,$$

and satisfies

$$(2.4) \quad \sup_{z \in K_{\Gamma}^\pm} \|I_j^* R_j(z) I_j\|_{B(\mathfrak{X}_j, \mathfrak{X}_j^*)} < \infty.$$

Here $B(\mathfrak{X}, \mathfrak{Y})$ denotes the Banach space of all bounded linear operators from a normed space \mathfrak{X} to a Banach space \mathfrak{Y} . (Condition (L.A.P.) is the so-called limiting absorption principle.)

Now put for $z \in \mathbf{C}^\pm$, $x, y \in \mathfrak{H}$, and $j=1, 2$,

$$(2.5) \quad \begin{aligned} e_j(z; x, y) &= \frac{\text{sgn}(\text{Im } z)}{2\pi i} (\{R_j(z) - R_j(\bar{z})\} x, y)_{\mathfrak{H}} \quad 7) \\ &= \frac{|\text{Im } z|}{\pi} (R_j(z)x, R_j(z)y)_{\mathfrak{H}}. \end{aligned}$$

Then if $x, y \in \mathfrak{X}_j$ and $\lambda \in \Gamma$, $e_j(\lambda \pm i\varepsilon; x, y)$ has a boundary value as $\varepsilon \rightarrow +0$. We denote it also by $e_j(\lambda; x, y)$ ⁸⁾ for $\lambda \in \Gamma$, $x, y \in \mathfrak{X}_j$, and $j=1, 2$. Then e_j satisfies

4) It is not indispensable to assume \mathfrak{X}_j is dense in \mathfrak{H} . It suffices to assume that the set $\{\sum_{k=1}^n E_{j,ac}(\Delta_k) x_k \mid \Delta_k \subset \Gamma, x_k \in \mathfrak{X}_j\}$, $j=1, 2$, is dense in $E_{j,ac}(\Gamma)\mathfrak{H}$. See §6.

5) \mathbf{C} =complex numbers and Im =imaginary part.

6) Re =real part.

7) $\text{sgn}(t)=1$ for $t>0$, and $=-1$ for $t<0$.

8) Notice that the boundary values of $e_j(\lambda \pm i\varepsilon; x, y)$ from the upper and the lower complex plain coincide with each other.

the following two properties:

1° For each $x, y \in \mathfrak{X}_j$, one has $e_j(\cdot; x, y) \in L^1(\Gamma)$ and

$$(2.6) \quad \int_{\mathcal{A}} e_j(\lambda; x, y) d\lambda = (E_{j,ac}(\mathcal{A})x, y)_{\mathfrak{H}}$$

for any Borel subset \mathcal{A} of Γ .

2° For each $\lambda \in \Gamma$, $e_j(\lambda; \cdot, \cdot)$ is a nonnegative Hermitian form on $\mathfrak{X}_j \times \mathfrak{X}_j$.

As to the family $\{X(t)\}_{t \in R^1}$ of self-adjoint operators in \mathfrak{H} , we suppose that the following conditions (X), (BC), and (XA) are satisfied:

(X) a) $H_1 e^{-iX(t)} \supseteq e^{-iX(t)} H_1$ for $t \in R^1$;

b) $e^{-iX(t)}u$ is continuous as an \mathfrak{H} -valued function of $t \in R^1$ for every $u \in \mathfrak{H}$.

(BC) For any $x \in \mathfrak{X}_1$ and $u \in \mathfrak{H}$, $(e^{-isH_1 - iX(s)}x, u)_{\mathfrak{H}}$ is of bounded variation in s on any finite interval of R^1 , and is continuous in $s \in R^1$.

(XA) For any $u \in \mathfrak{H}$ and $s \in R^1$, $\lim_{t \rightarrow \pm\infty} e^{i(X(t+s) - X(t))}u = u$.

Now for every $u, v \in \mathfrak{H}$ and $z \in C^\pm$, put

$$(2.7) \quad S^\pm(z)u = \pm i \int_{-\infty}^{\infty} \chi_\pm(t) e^{-iX(t)} e^{it(z - H_1)} u dt,$$

$$(2.8) \quad f^\pm(z; u, v) = \frac{|\text{Im } z|}{\pi} (S^\pm(z)u, S^\pm(z)v)_{\mathfrak{H}},$$

where χ_+ and χ_- denote the characteristic functions of $(0, \infty)$ and $(-\infty, 0)$, respectively.

It is clear that $S^\pm(z) \in B(\mathfrak{H}) \equiv B(\mathfrak{H}, \mathfrak{H})$ so that f^\pm is well-defined. Moreover, using the same argument as in the proof of (1.16) in the proof of Proposition 1.4 of [15], we can show that $S^\pm(z)u \in \mathcal{D}(H_1) = \mathcal{D}(H_2)$ if $u \in \mathcal{D}(H_1)$ and $z \in C^\pm$.

We now define for $u \in \mathcal{D}(H_1)$ and $z \in C^\pm$,

$$(2.9) \quad G^\pm(z)u = (H_2 - z)S^\pm(z)u,$$

$$(2.10) \quad Q^\pm(z)u = G^\pm(z)u - u,$$

and suppose that the following conditions (Q^\pm) and (f^\pm) are satisfied.

(Q^\pm) a) For every $z \in C^\pm$, the operator $Q^\pm(z)$ maps \mathfrak{X}_1 into \mathfrak{X}_2 ;

b) For every $\varepsilon > 0$ and $x \in \mathfrak{X}_1$, $Q^\pm(\lambda \pm i\varepsilon)x$ belongs to $L^2(R^1; \mathfrak{X}_2)$ as an \mathfrak{X}_2 -valued function of λ in R^1 , and the limit $\text{l.i.m.}_{\varepsilon \rightarrow +0} Q^\pm(\lambda \pm i\varepsilon)x$ exists in $L^2(R^1; \mathfrak{X}_2)$.

(f^\pm) For every $x, y \in \mathfrak{X}_1$, there exists the limit $\text{l.i.m.}_{\varepsilon \rightarrow +0} f^\pm(\lambda \pm i\varepsilon; x, y)$ in $L^2(\Gamma)$ as a C -valued function of λ in Γ and we have

$$(2.11) \quad \text{l.i.m.}_{\varepsilon \rightarrow +0} f^\pm(\lambda \pm i\varepsilon; x, y) = e_1(\lambda; x, y), \quad \text{a. e. } \lambda \in \Gamma.$$

Now we can state a theorem concerning the existence of the limit (2.1).

THEOREM 2.1. Suppose that conditions i)-iv), (L.A.P.), (X), (BC), (XA), (Q[±]), and (f[±]) are satisfied. Then the modified wave operator

$$(2.12) \quad W_{\mathfrak{D}}^{\pm}(\Gamma) = \mathbf{s}\text{-}\lim_{t \rightarrow \pm\infty} e^{itH_2} e^{-itH_1 - iX(t)} E_{1,ac}(\Gamma)$$

exists. This operator $W_{\mathfrak{D}}^{\pm}(\Gamma)$ is a partial isometry in \mathfrak{H} with initial set $\mathfrak{H}_{1,ac}(\Gamma)$ and final set contained in $\mathfrak{H}_{2,ac}(\Gamma)$, and satisfies the intertwining property: for any Borel set Δ in R^1 ,

$$(2.13) \quad W_{\mathfrak{D}}^{\pm}(\Gamma) E_1(\Delta) = E_2(\Delta) W_{\mathfrak{D}}^{\pm}(\Gamma).$$

Next we state a theorem concerning the completeness of $W_{\mathfrak{D}}^{\pm}(\Gamma)$.

THEOREM 2.2. Suppose that conditions i)-iv), (L.A.P.), (X), (BC), (XA), and (Q[±]) are satisfied. Suppose further that there exist a Hilbert space \mathfrak{h} and a linear operator $\mathfrak{F}_j^{\pm}(\lambda): \mathfrak{X}_j \rightarrow \mathfrak{h}$, $\lambda \in \Gamma$, $j=1, 2$, satisfying the following conditions:

(a) For every $\lambda \in \Gamma$ and $x, y \in \mathfrak{X}_j$,

$$(2.14) \quad (\mathfrak{F}_j^{\pm}(\lambda)x, \mathfrak{F}_j^{\pm}(\lambda)y)_{\mathfrak{h}} = e_j(\lambda; x, y).$$

(b) The range of $\mathfrak{F}_1^{\pm}(\lambda)$ is dense in \mathfrak{h} for every $\lambda \in \Gamma$.

(c) For every $x \in \mathfrak{X}_1$,

$$(2.15) \quad \mathfrak{F}_2^{\pm}(\lambda)(x + \text{l.i.m.}_{\varepsilon \rightarrow +0} Q^{\pm}(\lambda \pm i\varepsilon)x) = \mathfrak{F}_1^{\pm}(\lambda)x, \quad \text{a. e. } \lambda \in \Gamma.$$

Then the limit $W_{\mathfrak{D}}^{\pm}(\Gamma)$ in (2.12) exists and satisfies the same properties as in Theorem 2.1. Moreover $W_{\mathfrak{D}}^{\pm}(\Gamma)$ is complete, i. e. $\mathcal{R}(W_{\mathfrak{D}}^{\pm}(\Gamma)) = \mathfrak{H}_{2,ac}(\Gamma)^{9)}$.

We shall next discuss the invariance principle. Assume that Γ is bounded. Let φ be a real-valued Borel measurable function on R^1 which belongs to $C^2(I)$ on some bounded open interval I containing Γ . Take $\eta \in C_0^{\infty}(R^1)$ so that $\eta(\lambda) = 1$ on Γ and $\text{supp } \eta \subset I$. Put

$$(2.16) \quad a_{\varphi}(t, r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \eta(\lambda) e^{-it\varphi(\lambda) + ir\lambda} d\lambda.$$

Then $a_{\varphi}(t, \cdot) \in L^1(R^1)$ for any $t \in R^1$. Thus if we put

$$(2.17) \quad Q_{\varphi}(t)u = \int_{-\infty}^{\infty} a_{\varphi}(t, r) e^{-irH_1 - iX(r)} u dr$$

for $u \in \mathfrak{H}$ and $t \in R^1$, then $Q_{\varphi}(t) \in B(\mathfrak{H})$ for every $t \in R^1$.

THEOREM 2.3. Suppose that all the assumptions of Theorem 2.1 are satisfied and assume that Γ is bounded. Let $I, \varphi, \eta, a_{\varphi}, Q_{\varphi}(t)$ be as above and let $\varphi' > 0$ on I . Suppose furthermore that the following conditions (Q_φ^{as}) and (X₁) are satisfied:

(Q_φ^{as}) For any $t \in R^1$, there exists an isometric operator $Q_{\varphi}^{\text{as}}(t): \mathfrak{H}_{1,ac}(\Gamma) \rightarrow \mathfrak{H}$

9) $\mathcal{R}(T)$ denotes the range of an operator T .

such that

$$(2.18) \quad \lim_{t \rightarrow \pm\infty} (Q_\varphi^{as}(t)u - Q_\varphi(t)u) = 0 \quad \text{for } u \in \mathfrak{H}_{1,ac}(\Gamma).$$

$$(\mathfrak{K}_1) \quad \mathfrak{K}_1 \cap \mathfrak{H}_{1,ac}(\Gamma) \text{ is dense in } \mathfrak{H}_{1,ac}(\Gamma).$$

Then the limits

$$(2.19) \quad \begin{cases} W_\varphi^{\pm,as}(\Gamma) = \mathbf{s}\text{-}\lim_{t \rightarrow \pm\infty} e^{it\varphi(H_2)} Q_\varphi^{as}(t) E_{1,ac}(\Gamma), \\ W_\varphi^\pm(\Gamma) = \mathbf{s}\text{-}\lim_{t \rightarrow \pm\infty} e^{it\varphi(H_2)} Q_\varphi(t) E_{1,ac}(\Gamma) \end{cases}$$

exist and we have

$$(2.20) \quad W_{\tilde{D}}^\pm(\Gamma) = W_\varphi^{\pm,as}(\Gamma) = W_\varphi^\pm(\Gamma).$$

REMARK 2.4. Notice that in Theorem 2.2 we do not assume condition (f^\pm) . This is possible because we assume the existence of \mathfrak{h} and $\mathcal{F}_j^\pm(\lambda)$ satisfying (a), (b), and (c). (See § 6.4.) Similarly, we need not assume (f^\pm) in Theorem 2.3 if we assume the existence of such \mathfrak{h} and $\mathcal{F}_j^\pm(\lambda)$. (See § 6.5.)

REMARK 2.5. In the theorems given above our main results are described in the “so-called” time dependent way. Our methods in the present paper, however, are basically of “stationary” nature. Namely, under the assumptions of these theorems we shall first construct a stationary wave operator $W_{\tilde{F}}^\pm$ and then prove that $W_{\tilde{D}}^\pm(\Gamma)$, $W_\varphi^{\pm,as}(\Gamma)$, and $W_\varphi^\pm(\Gamma)$ exist and coincide with $W_{\tilde{F}}^\pm$. Especially, the completeness of $W_{\tilde{D}}^\pm(\Gamma)$ follows from that of $W_{\tilde{F}}^\pm$, the proof of which is essentially based on the “so-called” stationary (time independent) method. Unfortunately, it is not so simple to write down briefly the stationary definition of $W_{\tilde{F}}^\pm$. Thus, in the present subsection we preferred to state our main results in a form given above.

2.2. Some applications. In [15], we considered self-adjoint operators $H_1 = -\frac{1}{2}\Delta$ and $H_2 = H_1 + U$ in $\mathfrak{H} = L^2(\mathbb{R}^N)$ under Assumption 1.1 of [15] on $U = V_s + V$ and constructed the stationary wave operator $W_{\tilde{F}}^\pm$, $\Gamma = [a, b]$, $0 < a < b < \infty$, on the basis of the theorems which are almost identical with Theorems 4.1 and 4.4 of the present paper. Furthermore, we announced (see § 4 of [15]) that we can prove the existence of time dependent modified wave operators, its coincidence with $W_{\tilde{F}}^\pm$, and moreover the invariance principle. In this subsection we shall prove them using Theorems 2.1 and 2.3 given above and some results given in § 6 below.

Suppose that Assumption 1.1 of [15] is satisfied. Put $\mathfrak{K}_1 = \mathcal{F}^{-1}(C_0^\infty(\mathbb{R}^N - \{0\}))$ endowed with the norm of $L_\delta^2(\mathbb{R}^N) \equiv L^2(\mathbb{R}^N, (1 + |x|)^{2\delta} dx)$ and $\mathfrak{K}_2 = L_\delta^2(\mathbb{R}^N)$, where δ is taken as in page 320 of [15], and \mathcal{F} denotes the Fourier transformation in $L^2(\mathbb{R}^N)$. Then obviously \mathfrak{K}_j is dense in \mathfrak{H} and satisfies i)-iv) of the previous subsection. Furthermore (L.A.P.) is assured by Theorem 2.1 of [15] which is

due to Ikebe and Saitō [10]. (X) is obviously satisfied if we put $X(t) = \mathcal{F}^{-1} \left[\int_0^t V(s\xi) ds \cdot \right] \mathcal{F}$ as in [15]. Conditions (BC) and (XA) are obviously satisfied by the definition of $X(t)$ and Assumption 1.1 of [15]. (Q^\pm) and (f^\pm) hold by Propositions 2.3 and 2.4 of [15], respectively. Thus by Theorem 2.1 above, the limit

$$(2.21) \quad W_{\mathcal{D}}^\pm(\Gamma) = \mathbf{s}\text{-}\lim_{t \rightarrow \pm\infty} e^{itH_2} e^{-itH_1 - iX(t)} E_{1,ac}(\Gamma)$$

exists. Furthermore by the result of § 6.4 below, we have

$$(2.22) \quad W_{\mathcal{D}}^\pm(\Gamma) = W_{\mathcal{F}}^\pm,$$

where $W_{\mathcal{F}}^\pm$ is the stationary wave operator constructed in [15].

Next we shall prove the invariance principle. Let $I, \varphi, \eta, a_\varphi$, and $Q_\varphi(t)$ be as in the previous subsection for $\Gamma = [a, b]$ and $H_1 = -\frac{1}{2}\Delta$. Then the following lemma holds.

LEMMA 2.6 (due to Matveev [21]). *Let Assumption 1.1 of [15] be satisfied. Let $\varphi \in C^\infty(I)$ and $\varphi' > 0, \varphi'' \neq 0$ on I . Put*

$$Q_\varphi^{as}(t) = e^{-it\varphi(H_1) - iX(t\varphi'(H_1))} \quad \text{on } \mathfrak{H}_{1,ac}(\Gamma),$$

where $X(t\varphi'(H_1))$ is defined as follows:

$$X(t\varphi'(H_1)) = \mathcal{F}^{-1} \left[\int_0^{t\varphi'(|\xi|^{2/2})} V(s\xi) ds \cdot \right] \mathcal{F} \quad \text{on } \mathfrak{H}_{1,ac}(\Gamma).$$

Then $Q_\varphi^{as}(t) : \mathfrak{H}_{1,ac}(\Gamma) \rightarrow \mathfrak{H}$ is an isometric operator and satisfies

$$\lim_{t \rightarrow \pm\infty} \|(Q_\varphi^{as}(t) - Q_\varphi(t)) E_{1,ac}(\Gamma)\|_{B(\mathfrak{H})} = 0.$$

For the proof of this lemma, see the proof of Theorem 4 of Matveev [21].

Thus condition (Q_φ^{as}) is satisfied. Since (\mathfrak{K}_1) is obviously satisfied by the definition of \mathfrak{K}_1 , it follows from Theorem 2.3 that there exist the limits

$$\begin{cases} W_{\mathcal{F}}^{\pm, as}(\Gamma) = \mathbf{s}\text{-}\lim_{t \rightarrow \pm\infty} e^{it\varphi(H_2)} Q_\varphi^{as}(t) E_{1,ac}(\Gamma), \\ W_{\mathcal{F}}^\pm(\Gamma) = \mathbf{s}\text{-}\lim_{t \rightarrow \pm\infty} e^{it\varphi(H_2)} Q_\varphi(t) E_{1,ac}(\Gamma), \end{cases}$$

and we have

$$(2.23) \quad W_{\mathcal{F}}^{\pm, as}(\Gamma) = W_{\mathcal{F}}^\pm(\Gamma) = W_{\mathcal{D}}^\pm(\Gamma) = W_{\mathcal{F}}^\pm.$$

The completeness of $W_{\mathcal{D}}^\pm(\Gamma) = W_{\mathcal{F}}^\pm$ will be proved in Part II using Theorem 2.2 given above as one of the consequences derived from more general result including the long-range potential $V(x) = O(|x|^{-\varepsilon_0}), \varepsilon_0 > 0$.

§ 3. Some fundamental concepts.

In the subsequent two sections (i. e. §§ 4~5), we shall consider a stationary wave operator in a rather general context, and prove its completeness and uniqueness under certain assumptions. This section is devoted to reviewing several fundamental concepts due to Kato and Kuroda [13], [14] which will be used as our basic framework in those sections. We shall also prepare a proposition which will be used to verify the uniqueness theorem in § 5.

Let (Γ_0, \mathbf{B}) be a measurable space and m a σ -finite, complete, and non-negative scalar measure on (Γ_0, \mathbf{B}) . Let \mathfrak{H} be a complex Hilbert space and E a spectral measure with values in the set of all orthogonal projections in \mathfrak{H} . We denote by E_{ac} the absolutely continuous part of E . We call the 5-tuple $(\Gamma_0, \mathbf{B}, m, \mathfrak{H}, E)$ a spectral system.

Suppose that a subset $\Gamma \subset \Gamma_0$, $\Gamma \in \mathbf{B}$, and a linear subspace $\mathfrak{X} \subset \mathfrak{H}$ are given. A triple $(\Gamma, \mathfrak{X}, e)$ is called a spectral form for $(\Gamma_0, \mathbf{B}, m, \mathfrak{H}, E)$, if e is a complex valued function on $\Gamma \times \mathfrak{X} \times \mathfrak{X}$ satisfying the following two properties:

1° For each $x, y \in \mathfrak{X}$, $e(\cdot; x, y) \in L^1(\Gamma)$ and

$$(3.1) \quad \int_{\Delta} e(\lambda; x, y) m(d\lambda) = (E_{ac}(\Delta)x, y)_{\mathfrak{H}}$$

for any $\Delta \subset \Gamma$, $\Delta \in \mathbf{B}$.

2° For each $\lambda \in \Gamma$, $e(\lambda; \cdot, \cdot)$ is a nonnegative Hermitian form on $\mathfrak{X} \times \mathfrak{X}$.

Given a spectral form $(\Gamma, \mathfrak{X}, e)$, put $\mathfrak{N}(\lambda) = \{x \in \mathfrak{X} \mid e(\lambda; x, x) = 0\}$ and let $\mathfrak{X}(\lambda)$ be the quotient space $\mathfrak{X}/\mathfrak{N}(\lambda)$ for $\lambda \in \Gamma$. Then $\mathfrak{X}(\lambda)$ is a pre-Hilbert space with respect to the inner product induced by $e(\lambda; \cdot, \cdot)$. We denote by $\tilde{\mathfrak{X}}(\lambda)$ the completion of $\mathfrak{X}(\lambda)$, by $(\cdot, \cdot)_{\lambda}$ and $\|\cdot\|_{\lambda}$ the inner product and the norm of $\tilde{\mathfrak{X}}(\lambda)$, and by $J(\lambda)$ the canonical homomorphism from \mathfrak{X} to $\tilde{\mathfrak{X}}(\lambda)$. Then the range of $J(\lambda)$ is equal to $\mathfrak{X}(\lambda)$. Consider the product vector space $\tilde{\mathfrak{X}} = \prod_{\lambda \in \Gamma} \tilde{\mathfrak{X}}(\lambda)$ consisting of all vector fields $g = \{g(\lambda)\}_{\lambda \in \Gamma}$ with $g(\lambda) \in \tilde{\mathfrak{X}}(\lambda)$. We say two elements $g_1, g_2 \in \tilde{\mathfrak{X}}$ are equivalent, in symbol $g_1 \sim g_2$, if $g_1(\lambda) = g_2(\lambda)$ for m -a. e. $\lambda \in \Gamma$, and denote by $[g]$ the equivalence class of $g \in \tilde{\mathfrak{X}}$ by \sim . We define the space \mathcal{S} of all simple functions from Γ to \mathfrak{X} as follows:

$$(3.2) \quad \mathcal{S} = \left\{ h : \Gamma \rightarrow \mathfrak{X} \mid h(\lambda) = \sum_{k=1}^n \chi_{\Delta_k}(\lambda) x_k, x_k \in \mathfrak{X}, \Delta_k \in \mathbf{B}, \Delta_k \subset \Gamma \right\},$$

where χ_{Δ} denotes the characteristic function of $\Delta \in \mathbf{B}$. A vector field $g = \{g(\lambda)\}_{\lambda \in \Gamma} \in \tilde{\mathfrak{X}}$ is called e -measurable, if there exists some sequence $\{h_n\} \subset \mathcal{S}$ such that

$$(3.3) \quad \lim_{n \rightarrow \infty} \|g(\lambda) - J(\lambda)h_n(\lambda)\|_{\lambda} = 0, \quad m\text{-a. e. } \lambda \in \Gamma.$$

Put

$$(3.4) \quad \mathfrak{M} = \left\{ [g] \in \tilde{\mathfrak{X}} / \sim \mid g \text{ is } e\text{-measurable, } \|[g]\|_{\mathfrak{M}} = \left[\int_{\Gamma} \|g(\lambda)\|_{\lambda}^2 m(d\lambda) \right]^{1/2} < \infty \right\}.$$

Then \mathfrak{M} is a Hilbert space with respect to the inner product:

$$(3.5) \quad ([f], [g])_{\mathfrak{M}} = \int_{\Gamma} (f(\lambda), g(\lambda))_{\lambda} m(d\lambda), \quad [f], [g] \in \mathfrak{M}.$$

Furthermore $(JS)/\sim = \{ [\{ J(\lambda)h(\lambda) \}_{\lambda \in \Gamma}] \mid h \in \mathcal{S} \}$ is a dense linear subspace of \mathfrak{M} . Define

$$(3.6) \quad \mathfrak{L}_{ac}^{\circ}(\Gamma) = \left\{ \sum_{k=1}^n E_{ac}(\Delta_k) x_k \mid \Delta_k \subset \Gamma, \Delta_k \in \mathbf{B}, x_k \in \mathfrak{X} \right\}$$

and let $\mathfrak{L}_{ac}(\Gamma)$ be the closure of $\mathfrak{L}_{ac}^{\circ}(\Gamma)$ in \mathfrak{H} . Then $\mathfrak{L}_{ac}(\Gamma)$ is a closed linear subspace of $\mathfrak{H}_{ac}(\Gamma) \equiv E_{ac}(\Gamma)\mathfrak{H}$ and reduces E . Furthermore we denote by \mathfrak{L} the smallest closed subspace of \mathfrak{H} containing \mathfrak{X} and reducing E . Then we have $\mathfrak{L}_{ac}(\Gamma) = E_{ac}(\Gamma)\mathfrak{L}$. $\mathfrak{L}_{ac}(\Gamma)$ coincides with $\mathfrak{H}_{ac}(\Gamma)$ if \mathfrak{X} is dense in \mathfrak{H} . In the sequel, equivalent elements are to be identified in \mathfrak{M} . (As to the several facts stated here, see Kato and Kuroda [13], [14].)

Following Kato and Kuroda we now define a unitary operator $\pi: \mathfrak{L}_{ac}(\Gamma) \rightarrow \mathfrak{M}$ with the following properties:

- (a) For any $\Delta \in \mathbf{B}$ and $u \in \mathfrak{L}_{ac}(\Gamma)$,

$$(3.7) \quad \pi E_{ac}(\Delta)u = \{ \chi_{\Delta}(\lambda)(\pi u)(\lambda) \}_{\lambda \in \Gamma}.$$

- (b) For any $x \in \mathfrak{X}$,

$$(3.8) \quad \pi E_{ac}(\Gamma)x = \{ J(\lambda)x \}_{\lambda \in \Gamma}.$$

Thus π gives a spectral representation of E_{ac} . (See Kato and Kuroda [13], Theorem 1.11.)

Next we consider another spectral form $(\Gamma, \mathfrak{X}', e')$ for the same spectral system $(\Gamma_0, \mathbf{B}, m, \mathfrak{H}, E)$, and investigate some relation between $(\Gamma, \mathfrak{X}, e)$ and $(\Gamma, \mathfrak{X}', e')$ which will be used in §5 to prove the uniqueness theorem. For this purpose we use the obvious notations such as $\mathfrak{N}'(\lambda)$, $\mathfrak{X}'(\lambda)$, $\tilde{\mathfrak{X}}'(\lambda)$, etc., defined as above for $(\Gamma, \mathfrak{X}', e')$.

PROPOSITION 3.1. Assume that the following conditions (a)-(d) are satisfied:

- (a) \mathfrak{X} and \mathfrak{X}' are Hausdorff linear topological spaces;
 (b) $J(\lambda)$ and $J'(\lambda)$ are continuous for all $\lambda \in \Gamma$;
 (c) $\mathfrak{X}'' \equiv \mathfrak{X} \cap \mathfrak{X}'$ is dense in \mathfrak{X} and \mathfrak{X}' ;
 (d) $e(\lambda; x, y) = e'(\lambda; x, y)$ for all $x, y \in \mathfrak{X}''$ and $\lambda \in \Gamma$.

Then for any $\lambda \in \Gamma$ there exists a unitary operator $\tilde{I}(\lambda)$ from $\tilde{\mathfrak{X}}(\lambda)$ onto $\tilde{\mathfrak{X}}'(\lambda)$ such that for any $x \in \mathfrak{X}''$,

$$(3.9) \quad \tilde{I}(\lambda)J(\lambda)x = J'(\lambda)x.$$

Define $\tilde{I}g = \{\tilde{I}(\lambda)g(\lambda)\}_{\lambda \in \Gamma} \in \tilde{\mathfrak{X}}'$ for $g \in \tilde{\mathfrak{X}}$. Then $\tilde{I}|_{\mathfrak{M}}$ is a unitary operator from \mathfrak{M} onto \mathfrak{M}' , and we have $\mathfrak{L}_{ac}(\Gamma) = \mathfrak{L}'_{ac}(\Gamma)$ and $\tilde{I}|_{\mathfrak{M}}\pi = \pi'$.

PROOF. Put $e''(\lambda; x, y) \equiv e(\lambda; x, y) = e'(\lambda; x, y)$ for $\lambda \in \Gamma$ and $x, y \in \mathfrak{X}''$. Then $(\Gamma, \mathfrak{X}'', e'')$ is also a spectral form for $(\Gamma_0, \mathbf{B}, m, \mathfrak{G}, E)$. Thus we can define $\mathfrak{M}''(\lambda), \mathfrak{X}''(\lambda)$, etc.

Now define $K(\lambda)J''(\lambda)x = J(\lambda)x$ and $K'(\lambda)J''(\lambda)x = J'(\lambda)x$ for $x \in \mathfrak{X}''$ and $\lambda \in \Gamma$. Then it is easy to see that $K(\lambda)$ and $K'(\lambda)$ are well-defined and can be extended to isometric operators $\tilde{K}(\lambda): \tilde{\mathfrak{X}}''(\lambda) \rightarrow \tilde{\mathfrak{X}}(\lambda)$ and $\tilde{K}'(\lambda): \tilde{\mathfrak{X}}''(\lambda) \rightarrow \tilde{\mathfrak{X}}'(\lambda)$. Furthermore by (b) and (c) we can easily show that $\tilde{K}(\lambda)$ and $\tilde{K}'(\lambda)$ are unitary. Let \tilde{K} be defined by $\tilde{K}g'' = \{\tilde{K}(\lambda)g''(\lambda)\}_{\lambda \in \Gamma} \in \tilde{\mathfrak{X}}$ for $g'' \in \tilde{\mathfrak{X}}''$. Then \tilde{K} has an inverse \tilde{K}^{-1} such that $\tilde{K}^{-1}g = \{\tilde{K}(\lambda)^{-1}g(\lambda)\}_{\lambda \in \Gamma} \in \tilde{\mathfrak{X}}''$ for $g \in \tilde{\mathfrak{X}}$. It is clear that $\tilde{K}|_{\mathfrak{M}''}$ is an isometry from \mathfrak{M}'' to \mathfrak{M} , since $\mathcal{S}'' \subset \mathcal{S}$ and $\tilde{K}(\lambda)$ is unitary for $\lambda \in \Gamma$. Furthermore $\tilde{K}|_{\mathfrak{M}''}$ is unitary. In fact, from the relation $\tilde{K}(\lambda)J''(\lambda) = J(\lambda)$ on \mathfrak{X}'' , it can be easily seen that $\tilde{K}^{-1}(JS'') = J''S''$. Since $J\mathcal{S}''$ is dense in \mathfrak{M} by conditions (b) and (c), it follows from this that $\tilde{K}^{-1}\mathfrak{M} \subset \mathfrak{M}''$ because $\tilde{K}(\lambda)^{-1}$ is unitary for $\lambda \in \Gamma$. This proves the unitarity of $\tilde{K}|_{\mathfrak{M}''}$. Quite similarly we can define \tilde{K}' and prove that $\tilde{K}'|_{\mathfrak{M}''}: \mathfrak{M}'' \rightarrow \mathfrak{M}'$ is unitary.

Now we can prove the proposition. Put $\tilde{I}(\lambda) = \tilde{K}'(\lambda)\tilde{K}(\lambda)^{-1}$ for $\lambda \in \Gamma$. Then $\tilde{I}(\lambda)$ is a unitary operator from $\tilde{\mathfrak{X}}(\lambda)$ onto $\tilde{\mathfrak{X}}'(\lambda)$ and satisfies (3.9). Furthermore by definition we have $\tilde{I} = \tilde{K}'\tilde{K}^{-1}$. Thus $\tilde{I}|_{\mathfrak{M}}$ is a unitary operator from \mathfrak{M} onto \mathfrak{M}' . From (3.7), (3.8), and the definition of \tilde{K} , we get $\pi^{-1}\tilde{K}\pi''\mathring{\mathfrak{L}}''_{ac}(\Gamma) = \mathring{\mathfrak{L}}''_{ac}(\Gamma) \subset \mathfrak{L}_{ac}(\Gamma)$. Since π^{-1}, \tilde{K} , and π'' are unitary and $\mathring{\mathfrak{L}}''_{ac}(\Gamma)$ is dense in $\mathfrak{L}_{ac}(\Gamma)$, it follows from this that $\mathring{\mathfrak{L}}''_{ac}(\Gamma)$ is dense in $\mathfrak{L}_{ac}(\Gamma)$. Hence we have $\mathfrak{L}''_{ac}(\Gamma) = \mathfrak{L}_{ac}(\Gamma)$. Similarly we can prove $\mathfrak{L}''_{ac}(\Gamma) = \mathfrak{L}'_{ac}(\Gamma)$. Thus $\mathfrak{L}_{ac}(\Gamma) = \mathfrak{L}'_{ac}(\Gamma)$. $\tilde{I}\pi = \pi'$ follows easily from (3.7), (3.8), and (3.9). Q. E. D.

§ 4. Stationary wave operators.

In this and the next section we consider two spectral systems $(\Gamma_0, \mathbf{B}, m, \mathfrak{G}_j, E_j)$ with spectral forms $(\Gamma, \mathfrak{X}_j, e_j), j=1, 2$, where the measure space $(\Gamma_0, \mathbf{B}, m)$ and $\Gamma \in \mathbf{B}$ are common to the two systems. We use the obvious notations such as $E_{j,ac}, \mathfrak{N}_j(\lambda), \mathfrak{X}_j(\lambda), \tilde{\mathfrak{X}}_j(\lambda), (\cdot, \cdot)_{j,\lambda}, \| \cdot \|_{j,\lambda}, J_j(\lambda), \tilde{\mathfrak{X}}_j, \mathcal{S}_j, \sim_j, \mathfrak{M}_j, \mathring{\mathfrak{L}}_{j,ac}(\Gamma), \mathfrak{L}_{j,ac}(\Gamma), \mathfrak{G}_{j,ac}(\Gamma), \mathfrak{L}_j$, etc.

We denote by $\mathcal{M}(\Gamma; \mathfrak{X}_2)$ the linear space of all strongly \mathbf{B} -measurable functions on Γ to \mathfrak{X}_2 . The following theorem constructs an isometric operator \hat{G} which will be used to construct a stationary wave operator.

THEOREM 4.1. Let \mathfrak{X}_2 be a Banach space. Suppose that the following conditions (J2) and (Φ) are satisfied:

(J2) For every $\lambda \in \Gamma$, the linear operator $J_2(\lambda) : \mathfrak{X}_2 \rightarrow \tilde{\mathfrak{X}}_2(\lambda)$ is bounded¹⁰⁾.

(Φ) There exists a linear operator $\Phi : \mathfrak{X}_1 \rightarrow \mathcal{M}(\Gamma; \mathfrak{X}_2)$ which satisfies the following relation for all $x, y \in \mathfrak{X}_1$:

$$(4.1) \quad e_2(\lambda; (\Phi x)(\lambda), (\Phi y)(\lambda)) = e_1(\lambda; x, y), \quad m\text{-a. e. } \lambda \in \Gamma.$$

Here the exceptional m -null set may depend on x and y .

Then there exists a unique isometric operator $\hat{G} : \mathfrak{M}_1 \rightarrow \mathfrak{M}_2$ which satisfies the following two conditions:

(a) For any $\Delta \subset \Gamma$, $\Delta \in \mathbf{B}$ and any $u \in \mathfrak{M}_1$,

$$(4.2) \quad \hat{G} \{ \chi_\Delta(\lambda) u(\lambda) \}_{\lambda \in \Gamma} = \{ \chi_\Delta(\lambda) (\hat{G}u)(\lambda) \}_{\lambda \in \Gamma}.$$

(b) For every $x \in \mathfrak{X}_1$,

$$(4.3) \quad \hat{G} \{ J_1(\lambda)x \}_{\lambda \in \Gamma} = \{ J_2(\lambda)(\Phi x)(\lambda) \}_{\lambda \in \Gamma}.$$

PROOF. Quite similar to the proof of Theorem 2.8 of [15]. Q. E. D.

REMARK 4.2. In many cases of practical application, the mapping Φ is given by an operator of decomposable type:

$$(4.4) \quad (\Phi x)(\lambda) = G(\lambda)x, \quad \lambda \in \Gamma, x \in \mathfrak{X}_1.$$

Namely Assumption (Φ) follows immediately from the following condition (G) if we define Φ as in (4.4):

(G) For every $\lambda \in \Gamma$, there exists a linear operator $G(\lambda) : \mathfrak{X}_1 \rightarrow \mathfrak{X}_2$ such that:

(a) for every $\lambda \in \Gamma$ and every $x, y \in \mathfrak{X}_1$,

$$(4.5) \quad e_2(\lambda; G(\lambda)x, G(\lambda)y) = e_1(\lambda; x, y);$$

(b) for every $x \in \mathfrak{X}_1$, $G(\lambda)x$ is strongly \mathbf{B} -measurable as an \mathfrak{X}_2 -valued function of λ in Γ .

The conditions (J2) and (G) are essentially the same as the assumptions of Theorem 4.5 of Kato and Kuroda [14], so far as the assumptions needed to construct the isometric operator \hat{G} are concerned. Thus, as to the construction of \hat{G} , Theorem 4.1 is a generalization of Kato and Kuroda's Theorem 4.5. The construction of \hat{G} under condition (Φ) of Theorem 4.1 was actually used in [15], and we shall also use it in Part II of this paper when we consider modified wave operators of Alsholm's type (cf. [2]). But in our most important applications considered in Part II, the condition (G) will suffice.

REMARK 4.3. There may exist several Φ 's (or G 's) satisfying (Φ) (or (G)). In fact, in the long-range scattering, there actually exist at least two Φ 's, the one by us ([15] and Part II of this paper) and the one by Pinchuk [22] or Isozaki

10) This condition is equivalent to the continuity of the sesquibilinear form $e_2(\lambda; \cdot, \cdot)$ on $\mathfrak{X}_2 \times \mathfrak{X}_2$ for $\lambda \in \Gamma$.

[11]. This situation is quite different from the short-range case in which there is a natural choice of G given by $G^\pm(\lambda)=1+VR_1(\lambda\pm i0)$, where V is a short-range potential and $R_1(\lambda\pm i0)$ denotes the boundary value of the resolvent $R_1(\lambda\pm i\varepsilon)$, $\varepsilon>0$, of the unperturbed Schrödinger operator $-\Delta$ (cf. e. g., Kato and Kuroda [14], Kuroda [17], [18]). However, even in the long-range case, we can show that the isometric operator \hat{G} 's constructed as in Theorem 4.1 using our Φ and Pinchuk's or Isozaki's Φ coincide with each other except a factor of unitary operator, and hence the stationary wave operators W_Γ 's constructed in Theorem 4.4 below from those Φ 's essentially coincide. This will be proved in Part II by using the uniqueness theorem proved in the next section.

Now using \hat{G} constructed in Theorem 4.1, we can define a stationary wave operator.

THEOREM 4.4. *Let an isometric operator $\hat{G} : \mathfrak{M}_1 \rightarrow \mathfrak{M}_2$ satisfy (a) and (b) of Theorem 4.1, and put*

$$(4.6) \quad W_\Gamma = \begin{cases} \pi_2^{-1} \hat{G} \pi_1 & \text{on } \mathfrak{L}_{1,ac}(\Gamma), \\ 0 & \text{on } \mathfrak{H} \ominus \mathfrak{L}_{1,ac}(\Gamma). \end{cases}$$

Then W_Γ is a partially isometric operator in \mathfrak{H} with initial set $\mathfrak{L}_{1,ac}(\Gamma)$ and final set contained in $\mathfrak{L}_{2,ac}(\Gamma)$, and satisfies the following two conditions:

(a) For every $\Delta \in \mathbf{B}$,

$$(4.7) \quad W_\Gamma E_1(\Delta) = E_2(\Delta) W_\Gamma.$$

(b) For every $x \in \mathfrak{X}_1$, $y \in \mathfrak{X}_2$ and $\Delta_1, \Delta_2 \in \mathbf{B}$, $\Delta_1, \Delta_2 \subset \Gamma$,

$$(4.8) \quad (W_\Gamma E_{1,ac}(\Delta_1)x, E_{2,ac}(\Delta_2)y)_\mathfrak{H} = \int_{\Delta_1 \cap \Delta_2} e_2(\lambda; (\Phi x)(\lambda), y) m(d\lambda).$$

PROOF. Quite similar to the proof of Theorem 2.10 of [15]. Q. E. D.

DEFINITION 4.5. The partially isometric operator W_Γ constructed in Theorem 4.4 is called a *stationary wave operator* and is said to be *complete* if its range is equal to $\mathfrak{L}_{2,ac}(\Gamma)$.

Our next aim is to consider a sufficient condition for the completeness of W_Γ , and this will be discussed in the next section together with the uniqueness theorem concerning W_Γ referred to in Remark 4.3 above.

§ 5. Completeness and the uniqueness of stationary wave operators.

We continue to consider two spectral systems $(\Gamma_0, \mathbf{B}, m, \mathfrak{H}_j, E_j)$ with spectral forms $(\Gamma, \mathfrak{X}_j, e_j)$, $j=1, 2$, as in the previous section. To prove the completeness of stationary wave operator W_Γ constructed in Theorem 4.4, it suffices to show that \hat{G} constructed in Theorem 4.1 is a unitary operator from \mathfrak{M}_1 onto \mathfrak{M}_2 . In the following Theorem 5.1, we shall prove the unitarity of \hat{G}

under the assumption stronger than the one needed to construct \hat{G} in Theorem 4.1. Under this stronger assumption we can choose another way in constructing \hat{G} . As this way of construction clarifies situation and gives some more detailed informations, we adopt this way in proving the unitarity of \hat{G} . This proof will give an alternative proof of Theorem 4.1 under the assumption of Theorem 5.1. However the proof of completeness will be somewhat simpler if we use Theorem 4.1.

THEOREM 5.1. *Let \mathfrak{X}_1 be a separable normed space, \mathfrak{X}_2 a Banach space, and \mathfrak{H} a Hilbert space. Suppose that the following conditions (Jj) for $j=1, 2$, (\mathfrak{F}), and ($\mathfrak{F}\Phi$) are satisfied:*

(Jj) *For every $\lambda \in \Gamma$, the linear operator $J_j(\lambda): \mathfrak{X}_j \rightarrow \tilde{\mathfrak{X}}_j(\lambda)$ is bounded.*

(\mathfrak{F}) *For every $\lambda \in \Gamma$ and $j=1, 2$, there exists a linear operator $\mathfrak{F}_j(\lambda): \mathfrak{X}_j \rightarrow \mathfrak{H}$ such that:*

(a) *for every $\lambda \in \Gamma$ and $x, y \in \mathfrak{X}_j$,*

$$(5.1) \quad (\mathfrak{F}_j(\lambda)x, \mathfrak{F}_j(\lambda)y)_{\mathfrak{H}} = e_j(\lambda; x, y);$$

(b) *for every $\lambda \in \Gamma$, the range of $\mathfrak{F}_1(\lambda)$ is dense in \mathfrak{H} .*

($\mathfrak{F}\Phi$) *There exists a linear operator $\Phi: \mathfrak{X}_1 \rightarrow \mathcal{M}(\Gamma; \mathfrak{X}_2)$ which satisfies the following relation for every $x \in \mathfrak{X}_1$:*

$$(5.2) \quad \mathfrak{F}_2(\lambda)(\Phi x)(\lambda) = \mathfrak{F}_1(\lambda)x, \quad m\text{-a. e. } \lambda \in \Gamma.$$

Here the exceptional m -null set may depend on x , and $\mathcal{M}(\Gamma; \mathfrak{X}_2)$ is the same space as in § 4.

Then there exists a unique unitary operator $\hat{G}: \mathfrak{M}_1 \rightarrow \mathfrak{M}_2$ which satisfies conditions (a) and (b) of Theorem 4.1, and hence by Theorem 4.4 we can construct a complete stationary wave operator W_{Γ} which satisfies (a) and (b) of Theorem 4.4. (Notice that if Φ here coincides with Φ in Theorem 4.1, then \hat{G} and W_{Γ} here coincide with \hat{G} and W_{Γ} in Theorems 4.1 and 4.4.)

PROOF. By (5.1) and (5.2), we can prove (Φ) of Theorem 4.1 so that we can construct \hat{G} in the same way as in Theorem 4.1. But as stated before, we adopt another way. We divide the proof into 7 steps, and in each step we first state the result to be proved in that step.

1st step: For every $\lambda \in \Gamma$ and $j=1, 2$, there exists an isometric operator $F_j(\lambda): \tilde{\mathfrak{X}}_j(\lambda) \rightarrow \mathfrak{H}$ such that for every $\lambda \in \Gamma$ and every $x \in \mathfrak{X}_1$, $\mathfrak{F}_j(\lambda)x = F_j(\lambda)J_j(\lambda)x$.

Proof is easy by (5.1) and the denseness of $\mathfrak{X}_j(\lambda) = J_j(\lambda)\mathfrak{X}_j$ in $\tilde{\mathfrak{X}}_j(\lambda)$.

2nd step: For every $\lambda \in \Gamma$, $F_1(\lambda): \tilde{\mathfrak{X}}_1(\lambda) \rightarrow \mathfrak{H}$ is unitary.

This can be easily proved using (b) of (\mathfrak{F}).

3rd step: There exists a subset $A \subset \Gamma$, $A \in \mathcal{B}$ such that $m(\Gamma - A) = 0$ and such that for every $\lambda \in A$, $F_2(\lambda): \tilde{\mathfrak{X}}_2(\lambda) \rightarrow \mathfrak{H}$ is unitary.

Let $\{x_k\}_{k=1}^{\infty}$ be a countable dense subset of \mathfrak{X}_1 . It follows from ($\mathfrak{F}\Phi$) that there exists a subset $A \subset \Gamma$, $A \in \mathcal{B}$ such that $m(\Gamma - A) = 0$ and such that, for

every $\lambda \in A$ and k , one has $\mathcal{F}_2(\lambda)(\Phi x_k)(\lambda) = \mathcal{F}_1(\lambda)x_k$. Thus for every $\lambda \in A$, the range of $\mathcal{F}_2(\lambda)$ contains $\{\mathcal{F}_1(\lambda)x_k\}_{k=1}^\infty = \{F_1(\lambda)J_1(\lambda)x_k\}_{k=1}^\infty$, which is dense in \mathfrak{h} for every $\lambda \in \Gamma$ by the 2nd step, (J1), and the denseness of $\{x_k\}_{k=1}^\infty$ in \mathfrak{X}_1 . This completes the proof of 3rd step.

4th step: For every $\lambda \in \Gamma$, define $\hat{G}(\lambda) \equiv F_2(\lambda)*F_1(\lambda) : \tilde{\mathfrak{X}}_1(\lambda) \rightarrow \tilde{\mathfrak{X}}_2(\lambda)$. Then for every $\lambda \in A$, $\hat{G}(\lambda)$ is unitary.

This is obvious by the 2nd and the 3rd steps.

5th step: For every $\varphi \in J_1\mathcal{S}_1 \subset \mathfrak{M}_1$, put $\hat{G}\varphi = \{\hat{G}(\lambda)\varphi(\lambda)\}_{\lambda \in \Gamma} \in \tilde{\mathfrak{X}}_2$. Then one has $\hat{G}\varphi \in \mathfrak{M}_2$ and $\|\hat{G}\varphi\|_{\mathfrak{M}_2} = \|\varphi\|_{\mathfrak{M}_1}$. Thus \hat{G} can be extended to an isometric operator from \mathfrak{M}_1 to \mathfrak{M}_2 (which will be denoted also by \hat{G}).

Since $\hat{G}(\lambda)$ is unitary for every $\lambda \in A$, it suffices to see the e_2 -measurability of $\hat{G}J_1x$ for every $x \in \mathfrak{X}_1$ because every $\varphi \in J_1\mathcal{S}_1$ can be represented as $\varphi = J_1h$, where $h \in \mathcal{S}_1$ is finitely-valued. But we have $\hat{G}J_1x = \{\hat{G}(\lambda)J_1(\lambda)x\}_{\lambda \in \Gamma} = \{F_2(\lambda)*\mathcal{F}_1(\lambda)x\}_{\lambda \in \Gamma} = \{F_2(\lambda)*\mathcal{F}_2(\lambda)(\Phi x)(\lambda)\}_{\lambda \in \Gamma} = \{J_2(\lambda)(\Phi x)(\lambda)\}_{\lambda \in \Gamma}$. Thus by $\Phi x \in \mathcal{M}(\Gamma; \mathfrak{X}_2)$ and (J2), $\hat{G}J_1x$ is e_2 -measurable as desired.

6th step: The isometric operator $\hat{G} : \mathfrak{M}_1 \rightarrow \mathfrak{M}_2$ is unitary.

Let $v \in \mathfrak{M}_2$, and let $(\hat{G}\varphi, v)_{\mathfrak{M}_2} = 0$ for every $\varphi \in J_1\mathcal{S}_1$. We have to prove $v = 0$ in \mathfrak{M}_2 . Take a countable dense subset $\{x_k\}_{k=1}^\infty$ of \mathfrak{X}_1 . Then we have $(\hat{G}J_1\chi_{\Delta}x_k, v)_{\mathfrak{M}_2} = 0$, namely, $\int_{\Delta} (\hat{G}(\lambda)J_1(\lambda)x_k, v(\lambda))_{\mathfrak{h}} m(d\lambda) = 0$ for every k and $\Delta \subset \Gamma$, $\Delta \in \mathcal{B}$. Thus there exists a subset $B \subset A$, $B \in \mathcal{B}$ such that $m(\Gamma - B) = 0$ and such that $(\hat{G}(\lambda)J_1(\lambda)x_k, v(\lambda))_{\mathfrak{h}} = 0$ for every $\lambda \in B$ and k . Since $\hat{G}(\lambda)$ is unitary for every $\lambda \in B \subset A$ and since $\{J_1(\lambda)x_k\}_{k=1}^\infty$ is dense in $\tilde{\mathfrak{X}}_1(\lambda)$ for every $\lambda \in \Gamma$ by (J1), it follows from this that $v(\lambda) = 0$ for every $\lambda \in B$ and hence $v = 0$ in \mathfrak{M}_2 .

7th step: The unitary operator \hat{G} satisfies (a) and (b) of Theorem 4.1.

(a) is clear by the definition of \hat{G} . (b) has already been proved in the 5th step. Q. E. D.

REMARK 5.2. Assumption $(\mathcal{F}\Phi)$ of Theorem 5.1 follows immediately from the following condition $(\mathcal{F}G)$ if we put $(\Phi x)(\lambda) = G(\lambda)x$ for $\lambda \in \Gamma$ and $x \in \mathfrak{X}_1$:

$(\mathcal{F}G)$ For every $\lambda \in \Gamma$, there exists a linear operator $G(\lambda) : \mathfrak{X}_1 \rightarrow \mathfrak{X}_2$ such that:

(a) for every $\lambda \in \Gamma$ and $x \in \mathfrak{X}_1$,

$$(5.3) \quad \mathcal{F}_2(\lambda)G(\lambda)x = \mathcal{F}_1(\lambda)x;$$

(b) for every $x \in \mathfrak{X}_1$, $G(\lambda)x$ is strongly \mathcal{B} -measurable as an \mathfrak{X}_2 -valued function of λ in Γ .

If we assume this condition $(\mathcal{F}G)$ instead of $(\mathcal{F}\Phi)$ in Theorem 5.1, the results of the 3rd and the 4th steps in the proof are improved as follows:

3rd step: For every $\lambda \in \Gamma$, $F_2(\lambda) : \tilde{\mathfrak{X}}_2(\lambda) \rightarrow \mathfrak{h}$ is unitary.

4th step: For every $\lambda \in \Gamma$, $\hat{G}(\lambda) : \tilde{\mathfrak{X}}_1(\lambda) \rightarrow \tilde{\mathfrak{X}}_2(\lambda)$ is unitary.

In our main applications considered in Part II, the condition $(\mathcal{F}G)$ will be

satisfied.

REMARK 5.3. Let us consider the short-range case. Then, for $G^\pm(\lambda)$ referred to in Remark 4.3, the relation (5.3) can be written as $\mathcal{F}_2(\lambda)G^\pm(\lambda)x = \mathcal{F}_1(\lambda)x$ and hence

$$\mathcal{F}_2(\lambda)x + \mathcal{F}_2(\lambda)VR_1(\lambda \pm i0)x = \mathcal{F}_1(\lambda)x.$$

In actual applications considered in Part II, $\mathcal{F}_j(\lambda)$ corresponds to the eigenoperator constructed by Ikebe [9] or Saitō [25]~[28]. Thus the relation (5.2) or (5.3) can be regarded as the weak form of the Lippmann-Schwinger equation. Thus in the case of long-range scattering, we may call the relation (5.2) or (5.3) the modified Lippmann-Schwinger equation, because in the long-range case Φ or G has a form somewhat different from $G^\pm(\lambda)$. Incidentally, we add a comment that, even in the short-range case, our Theorem 5.1 can be applied to prove the completeness of ordinary wave operators.

Next let us prove a certain kind of uniqueness theorem concerning W_Γ referred to in Remark 4.3. Let $(\Gamma, \mathfrak{X}'_j, e'_j)$ be another spectral form for the same spectral system $(\Gamma_0, \mathbf{B}, m, \mathfrak{H}_j, E_j)$ for $j=1, 2$. Then we can define $\mathfrak{X}'_j(\lambda)$, $\tilde{\mathfrak{X}}'_j(\lambda)$, etc., as in § 3.

THEOREM 5.4. *Suppose that all the assumptions of Theorem 5.1 are satisfied not only for \mathfrak{X}_j and e_j but also for \mathfrak{X}'_j and e'_j with Φ and $\mathcal{F}_j(\lambda)$ replaced by some other Φ' and $\mathcal{F}'_j(\lambda)$ with the same Hilbert space \mathfrak{h} . Suppose furthermore that the following conditions are satisfied for $j=1, 2$:*

- (a) $\mathfrak{X}'_j \equiv \mathfrak{X}_j \cap \mathfrak{X}'_j$ is dense in \mathfrak{X}_j and \mathfrak{X}'_j ;
- (b) for every $\lambda \in \Gamma$ and $x, y \in \mathfrak{X}'_j$, $e_j(\lambda; x, y) = e'_j(\lambda; x, y)$;
- (c) for every $\lambda \in \Gamma$ and $x \in \mathfrak{X}'_j$, $\mathcal{F}_1(\lambda)x = \mathcal{F}'_1(\lambda)x$;
- (d) for every $\lambda \in \Gamma$ there exists a unitary operator $U(\lambda)$ in \mathfrak{h} such that $\mathcal{F}_2(\lambda)x = U(\lambda)\mathcal{F}'_2(\lambda)x$ for every $x \in \mathfrak{X}'_2$ and $\lambda \in \Gamma$.

Then, according to Theorem 5.1, we can construct unitary operators \hat{G} and \hat{G}' , and complete stationary wave operators W_Γ and W'_Γ . Moreover, by Proposition 3.1, we can construct a unitary operator $\tilde{I}_j \equiv \tilde{I}_j|_{\mathfrak{M}_j}$ from \mathfrak{M}_j onto \mathfrak{M}'_j for $j=1, 2$. Now put $M(\lambda) = \tilde{I}_2(\lambda) * F'_2(\lambda) * U(\lambda) F_2(\lambda) \tilde{I}_2(\lambda)$ for $\lambda \in \Gamma$ and define $M\varphi = \{M(\lambda)\varphi(\lambda)\}_{\lambda \in \Gamma}$ for $\varphi \in J_2\mathcal{S}_2$. Then M can be extended to a unitary operator in \mathfrak{M}_2 (which will be denoted also by M). Define $L = \pi_2^{-1}M\pi_2$. Then, for these operators, we have

$$(5.4) \quad \tilde{I}_2 M \hat{G} = \hat{G}' \tilde{I}_1,$$

$$(5.5) \quad L W_\Gamma = W'_\Gamma.$$

PROOF. From the definition of $M(\lambda)$, (d), (c), and the denseness of $J_j(\lambda)\mathfrak{X}'_j$ in $\tilde{\mathfrak{X}}_j(\lambda)$, it follows that $M(\lambda) * \tilde{I}_2(\lambda) * \hat{G}'(\lambda) \tilde{I}_1(\lambda) = \tilde{I}_2(\lambda) * F'_2(\lambda) * U(\lambda) * F_2(\lambda) \hat{G}'(\lambda) \tilde{I}_1(\lambda) = (U(\lambda) F'_2(\lambda) \tilde{I}_2(\lambda)) * F'_1(\lambda) \tilde{I}_1(\lambda) = F_2(\lambda) * F'_1(\lambda) \tilde{I}_1(\lambda) = F_2(\lambda) * F_1(\lambda) = \hat{G}(\lambda)$ for every $\lambda \in \Gamma$.

Therefore M can be written as $M = \tilde{I}_2^* \hat{G}' \tilde{I}_1 \hat{G}^*$ and hence is a unitary operator in \mathfrak{M}_2 . Moreover we have proved (5.4). Relation (5.5) follows from (5.4) at once by Proposition 3.1, (4.6), and the definition of L . Q. E. D.

§ 6. Time dependent wave operators and the invariance principle (proof of Theorems 2.1~2.3).

In this section, we consider as in § 2.1 two self-adjoint operators H_1 and H_2 in a Hilbert space \mathfrak{H} with $\mathcal{D}(H_1) = \mathcal{D}(H_2)$. Let \mathcal{B} be the Borel field on R^1 , m the Lebesgue measure on R^1 , and E_j the spectral measure associated with H_j for $j=1, 2$. Then $(R^1, \mathcal{B}, m, \mathfrak{H}, E_j)$ becomes a spectral system. The purpose of the present section is to construct, using the results of §§ 4~5, a stationary wave operator for the spectral systems $(R^1, \mathcal{B}, m, \mathfrak{H}, E_j)$, $j=1, 2$, under some of the assumptions on H_j stated in § 2.1, and then to prove Theorems 2.1~2.3.

As shown in Theorem 4.1, the operator Φ plays a crucial role in the construction of stationary wave operators, and as mentioned in Remark 4.3, there may exist several Φ 's satisfying (Φ) . But in order to establish a connection with the time dependent wave operator, we must choose Φ suitably. Our assumptions stated in § 2.1 are selected on the basis of the form of time dependent modified wave operators for long-range scattering (cf. Buslaev and Matveev [5], Alsholm and Kato [3], Alsholm [2], Hörmander [8], etc.), and are suitable for constructing such Φ as will be shown below. (As to the motivation of our approach, see the introduction of [15].)

6.1. Construction of stationary wave operators. We shall construct a stationary wave operator. Let Γ be a Borel set in R^1 , \mathfrak{X}_j a linear subspace¹¹⁾ of \mathfrak{H} for $j=1, 2$ satisfying i)-iv)¹²⁾ of § 2.1, and suppose that (L.A.P.) of § 2.1 is satisfied. Then by 1° and 2° of § 2.1, $(\Gamma, \mathfrak{X}_j, e_j)$ is a spectral form for $(R^1, \mathcal{B}, m, \mathfrak{H}, E_j)$ for $j=1, 2$, where e_j is the complex valued function on $\Gamma \times \mathfrak{X}_j \times \mathfrak{X}_j$ defined in § 2.1. Furthermore (Jj) of § 5 is satisfied for $j=1, 2$ by (2.4). Suppose that condition (X) of § 2.1 is satisfied. Then it is easy to see that the following relation holds for every $u, v \in \mathcal{D}(H_1)$ and $z \in C^\pm$,

$$(6.1) \quad e_2(z; G^\pm(z)u, G^\pm(z)v) = f^\pm(z; u, v),$$

where e_j is defined by (2.5), $G^\pm(z)$ by (2.9), and f^\pm by (2.8) (cf. Proposition 1.4 of [15]). Thus if we assume (Q^\pm) and (f^\pm) of § 2.1, and put for $x \in \mathfrak{X}_1$ and $\lambda \in \Gamma$,

11) Note that we do not assume the denseness of \mathfrak{X}_j in \mathfrak{H} here.

12) Throughout the remainder of this section, we assume i)-iv) of § 2.1 for \mathfrak{X}_j , $j=1, 2$.

$$(6.2) \quad (\Phi^\pm x)(\lambda) = x + \text{l.i.m.}_{\varepsilon \rightarrow +0} Q^\pm(\lambda \pm i\varepsilon)x,$$

then $\Phi = \Phi^\pm$ satisfies (Φ) of §4. Therefore we have proved the following theorem.

THEOREM 6.1. *Suppose that conditions i)-iv), (L.A.P.), (X) , (Q^\pm) , and (f^\pm) of §2.1 are satisfied. Then there exists a unique isometric operator $G^\pm : \mathfrak{M}_1 \rightarrow \mathfrak{M}_2$ satisfying (a) and (b) of Theorem 4.1 with $\Phi = \Phi^\pm$, where Φ^\pm is defined by (6.2). Hence we can define a stationary wave operator W_{\mp}^\pm following Theorem 4.4, which is a partial isometry in \mathfrak{H} with initial set $\mathfrak{S}_{1,ac}(I)$ and final set contained in $\mathfrak{S}_{2,ac}(I)$, and satisfies (a) and (b) of Theorem 4.4 with $\Phi = \Phi^\pm$.*

REMARK 6.2. When (\mathcal{F}) and $(\mathcal{F}\Phi)$ of Theorem 5.1 are satisfied with $\Phi = \Phi^\pm$, we need not assume condition (f^\pm) in the above theorem, because (\mathcal{F}) and $(\mathcal{F}\Phi)$ imply condition (Φ) . Furthermore W_{\mp}^\pm above becomes complete (see Theorem 5.1).

REMARK 6.3. Suppose that in addition to a), b) of (Q^\pm) , the following condition c) is satisfied:

c) For every $\lambda \in I$ and $x \in \mathfrak{X}_1$, the limit $\lim_{\varepsilon \rightarrow +0} Q^\pm(\lambda \pm i\varepsilon)x$ exists in \mathfrak{X}_2 .

Then we can define a linear operator $G^\pm(\lambda) : \mathfrak{X}_1 \rightarrow \mathfrak{X}_2$ as $G^\pm(\lambda)x = x + \lim_{\varepsilon \rightarrow +0} Q^\pm(\lambda \pm i\varepsilon)x$ for $\lambda \in I$ and $x \in \mathfrak{X}_1$. This $G^\pm(\lambda)$ obviously satisfies (b) of $(\mathcal{F}G)$ in Remark 5.2 (cf. b) of (Q^\pm)). Thus if in addition to condition (\mathcal{F}) of Theorem 5.1 relation (5.3) is satisfied for $\lambda \in I$ and $x \in \mathfrak{X}_1$, we have $(\mathcal{F}G)$. Moreover (G) of Remark 4.2 holds, since (\mathcal{F}) and $(\mathcal{F}G)$ are satisfied. What we wanted to say in Remarks 4.2 and 5.2 is that this is the case in our main applications considered in Part II.

6.2. Preparatory lemmas and some definitions. We state and prove several lemmas which we need in the next subsection. We denote by $H^2(C^\pm; \mathfrak{X})$ an \mathfrak{X} -valued Hardy class on C^\pm , where \mathfrak{X} is a separable Hilbert space.

LEMMA 6.4. *Let (L.A.P.) be satisfied. Put*

$$(6.3) \quad \delta_2(z) = -\frac{\text{sgn}(\text{Im } z)}{2\pi i} (R_2(z) - R_2(\bar{z}))$$

for $z \in C^\pm$. Then for every $u \in \mathfrak{H}$, we have

$$(6.4) \quad I_{\frac{1}{2}}^* \delta_2(\cdot) E_2(I) u \in H^2(C^\pm; \mathfrak{X}_{\frac{1}{2}}^*).$$

PROOF. Since $I_{\frac{1}{2}}^* \delta_2(\cdot) E_2(I) u : C^\pm \rightarrow \mathfrak{X}_{\frac{1}{2}}^*$ is holomorphic, we have only to prove that for some positive constant C ,

$$(6.5) \quad \sup_{\varepsilon > 0} \int_{-\infty}^{\infty} \|I_{\frac{1}{2}}^* E_2(I) R_2(\lambda \pm i\varepsilon) u\|_{\mathfrak{X}_{\frac{1}{2}}^*}^2 d\lambda \leq C \|u\|_{\mathfrak{H}}^2.$$

But this means that $I_{\frac{1}{2}}^*$ is H_2 -smooth on I in the terminology of Lavine [20],

and hence follows from the estimate (2.4) of (L.A.P.) (cf. Lavine [19], Theorem 1, and [20], Lemma 2.1). Q.E.D.

LEMMA 6.5. *Let (L.A.P.) be satisfied. Suppose that Γ is bounded. Then for any bounded Borel measurable function α on R^1 with $\text{supp } \alpha \subset \Gamma$, and any $x, y \in \mathfrak{X}_2$, we have*

$$(6.6) \quad e_2(\lambda \pm i\varepsilon; x, \alpha(H_{2,ac})y) \in L^1(R^1_\lambda), \quad \varepsilon > 0$$

and

$$(6.7) \quad e_2(\lambda \pm i\varepsilon; x, \alpha(H_{2,ac})y) \xrightarrow{(\varepsilon \rightarrow +0)} \overline{\alpha(\lambda)} e_2(\lambda; x, y) \quad \text{in } L^1(R^1_\lambda).$$

Furthermore, the family $\{e_2(\lambda \pm i\varepsilon; \cdot, \alpha(H_{2,ac})y) \mid \lambda \in \Gamma, \varepsilon > 0\}$ of linear forms on \mathfrak{X}_2 is equi-continuous.

PROOF. Since e_2 is a spectral form for H_2 , we have

$$(6.8) \quad e_2(\lambda \pm i\varepsilon; x, \alpha(H_{2,ac})y) = \frac{\varepsilon}{\pi} \int_{-\infty}^{\infty} \frac{1}{(\lambda - \mu)^2 + \varepsilon^2} \overline{\alpha(\mu)} e_2(\mu; x, y) d\mu.$$

Hence, $e_2(\lambda \pm i\varepsilon; x, \alpha(H_{2,ac})y)$ is the Poisson integral of $f(\mu) = \overline{\alpha(\mu)} e_2(\mu; x, y)$, where $f \in L^1(R^1)$ by $\text{supp } \alpha \subset \Gamma$ and (2.4). (6.6) and (6.7) are then immediate consequences of well-known property of the Poisson integral. The equi-continuity of the family in the lemma is obvious by (L.A.P.) and (6.8). Q.E.D.

LEMMA 6.6. *Let (L.A.P.), (X), and (Q^\pm) be satisfied. Let Γ be bounded and let α be as in Lemma 6.5. Then for any $x \in \mathfrak{X}_1$ and $y \in \mathfrak{X}_2$, there exist a sequence of positive numbers $\{\varepsilon_m\}$, a Borel subset D of Γ and a Borel set A in R^1 which satisfy the following conditions (a)-(d):*

- (a) $\lim_{m \rightarrow \infty} \varepsilon_m = 0$.
- (b) $m(\Gamma - D) = 0, m(R^1 - A) = 0$ and $D \subset A$.
- (c) For each $\lambda \in A$, there exist the following three limits:

$$\lim_{m \rightarrow \infty} Q^\pm(\lambda \pm i\varepsilon_m)x \quad \text{in } \mathfrak{X}_2,$$

$$\lim_{m \rightarrow \infty} I_2^* \delta_2(\lambda \pm i\varepsilon_m) \alpha(H_{2,ac})y \quad \text{in } \mathfrak{X}_2^*,$$

$$\lim_{m \rightarrow \infty} e_2(\lambda \pm i\varepsilon_m; Q^\pm(\lambda \pm i\varepsilon_m)x, \alpha(H_{2,ac})y) \quad \text{in } \mathbb{C}.$$

- (d) For each $\lambda \in D$ ($\subset A$), we have

$$(6.9) \quad \begin{aligned} & \lim_{m \rightarrow \infty} e_2(\lambda \pm i\varepsilon_m; Q^\pm(\lambda \pm i\varepsilon_m)x, \alpha(H_{2,ac})y) \\ & = \overline{\alpha(\lambda)} e_2(\lambda; \lim_{m \rightarrow \infty} Q^\pm(\lambda \pm i\varepsilon_m)x, y). \end{aligned}$$

PROOF. We know by (Q^\pm) and Lemma 6.4 that $\text{l.i.m.}_{\varepsilon \rightarrow +0} Q^\pm(\lambda \pm i\varepsilon)x$ in $L^2(R^1; \mathfrak{X}_2)$ and $\text{l.i.m.}_{\varepsilon \rightarrow +0} I_2^* \delta_2(\lambda \pm i\varepsilon) \alpha(H_{2,ac})y$ in $L^2(R^1; \mathfrak{X}_2^*)$ exist. Hence there exist a sequence

$\{\varepsilon_m\}$ of positive numbers and a Borel set A in R^1 such that $\lim_{m \rightarrow \infty} \varepsilon_m = 0$, $m(R^1 - A) = 0$, and (c) holds. Thus it remains to prove the existence of $D \subset \Gamma$ that satisfies (b) and (d). But this follows immediately from the equi-continuity of $\{e_2(\lambda \pm i\varepsilon_m; \cdot, \alpha(H_{2,ac})y) \mid \lambda \in \Gamma, m=1, 2, \dots\}$, the existence of the limit $\lim_{m \rightarrow \infty} Q^\pm(\lambda \pm i\varepsilon_m)x$, and (6.7). (We switch to another subsequence, if necessary.)

Q. E. D.

LEMMA 6.7. *Let (L.A.P.), (X), and (Q^\pm) be satisfied. Let Γ be bounded and put $\Gamma' = R^1 - \Gamma$. Let α be as in Lemma 6.5. Then for any $x \in \mathfrak{X}_1$, $y \in \mathfrak{X}_2$, and $\phi \in L^\infty(\Gamma')$, we have*

$$(6.10) \quad L_\varepsilon^\pm \equiv \int_{\Gamma'} \phi(\lambda) e_2(\lambda \pm i\varepsilon; Q^\pm(\lambda \pm i\varepsilon)x, \alpha(H_{2,ac})y) d\lambda \xrightarrow{(\varepsilon \rightarrow +0)} 0.$$

PROOF. From the definition of e_2 (i. e. (2.5)), we see that the integrand of (6.10) is equal to

$$\phi(\lambda)(Q^\pm(\lambda \pm i\varepsilon)x, \delta_2(\lambda \pm i\varepsilon)\alpha(H_{2,ac})y)_\mathfrak{F}.$$

Hence we know by (Q^\pm) and Lemma 6.4 that $\lim_{\varepsilon \rightarrow +0} L_\varepsilon^\pm$ exists. Thus using Lemma 6.6, we obtain

$$(6.11) \quad \begin{aligned} \left| \lim_{\varepsilon \rightarrow +0} L_\varepsilon^\pm \right| &\leq \|\phi\|_{L^\infty(\Gamma')} \left[\int_{\Gamma'} \left\| \lim_{m \rightarrow \infty} Q^\pm(\lambda \pm i\varepsilon_m)x \right\|_{\mathfrak{X}_2}^2 d\lambda \right]^{1/2} \\ &\quad \times \left[\int_{\Gamma'} \left\| \lim_{m \rightarrow \infty} I_2^* \delta_2(\lambda \pm i\varepsilon_m)\alpha(H_{2,ac})y \right\|_{\mathfrak{X}_2}^2 d\lambda \right]^{1/2}. \end{aligned}$$

On the other hand, by Lemma 6.5, $(\delta_2(\lambda \pm i\varepsilon)\alpha(H_{2,ac})y, z)_\mathfrak{F} = e_2(\lambda \pm i\varepsilon; y, \bar{\alpha}(H_{2,ac})z)$ converges to $\alpha(\lambda)e_2(\lambda; y, z)$ in $L^1(R^1)$ as $\varepsilon \rightarrow +0$ for each $z \in \mathfrak{X}_2$. Thus for any $z \in \mathfrak{X}_2$ and any Borel subset \mathcal{A} of Γ' such that $m(\mathcal{A}) < \infty$, we have

$$\begin{aligned} &\int_{\Gamma'} \langle \lim_{m \rightarrow \infty} I_2^* \delta_2(\lambda \pm i\varepsilon_m)\alpha(H_{2,ac})y, \chi_{\mathcal{A}}(\lambda)z \rangle d\lambda \\ &= \int_{\Gamma'} \chi_{\mathcal{A}}(\lambda) \lim_{m \rightarrow \infty} (\delta_2(\lambda \pm i\varepsilon_m)\alpha(H_{2,ac})y, z)_\mathfrak{F} d\lambda \\ &= \int_{\Gamma'} \chi_{\mathcal{A}}(\lambda) \alpha(\lambda) e_2(\lambda; y, z) d\lambda = 0 \end{aligned}$$

since $\alpha = 0$ on Γ' . Thus, from the fact that $\{\chi_{\mathcal{A}}(\lambda)z \mid \mathcal{A} \subset \Gamma', m(\mathcal{A}) < \infty, z \in \mathfrak{X}_2\}$ is a fundamental subset of $L^2(\Gamma'; \mathfrak{X}_2)$, we obtain

$$(6.12) \quad \lim_{m \rightarrow \infty} I_2^* \delta_2(\lambda \pm i\varepsilon_m)\alpha(H_{2,ac})y = 0 \quad \text{in } L^2(\Gamma'; \mathfrak{X}_2^*).$$

From (6.11) and (6.12), we obtain the desired result.

Q. E. D.

Now in order to formulate and prove a proposition in the next subsection, let us introduce some notations. Let (X) be satisfied. For each $z \in \mathcal{C}^\pm$, $u \in \mathfrak{F}$,

and $v \in \mathcal{D}(H_1) = \mathcal{D}(H_2)$, put

$$(6.13) \quad \tau^\pm(z)u = (S^\pm(z) - S^\mp(\bar{z}))u / 2\pi i,$$

$$(6.14) \quad D^\pm(z)v = (G^\pm(z) - G^\mp(\bar{z}))v / 2\pi i = (Q^\pm(z) - Q^\mp(\bar{z}))v / 2\pi i.$$

Then by (2.9) we have for $z \in C^\pm$ and $u \in \mathcal{D}(H_1)$,

$$(6.15) \quad \tau^\pm(z)u = \pm \delta_2(z)G^\pm(z)u + R_2(\bar{z})D^\pm(z)u.$$

6.3. A proposition. Our aim in this subsection is to prove a proposition which will play an important role in the proof of the invariance principle, etc. Let (L.A.P.) be satisfied. Then it can be easily seen that there exists a non-negative, self-adjoint and bounded operator $K_2(\lambda)$ in \mathfrak{X}_2 for any $\lambda \in \Gamma$ which satisfies the following three conditions:

$$(6.16) \quad K_2(\cdot)y: \Gamma \rightarrow \mathfrak{X}_2 \text{ is strongly continuous for each } y \in \mathfrak{X}_2;$$

$$(6.17) \quad e_2(\lambda; z, y) = (K_2(\lambda)z, y)_{\mathfrak{X}_2} = (z, K_2(\lambda)y)_{\mathfrak{X}_2} \quad \text{for every } z, y \in \mathfrak{X}_2$$

and $\lambda \in \Gamma$; and

$$(6.18) \quad \sup_{\lambda \in \Gamma} \|K_2(\lambda)\|_{B(\mathfrak{X}_2)} < \infty.$$

Now we can prove the following proposition.

PROPOSITION 6.8. *Let i)-iv), (L.A.P.), (X) and (Q[±]) be satisfied and let Γ be bounded. Let α be a bounded Borel measurable function on R¹ such that supp α ⊂ Γ. Then for any x ∈ X₁ and y ∈ X₂, we have*

$$(6.19) \quad (\tau^\pm(\lambda \pm i\varepsilon)x, \alpha(H_{2,ac})y)_\mathfrak{S} \in L^1(R^1_\lambda)$$

for any $\varepsilon > 0$. Furthermore let $\Psi \in L^\infty(R^1)$. Then the limit in the following inequality exists and we have

$$(6.20) \quad \left| \lim_{\varepsilon \rightarrow +0} \int_{-\infty}^{\infty} \Psi(\lambda) (\tau^\pm(\lambda \pm i\varepsilon)x, \alpha(H_{2,ac})y)_\mathfrak{S} d\lambda \mp \int_\Gamma \Psi(\lambda) \overline{\alpha(\lambda)} e_2(\lambda; (\Phi^\pm x)(\lambda), y) d\lambda \right|^2 \leq C(x) \|\Psi\|_{L^\infty(R^1)}^2 \int_0^\infty \left\| \int_\Gamma \alpha(\lambda) e^{\mp is\lambda} K_2(\lambda)y d\lambda \right\|_{\mathfrak{X}_2}^2 ds.$$

Here $C(x)$ is a positive constant depending only on x , and Φ^\pm is defined by (6.2).

PROOF. From (6.15) and (2.10), we have

$$(6.21) \quad (\tau^\pm(\lambda \pm i\varepsilon)x, \alpha(H_{2,ac})y)_\mathfrak{S} = \pm A_{\mp, \varepsilon}^\pm(\lambda) \pm A_{\pm, \varepsilon}^\pm(\lambda) + A_{\pm, \varepsilon}^\pm(\lambda),$$

where

$$(6.22) \quad \begin{cases} A_{1,\varepsilon}^\pm(\lambda) = e_2(\lambda \pm i\varepsilon; x, \alpha(H_{2,ac})y), \\ A_{2,\varepsilon}^\pm(\lambda) = e_2(\lambda \pm i\varepsilon; Q^\pm(\lambda \pm i\varepsilon)x, \alpha(H_{2,ac})y), \\ A_{3,\varepsilon}^\pm(\lambda) = (D^\pm(\lambda \pm i\varepsilon)x, R_2(\lambda \pm i\varepsilon)\alpha(H_{2,ac})y)_\mathfrak{F}. \end{cases}$$

Let us investigate some properties of $A_{j,\varepsilon}^\pm$ ($j=1, 2, 3$) for the moment.

$A_{1,\varepsilon}^\pm$: By Lemma 6.5, we have $A_{1,\varepsilon}^\pm \in L^1(R^1)$ for $\varepsilon > 0$ and

$$(6.23) \quad \lim_{\varepsilon \rightarrow +\infty} \int_{-\infty}^{\infty} \Psi(\lambda) A_{1,\varepsilon}^\pm(\lambda) d\lambda = \int_{\Gamma} \Psi(\lambda) \overline{\alpha(\bar{\lambda})} e_2(\lambda; x, y) d\lambda.$$

$A_{2,\varepsilon}^\pm$: Lemma 6.4 and (Q^\pm) show that $A_{2,\varepsilon}^\pm \in L^1(R^1)$ for $\varepsilon > 0$ and that $\lim_{\varepsilon \rightarrow +0} A_{2,\varepsilon}^\pm$ exists in $L^1(R^1)$. Thus we have by Lemma 6.7,

$$(6.24) \quad \begin{aligned} \lim_{\varepsilon \rightarrow +0} \int_{-\infty}^{\infty} \Psi(\lambda) A_{2,\varepsilon}^\pm(\lambda) d\lambda &= \lim_{\varepsilon \rightarrow +0} \left(\int_{\Gamma} + \int_{\Gamma'} \right) \Psi(\lambda) A_{2,\varepsilon}^\pm(\lambda) d\lambda \\ &= \int_{\Gamma} \Psi(\lambda) \lim_{\varepsilon \rightarrow +0} A_{2,\varepsilon}^\pm(\lambda) d\lambda. \end{aligned}$$

Lemma 6.6 shows that this is equal to

$$(6.25) \quad \begin{aligned} &\int_{\Gamma} \Psi(\lambda) \lim_{m \rightarrow \infty} A_{2,\varepsilon_m}^\pm(\lambda) d\lambda \\ &= \int_{\Gamma} \Psi(\lambda) \overline{\alpha(\bar{\lambda})} e_2(\lambda; \lim_{m \rightarrow \infty} Q^\pm(\lambda \pm i\varepsilon_m)x, y) d\lambda \end{aligned}$$

where $\{\varepsilon_m\}$ is the sequence appearing in Lemma 6.6.

$A_{3,\varepsilon}^\pm$: Since $D^\pm(\lambda \pm i\varepsilon)x \in \mathfrak{X}_2$ for all $\lambda \in R^1$ and $\varepsilon > 0$ by (Q^\pm) and (6.14), and since e_2 is a spectral form for H_2 , we have for $\lambda \in R^1$ and $\varepsilon > 0$

$$(6.26) \quad A_{3,\varepsilon}^\pm(\lambda) = \int_{\Gamma} \frac{\overline{\alpha(\mu)}}{\mu - (\lambda \mp i\varepsilon)} e_2(\mu; D^\pm(\lambda \pm i\varepsilon)x, y) d\mu.$$

By (6.18), we have $\alpha(\cdot)K_2(\cdot)y \in L^1(R^1; \mathfrak{X}_2) \cap L^2(R^1; \mathfrak{X}_2)$. Thus the integral

$$(6.27) \quad h^\pm(\lambda \pm i\varepsilon) = \int_{\Gamma} \frac{\alpha(\mu)}{\mu - (\lambda \pm i\varepsilon)} K_2(\mu)y d\mu$$

converges in \mathfrak{X}_2 for any $\lambda \in R^1$ and $\varepsilon > 0$, and moreover we have $h^\pm \in H^2(C^\pm; \mathfrak{X}_2)$. Therefore using (6.17), we can write

$$(6.28) \quad A_{3,\varepsilon}^\pm(\lambda) = (D^\pm(\lambda \pm i\varepsilon)x, h^\pm(\lambda \pm i\varepsilon))_{\mathfrak{X}_2}$$

for $\lambda \in R^1$ and $\varepsilon > 0$. Hence by (Q^\pm) and $h^\pm \in H^2(C^\pm; \mathfrak{X}_2)$, we have $A_{3,\varepsilon}^\pm \in L^1(R^1)$ and

$$(6.29) \quad \begin{aligned} &\lim_{\varepsilon \rightarrow +0} \int_{-\infty}^{\infty} \Psi(\lambda) A_{3,\varepsilon}^\pm(\lambda) d\lambda \\ &= \int_{-\infty}^{\infty} \Psi(\lambda) (\text{l.i.m.}_{\varepsilon \rightarrow +0} D^\pm(\lambda \pm i\varepsilon)x, h^\pm(\lambda))_{\mathfrak{X}_2} d\lambda, \end{aligned}$$

where $h^\pm(\lambda) = \text{l.i.m.}_{\varepsilon \rightarrow +0} h^\pm(\lambda \pm i\varepsilon)$ in $L^2(R^1; \mathfrak{X}_2)$. On the other hand, by the theory of Fourier transformation for Hilbert space valued functions, we get

$$(6.30) \quad \int_{-\infty}^{\infty} \|h^\pm(\lambda)\|_{\mathfrak{X}_2}^2 d\lambda = 2\pi \int_0^\infty \left\| \int_{\Gamma} \alpha(\lambda) e^{\mp i s \lambda} K_2(\lambda) y d\lambda \right\|_{\mathfrak{X}_2}^2 ds.$$

Thus using Schwarz inequality we obtain

$$(6.31) \quad \left| \lim_{\varepsilon \rightarrow +0} \int_{-\infty}^{\infty} \Psi(\lambda) A_{3,\varepsilon}^\pm(\lambda) d\lambda \right|^2 \leq C(x) \|\Psi\|_{L^\infty(R^1)}^2 \int_0^\infty \left\| \int_{\Gamma} \alpha(\lambda) e^{\mp i s \lambda} K_2(\lambda) y d\lambda \right\|_{\mathfrak{X}_2}^2 ds,$$

where $C(x) = 2\pi \int_{-\infty}^{\infty} \|\text{l.i.m.}_{\varepsilon \rightarrow +0} D^\pm(\lambda \pm i\varepsilon)x\|_{\mathfrak{X}_2}^2 d\lambda$.

Now it is easy to see that Proposition 6.8 holds. The relation (6.19) follows from $A_{j,\varepsilon}^\pm \in L^1(R^1)$ ($j=1, 2, 3, \varepsilon>0$). The inequality (6.20) can be easily proved by using (6.21), (6.23), (6.24), (6.25), and (6.31). Q. E. D.

6.4. Time dependent wave operators. In this subsection we shall prove Theorems 2.1 and 2.2. In the proof we do not assume \mathfrak{X}_j is dense in \mathfrak{H} (cf. footnote 4)). For this purpose, it suffices to prove that, under the assumptions of Theorem 2.1 or 2.2, we have

$$(6.32) \quad W_{\tilde{F}}^\pm u = \lim_{t \rightarrow \pm\infty} W_D(t)u \quad \text{for } u \in \mathfrak{L}_{1,ac}(\Gamma),$$

where $W_D(t) = e^{itH_2} e^{-itH_1 - iX(t)}$, and $W_{\tilde{F}}^\pm$ is the stationary wave operator in Theorem 6.1. In fact, Theorem 2.1 follows from Theorem 6.1 and (6.32), and Theorem 2.2 from Theorem 6.1, Remark 6.2 and (6.32), because conditions (a)-(c) of Theorem 2.2 are equivalent to (\mathcal{F}) and $(\mathcal{F}\Phi)$ of Theorem 5.1 with $\Phi = \Phi^\pm$.

Now let us prove (6.32). Since $W_D(t)$ is unitary and $\bigcup_{\tilde{F} \subset \Gamma} \mathfrak{L}_{1,ac}(\tilde{F})$ is fundamental in $\mathfrak{L}_{1,ac}(\Gamma)$, where \tilde{F} varies over the set of all bounded Borel subsets of Γ , we may and shall assume Γ is bounded.

Let $x \in \mathfrak{X}_1$, $y \in \mathfrak{X}_2$ and a Borel subset Δ of Γ be arbitrary but fixed. In Proposition 6.8, put $\Psi(\lambda) = e^{-it\lambda}$ and $\alpha(\lambda) = e^{-it\lambda} \chi_\Delta(\lambda)$. Then from (6.20), we obtain for all $t \in R^1$,

$$(6.33) \quad \left| \lim_{\varepsilon \rightarrow +0} \int_{-\infty}^{\infty} e^{-it\lambda} (\tau^\pm(\lambda \pm i\varepsilon)x, e^{-itH_2} E_{2,ac}(\Delta)y)_{\mathfrak{H}} d\lambda \mp \int_{\Gamma} \chi_\Delta(\lambda) e_2(\lambda; (\Phi^\pm x)(\lambda), y) d\lambda \right|^2 \leq C(x) \int_0^\infty \left\| \int_{\Gamma} \chi_\Delta(\lambda) e^{-it\lambda \mp is\lambda} K_2(\lambda) y d\lambda \right\|_{\mathfrak{X}_2}^2 ds,$$

where $C(x)$ is a positive constant depending only on x .

Now let us put for $t \in R^1$ and $\varepsilon > 0$,

$$f_{\varepsilon,t}(s) = e^{-\varepsilon|s|} (e^{-isH_1 - iX(s)}x, e^{-itH_2}E_{2,ac}(\mathcal{A})y)_{\mathfrak{F}}.$$

Then for any $t \in R^1$ and $\varepsilon > 0$, we have $f_{\varepsilon,t} \in L^1(R^1)$. Thus from the definitions of τ^{\pm} and S^{\pm} , and (6.19), we get

$$-\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{is\lambda} f_{\varepsilon,t}(s) ds = \pm (\tau^{\pm}(\lambda \pm i\varepsilon)x, e^{-itH_2}E_{2,ac}(\mathcal{A})y)_{\mathfrak{F}} \in L^1(R^1)$$

for $t \in R^1$ and $\varepsilon > 0$. Moreover by (BC), $f_{\varepsilon,t}(s)$ is of bounded variation in s on any finite interval and continuous in s . Thus we can apply the inversion formula for Fourier transforms to $f_{\varepsilon,t}$ and we obtain

$$(6.34) \quad f_{\varepsilon,t}(s) = \pm \int_{-\infty}^{\infty} e^{-is\lambda} (\tau^{\pm}(\lambda \pm i\varepsilon)x, e^{-itH_2}E_{2,ac}(\mathcal{A})y)_{\mathfrak{F}} d\lambda$$

for any $t, s \in R^1$ and $\varepsilon > 0$. Therefore, putting $s=t$, letting $\varepsilon \rightarrow +0$ in (6.34), and remembering the definition of $f_{\varepsilon,t}(s)$, we have for any $t \in R^1$

$$(6.35) \quad \lim_{\varepsilon \rightarrow +0} \int_{-\infty}^{\infty} e^{-it\lambda} (\tau^{\pm}(\lambda \pm i\varepsilon)x, e^{-itH_2}E_{2,ac}(\mathcal{A})y)_{\mathfrak{F}} d\lambda \\ = \pm (W_D(t)x, E_{2,ac}(\mathcal{A})y)_{\mathfrak{F}}.$$

On the other hand, by Theorem 6.1 and Remark 6.2, we have

$$(6.36) \quad \int_{\Gamma} \chi_{\mathcal{A}}(\lambda) e_2(\lambda; (\Phi^{\pm}x)(\lambda), y) d\lambda = (W_{\mathfrak{F}}^{\pm}x, E_{2,ac}(\mathcal{A})y)_{\mathfrak{F}},$$

under either assumptions of Theorem 2.1 or Theorem 2.2. Thus from (6.33), (6.35), (6.36), and the fact that the right-hand side of (6.33) converges to zero as $t \rightarrow \pm\infty$, we obtain

$$(6.37) \quad \lim_{t \rightarrow \pm\infty} ((W_D(t) - W_{\mathfrak{F}}^{\pm})x, E_{2,ac}(\mathcal{A})y)_{\mathfrak{F}} = 0$$

for any $x \in \mathfrak{X}_1$, $y \in \mathfrak{X}_2$, and $\mathcal{A} \subset \Gamma$, $\mathcal{A} \in \mathcal{B}$.

Now from (6.37), we can show that for any $u \in \mathfrak{X}_1$ and $w \in \mathfrak{X}_{2,ac}(\Gamma)$,

$$(6.38) \quad \lim_{t \rightarrow \pm\infty} (W_D(t)u, w)_{\mathfrak{F}} = (W_{\mathfrak{F}}^{\pm}u, w)_{\mathfrak{F}}.$$

In fact, since $W_D(t)$ and $W_{\mathfrak{F}}^{\pm}$ are uniformly bounded and the vectors of the form $E_{2,ac}(\mathcal{A})y$, $y \in \mathfrak{X}_2$, $\mathcal{A} \subset \Gamma$, $\mathcal{A} \in \mathcal{B}$, are fundamental in $\mathfrak{X}_{2,ac}(\Gamma)$, it is easy to see that (6.38) holds for any $u \in \mathfrak{X}_1$ and $w \in \mathfrak{X}_{2,ac}(\Gamma)$. To prove (6.38), it now suffices to consider the case $u = e^{-isH_1}x$, $x \in \mathfrak{X}_1$, $s \in R^1$, and $w \in \mathfrak{X}_{2,ac}(\Gamma)$, for such u 's are fundamental in \mathfrak{X}_1 . But using (XA), $e^{isH_2}w \in \mathfrak{X}_{2,ac}(\Gamma)$, and the intertwining property (4.7) of $W_{\mathfrak{F}}^{\pm}$, we obtain $\lim_{t \rightarrow \pm\infty} (W_D(t)e^{-isH_1}x, w)_{\mathfrak{F}} =$

$$\lim_{t \rightarrow \pm\infty} (W_D(t+s)e^{i(X(t+s)-X(t))}x, e^{isH_2}w)_{\mathfrak{F}} = \lim_{t \rightarrow \pm\infty} (W_D(t+s)x, e^{isH_2}w)_{\mathfrak{F}} = (W_{\mathfrak{F}}^{\pm}x, e^{isH_2}w)_{\mathfrak{F}} =$$

$(W_{\mp}^{\pm} e^{-isH_1} x, w)_{\mathfrak{H}}$, as desired.

Suppose now that $u \in \mathfrak{L}_{1,ac}(\Gamma) \subset \mathfrak{L}_1$. Since $\|W_D(t)\|_{B(\mathfrak{H})} \leq 1$ and $\|W_{\mp}^{\pm} u\|_{\mathfrak{H}} = \|u\|_{\mathfrak{H}}$, we have $\|W_D(t)u - W_{\mp}^{\pm} u\|_{\mathfrak{H}}^2 \leq 2 \operatorname{Re} (W_{\mp}^{\pm} u - W_D(t)u, W_{\mp}^{\pm} u)_{\mathfrak{H}}$. The right-hand side tends to zero as $t \rightarrow \pm\infty$ by (6.38), since $W_{\mp}^{\pm} u \in \mathfrak{L}_{2,ac}(\Gamma)$. Thus we have proved (6.32).

6.5. The invariance principle. We shall prove Theorem 2.3 and the last half of Remark 2.4 without assuming the denseness of \mathfrak{X}_j in \mathfrak{H} as in the previous subsection. For this purpose, it is sufficient to prove

$$(6.39) \quad W_{\mp}^{\pm} u = \lim_{t \rightarrow \pm\infty} W_{\varphi}^{as}(t)u = \lim_{t \rightarrow \pm\infty} W_{\varphi}(t)u \quad \text{for } u \in \mathfrak{L}_{1,ac}(\Gamma),$$

where $W_{\varphi}^{as}(t) = e^{it\varphi(H_2)} Q_{\varphi}^{as}(t)$ and $W_{\varphi}(t) = e^{it\varphi(H_2)} Q_{\varphi}(t)$, with $\mathfrak{L}_{1,ac}(\Gamma)$ in conditions (Q_{φ}^{as}) and (\mathfrak{X}_1) replaced by $\mathfrak{L}_{1,ac}(\Gamma)$.

Let $x \in \mathfrak{X}_1$, $y \in \mathfrak{X}_2$ and $\Delta \subset \Gamma$, $\Delta \in \mathcal{B}$ be arbitrarily fixed. In Proposition 6.8, put $\Psi(\lambda) = e^{-it\varphi(\lambda)} \eta(\lambda)$ and $\alpha(\lambda) = e^{-it\varphi(\lambda)} \chi_{\Delta}(\lambda)$, where η is taken as in § 2.1. Then from (6.20), we obtain for all $t \in R^1$

$$(6.40) \quad \left| \lim_{\varepsilon \rightarrow +0} \int_{-\infty}^{\infty} \eta(\lambda) e^{-it\varphi(\lambda)} (\tau^{\pm}(\lambda \pm i\varepsilon)x, e^{-it\varphi(H_2)} E_{2,ac}(\Delta)y)_{\mathfrak{H}} d\lambda \right. \\ \left. \mp \int_{\Gamma} \chi_{\Delta}(\lambda) e_2(\lambda; (\Phi^{\pm}x)(\lambda), y) d\lambda \right|^2 \\ \leq C(x) \int_0^{\infty} \left\| \int_{\Gamma} \chi_{\Delta}(\lambda) e^{-it\varphi(\lambda) \mp is\lambda} K_2(\lambda)y d\lambda \right\|_{\mathfrak{X}_2}^2 ds.$$

Since $\sup_{\lambda \in R^1} \|\tau^{\pm}(\lambda \pm i\varepsilon)\|_{B(\mathfrak{H})} < \infty$ for any $\varepsilon > 0$ by definition, the following integral converges in \mathfrak{H} for any $\varepsilon > 0$ and $u \in \mathfrak{H}$:

$$\int_{-\infty}^{\infty} \eta(\lambda) e^{-it\varphi(\lambda)} \tau^{\pm}(\lambda \pm i\varepsilon)u d\lambda.$$

By Fubini's theorem and the definition of τ^{\pm} , this becomes equal to

$$\pm \int_{-\infty}^{\infty} a_{\varphi}(t, r) (e^{-\varepsilon|r|} e^{-irH_1 - iX(r)} u) dr,$$

and hence converges to $\pm Q_{\varphi}(t)u$ as $\varepsilon \rightarrow +0$, since $a_{\varphi}(t, \cdot) \in L^1(R^1)$ for $t \in R^1$. Thus for every $t \in R^1$, we have

$$(6.41) \quad \lim_{\varepsilon \rightarrow +0} \int_{-\infty}^{\infty} \eta(\lambda) e^{-it\varphi(\lambda)} (\tau^{\pm}(\lambda \pm i\varepsilon)x, e^{-it\varphi(H_2)} E_{2,ac}(\Delta)y)_{\mathfrak{H}} d\lambda \\ = \pm (W_{\varphi}(t)x, E_{2,ac}(\Delta)y)_{\mathfrak{H}}.$$

On the other hand, under either assumptions of Theorem 2.3 or the last half of Remark 2.4, relation (6.36) holds. Thus from (6.40), (6.41), (6.36), and the fact that the right-hand side of (6.40) converges to zero as $t \rightarrow \pm\infty$ since $\varphi' > 0$

on I (cf. Lemma 4.6 of Chapt. X of Kato [12], and Lemma 7.6 of Kato and Kuroda [13]), we obtain

$$(6.42) \quad \lim_{t \rightarrow \pm\infty} (W_\varphi(t)x - W_{\tilde{F}}^\pm x, E_{2,ac}(\mathcal{A})y)_\mathfrak{B} = 0$$

for all $x \in \mathfrak{X}_1$, $y \in \mathfrak{X}_2$, and $\mathcal{A} \subset \Gamma$, $\mathcal{A} \in \mathcal{B}$. From this and (2.18) in (Q_φ^{as}) , we get

$$(6.43) \quad \lim_{t \rightarrow \pm\infty} (W_\varphi^{as}(t)x - W_{\tilde{F}}^\pm x, E_{2,ac}(\mathcal{A})y)_\mathfrak{B} = 0$$

for any $x \in \mathfrak{X}_1 \cap \mathfrak{D}_{1,ac}(\Gamma)$, $y \in \mathfrak{X}_2$ and $\mathcal{A} \subset \Gamma$, $\mathcal{A} \in \mathcal{B}$. Thus by $\|W_\varphi^{as}(t)\|_{B(\mathfrak{D}_{1,ac}(\Gamma), \mathfrak{B})} \leq 1$ and (\mathfrak{X}_1) , we can easily prove (6.39) in a way similar to the last part of § 6.4.

References

- [1] S. Agmon, Spectral properties of Schrödinger operators and scattering theory, *Ann. Scuola Norm. Sup. Pisa*, (4) 2 (1975), 151-218.
- [2] P. Alsholm, Wave operators for long-range scattering, (preprint, 1976).
- [3] P. Alsholm and T. Kato, Scattering with long-range potentials, *Proc. Symp. Pure Math.*, 23 (1973), 393-399.
- [4] W.O. Amrein, Ph. A. Martin and B. Misra, On the asymptotic condition of scattering theory, *Helv. Phys. Acta*, 43 (1970), 313-344.
- [5] V.S. Buslaev and V. B. Matveev, Wave operators for the Schrödinger equation with a slowly decreasing potential, *Theoret. and Math. Phys.*, 2 (1970), 266-274 (English translation from Russian).
- [6] C. Chandler and A.G. Gibson, Invariance principle for scattering with long-range (and other) potentials, *Indiana Univ. Math. J.*, 25 (1976), 443-460.
- [7] J.D. Dollard, Asymptotic convergence and the Coulomb interaction, *J. Math. Phys.*, 5 (1964), 729-738.
- [8] L. Hörmander, The existence of wave operators in scattering theory, *Math. Z.*, 146 (1976), 69-91.
- [9] T. Ikebe, Spectral representation for Schrödinger operators with long-range potentials, *J. Functional Analysis*, 20 (1975), 158-177.
- [10] T. Ikebe and Y. Saitō, Limiting absorption method and absolute continuity for the Schrödinger operator, *J. Math. Kyoto Univ.*, 12 (1972), 513-542.
- [11] H. Isozaki, On the long-range stationary wave operator, (to appear).
- [12] T. Kato, *Perturbation Theory for Linear Operators*, 2nd ed., Springer-Verlag, Berlin, Heiderberg, and New York, 1976.
- [13] T. Kato and S.T. Kuroda, Theory of simple scattering and eigenfunction expansions, *Functional Analysis and Related Fields*, Springer-Verlag, Berlin, Heiderberg, and New York, 1970, 99-131.
- [14] T. Kato and S.T. Kuroda, The abstract theory of scattering, *Rocky Mount. J. Math.*, 1 (1971), 127-171.
- [15] H. Kitada, A stationary approach to long-range scattering, *Osaka J. Math.*, 13 (1976), 311-333.
- [16] H. Kitada, On the completeness of modified wave operators, *Proc. Japan Acad.*, 52 (1976), 409-412.
- [17] S.T. Kuroda, Scattering theory for differential operators, I, operator theory, *J.*

- Math. Soc. Japan, 25 (1973), 75-104.
- [18] S. T. Kuroda, Scattering theory for differential operators, II, self-adjoint elliptic operators, J. Math. Soc. Japan, 25 (1973), 222-234.
- [19] R. Lavine, Absolute continuity of positive spectrum for Schrödinger operators with long-range potentials, J. Functional Analysis, 12 (1973), 30-54.
- [20] R. Lavine, Commutators and scattering theory, II, Indiana Univ. Math. J., 21 (1972), 643-656.
- [21] V. B. Matveev, The invariance principle for generalized wave operators, Topics in Mathematics, Consultants Bureau, New York, London, 1972, 77-85 (English translation from Russian).
- [22] G. Pinchuk, Abstract time-independent wave operator theory for long-range potentials, (preprint, 1975).
- [23] Y. Saitō, The principle of limiting absorption for second-order differential equations with operator-valued coefficients, Publ. RIMS, Kyoto Univ., 7 (1971/72), 581-619.
- [24] Y. Saitō, Spectral and scattering theory for second-order differential operators with operator-valued coefficients, Osaka J. Math., 9 (1972), 463-498.
- [25] Y. Saitō, Spectral theory for second-order differential operators with long-range operator-valued coefficients, I, limiting absorption principle, Japan. J. Math., 1 (1975), 311-349.
- [26] Y. Saitō, Spectral theory for second-order differential operators with long-range operator-valued coefficients, II, eigenfunction expansions and the Schrödinger operators with long-range potentials, Japan. J. Math., 1 (1975), 351-382.
- [27] Y. Saitō, On the asymptotic behavior of the solutions of the Schrödinger equation $(-\Delta + Q(y) - k^2)V = F$, Osaka J. Math., 14 (1977), 11-35.
- [28] Y. Saitō, Eigenfunction expansions for the Schrödinger operators with long-range potentials $Q(y) = O(|y|^{-\varepsilon})$, $\varepsilon > 0$, Osaka J. Math., 14 (1977), 37-53.

Hitoshi KITADA

Department of Pure and Applied Sciences
College of General Education
University of Tokyo
Komaba, Meguro-ku
Tokyo, Japan