J. Math. Soc. Japan
Vol. 29, No. 4, 1977

On Čech homology and a stability theorem in shape theory

By Tadashi WATANABE

(Received Feb. 16, 1976)
(Revised May 31, 1976)

§ 1. Introduction and theorems.

The notion of shape was originally introduced by Borsuk and has been extended by many authors (see Borsuk [2] and [3] for shape theory). In this paper we shall use the ANR-systems approach to shape theory of Mardešić and Segal [9].

In [1], Borsuk introduced the notion of movability and raised the following problem: Let X and A be movable compact metric spaces such that $A\subset X$. Is the Čech homology sequence of such a pair (X, A) necessarily exact? Concerning this problem, Overton [11] constructed a counter-example and proved that Borsuk's problem is true if a pair (X, A) is movable. In this paper we shall consider Borsuk's problem under a different condition. Specifically we show the following theorem.

THEOREM 1. *Let X and A be movable compact metric spaces such that $A\subset X$. If the n-th Čech homology of X is a countable group for each n, then the Čech homology sequence of (X, A) is exact.*

Recently Edward and Geoghegan [6] proved a very important theorem, "stability theorem", which gives algebraic characterizations of FANR-continua in terms of the category of pro-groups or by using topologies on shape groups. These characterizations, however, are not simple. Therefore, in this paper we shall give some improvements on these characterizations. That is, we shall show the following theorem.

THEOREM 2. *Let (X, x) be a pointed compact connected metric space with finite dimension. Then the following conditions are equivalent:*

(A) *(X, x) is pointed movable and the n-th shape group of (X, x) is a countable group for each $n\geqq 1$.*

(B) *The n-th pro-homotopy groups of (X, x) satisfies Mittag-Leffler condition and the n-th shape group of (X, x) is a countable group for each $n\geqq 1$.*

(C) *(X, x) is a pointed FANR-space.*

§ 2. Mittag-Leffler condition and lemmas.

In this section we consider only inverse sequences of groups, that is, inverse systems of groups directed by all positive integers, simply denoted by $\{G_i\}$.

DEFINITION 1. Let $\{G_i, \pi_i^j\}$ be an inverse sequence of groups. It is said to be a *normal inverse sequence* if $\pi_i^j(G_j)$ is a normal subgroup of G_i for each $i, j, i \leqq j$.

DEFINITION 2. An inverse sequence $\{G_i, \pi_i^j\}$ of groups satisfies *Mittag-Leffler condition* if for each i, there exists j, $j \geqq i$ such that $\pi_i^j(G_j) = \pi_i^k(G_k)$ for each k, $k \geqq j$.

Let $\{G_i, \pi_i^j\}$ and $\{H_i, \rho_i^j\}$ be inverse sequences of groups. Then $\{g_i\}$ is said to be a *map* from $\{G_i\}$ to $\{H_i\}$ if each $g_i : G_i \to H_i$ is a group homomorphism such that $g_i \pi_i^{i+1} = \rho_i^{i+1} g_{i+1}$ for each i (in notation $\{g_i\} : \{G_i\} \to \{H_i\}$). A sequence $* \longrightarrow \{K_i\} \xrightarrow{\{f_i\}} \{G_i\} \xrightarrow{\{g_i\}} \{H_i\} \longrightarrow *$ is said to be exact if for each i, $* \longrightarrow K_i \xrightarrow{f_i} G_i \xrightarrow{g_i} H_i \longrightarrow *$ is exact in the usual sense where $*$ means the null group.

The following lemma is easily proved by the definition of Mittag-Leffler condition.

LEMMA 1. *Let* $\{G_i\} \xrightarrow{\{g_i\}} \{H_i\} \longrightarrow *$ *be exact. If* $\{G_i\}$ *satisfies Mittag-Leffler condition, then* $\{H_i\}$ *satisfies Mittag-Leffler condition.*

Next, we investigate the functors $\varprojlim$ and $\varprojlim^1$ for inverse sequences of groups. In this paper we use the definition of Bousfield and Kan [4, p. 251], because we have to consider the non-abelian groups. Let $\{G_i, \pi_i^j\}$ be an inverse sequence of groups. Then $\varprojlim \{G_i\}$ means the inverse limit group of $\{G_i\}$, and $\varprojlim^1\{G_i\}$ is the pointed set as follows; $\varprojlim^1\{G_i\}$ consists of the equivalence classes of $\prod_{i=1}^{\infty} G_i$ under the equivalence relation given by $x \sim y$ if and only if $y = g \circ x$ for some $g \in \prod_{i=1}^{\infty} G_i$, where $(g_1, g_2, \cdots) \circ (x_1, x_2, \cdots) = (g_1 x_1 \pi_1^2 g_2^{-1}, g_2 x_2 \pi_2^3 g_3^{-1}, \cdots)$ and the class represented by $e = (e_1, e_2, \cdots)$ is a base point where each e_i is the unit element of G_i. Let $\rho : \prod_{i=1}^{\infty} G_i \to \varprojlim^1\{G_i\}$ be the natural projection.

The following lemma is due to [4, p. 252].

LEMMA 2. *Let* $* \longrightarrow \{K_i\} \xrightarrow{\{f_i\}} \{G_i, \pi_i^j\} \xrightarrow{\{g_i\}} \{H_i\} \longrightarrow *$ *be exact. Then we have the following exact sequence;* $* \longrightarrow \varprojlim \{K_i\} \xrightarrow{f} \varprojlim \{G_i\} \xrightarrow{g} \varprojlim \{H_i\} \xrightarrow{\delta} \varprojlim^1\{K_i\} \xrightarrow{f'} \varprojlim^1\{G_i\} \xrightarrow{g'} \varprojlim^1\{H_i\} \longrightarrow *$, *where* f, g, f' *and* g' *are induced* *maps and δ is the connecting map.*

Now, we recall the definition of the connecting map δ. Let (x_i) be an element of $\varprojlim \{H_i\}$. Since the given sequence is exact, there exists $y_i \in G_i$ such that $g_i(y_i)=x_i$ for each i. Since $(x_i) \in \varprojlim \{H_i\}$, $g_i(y_i \pi_i^{i+1}(y_{i+1}^{-1}))=*$ for each i. Hence there exists $z_i \in K_i$ such that $f_i(z_i)=y_i \pi_i^{i+1}(y_{i+1}^{-1})$ for each i by the exactness of the given sequence. The connecting map δ is well-defined by $\delta(x_i)=\rho(z_1, z_2, \cdots)$. This description of the connecting map δ is used in the latter lemma.

The following property of Mittag-Leffler condition is very important, which is due to [**4**, p. 256].

LEMMA 3. *If $\{G_i\}$ satisfies Mittag-Leffler condition, then $\varprojlim^1\{G_i\}=*$.*

Furthermore the converse of Lemma 3 is true for special inverse sequences. For it we have to calculate cardinalities of sets. Let $|A|$ be the cardinality of a set A.

LEMMA 4. *Let $\{G_i, \pi_i^j\}$ be a normal inverse sequence of groups such that G_i is a countable group for each i. Then we have the followings:*

(1) *$\{G_i\}$ satisfies Mittag-Leffler condition if and only if $\varprojlim^1\{G_i\}=*$, that is, $|\varprojlim^1\{G_i\}|=1$.*

(2) *$\{G_i\}$ does not satisfy Mittag-Leffler condition if and only if $|\varprojlim^1\{G_i\}| > \aleph_0$.*

PROOF. Since we have Lemma 3, it is enough for our proof to show that if $\{G_i\}$ does not satisfy Mittag-Leffler condition, then $|\varprojlim^1\{G_i\}| > \aleph_0$. Therefore we suppose that $\{G_i\}$ does not satisfy Mittag-Leffler condition. Then there exists n_0 such that for infinitely many n, $n \geqq n_0$, $\pi_{n_0}^n(G_n) \supsetneqq \pi_{n0}^{n+1}(G_{n+1})$. Let $H_n=\pi_{n_0}^n(G_n)$ for each n, $n \geqq n_0$. Then $\{H_n\}_{n \geqq n_0}$ forms an inverse sequence whose bonding maps are inclusions. Next, we define a map $f_n : G_n \to H_n$ such that $f_n(x)=\pi_{n_0}^n(x)$ for each $x \in G_n$, $n \geqq n_0$. We obtain the following exact sequence by Lemma 2;

$$\varprojlim^1\{G_n\}_{n \geqq n_0} \xrightarrow{f'} \varprojlim^1\{H_n\}_{n \geqq n_0} \longrightarrow * .$$

Since $\varprojlim^1 \{G_n\} = \varprojlim^1 \{G_n\}_{n \geqq n_0}$, it is enough for our proof to show that $|\varprojlim^1\{H_n\}_{n \geqq n_0}| > \aleph_0$. This fact is reduced to the following lemma.

LEMMA 5. *Let H_i be a countable group such that $H_i \supset H_{i+1}$ for each i, each H_i is a normal subgroup of H_1 and for infinitely many i, $H_i \supsetneqq H_{i+1}$. Then $|\varprojlim^1\{H_i\}| > \aleph_0$ where $\{H_i\}$ is an inverse sequence whose bonding maps are inclusions.*

PROOF. Since each H_i is a normal subgroup of H_1, we can consider the quotient group H_1/H_i. Let $\pi_i : H_1 \to H_1/H_i$ be the natural projection. Thus we obtain the following exact sequence by Lemma 2;

$$(1) \qquad * \longrightarrow \varprojlim \{H_i\} \longrightarrow \varprojlim \{H_1\} \xrightarrow{\pi} \varprojlim \{H_1/H_i\} \xrightarrow{\delta} \varprojlim{}^1\{H_i\} \longrightarrow$$
$$\longrightarrow \varprojlim{}^1\{H_1\} \longrightarrow \varprojlim{}^1\{H_1/H_i\} \longrightarrow * \qquad \text{where } \pi=\varprojlim \{\pi_i\}\,.$$

Since each bonding map of H_1 is the identity map, then $\{H_1\}$ satisfies Mittag-Leffler condition. Hence $\varprojlim{}^1\{H_1\}=*$ by Lemma 3. Therefore we obtain the following exact sequence from (1);

$$(2) \qquad * \longrightarrow \bigcap_{i=1}^{\infty} H_i \longrightarrow H_1 \xrightarrow{\pi} \varprojlim \{H_1/H_i\} \xrightarrow{\delta} \varprojlim{}^1\{H_i\} \longrightarrow *\,.$$

Here, again we recall the definition of the connecting map δ. It is easily proved by the definition that if $y\in H_1$ and $x=(x_1, x_2, \cdots)\in\varprojlim \{H_1/H_i\}$, then $\delta(x)=\delta(x_1\pi_1(y), x_2\pi_2(y), \cdots)=\delta(x\pi(y))$. Hence naturally we can define the map $\hat{\delta}: \hat{H} \longrightarrow \varprojlim{}^1\{H_i\}$ by δ where $\hat{H}$ is the right coset of the group $\varprojlim \{H_1/H_i\}$ by the subgroup $\pi(H_1)$.

CLAIM 1. *$\hat{\delta}$ is bijective.*

PROOF OF CLAIM 1. Since δ is onto by (2), then $\hat{\delta}$ is onto. Next, let $x=(x_1, x_2, \cdots)$, $\tilde{x}=(\tilde{x}_1, \tilde{x}_2, \cdots)$ be elements of $\varprojlim \{H_1/H_i\}$ such that $\delta(x)=\delta(\tilde{x})$. Let $y_i, \tilde{y}_i$ be elements of H_1 such that $\pi_i(y_i)=x_i$, $\pi_i(\tilde{y}_i)=\tilde{x}_i$ for each i. Hence $\delta(x)=\rho(y_1y_2^{-1}, y_2y_3^{-1}, \cdots)$ and $\delta(\tilde{x})=\rho(\tilde{y}_1\tilde{y}_2^{-1}, \tilde{y}_2\tilde{y}_3^{-1}, \cdots)$ by the definition of δ. Since $\delta(x)=\delta(\tilde{x})$, there exists $z_i\in H_i$ such that $y_iy_{i+1}^{-1}=z_i\tilde{y}_i\tilde{y}_{i+1}^{-1}z_{i+1}^{-1}$ for each i. Thus $y_i^{-1}z_i\tilde{y}_i=y_{i+1}^{-1}z_{i+1}\tilde{y}_{i+1}$ for each i. This means that for each i, j, $y_i^{-1}z_i\tilde{y}_i=y_j^{-1}z_j\tilde{y}_j$. Let $z=y_i^{-1}z_i\tilde{y}_i\in H_1$. Since $z_i\in H_i$, we obtain that $\pi_i(z)=\pi_i(y_i^{-1})\pi_i(z_i)\pi_i(\tilde{y}_i)=\pi_i(y_i^{-1})\pi_i(\tilde{y}_i)$ for each i. This means that $\tilde{x}_i=x_i\pi_i(z)$ for each i. Therefore $\tilde{x}=x\pi(z)$. That is, $\hat{\delta}$ is injective. This completes the proof of Claim 1.

CLAIM 2. $|\varprojlim \{H_1/H_i\}|>\aleph_0$.

PROOF OF CLAIM 2. If we forget the group structure of $\varprojlim \{H_1/H_i\}$, then $\varprojlim\{H_1/H_i\}=\prod_{i=1}^{\infty}(H_i/H_{i+1})$ as sets. Since for infinitely many i, $H_i\supsetneqq H_{i+1}$ $|\varprojlim\{H_1/H_i\}|=\prod_{i=1}^{\infty}|H_i/H_{i+1}|>\aleph_0$. This completes the proof of Claim 2.

Finally we calculate the cardinality of $\varprojlim{}^1\{H_i\}$. By Claim 1, $|\varprojlim{}^1\{H_i\}|=|\hat{H}|$. By the definition of $\hat{H}$, $|\varprojlim\{H_1/H_i\}|=|\hat{H}|\,|\pi(H_1)|$, and by Claim 2, $|\varprojlim\{H_1/H_i\}|>\aleph_0$. Hence $|\varprojlim{}^1\{H_i\}|>\aleph_0$, because $|H_1|\leqq\aleph_0$. This completes the proof of Lemma 5, and hence completes the proof of Lemma 4.

REMARK 1. The (1) of Lemma 4 was obtained by Gray [7], but he did not point out the (2) of Lemma 4. However our proof depends on his techniques. In [13], the author used the (2) of Lemma 4 without detailed proof.

REMARK 2(*). In Lemma 4 we have assumed that an inverse sequence $\{G_i\}$ is normal. However Lemma 4 is true without the assumption of normality, because essentially in our proof we have not used the group structure of $\{H_1/H_i\}$.

§3. Proof of Theorem 1.

In this section we prove Theorem 1. For our purpose we need the following lemma.

LEMMA 6. *Let $\{K_i\}$, $\{G_i\}$ and $\{H_i\}$ be inverse sequences of abelian groups such that each K_i is a countable group. Let $* \longrightarrow \{K_i\} \xrightarrow{\{f_i\}} \{G_i\} \xrightarrow{\{g_i\}} \{H_i\} \longrightarrow *$ be exact. If $\{G_i\}$ satisfies Mittag-Leffler condition and $\varprojlim\{H_i\}$ is a countable group, then $\{K_i\}$ satisfies Mittag-Leffler condition and $* \longrightarrow \varprojlim\{K_i\} \xrightarrow{f} \varprojlim\{G_i\} \xrightarrow{g} \varprojlim\{H_i\} \longrightarrow *$ is exact.*

PROOF. Since $\{G_i\}$ satisfies Mittag-Leffler condition, then $\varprojlim^1\{G_i\}=*$ by Lemma 3. Thus we obtain the following exact sequence from Lemma 2; $* \longrightarrow \varprojlim\{K_i\} \xrightarrow{f} \varprojlim\{G_i\} \xrightarrow{g} \varprojlim\{H_i\} \xrightarrow{\delta} \varprojlim^1\{K_i\} \longrightarrow *$. Since $\varprojlim\{H_i\}$ is a countable group, $|\varprojlim^1\{K_i\}| \leqq \aleph_0$ by the above sequence. Therefore by Lemma 4, $\{K_i\}$ satisfies Mittag-Leffler condition and $\varprojlim^1\{K_i\}=*$. This completes the proof of Lemma 6.

Now, we are going to prove Theorem 1.

PROOF OF THEOREM 1. Since (X, A) is a compact metric pair, there exists an inverse sequence $\{(X_i, A_i)\}$ such that each X_i is a finite simplicial complex, each A_i is a subcomplex of X_i and $\varprojlim\{(X_i, A_i)\}=(X, A)$. Let H_* be the usual homology with integral coefficients. Then for each i, we obtain the following usual exact homology sequence of (X_i, A_i);

$$\cdots \xrightarrow{\kappa_{n+1\,i}} H_{n+1}(X_i, A_i) \xrightarrow{\delta_{n\,i}} H_n(A_i) \xrightarrow{\mu_{n\,i}} H_n(X_i) \xrightarrow{\kappa_{n\,i}} H_n(X_i, A_i) \xrightarrow{\delta_{n-1\,i}} \cdots .$$

Thus we obtain the following exact sequence;

$$\begin{aligned}(1)\qquad &\cdots \xrightarrow{\{\kappa_{n+1\,i}\}} \{H_n(X_i, A_i)\} \xrightarrow{\{\delta_{n\,i}\}} \{H_n(A_i)\} \xrightarrow{\{\mu_{n\,i}\}} \\ &\{H_n(X_i)\} \xrightarrow{\{\kappa_{n\,i}\}} \{H_n(X_i, A_i)\} \xrightarrow{\{\delta_{n-1\,i}\}} \cdots .\end{aligned}$$

(*) The author thanks the referee who pointed out this fact.

By taking inverse limits of (1), we obtain the following sequence of Čech homology by the continuity Theorem of Čech homology;

$$\text{(2)} \qquad \cdots \xrightarrow{\kappa_{n+1}} \check{H}_n(X, A) \xrightarrow{\delta_n} \check{H}_n(A) \xrightarrow{\mu_n} \check{H}_n(X) \xrightarrow{\kappa_n} \check{H}_n(X, A) \xrightarrow{\delta_{n-1}} \cdots$$

where $\check{H}_*$ means Čech homology functor, κ_n, δ_n and μ_n are induced maps. We have to show that (2) is exact. Since X and A are movable, it is easily proved by the definition of movability that

(3) $\{H_n(X_i)\}$ and $\{H_n(A_i)\}$ satisfy Mittag-Leffler condition for each $n \geqq 0$ (see [11]).

From (1) we obtain the following exact sequences;

$$\text{(4)} \qquad \begin{aligned} &* \longrightarrow \{\mathrm{Ker}\, \delta_{n\,i}\} \longrightarrow \{H_{n+1}(X_i, A_i)\} \longrightarrow \{\mathrm{Im}\, \delta_{n\,i}\} \longrightarrow *, \\ &* \longrightarrow \{\mathrm{Ker}\, \kappa_{n\,i}\} \longrightarrow \{H_n(X_i)\} \longrightarrow \{\mathrm{Im}\, \kappa_{n\,i}\} \longrightarrow *, \\ &* \longrightarrow \{\mathrm{Ker}\, \mu_{n\,i}\} \longrightarrow \{H_n(A_i)\} \longrightarrow \{\mathrm{Im}\, \mu_{n\,i}\} \longrightarrow *, \end{aligned}$$

where $\mathrm{Im} f$, $\mathrm{Ker} f$ denotes the image of f, the kernel of f for a group homomorphism f respectively. Since $\{\mathrm{Ker}\, \delta_{n\,i}\} = \{\mathrm{Im}\, \kappa_{n+1\,i}\}$ and $\{\mathrm{Ker}\, \kappa_{n\,i}\} = \{\mathrm{Im}\, \mu_{n\,i}\}$ by (1), $\{\mathrm{Ker}\, \delta_{n\,i}\}$ and $\{\mathrm{Ker}\, \kappa_{n\,i}\}$ satisfy Mittag-Leffler condition by (3) and Lemma 1. Therefore we obtain the following exact sequences by Lemmas 2 and 3;

$$\text{(5)} \qquad \left\{ \begin{aligned} &* \longrightarrow \varprojlim \{\mathrm{Ker}\, \delta_{n\,i}\} \longrightarrow \check{H}_{n+1}(X, A) \longrightarrow \varprojlim \{\mathrm{Im}\, \delta_{n\,i}\} \longrightarrow *, \\ &* \longrightarrow \varprojlim \{\mathrm{Ker}\, \kappa_{n\,i}\} \longrightarrow \check{H}_n(X) \longrightarrow \varprojlim \{\mathrm{Im}\, \kappa_{n\,i}\} \longrightarrow *. \end{aligned} \right.$$

Since $\check{H}_n(X)$ is a countable group, then $\varprojlim \{\mathrm{Im}\, \mu_{n\,i}\} = \varprojlim \{\mathrm{Ker}\, \kappa_{n\,i}\}$ is a countable group. Hence we obtain the following exact sequence from (4) by Lemma 6 and (3);

$$\text{(6)} \qquad * \longrightarrow \varprojlim \{\mathrm{Ker}\, \mu_{n\,i}\} \longrightarrow \check{H}_n(A) \longrightarrow \varprojlim \{\mathrm{Im}\, \mu_{n\,i}\} \longrightarrow *.$$

By using the arguments of [11], it is easily proved by (5) and (6) that (2) is exact. This completes the proof of Theorem 1.

Since every compact metric FANR-space is movable and its n-th Čech homology is a countable group for each n, then we obtain the following.

COROLLARY 1. *Let X be a compact metric FANR-space and A be a movable compact subset of X. Then the Čech homology sequence of (X, A) is exact.*

REMARK 3. The countability condition of Theorem 1 is essential, because we have the Overton's example (see [**11**, Th. 2]).

REMARK 4. In the next section, we shall show that if X is a movable metric space such that $\check{H}_n(X)$ is a countable group, then $\check{H}_n(X)$ is finitely generated.

REMARK 5. By the same argument as in the proof of Theorem 1, the following is easily proved by the fact of Remark 2.

THEOREM 1′. *Let (A, p) and (X, p) be pointed movable compact connected metric spaces such that $p \in A \subset X$. If the n-th shape group of (X, p) is a countable group for each n, then the shape groups sequence of (X, A, p) is exact.*

§ 4. Proof of Theorem 2.

In this section we prove Theorem 2. For our purpose we need the following theorem (see [**5**], [8] and [**10**] for the category of pro-groups).

THEOREM 3. *Let $\{G_i, \pi_i^j\}$ be an inverse sequence of groups. If $\{G_i\}$ satisfies Mittag-Leffler condition and $\varprojlim\{G_i\}$ is a countable group, then $\{G_i\}$ is isomorphic to $\varprojlim\{G_i\}$ in the category of pro-groups.*

PROOF. Let G be the inverse limit group of $\{G_i\}$ and let $\pi_i : G \to G_i$ be the natural projection for each i. Then $\{\pi_i(G)\}$ forms an inverse sequence. Since $\{G_i\}$ satisfies Mittag-Leffler condition, the following claim is easily proved by Morita's diagonal Theorem ([**10**, Th. 1.1]).

CLAIM 1. *The inclusion map from $\{\pi_i(G)\}$ to $\{G_i\}$ is an isomorphism in the category of pro-groups.*

Next, we show the following claim.

CLAIM 2. *$\{\pi_i(G)\}$ is isomorphic to $\varprojlim\{G_i\}$ in the category of pro-groups.*

PROOF OF CLAIM 2. Let K_i be the kernel of π_i for each i. Then by Lemma 2, we obtain the following exact sequence;

$$(1) \qquad * \longrightarrow \varprojlim\{K_i\} \longrightarrow \varprojlim\{G\} \xrightarrow{\pi} \varprojlim\{\pi_i(G)\} \longrightarrow \varprojlim{}^1\{K_i\} \longrightarrow$$
$$\longrightarrow \varprojlim{}^1\{G\} \longrightarrow \varprojlim{}^1\{\pi_i(G)\} \longrightarrow * , \qquad \text{where } \pi = \varprojlim\{\pi_i\} .$$

Since each bonding map of $\{G\}$ is the identity map, then $\varprojlim\{G\} = G$ and $\varprojlim{}^1\{G\} = *$ by Lemma 3. It is easily proved by the definition of $\pi = \varprojlim\{\pi_i\}$ that π is the identity map. Therefore it follows from (1) that $\varprojlim\{K_i\} = *$ and $\varprojlim{}^1\{K_i\} = *$. It is immediate from the definition of K_i that $\{K_i\}$ is a normal inverse sequence such that each K_i is a countable group. Hence we can apply Lemma 4 for $\{K_i\}$. Then $\{K_i\}$ satisfies Mittag-Leffler condition. Since $K_1 \supset K_2 \supset K_3 \cdots$, there exists n_0 such that $K_{n_0} = K_m$ for each $m, m \geqq n_0$. Since $\varprojlim\{K_i\} = \bigcap_{i=1}^{\infty} K_i = *$, we obtain that $* = \bigcap_{i=1}^{\infty} K_i = K_{n_0} = K_m$ for each $m, m \geqq n_0$. This means that π_i is injective for each i, $i \geqq n_0$. Since $\pi_i : G \to \pi_i(G)$ is onto, hence we have the following;

(2) there exists n_0 such that $\pi_i : G \to \pi_i(G)$ is an isomorphism for each i, $i \geqq n_0$.

Since $\pi_i = \pi_i^j \pi_j$ for each i, j, $i \leqq j$, it is easily proved by (2) that $\{\pi_i\} : G \to \{\pi_i(G)\}$ is an isomorphism in the category of pro-groups. This completes the proof of Claim 2.

Now, our Theorem 3 is completely proved by Claims 1 and 2.

Next, we consider special cases of Theorem 3.

COROLLARY 2. *Let X be a movable compact metric space. If $\check{H}_n(X)$ is a countable group, then $\check{H}_n(X)$ is finitely generated.*

PROOF. Let $\{X_i\}$ be an inverse sequence such that each X_i is a finite simplicial complex and $\varprojlim \{X_i\} = X$. Since X is movable, $\{H_n(X_i)\}$ satisfies Mittag-Leffler condition. By the continuity Theorem of Čech homology, $\check{H}_n(X) = \varprojlim \{H_n(X_i)\}$. Since $\check{H}_n(X)$ is a countable group, then by Theorem 3 there exists k such that $\check{H}_n(X)$ is isomorphic to a subgroup of $H_n(X_k)$ (see the proof of Theorem 3). Since X_k is a finite simplicial complex hence $H_n(X_k)$ is finitely generated. Therefore $\check{H}_n(X)$ is finitely generated. This completes the proof of Corollary 2.

By a similar argument we obtain the following.

COROLLARY 3. *Let X be a compact metric space and x be a point of X such that (X, x) is pointed movable. Then we have the followings.*

(1) *If the first shape group of (X, x) is a countable group, then it is finitely generated.*

(2) *If the first shape group of (X, x) is a null group and the n-th shape group of (X, x) is a countable group, then the n-th shape group is finitely generated.*

Now, we are going to show Theorem 2.

PROOF OF THEOREM 2. First we note that every finite dimensional compact metric space is always embedded in a finite dimensional Euclidian space by well-known Menger's embedding theorem. Thus the assumption of Theorem 5.1 of [**6**] is not essential. Now we show that (C) implies (A). Since (X, x) is a pointed FANR, there exists a finite simplicial complex P such that (X, x) is shape dominated by (P, p) where p is a point of P. Since P is a simplicial complex, then $\check{\pi}_n(P, p) = \pi_n(P, p)$ for each n, where $\check{\pi}_*$, π_* mean shape group functor, usual homotopy group functor respectively. Since P is a finite simplicial complex, it is well known that $\pi_n(P, p)$ is a countable group for each n. Since (X, x) is shape dominated by (P, p), then $\check{\pi}_n(X, x)$ is dominated by $\check{\pi}_n(P, p) = \pi_n(P, p)$ for each n. Hence $\check{\pi}_n(X, x)$ is a countable group for each n. It is trivial that (X, x) is pointed movable, because (X, x) is shape dominated by the pointed movable (P, p). This means that (C) implies (A). Next, we show that (A) implies (B). However it is trivial, because in general if (X, x) is pointed movable, then the n-th pro-homotopy groups of it satisfies Mittag-Leffler condition for each n. Finally we show that (B) implies (C). According to [**6**, Th. 5.1] our condition (C) is equivalent to the following condition; for each i, the i-th pro-homotopy groups of (X, x) is isomorphic to the i-th shape group of (X, x) in the category of pro-groups. (Note the i-th shape group is the inverse limit group of the i-th pro-homotopy groups.) Therefore the implication, (B) implies (C), is obvious by Theorem 3. This completes the proof of Theorem 2.

REMARK 6. The condition (A) of Theorem 2 is an improvement of the condition (iv) of [**6**, Th. 5.1] and also (B) is an improvement of the condition (i) of [**6**, Th. 5.1].

Now, we consider the following special case.

COROLLARY 4. *Let (X, x) be a pointed compact connected metric space with finite dimension. If (X, x) is 1-shape connected, then the following conditions are equivalent:*

(A′) *(X, x) is pointed movable and the n-th shape group of (X, x) is finitely generated for each $n \geqq 1$.*

(B′) *The n-th pro-homotopy groups of (X, x) satisfies Mittag-Leffler condition and the n-th shape group of (X, x) is finitely generated for each $n \geqq 1$.*

(C′) *(X, x) has the pointed shape of a finite simplicial complex.*

PROOF. First we show that (C′) implies (A′). Let (P, p) be a pointed finite simplicial complex such that (X, x) has the same shape of (P, p). Then $\check{\pi}_n(X, x) = \check{\pi}_n(P, p) = \pi_n(P, p)$ for each $n \geqq 0$. Note P is connected. Since P is finite simplicial complex, then $H_n(P)$ is finitely generated for each n. Hence by 16 Corollary of [**11**, p. 509], each $\pi_n(P, p)$ is finitely generated. Therefore $\check{\pi}_n(X, x) = \pi_n(P, p)$ is finitely generated. This means that (C′) implies (A′). It is obvious that (A′) implies (B′). Moreover it is easily proved by Corollary 5.2 of [**6**] and Theorem 3 that (B′) implies (C′). This completes the proof of Corollary 4.

References

[1] K. Borsuk, On movable compacta, Fund. Math., **66** (1969), 137-146.
[2] K. Borsuk, Theory of shape, Lecture notes series No. 28, Aarhus Univ., 1971.
[3] K. Bosuk, Theory of shape, Monografie Math, Tom 59, PWN, Warszawa, 1975.
[4] A.K. Bousfield and D.M. Kan, Homotopy limits, completions and localizations, Springer lecture note series No. 304, 1972.
[5] D.A. Edward and R. Geoghegan, Compacta weak shape equivalent to ANR's, Fund. Math., **90** (1976) 115-124.

[6] D. A. Edward and R. Geoghegan, The stability problem in shape and a Whitehead theorem in pro-homotopy, Trans. Amer. Math. Soc., **214** (1975), 261-277.
[7] B. I. Gray, Spaces of the same n-type, for all n, Topology, **5** (1966), 241-243.
[8] S. Mardešić, On the Whitehead theorem in shape theory I, Fund. Math., **91** (1976), 51-64.
[9] S. Mardešić and J. Segal, Shapes of compacta and ANR-systems, Fund. Math., **72** (1971), 41-59.
[10] K. Morita, The Hurewicz and Whitehead theorems in shape theory, Sci. Rep. Tokyo Kyoiku Daigaku, sect. A, **12** (1974), 246-258.
[11] R. H. Overton, Čech homology for movable compacta, Fund. Math., **77** (1973), 241-251.
[12] E. H. Spanier, Algebraic topology, McGraw-Hill Book Company, 1966.
[13] T. Watanabe, A note on the Hurewicz theorem in shape theory, Proc. Amer. Matn. Soc., **61** (1976), 137-140.

Tadashi WATANABE
Department of Mathematics
Tokyo University of Education

Present address:
Department of Mathematics
Faculty of Education
University of Yamaguchi
Yamaguchi, Japan