J. Math. Soc. Japan
Vol. 29, No. 3, 1977

The formal system for various 3-valued logics II

By Toshio NISHIMURA and Takemasa OHYA

(Received May 31, 1976)
(Revised Oct. 15, 1976)

§ 1. Introduction and semantics.

We gave the Gentzen-type formal system of Kleene's 3-valued logic and McCarthy's 3-valued logic interpreted into the system in [3]. In this paper we shall give the Gentzen-type formal system of McCarthy's logic itself. After that, we shall give the formal system in which McCarthy's and Kleene's are joined. In this system, serial or parallel evaluation is mixed.

We shall use the same terminology in [3]. Especially we use the symbols $+$, $\cdot$ or $\supset_M$ for *'or'*, *'and'* or *'implies'* in McCarthy's sense respectively and $\vee$, $\wedge$ or $\supset$ for them in Kleene's sense respectively, and use $\neg$ for *'not'* in common.

As we see from the following truth tables, formulas are evaluated serially from left to right in McCarthy's logic while in parallel in Kleene's logic.

$A+B$

$A\backslash B$	t	ω	f
t	t	t	t
ω	ω	ω	ω
f	t	ω	f

$A\cdot B$

$A\backslash B$	t	ω	f
t	t	ω	f
ω	ω	ω	ω
f	f	f	f

$A\supset_M B$

$A\backslash B$	t	ω	f
t	t	ω	f
ω	ω	ω	ω
f	t	t	t

$A\vee B$

$A\backslash B$	t	ω	f
t	t	t	t
ω	t	ω	ω
f	t	ω	f

$A\wedge B$

$A\backslash B$	t	ω	f
t	t	ω	f
ω	ω	ω	f
f	f	f	f

$A\supset B$

$A\backslash B$	t	ω	f
t	t	ω	f
ω	t	ω	ω
f	t	t	t

$\neg A$

A	
t	f
ω	ω
f	t

Here t, f or ω means *'true'*, *'false'* or *'undefined'* respectively.

It is clear that $P_1\cdot P_2\cdot \cdots \cdot P_n$ (o1 $P_1+P_2+\cdots+P_n$) has the value t (or f) if and only if every P_i has the value $\boldsymbol{t}$ (or $\boldsymbol{f}$). So $A\cdot\omega\cdot B$ or $A\cdot\boldsymbol{f}\cdot B$ has never the value $\boldsymbol{t}$ and $A+\omega+B$ or $A+\boldsymbol{t}+B$ has never the value $\boldsymbol{f}$.

Now we can easily verify the following equalities and inequalities, where $A=B$ means that A has always the same value as B and $A\leqq B$ means that A has always the value less than or equal to the value of B with respect to the order $\boldsymbol{f}<\omega<\boldsymbol{t}$. In what follows we shall promise that '$\cdot$' is stronger than the other connectives with respect to the order of combination, that is, $A+B\cdot C$ means $A+(B\cdot C)$. We note that $A=B$ and $F=G$ cause $F^*=G^*$ where F^* or G^* is that obtained from F or G by replacing A by B respectively.

(1) (the law of double negation)

$$\neg\neg A=A\,.$$

We shall occasionally regard $\neg\neg A$ as A itself. To clarify this we use the notation $\bar{A}$. When A is of the form $\neg B$, $\bar{A}$ is B, otherwise $\bar{A}$ is $\neg A$.

(2) (the associative law)

$$A\cdot(B\cdot C)=(A\cdot B)\cdot C\,,\qquad A+(B+C)=(A+B)+C\,.$$

We shall write simply $A\cdot B\cdot C$ for $A\cdot(B\cdot C)$ or $(A\cdot B)\cdot C$, and $A+B+C$ for $A+(B+C)$ or $(A+B)+C$.

(3) (the first absorption law)

$$A\cdot L\cdot A=A\cdot L\,,\qquad A+R+A=A+R\,.$$

(4)
$$A\cdot L\cdot\bar{A}\leqq\omega\,,\qquad A+R+\bar{A}\geqq\omega\,.$$

(5) (the second absorption law)

$$A\cdot L\cdot\bar{A}\cdot K=A\cdot L\cdot\bar{A}\,,\qquad A+R+\bar{A}+S=A+R+\bar{A}\,.$$

(6) (the left distributive law)

$$A\cdot(B+C)=(A\cdot B)+(A\cdot C)\,,\qquad A+(B\cdot C)=(A+B)\cdot(A+C)\,.$$

(7) (the right distributive law)

$$(B+C)\cdot D=(B\cdot D)+(\bar{B}\cdot C\cdot D)\,,\qquad (B\cdot C)+D=(B+D)\cdot(\bar{B}+C+D)\,.$$

Combining (6) and (7) we obtain

(8)
$$A\cdot(B+C)\cdot D=(A\cdot B\cdot D)+(A\cdot\bar{B}\cdot C\cdot D)=(A\cdot B\cdot D)\vee(A\cdot\bar{B}\cdot C\cdot D)\,,$$
$$A+(B\cdot C)+D=(A+B+D)\cdot(A+\bar{B}+C+D)=(A+B+D)\wedge(A+\bar{B}+C+D)\,.$$

(9) (de Morgan's law)

$$\neg(A\cdot B)=\neg A+\neg B, \qquad \neg(A+B)=\neg A\cdot\neg B.$$

(10) $$A\supset_M B=\neg A+B.$$

(11) $$A\cdot(B\wedge C)=(A\cdot B)\wedge(A\cdot C), \qquad A+(B\vee C)=(A+B)\vee(A+C).$$

(12) $$(B\wedge C)\cdot D=(B\cdot C\cdot D)\wedge(C\cdot B\cdot D), \qquad (B\vee C)+D=(B+C+D)\vee(C+B+D).$$

Combining (11) and (12)

(13) $$A\cdot(B\wedge C)\cdot D=(A\cdot B\cdot C\cdot D)\wedge(A\cdot C\cdot B\cdot D),$$
$$A+(B\vee C)+D=(A+B+C+D)\vee(A+C+B+D).$$

(14) $$A\cdot(B\vee C)=(A\cdot B)\vee(A\cdot C), \qquad A+(B\wedge C)=(A+B)\wedge(A+C).$$

(15) $$(B\vee C)\cdot D=((B\cdot D)\wedge\bar{C})\vee((C\cdot D)\wedge\bar{B})\vee(B\wedge D)\vee(C\wedge D),$$
$$(B\wedge C)+D=((B+D)\vee\bar{C})\wedge((C+D)\vee\bar{B})\wedge(B\vee D)\wedge(C\vee D).$$

Combining (14) and (15),

(16) $$A\cdot(B\vee C)\cdot D=((A\cdot B\cdot D)\wedge(A\cdot\bar{C}))\vee((A\cdot C\cdot D)\wedge(A\cdot\bar{B}))$$
$$\vee((A\cdot B)\wedge(A\cdot D))\vee((A\cdot C)\wedge(A\cdot D)),$$
$$A+(B\wedge C)+D=((A+B+D)\vee(A+\bar{C}))\wedge((A+C+D)\vee(A+\bar{B}))$$
$$\wedge((A+B)\vee(A+D))\wedge((A+C)\vee(A+D)).$$

We shall use the same terminologies, a formula, a literal, etc., as those in [3]. However, the antecedent or the succedent of a sequent shall be a sequence of formulas rather than a set. And the antecedent means the McCarthy's conjunction and the succedent means the McCarthy's disjunction. In order to emphasize it we shall write a sequent as $\langle A_1,\cdots,A_m\rangle\Rightarrow\langle B_1,\cdots,B_n\rangle$. Indeed in the sections § 5, § 6 the notation $\langle A_1,\cdots,A_m\rangle$ shall be used as abbreviation of $A_1\cdot\cdots\cdot A_m$ or $A_1+\cdots+A_m$ according as it occurs in the antecedent or the succedent. In this paper γ, δ, etc. denote a sequence of formulas.

§ 2. Formal system for McCarthy's logic.

In this section we shall give the formal system for McCarthy's and show its plausibility.

Before giving axioms, we consider the following four conditions concerning a sequent $\langle\gamma\rangle\Rightarrow\langle\delta\rangle$.

(a) γ contains a formula and its negation.

(b) γ and δ contain a formula in common. Let B be the leftmost formula in δ which occurs in γ too. Then $\langle\delta\rangle$ is of the form $\langle\delta_1, B, \delta_2\rangle$. Under this assumption γ contains the negation $\bar{C}$ of every formula C in δ_1.

(c) δ contains a formula and its negation.

(d) γ and δ contain a formula in common. Let A be the leftmost formula in γ which occurs in δ too. Then $\langle\gamma\rangle$ is of the form $\langle\gamma_1, A, \gamma_2\rangle$. And δ contains the negation $\bar{C}$ of every formula C in γ_1.

Now we admit as axioms those sequents:

i) which satisfy the conditions (a) and (c),

ii) which satisfy the conditions (a) and (d),

iii) which satisfy the conditions (b) and (c),

or iv) which satisfy the conditions (b) and (d).

That is, a sequent is an axiom if and only if it satisfies (a) or (b) and it satisfies (c) or (d).

These axioms are valid. For example, suppose that $\langle\gamma\rangle\Rightarrow\langle\delta\rangle$ is an axiom on the ground of (b) and (d). Let $\langle\gamma_1, A, \gamma_2\rangle$ and $\langle\delta_1, B, \delta_2\rangle$ be those such as in the assumption of (b) and (d). When γ takes $\boldsymbol{t}$, δ_1 takes $\boldsymbol{f}$ and B takes $\boldsymbol{t}$ by the condition (b). (To be exact, when the McCarthy's conjunction of γ takes $\boldsymbol{t}$, the McCarthy's disjunction of δ_1 takes $\boldsymbol{f}$ and B takes $\boldsymbol{t}$.) Hence δ takes $\boldsymbol{t}$. When γ takes ω, either γ_1 takes ω or γ_1 takes $\boldsymbol{t}$ and A takes ω or $\boldsymbol{t}$. In that case δ contains a formula taking ω by the condition (d). In this case δ contains A which takes ω or $\boldsymbol{t}$. Hence δ takes ω or $\boldsymbol{t}$ in either case. When γ takes $\boldsymbol{f}$, the value of δ is not mattered.

We shall give the rules of inference.

$(\cdot\Rightarrow)$ $$\frac{\langle\gamma_1, A, B, \gamma_2\rangle\Rightarrow\langle\delta\rangle}{\langle\gamma_1, A\cdot B, \gamma_2\rangle\Rightarrow\langle\delta\rangle},$$

$(\Rightarrow\cdot)$ $$\frac{\langle\gamma\rangle\Rightarrow\langle\delta_1, A, \delta_2\rangle \quad \langle\gamma\rangle\Rightarrow\langle\delta_1, \neg A, B, \delta_2\rangle}{\langle\gamma\rangle\Rightarrow\langle\delta_1, A\cdot B, \delta_2\rangle},$$

$(+\Rightarrow)$ $$\frac{\langle\gamma_1, A, \gamma_2\rangle\Rightarrow\langle\delta\rangle \quad \langle\gamma_1, \neg A, B, \gamma_2\rangle\Rightarrow\langle\delta\rangle}{\langle\gamma_1, A+B, \gamma_2\rangle\Rightarrow\langle\delta\rangle},$$

$(\Rightarrow+)$ $$\frac{\langle\gamma\rangle\Rightarrow\langle\delta_1, A, B, \delta_2\rangle}{\langle\gamma\rangle\Rightarrow\langle\delta_1, A+B, \delta_2\rangle},$$

$(\supset_M\Rightarrow)$ $$\frac{\langle\gamma_1, \neg A, \gamma_2\rangle\Rightarrow\langle\delta\rangle \quad \langle\gamma_1, A, B, \gamma_2\rangle\Rightarrow\langle\delta\rangle}{\langle\gamma_1, A\supset_M B, \gamma_2\rangle\Rightarrow\langle\delta\rangle},$$

$(\Rightarrow\supset_M)$ $$\frac{\langle\gamma\rangle\Rightarrow\langle\delta_1, \neg A, B, \delta_2\rangle}{\langle\gamma\rangle\Rightarrow\langle\delta_1, A\supset_M B, \delta_2\rangle},$$

$(\neg\neg\Rightarrow)$ $$\frac{\langle\gamma_1, A, \gamma_2\rangle\Rightarrow\langle\delta\rangle}{\langle\gamma_1, \neg\neg A, \gamma_2\rangle\Rightarrow\langle\delta\rangle},$$

$(\Rightarrow\neg\neg)$ $$\frac{\langle\gamma\rangle\Rightarrow\langle\delta_1, A, \delta_2\rangle}{\langle\gamma\rangle\Rightarrow\langle\delta_1, \neg\neg A, \delta_2\rangle},$$

$(\neg\cdot\Rightarrow)$ $$\frac{\langle\gamma_1, \neg A, \gamma_2\rangle\Rightarrow\langle\delta\rangle \quad \langle\gamma_1, A, \neg B, \gamma_2\rangle\Rightarrow\langle\delta\rangle}{\langle\gamma_1, \neg(A\cdot B), \gamma_2\rangle\Rightarrow\langle\delta\rangle},$$

$(\Rightarrow\neg\cdot)$ $$\frac{\langle\gamma\rangle\Rightarrow\langle\delta_1, \neg A, \neg B, \delta_2\rangle}{\langle\gamma\rangle\Rightarrow\langle\delta_1, \neg(A\cdot B), \delta_2\rangle},$$

$(\neg+\Rightarrow)$ $$\frac{\langle\gamma_1, \neg A, \neg B, \gamma_2\rangle\Rightarrow\langle\delta\rangle}{\langle\gamma_1, \neg(A+B), \gamma_2\rangle\Rightarrow\langle\delta\rangle},$$

$(\Rightarrow\neg+)$ $$\frac{\langle\gamma\rangle\Rightarrow\langle\delta_1, \neg A, \delta_2\rangle \quad \langle\gamma\rangle\Rightarrow\langle\delta_1, A, \neg B, \delta_2\rangle}{\langle\gamma\rangle\Rightarrow\langle\delta_1, \neg(A+B), \delta_2\rangle},$$

$(\supset_M\Rightarrow)$ $$\frac{\langle\gamma_1, A, \neg B, \gamma_2\rangle\Rightarrow\langle\delta\rangle}{\langle\gamma_1, \neg(A\supset_M B), \gamma_2\rangle\Rightarrow\langle\delta\rangle},$$

$(\Rightarrow\supset_M)$ $$\frac{\langle\gamma\rangle\Rightarrow\langle\delta_1, A, \delta_2\rangle \quad \langle\gamma\rangle\Rightarrow\langle\delta_1, \neg A, \neg B, \delta_2\rangle}{\langle\gamma\rangle\Rightarrow\langle\delta_1, \neg(A\supset_M B), \delta_2\rangle}.$$

The following rules are not essential but useful.

$$\frac{\langle\gamma_1, A, \gamma_2, \gamma_3\rangle\Rightarrow\langle\delta\rangle}{\langle\gamma_1, A, \gamma_2, A, \gamma_3\rangle\Rightarrow\langle\delta\rangle}, \qquad \frac{\langle\gamma\rangle\Rightarrow\langle\delta_1, A, \delta_2, \delta_3\rangle}{\langle\gamma\rangle\Rightarrow\langle\delta_1, A, \delta_2, A, \delta_3\rangle},$$

$$\frac{\langle\gamma_1, A, \gamma_2, \bar{A}\rangle\Rightarrow\langle\delta\rangle}{\langle\gamma_1, A, \gamma_2, \bar{A}, \gamma_3\rangle\Rightarrow\langle\delta\rangle}, \qquad \frac{\langle\gamma\rangle\Rightarrow\langle\delta_1, A, \delta_2, \bar{A}\rangle}{\langle\gamma\rangle\Rightarrow\langle\delta_1, A, \delta_2, \bar{A}, \delta_3\rangle},$$

$$\frac{\langle\gamma_1\rangle\Rightarrow\langle\delta\rangle}{\langle\gamma_1, \gamma_2\rangle\Rightarrow\langle\delta\rangle}, \qquad \frac{\langle\gamma\rangle\Rightarrow\langle\delta_1\rangle}{\langle\gamma\rangle\Rightarrow\langle\delta_1, \delta_2\rangle}.$$

These rules preserve the validity. We shall show it for the rule $(+\Rightarrow)$ and $(\Rightarrow\neg\cdot)$ as examples.

Suppose that an assignment does not satisfy the lower sequent of the rule $(+\Rightarrow)$, then either $\langle\gamma_1, A+B, \gamma_2\rangle$ takes ω and δ takes $\boldsymbol{f}$ or $\langle\gamma_1, A+B, \gamma_2\rangle$ takes $\boldsymbol{t}$ and δ takes $\boldsymbol{f}$ or ω. That $\langle\gamma_1, A+B, \gamma_2\rangle$ takes ω implies one of the following three cases. i) If γ_1 takes ω, then so does $\langle\gamma_1, A, \gamma_2\rangle$. ii) If γ_1 takes $\boldsymbol{t}$ and $A+B$ takes ω, either A takes ω or A takes $\boldsymbol{f}$ and B takes ω. The former makes $\langle\gamma_1, A, \gamma_2\rangle$ take ω and the latter makes $\langle\gamma_1, \neg A, B, \gamma_2\rangle$ take ω. iii) If γ_1 and $A+B$ take $\boldsymbol{t}$ and γ_2 takes ω, either A takes $\boldsymbol{t}$ or A takes $\boldsymbol{f}$ and B takes $\boldsymbol{t}$. Then $\langle\gamma_1, A, \gamma_2\rangle$ or $\langle\gamma_1, \neg A, B, \gamma_2\rangle$ takes ω respectively. In each case one of upper sequents fails to be satisfied. We shall prove the rest by using a table as follows. It is divided into several rows according to assignments which does not satisfy the lower sequent of the rule. The last column indicates which upper sequent fails to be satisfied by such assignment.

For the rule $(+\Rightarrow)$.

$\langle\gamma_1, A+B, \gamma_2\rangle$	δ	γ_1	$A+B$	γ_2	A	B	upper sequent
ω	f	ω	—	—	—	—	both
		t	ω	—	ω	—	first
		t	ω	—	f	ω	second
		t	t	ω	t	—	first
		t	t	ω	f	t	second
t	ω	t	t	t	t	—	first
		t	t	t	f	t	second
t	f	t	t	t	t	—	first
		t	t	t	f	t	second

For the rule $(\Rightarrow\neg\cdot)$.

γ	$\langle\delta_1, \neg(A\cdot B), \delta_2\rangle$	δ_1	$\neg(A\cdot B)$	δ_2	A	B	upper sequent
t	f	f	f	f	t	t	first
ω	f	f	f	f	t	t	first
t	ω	ω	—	—	—	—	first
		f	ω	—	ω	—	first
		f	ω	—	t	ω	first
		f	f	ω	t	t	first

Thus the following theorem holds.

PLAUSIBILITY THEOREM. *Every provable sequent is valid.*

As an example of the formal proof, we show the equivalence of $(A\supset_M B)\cdot(\neg A\supset_M C)$ and $(A\cdot B)+(\neg A\cdot C)$.

(b) (c) (b) (d)

$$\langle \neg A, \neg A, C\rangle \Rightarrow \langle A, \neg A\rangle \qquad \langle \neg A, \neg A, C\rangle \Rightarrow \langle A, \neg\neg A, C\rangle$$

$$\langle \neg A, \neg A, C\rangle \Rightarrow \langle A, \neg A\cdot C\rangle$$

(a) (d)

$$\langle \neg A, A\rangle \Rightarrow \langle A, \neg A\cdot C\rangle$$

$$\langle \neg A, \neg\neg A\rangle \Rightarrow \langle A, \neg A\cdot C\rangle$$

(b) (d)

$$\langle \neg A, \neg A\supset_M C\rangle \Rightarrow \langle A, \neg A\cdot C\rangle \qquad \langle \neg A, \neg A\supset_M C\rangle \Rightarrow \langle \neg A, B, \neg A\cdot C\rangle$$

$$\langle \neg A, \neg A\supset_M C\rangle \Rightarrow \langle A\cdot B, \neg A\cdot C\rangle$$

(b) (d) (b) (d)

$$\langle A, B, \neg A\supset_M C\rangle \Rightarrow \langle A, \neg A\cdot C\rangle \qquad \langle A, B, \neg A\supset_M C\rangle \Rightarrow \langle \neg A, B, \neg A\cdot C\rangle$$

$$\langle A, B, \neg A\supset_M C\rangle \Rightarrow \langle A\cdot B, \neg A\cdot C\rangle$$

$$\langle A\supset_M B, \neg A\supset_M C\rangle \Rightarrow \langle A\cdot B, \neg A\cdot C\rangle$$

$$\langle (A\supset_M B)\cdot(\neg A\supset_M C)\rangle \Rightarrow \langle A\cdot B, \neg A\cdot C\rangle$$

$$\langle (A\supset_M B)\cdot(\neg A\supset_M C)\rangle \Rightarrow \langle (A\cdot B)+(\neg A\cdot C)\rangle$$

Conversely

(b) (d) (a) (d)

$$\langle \neg A, \neg A, C\rangle \Rightarrow \langle \neg A, B\rangle \qquad \langle A, \neg B, \neg A, C\rangle \Rightarrow \langle \neg A, B\rangle$$

(b) (d)

$$\langle A, B\rangle \Rightarrow \langle \neg A, B\rangle \qquad \langle \neg(A\cdot B), \neg A, C\rangle \Rightarrow \langle \neg A, B\rangle$$

$$\langle A\cdot B\rangle \Rightarrow \langle \neg A, B\rangle \qquad \langle \neg(A\cdot B), \neg A\cdot C\rangle \Rightarrow \langle \neg A, B\rangle$$

$$\langle (A\cdot B)+(\neg A\cdot C)\rangle \Rightarrow \langle \neg A, B\rangle$$

$$\langle (A\cdot B)+(\neg A\cdot C)\rangle \Rightarrow \langle A\supset_M B\rangle$$

(b) (d) (a) (d)

$$\langle \neg A, \neg A, C\rangle \Rightarrow \langle A, C\rangle \qquad \langle A, \neg B, \neg A, C\rangle \Rightarrow \langle A, C\rangle$$

(b) (d)

$$\langle A, B\rangle \Rightarrow \langle A, C\rangle \qquad \langle \neg(A\cdot B), \neg A, C\rangle \Rightarrow \langle A, C\rangle$$

$$\langle A\cdot B\rangle \Rightarrow \langle A, C\rangle \qquad \langle \neg(A\cdot B), \neg A\cdot C\rangle \Rightarrow \langle A, C\rangle$$

$$\langle (A\cdot B)+(\neg A\cdot C)\rangle \Rightarrow \langle A, C\rangle$$

$$\langle (A\cdot B)+(\neg A\cdot C)\rangle \Rightarrow \langle A, A, C\rangle$$

(b) (c) (b) (c)

$$\langle A, B\rangle \Rightarrow \langle \neg A, \neg B, A, C\rangle \qquad \langle \neg(A\cdot B), \neg A, C\rangle \Rightarrow \langle \neg A, \neg B, A, C\rangle$$

$$\langle A\cdot B\rangle \Rightarrow \langle \neg A, \neg B, A, C\rangle \qquad \langle \neg(A\cdot B), \neg A\cdot C\rangle \Rightarrow \langle \neg A, \neg B, A, C\rangle$$

$$\langle (A\cdot B)+(\neg A\cdot C)\rangle \Rightarrow \langle \neg A, \neg B, A, C\rangle$$

$$\langle (A\cdot B)+(\neg A\cdot C)\rangle \Rightarrow \langle \neg(A\supset_M B), A, C\rangle$$

$$\langle (A\cdot B)+(\neg A\cdot C)\rangle \Rightarrow \langle \neg(A\supset_M B), \neg\neg A, C\rangle$$

$$\langle (A\cdot B)+(\neg A\cdot C)\rangle \Rightarrow \langle \neg(A\supset_M B), \neg A\supset_M C\rangle$$

$$\langle (A\cdot B)+(\neg A\cdot C)\rangle \Rightarrow \langle (A\supset_M B)\cdot(\neg A\supset_M C)\rangle$$

§ 3. The completeness of the previous system.

In this section, we shall show the completeness of the system defined in the previous section.

We define the degree of a formula or a sequent as the sum of the number of negation symbols and twice the number of other symbols occuring in it.

We can easily verify that the degree of the lower sequent is greater than that of each upper sequent in every rule of inference.

Now we shall show the theorem.

COMPLETENESS THEOREM. *Every valid sequent is provable.*

Suppose that a sequent is not provable, and we shall find an assignment which does not satisfy the given sequent.

We make the decomposition of it, that is, we construct strings of sequents as follows. i) The first sequent is the given sequent. ii) If either the n-th sequent is an axiom or it is no longer decomposable, that is, it is of the form $\langle P_1, \cdots, P_m\rangle \Rightarrow \langle Q_1, \cdots, Q_n\rangle$ with literals $P_1, \cdots, Q_n$, it is the end of the string. iii) If the n-th sequent is not an axiom but decomposable, then the $(n+1)$-st sequent is one of upper sequents of the rule whose lower sequent is the n-th sequent.

Since the $(n+1)$-st sequent has the degree less than the n-th sequent, every string is finite. If every string ends in an axiom, the first sequent is clearly provable. So there must be a string which does not end in an axiom since the given sequent is not provable. Let $\langle\gamma\rangle \Rightarrow \langle\delta\rangle$ be the last sequent of the string, then it satisfies neither the conditions (a) nor (b), or neither (c) nor (d). We may assume the former case without loss of generality.

Take the assignment which assigns $\boldsymbol{t}$ to every prime formula in γ, $\boldsymbol{f}$ to every prime formula whose negation occurs in γ, and $\boldsymbol{\omega}$ to every other prime formula. Such assignment is well-defined since γ does not contain both a prime formula and its negation.

It will be shown by the induction on the degree that every sequent in the string is not satisfied by the assignment, indeed it assigns $\boldsymbol{t}$ to each antecedent and $\boldsymbol{\omega}$ or $\boldsymbol{f}$ to succedent. i) The last sequent $\langle\gamma\rangle \Rightarrow \langle\delta\rangle$ mentioned above is not satisfied. It is clear that γ takes $\boldsymbol{t}$ and that δ takes $\boldsymbol{f}$ or $\boldsymbol{\omega}$ when there is no formula in common. When there is a formula in common, let δ_1, B and δ_2 be as in the condition (b). Then δ_1 contains a formula whose negation does not occur in γ and to which $\boldsymbol{\omega}$ is assigned. Hence δ takes $\boldsymbol{\omega}$. ii) To show the induction step, it is sufficient to show that the value assigned to the antecedent of the lower sequent is greater than or equal to that of each upper sequent and the value assigned to the succedent of the lower sequent is less than or equal to that of each upper sequent for every rule for connective. And this is easily shown by the equalities (1), (8), (9) and (10) in § 1.

Given the sequent $\langle(A\cdot B)+C\rangle \Rightarrow \langle(A+C)\cdot(B+C)\rangle$ as an example, we find the following string.

$$\langle(A\cdot B)+C\rangle \Rightarrow \langle(A+C)\cdot(B+C)\rangle,$$

$$\langle(A\cdot B)+C\rangle \Rightarrow \langle \urcorner(A+C), B+C\rangle,$$

$$\langle(A\cdot B)+C\rangle \Rightarrow \langle \urcorner(A+C), B, C\rangle,$$

$$\langle (A\cdot B)+C\rangle \Rightarrow \langle A,\ \neg C, B, C\rangle,$$

$$\langle \neg(A\cdot B), C\rangle \Rightarrow \langle A,\ \neg C, B, C\rangle,$$

$$\langle \neg A, C\rangle \Rightarrow \langle A,\ \neg C, B, C\rangle.$$

Then the assignment that assigns $\boldsymbol{f}$ to A, $\boldsymbol{t}$ to C and $\boldsymbol{\omega}$ to B does not satisfy $\langle (A\cdot B)+C\rangle \Rightarrow \langle (A+C)\cdot(B+C)\rangle$.

§ 4. Correlation between the present formulation and Takahashi's.

We shall compare our formulation with that using Takahashi's general method which formulates many valued logics using '*matrix*' (see [4]).

Axioms in his system are of the form $\{\Gamma, A\}_f \cup \{\Lambda, A\}_\omega \cup \{\Delta, A\}_t$.

Rules are

$(\neg \boldsymbol{f})$ $$\frac{\{\Gamma\}_f \cup \{\Lambda\}_\omega \cup \{\Delta, A\}_t}{\{\Gamma, \neg A\}_f \cup \{\Lambda\}_\omega \cup \{\Delta\}_t},$$

$(\neg \boldsymbol{\omega})$ $$\frac{\{\Gamma\}_f \cup \{\Lambda, A\}_\omega \cup \{\Delta\}_t}{\{\Gamma\}_f \cup \{\Lambda, \neg A\}_\omega \cup \{\Delta\}_t},$$

$(\neg \boldsymbol{t})$ $$\frac{\{\Gamma, A\}_f \cup \{\Lambda\}_\omega \cup \{\Delta\}_t}{\{\Gamma\}_f \cup \{\Lambda\}_\omega \cup \{\Delta, \neg A\}_t},$$

$(\cdot \boldsymbol{f})$ $$\frac{\{\Gamma, A\}_f \cup \{\Lambda\}_\omega \cup \{\Delta\}_t}{\{\Gamma, A\cdot B\}_f \cup \{\Lambda\}_\omega \cup \{\Delta\}_t}, \quad \frac{\{\Gamma\}_f \cup \{\Lambda\}_\omega \cup \{\Delta, A\}_t \quad \{\Gamma, B\}_f \cup \{\Lambda\}_\omega \cup \{\Delta\}_t}{\{\Gamma, A\cdot B\}_f \cup \{\Lambda\}_\omega \cup \{\Delta\}_t}$$

$(\cdot \boldsymbol{\omega})$ $$\frac{\{\Gamma\}_f \cup \{\Lambda, A\}_\omega \cup \{\Delta\}_t}{\{\Gamma\}_f \cup \{\Lambda, A\cdot B\}_\omega \cup \{\Delta\}_t}, \quad \frac{\{\Gamma\}_f \cup \{\Lambda\}_\omega \cup \{\Delta, A\}_t \quad \{\Gamma\}_f \cup \{\Lambda, B\}_\omega \cup \{\Delta\}_t}{\{\Gamma\}_f \cup \{\Lambda, A\cdot B\}_\omega \cup \{\Delta\}_t}$$

$(\cdot \boldsymbol{t})$ $$\frac{\{\Gamma\}_f \cup \{\Lambda\}_\omega \cup \{\Delta, A\}_t \quad \{\Gamma\}_f \cup \{\Lambda\}_\omega \cup \{\Delta, B\}_t}{\{\Gamma\}_f \cup \{\Lambda\}_\omega \cup \{\Delta, A\cdot B\}_t},$$

$(+\boldsymbol{f})$ $$\frac{\{\Gamma, A\}_f \cup \{\Lambda\}_\omega \cup \{\Delta\}_t \quad \{\Gamma, B\}_f \cup \{\Lambda\}_\omega \cup \{\Delta\}_t}{\{\Gamma, A+B\}_f \cup \{\Lambda\}_\omega \cup \{\Delta\}_t},$$

$(+\boldsymbol{\omega})$ $$\frac{\{\Gamma\}_f \cup \{\Lambda, A\}_\omega \cup \{\Delta\}_t}{\{\Gamma\}_f \cup \{\Lambda, A+B\}_\omega \cup \{\Delta\}_t}, \quad \frac{\{\Gamma, A\}_f \cup \{\Lambda\}_\omega \cup \{\Delta\}_t \quad \{\Gamma\}_f \cup \{\Lambda, B\}_\omega \cup \{\Delta\}_t}{\{\Gamma\}_f \cup \{\Lambda, A+B\}_\omega \cup \{\Delta\}_t}$$

$(+\boldsymbol{t})$ $$\frac{\{\Gamma\}_f \cup \{\Lambda\}_\omega \cup \{\Delta, A\}_t}{\{\Gamma\}_f \cup \{\Lambda\}_\omega \cup \{\Delta, A+B\}_t}, \quad \frac{\{\Gamma, A\}_f \cup \{\Lambda\}_\omega \cup \{\Delta\}_t \quad \{\Gamma\}_f \cup \{\Lambda\}_\omega \cup \{\Delta, B\}_t}{\{\Gamma\}_f \cup \{\Lambda\}_\omega \cup \{\Delta, A+B\}_t}$$

Rules for other symbols, $(\supset_M \boldsymbol{f})$, $(\supset_M \boldsymbol{\omega})$ and $(\supset_M \boldsymbol{t})$ are similarly given.

THEOREM. *A sequent* $\langle A_1, \cdots, A_m\rangle \Rightarrow \langle B_1, \cdots, B_n\rangle$ *is provable in our system if and only if the matrices*

$$\{A_1, \cdots, A_i, \neg A_i\}_{f,\omega} \cup \{B_1, \cdots, B_n\}_t \qquad (i=1, \cdots, m-1),$$

$$\{A_1, \cdots, A_i, \neg A_i\}_f \cup \{B_1, \cdots, B_n\}_{\omega,t} \qquad (i=1, \cdots, m-1),$$

$$\{A_1, \cdots, A_m\}_{f,\omega} \cup \{B_1, \cdots, B_j, \neg B_j\}_t \qquad (j=1, \cdots, n-1),$$

$$\{A_1, \cdots, A_m\}_f \cup \{B_1, \cdots, B_j, \neg B_j\}_{\omega,t} \qquad (j=1, \cdots, n-1),$$

$$\{A_1, \cdots, A_m\}_{f,\omega} \cup \{B_1, \cdots, B_n\}_t,$$

and $\{A_1, \cdots, A_m\}_f \cup \{B_1, \cdots, B_n\}_{\omega,t}$ *are all provable in his system.*

§ 5. Serial and parallel logic SPL.

Combining McCarthy's logic and Kleene's, we shall construct a new logic. We call it the serial and parallel logic, abbreviated as SPL.

In what follows, a formula may contain both Kleene's symbols and McCarthy's. On a sequent $L_1, \cdots, L_m \Rightarrow R_1, \cdots, R_n$ the antecedent means the Kleene's conjunction and the succedent means the Kleene's disjunction. And each L_i or R_j is of the form $\langle A_1, \cdots, A_k\rangle$ or $\langle B_1, \cdots, B_h\rangle$ which is an abbreviation of $A_1 \cdot \cdots \cdot A_k$ or $B_1 + \cdots + B_h$ respectively. Γ, Δ, etc. denote sets of formulas and γ, δ, etc. denote sequences of formulas. When Γ is $\langle A_{1,1}, \cdots, A_{1,k_1}\rangle, \cdots, \langle A_{m,1}, \cdots, A_{m,km}\rangle$, $\tilde{\Gamma}$ is the set $\{A_{1,1}, \cdots, A_{1,k_1}, \cdots A_{m,1}, \cdots, A_{m,km}\}$.

We consider the following four conditions concerning a sequent $\Gamma \Rightarrow \Delta$:

a) $\tilde{\Gamma}$ contains a formula and its negation.

b) There is a formula $\langle B_1, \cdots, B_j, \cdots, B_k\rangle$ in Δ such that $\tilde{\Gamma}$ contains B_j and every negation of $B_1, \cdots, B_{j-1}$.

c) $\tilde{\Delta}$ contains a formula and its negaiton.

d) There is a formula $\langle A_1, \cdots, A_i, \cdots, A_k\rangle$ in Γ such that $\tilde{\Delta}$ contains A_i and every negation of $A_1, \cdots, A_{i-1}$.

The axioms in SPL are those sequents which satisfy the conditions (a) and (c), (a) and (d), (b) and (c), or (b) and (d).

The rules of inference in SPL are as follows.

$$(\cdot\Rightarrow) \qquad \frac{\Gamma, \langle \gamma_1, A, B, \gamma_2\rangle \Rightarrow \Delta}{\Gamma, \langle \gamma_1, A \cdot B, \gamma_2\rangle \Rightarrow \Delta},$$

$$(\Rightarrow\cdot) \qquad \frac{\Gamma \Rightarrow \Delta, \langle \delta_1, A, \delta_2\rangle \quad \Gamma \Rightarrow \Delta, \langle \delta_1, \neg A, B, \delta_2\rangle}{\Gamma \Rightarrow \Delta, \langle \delta_1, A \cdot B, \delta_2\rangle}.$$

The other rules for McCarthy's connective are similar to those in the previous system and so we omit to list up them.

$(\wedge\Rightarrow 1)$ $$\frac{\Gamma, \langle\gamma_1, A\rangle, \langle\gamma_1 B\rangle \Rightarrow \Delta}{\Gamma, \langle\gamma_1, A\wedge B\rangle \Rightarrow \Delta},$$

$(\wedge\Rightarrow 2)$ $$\frac{\Gamma, \langle\gamma_1, A, B, \gamma_2\rangle, \langle\gamma_1, B, A, \gamma_2\rangle \Rightarrow \Delta}{\Gamma, \langle\gamma_1, A\wedge B, \gamma_2\rangle \Rightarrow \Delta},$$

$(\Rightarrow\wedge 1)$ $$\frac{\Gamma \Rightarrow \Delta, \langle\delta_1, A\rangle \quad \Gamma \Rightarrow \Delta, \langle\delta_1, B\rangle}{\Gamma \Rightarrow \Delta, \langle\delta_1, A\wedge B\rangle},$$

$(\Rightarrow\wedge 2)$ $$\frac{\begin{array}{ll}\Gamma \Rightarrow \Delta, \langle\delta_1, \neg A\rangle, \langle\delta_1, B, \delta_2\rangle & \Gamma \Rightarrow \Delta, \langle\delta_1, A\rangle, \langle\delta_1, \delta_2\rangle \\ \Gamma \Rightarrow \Delta, \langle\delta_1, \neg B\rangle, \langle\delta_1, A, \delta_2\rangle & \Gamma \Rightarrow \Delta, \langle\delta_1, B\rangle, \langle\delta_1, \delta_2\rangle\end{array}}{\Gamma \Rightarrow \Delta, \langle\delta_1, A\wedge B, \delta_2\rangle}$$

$(\vee\Rightarrow 1)$ $$\frac{\Gamma, \langle\gamma_1, A\rangle \Rightarrow \Delta \quad \Gamma, \langle\gamma_1, B\rangle \Rightarrow \Delta}{\Gamma, \langle\gamma_1, A\vee B\rangle \Rightarrow \Delta},$$

$(\vee\Rightarrow 2)$ $$\frac{\begin{array}{ll}\Gamma, \langle\gamma_1, \neg A\rangle, \langle\gamma_1, B, \gamma_2\rangle \Rightarrow \Delta & \Gamma, \langle\gamma_1, A\rangle, \langle\gamma_1, \gamma_2\rangle \Rightarrow \Delta \\ \Gamma, \langle\gamma_1, \neg B\rangle, \langle\gamma_1, A, \gamma_2\rangle \Rightarrow \Delta & \Gamma, \langle\gamma_1, B\rangle, \langle\gamma_1, \gamma_2\rangle \Rightarrow \Delta\end{array}}{\Gamma, \langle\gamma_1, A\vee B, \gamma_2\rangle \Rightarrow \Delta},$$

$(\Rightarrow\vee 1)$ $$\frac{\Gamma \Rightarrow \Delta, \langle\delta_1, A\rangle, \langle\delta_1, B\rangle}{\Gamma \Rightarrow \Delta, \langle\delta_1, A\vee B\rangle},$$

$(\Rightarrow\vee 2)$ $$\frac{\Gamma \Rightarrow \Delta, \langle\delta_1, A, B, \delta_2\rangle, \langle\delta_1, B, A, \delta_2\rangle}{\Gamma \Rightarrow \Delta, \langle\delta_1, A\vee B, \delta_2\rangle},$$

$(\supset\Rightarrow 1)$ $$\frac{\Gamma, \langle\gamma_1, \neg A\rangle \Rightarrow \Delta \quad \Gamma, \langle\gamma_1, B\rangle \Rightarrow \Delta}{\Gamma, \langle\gamma_1, A\supset B\rangle \Rightarrow \Delta},$$

$(\supset\Rightarrow 2)$ $$\frac{\begin{array}{ll}\Gamma, \langle\gamma_1, A\rangle, \langle\gamma_1, B, \gamma_2\rangle \Rightarrow \Delta & \Gamma, \langle\gamma_1, \neg A\rangle, \langle\gamma_1, \gamma_2\rangle \Rightarrow \Delta \\ \Gamma, \langle\gamma_1, \neg B\rangle, \langle\gamma_1, \neg A, \gamma_2\rangle \Rightarrow \Delta & \Gamma, \langle\gamma_1, B\rangle. \langle\gamma_1, \gamma_2\rangle \Rightarrow \Delta\end{array}}{\Gamma, \langle\gamma_1, A\supset B, \gamma_2\rangle \Rightarrow \Delta},$$

$(\Rightarrow\supset 1)$ $$\frac{\Gamma \Rightarrow \Delta, \langle\delta_1, \neg A\rangle, \langle\delta_1, B\rangle}{\Gamma \Rightarrow \Delta, \langle\delta_1, A\supset B\rangle},$$

$(\Rightarrow\supset 2)$ $$\frac{\Gamma \Rightarrow \Delta, \langle\delta_1, \neg A, B, \delta_2\rangle, \langle\delta_1, B, \neg A, \delta_2\rangle}{\Gamma \Rightarrow \Delta, \langle\delta_1, A\supset B, \delta_2\rangle}.$$

The rules $(\neg\wedge\Rightarrow 1)$, $(\neg\wedge\Rightarrow 2)$, $(\Rightarrow\neg\wedge 1)$, $(\Rightarrow\neg\wedge 2)$, $(\neg\vee\Rightarrow 1)$, $(\neg\vee\Rightarrow 2)$, $(\Rightarrow\neg\vee 1)$, $(\Rightarrow\neg\vee 2)$, $(\neg\supset\Rightarrow 1)$, $(\neg\supset\Rightarrow 2)$, $(\Rightarrow\neg\supset 1)$ and $(\Rightarrow\neg\supset 2)$ are similarly given.

It is easily seen that these rules preserve the validity. For example, we shall show the case $(\Rightarrow\wedge 2)$ using the table as in § 2.

Supposing that the lower sequent is not valid, the following cases happen.

Γ	Δ	$\langle\delta_1, A\wedge B, \delta_2\rangle$	δ_1	A	B	δ_2	upper sequent
t	f or ω	ω	ω	—	—	—	all
			f	ω	t	—	second
			f	ω	ω	—	first, second
			f	t	ω	—	first
			f	f	—	ω	third
			f	—	f	ω	fourth
		f	f	f	—	f	third
			f	—	f	f	fourth
ω	f	f	f	f	—	f	third
			f	—	f	f	fourth

We also define the degree of a sequent as the sum of the number of negation symbols and twice the number of other symbols occuring in it. Now the completenss of SPL will be shown in quite similar way as in § 3.

§ 6. The formal system for the extended SPL.

We shall extend SPL to the predicate calculus with infinitary connectives.

$\forall xA(x)$ is true if and only if $A(t)$ is true for every term t, it is false if and only if $A(t)$ is false for some term t, and it is undefined if and only if none of $A(t)$'s are false and some of them are undefined. $\wedge(A_1, A_2, \cdots)$, the infinitary conjunction in Kleene's sense, is true if and only if every A_n is true, it is false if and only if some A_n is false, and it is undefined if and only if none of A_n's are false and some of them are undefined. While $\Pi(A_1, A_2, \cdots)$, the infinitary conjunction in McCarthy's sense, is true if and only if every A_n is true, it is false if and only if there exists an n such that $A_1, \cdots, A_{n-1}$ are true and A_n is false, and it is undefined if and only if there exists an n such that $A_1, \cdots, A_{n-1}$ are true and A_n is undefined even if some A_{n+1} is false. $\exists xA(x)$, $\vee(A_1, A_2, \cdots)$ and $\Sigma(A_1, A_2, \cdots)$ are interpreted likewise.

The following rules for these symbols are added to SPL.

$$(\forall\Rightarrow 1)\qquad \frac{\Gamma, \langle\gamma_1, A(t_1)\rangle, \cdots, \langle\gamma_1, A(t_m)\rangle, \langle\gamma_1, \forall xA(x)\rangle \Rightarrow \Delta}{\Gamma, \langle\gamma_1, \forall xA(x)\rangle \Rightarrow \Delta},$$

$$(\forall\Rightarrow 2)\qquad \frac{\begin{array}{c}\Gamma, \langle\gamma_1, A(t_1)\rangle, \cdots, \langle\gamma_1, A(t_m)\rangle, \langle\gamma_1, \forall xA(x)\rangle, \langle\gamma_1, \neg\forall xA(x)\rangle \Rightarrow \Delta \\ \Gamma, \langle\gamma_1, A(t_1)\rangle, \cdots, \langle\gamma_1, A(t_m)\rangle, \langle\gamma_1, \forall xA(x)\rangle, \langle\gamma_1, \gamma_2\rangle \Rightarrow \Delta\end{array}}{\Gamma, \langle\gamma_1, \forall xA(x), \gamma_2\rangle \Rightarrow \Delta},$$

where $t_1, \cdots, t_m$ are arbitrary terms.

(⇒∀1) $$\frac{\Gamma\Rightarrow\Delta, \langle\delta_1, A(a)\rangle}{\Gamma\Rightarrow\Delta, \langle\delta_1, \forall xA(x)\rangle},$$

(⇒∀2) $$\frac{\Gamma\Rightarrow\Delta, \langle\delta_1, A(a)\rangle, \langle\delta_1, \neg\forall xA(x)\rangle \quad \Gamma\Rightarrow\Delta, \langle\delta_1, A(a)\rangle, \langle\delta_1, \delta_2\rangle}{\Gamma\Rightarrow\Delta, \langle\delta_1, \forall xA(x), \delta_2\rangle},$$

where a is an eigen-variable, that is, a free valiable not occuring in the lower sequent.

(∧⇒1) $$\frac{\Gamma, \langle\gamma_1, A_1\rangle, \cdots, \langle\gamma_1, A_m\rangle, \langle\gamma_1, \wedge(A_1, A_2, \cdots)\rangle\Rightarrow\Delta \text{ for some } m}{\Gamma, \langle\gamma_1, \wedge(A_1, A_2, \cdots)\rangle\Rightarrow\Delta},$$

(∧⇒2) $$\frac{\begin{array}{c}\Gamma, \langle\gamma_1, A_1\rangle, \cdots, \langle\gamma_1, A_m\rangle, \langle\gamma_1, \wedge(A_1A_2, \cdots)\rangle, \langle\gamma_1, \neg\wedge(A_1, A_2, \cdots)\rangle\Rightarrow\Delta \\ \Gamma, \langle\gamma_1, A_1\rangle, \cdots, \langle\gamma_1, A_m\rangle, \langle\gamma_1, \wedge(A_1, A_2, \cdots)\rangle, \langle\gamma_1, \gamma_2\rangle\Rightarrow\Delta \text{ for some } m\end{array}}{\Gamma, \langle\gamma_1, \wedge(A_1, A_2, \cdots), \gamma_2\rangle\Rightarrow\Delta},$$

(⇒∧1) $$\frac{\Gamma\Rightarrow\Delta, \langle\delta_1, A_1\rangle \cdots \Gamma\Rightarrow\Delta, \langle\delta_1, A_n\rangle \cdots}{\Gamma\Rightarrow\Delta, \langle\delta_1, \wedge(A_1, A_2, \cdots)\rangle},$$

(⇒∧2) $$\frac{\begin{array}{cc}\Gamma\Rightarrow\Delta, \langle\delta_1, A_1\rangle, \langle\delta_1, \neg\wedge(A_1, A_2, \cdots)\rangle & \Gamma\Rightarrow\Delta, \langle\delta_1, A_1\rangle, \langle\delta_1, \delta_2\rangle \\ \cdots\cdots & \cdots\cdots \\ \Gamma\Rightarrow\Delta, \langle\delta_1, A_n\rangle, \langle\delta_1, \neg\wedge(A_1, A_2, \cdots)\rangle & \Gamma\Rightarrow\Delta, \langle\delta_1, A_n\rangle, \langle\delta_1, \delta_2\rangle \\ \cdots\cdots & \cdots\cdots\end{array}}{\Gamma\Rightarrow\Delta, \langle\delta_1, \wedge(A_1, A_2, \cdots), \delta_2\rangle},$$

(Π⇒1) $$\frac{\Gamma, \langle\gamma_1, A_1, \cdots, A_m, \Pi(A_1, A_2, \cdots)\rangle\Rightarrow\Delta \text{ for some } m}{\Gamma, \langle\gamma_1, \Pi(A_1, A_2, \cdots)\rangle\Rightarrow\Delta},$$

(Π⇒2) $$\frac{\begin{array}{c}\Gamma, \langle\gamma_1, A_1, \cdots, A_m, \Pi(A_1, A_2, \cdots)\rangle, \langle\gamma_1, \neg\Pi(A_1, A_2, \cdots)\rangle\Rightarrow\Delta \\ \Gamma, \langle\gamma_1, A_1, \cdots, A_m, \Pi(A_1, A_2, \cdots)\rangle, \langle\gamma_1, \gamma_2\rangle\Rightarrow\Delta \text{ for some } m\end{array}}{\Gamma, \langle\gamma_1, \Pi(A_1, A_2, \cdots), \gamma_2\rangle\Rightarrow\Delta},$$

(⇒Π1) $$\frac{\Gamma\Rightarrow\Delta, \langle\delta_1, A_1\rangle \cdots \Gamma\Rightarrow\Delta, \langle\delta_1, \neg A_1, \cdots, \neg A_{n-1}, A_n\rangle \cdots}{\Gamma\Rightarrow\Delta, \langle\delta_1, \Pi(A_1, A_2, \cdots)\rangle},$$

(⇒Π2) $$\frac{\begin{array}{l}\Gamma\Rightarrow\Delta, \langle\delta_1, A_1\rangle, \langle\delta_1, \neg\Pi(A_1, A_2, \cdots)\rangle \\ \quad\Gamma\Rightarrow\Delta, \langle\delta_1, A_1\rangle, \langle\delta_1, \delta_2\rangle \\ \quad\cdots\cdots \\ \Gamma\Rightarrow\Delta, \langle\delta_1, \neg A_1, \cdots, \neg A_{n-1}, A_n\rangle, \langle\delta_1, \neg\Pi(A_1, A_2, \cdots)\rangle \\ \quad\Gamma\Rightarrow\Delta, \langle\delta_1, \neg A_1, \cdots, \neg A_{n-1}, A_n\rangle, \langle\delta_1, \delta_2\rangle \\ \quad\cdots\cdots\end{array}}{\Gamma\Rightarrow\Delta, \langle\delta_1, \Pi(A_1, A_2, \cdots), \delta_2\rangle}.$$

The other rules for $\exists$, $\vee$, Σ, $\neg\forall$, $\neg\wedge$, $\neg\Pi$, $\neg\exists$, $\neg\vee$, and $\neg\Sigma$ are similarly given.

PLAUSIBILITY THEOREM. *Every provable sequent is valid.*

This is easily verified as in § 3.

The degrees of formulas are recursively defined as follows.

1) $d(A)=2$ for prime formula A.

2) $d(\neg A)=d(A)+1$.

3) $d(A\wedge B)=d(A\vee B)=d(A\supset B)=d(A\cdot B)=d(A+B)=d(A\supset_M B)=(d(A)+1)\sharp(d(B)+1)$, where $\sharp$ is the natural sum of $d(A)+1$ and $d(B)+1$.

4) $d(\forall xA)=d(\exists xA)=d(A)+2$.

5) $d(\wedge(A_1, A_2, \cdots))=d(\vee(A_1, A_2, \cdots))=d(\Pi(A_1, A_2, \cdots))=d(\Sigma(A_1, A_2, \cdots))=\sup\{(d(A_1)+1)\sharp\cdots\sharp(d(A_n)+1)\mid n=1, 2, \cdots\}$

COMPLETENESS THEOREM. *Every valid sequent is provable.*

When a sequent is given, we decompose it and construct strings of sequents as the proof in § 4 in [3]. The strings satisfies the conditions:

i) S_1 is the given sequent. ii) If S_n is an axiom, it is the end of the string. iii) If S_n is not an axiom, let p and q be the maximum number such that $2^p\cdot 3^q$ divides n. If F_p, the p-th formula of a fixed enumeration of all formulas, occurs at least q times in S_n, then S_{n+1} is one of the upper sequents of the rule of inference whose principal formula is the q-th F_p in S_n. Moreover if the rule is $(\forall\Rightarrow)$, $(\Rightarrow\exists)$, $(\Rightarrow\neg\forall)$ or $(\neg\exists\Rightarrow)$, then we must select the first n terms of a fixed enumeration of all terms as $t_1, \cdots, t_m$ described in the above schema. And if the rule is $(\wedge\Rightarrow)$, $(\Rightarrow\vee)$, $(\Rightarrow\neg\wedge)$, $(\neg\vee\Rightarrow)$, $(\Pi\Rightarrow)$, $(\Rightarrow\Sigma)$, $(\Rightarrow\neg\Pi)$ or $(\neg\Sigma\Rightarrow)$, then we must take n as m described in the above schema, that is, we must select the first n components of the principal formula. iv) In other cases S_{n+1} is S_n. We call this production S_{n+1} from S_n the n-th decomposition on the string.

When every string is finite, S_1 is provable. Hence if the given sequent is not provable, there is an infinite string $\mathfrak{S}$. Let Γ or Δ be the set of all formulas in the antecedents or in the succedents of the sequents in $\mathfrak{S}$. Suppose Γ and Δ satisfy the conditions (a) or (b) and satisfy (c) or (d) described in § 4, then we can find an axiom in $\mathfrak{S}$ contradicting the assumption. We may assume that they satisfy neither (a) nor (b) without loss of generality. We assign $\boldsymbol{t}$ to prime formulas which occur in $\tilde{\Gamma}$, $\boldsymbol{f}$ to prime formulas whose negations occur in $\tilde{\Gamma}$, and $\boldsymbol{\omega}$ to all other primes. The following two lemmas imply that the given sequent S_1 is not valid.

LEMMA. *Every formula in Γ takes $\boldsymbol{t}$. That is, every formula in $\tilde{\Gamma}$ takes $\boldsymbol{t}$.*

This is shown by the induction on the degree of the formula in $\tilde{\Gamma}$. We only note that if S_{n+1} were chosen the first upper sequent of the rule $(\forall\Rightarrow 2)$, $(\Rightarrow\forall 2)$, etc. at the n-th decomposition on $\mathfrak{S}$, then Γ must satisfy the condition (a) contradicting the assumption.

LEMMA. *Every formula in Δ takes $\boldsymbol{f}$ or $\boldsymbol{\omega}$. That is, Δ contains no formula* *of the form* $\langle A_1, \cdots A_k, \cdots, A_m\rangle$ *where* $A_1, \cdots, A_{k-1}$ *takes* $\boldsymbol{f}$ *and* A_k *takes* $\boldsymbol{t}$.

This is proved by the induction on the rank $\omega_1 \cdot d + d(A_k)$, where d is the natural sum of $d(A_1), \cdots, d(A_k)$, and ω_1 is the least uncountable ordinal.

If all $A_1, \cdots, A_k$ are literals, $\bar{A}_1, \cdots, \bar{A}_{k-1}$ and A_k must occur in $\tilde{\Gamma}$ since $\boldsymbol{t}$ is assigned to them. Therefore Γ and Δ satisfy (b) contradicting the assumption.

If $\langle A_1, \cdots, A_k, \delta\rangle$ occurs in S_n and the n-th decomposition is concerned with a formula other than these $A_1, \cdots, A_k$, then $\langle A_1, \cdots, A_k, \delta'\rangle$ occurs in S_{n+1}. So at last we find the n-th decomposition concerned with one of $A_1, \cdots, A_{k-1}$ or with A_k. 1) The case it is one of $A_1, \cdots, A_{k-1}$, say A_i. 1.1) When A_i is of the form $B \cdot C$, then either B takes $\boldsymbol{f}$ or B takes $\boldsymbol{t}$ and C takes $\boldsymbol{f}$. In either case both $\langle A_1, \cdots, A_{i-1}, B, A_{i+1}, \cdots, A_k, \delta\rangle$ and $\langle A_1, \cdots, A_{i-1}, \neg B, C, A_{i+1}, \cdots, A_k, \delta\rangle$ have the smaller rank. And one of them appears in the succedent of S_{n+1}. 1.2) When A_i is of the form $\forall x B(x)$, then S_{n+1} contains $\langle A_1, \cdots, A_{i-1}, \neg A_i\rangle$ or $\langle A_1, \cdots, A_{i-1}, A_{i+1}, \cdots A_k, \delta\rangle$. They have the smaller rank. It is similar in other cases. 2) The case the n-th decomposition is concerned with A_k. 2.1) When A_k is of the form $\Sigma(B_1, B_2, \cdots)$ then $\boldsymbol{t}$ is assigned to all B_m's and all $\langle A_1, \cdots, A_{k-1}, \neg B_1, \cdots, \neg B_{m-1}, B_m\rangle$'s have the smaller rank. 2.2) When A_k is of the form $\exists x B(x)$, then $\boldsymbol{t}$ is assigned to $B(t_m)$ for some term t_m, the m-th term of the enumeration. If $m \leqq n$, then S_{n+1} contains $\langle A_1, \cdots, A_{k-1}, B(t_m)\rangle$ whose rank is smaller. If $m > n$, $\langle A_1, \cdots, A_{k-1}, \exists x B(x)\rangle$ remains in S_{n+1}. There are two cases. 2.3) It remains throughout the string, and at the sufficiently large n-th decomposition $\langle A_1, \cdots, A_{k-1}, B(t_m)\rangle$ appears in S_{n+1}. 2.4) It vanishes on the way. This happens only when some of $A_1, \cdots, A_{k-1}$ is decomposed. So it results in the case 1). It is similar in other cases.

References

[1] S. C. Kleene, Introduction to metamathematics, Van Nostrand Company, Inc., Princeton, N. J., 1952.

[2] J. McCarthy, Predicate calculus with "undefined" as a truth value, Stanford Artificial Intelligence Project, Memo 1, 1963.

[3] T. Nishimura and T. Ohya, The formal system for various 3-valued logics I, Science Reports of the Tokyo Kyoiku Daigaku, Section A, Vol. 13, No. 348 (1975), 7-22.

[4] M. Takahashi, Many valued logics of extended Gentzen style, Science Reports of the Tokyo Kyoiku Daigaku, Section A, Vol. 9, No. 237 (1967), 271-292.

Toshio NISHIMURA
Department of Mathematics
University of Tsukuba
Sakura-mura, Ibaraki
Japan

Takemasa OHYA
Department of Mathematics
Tokai University
Hiratsuka, Kanagawa
Japan